KOITER SHELL GOVERNED BY STRONGLY MONOTONE CONSTITUTIVE EQUATIONS

PIOTR KALITA*

* Jagiellonian University, Institute of Computer Science
ul. Nawojki 11, 30–072 Kraków, Poland
e-mail: kalita@softlab.ii.uj.edu.pl

In this paper we use the theory of monotone operators to generalize the linear shell model presented in (Blouza and Le Dret, 1999) to a class of physically nonlinear models. We present a family of nonlinear constitutive equations, for which we prove the existence and uniqueness of the solution of the presented nonlinear model, as well as the convergence of the Galerkin method. We also present the physical discussion of the model.

Keywords: Koiter shell, physical nonlinearity, strongly monotone operators

1. Introduction and Motivation

Koiter (1970) formulated a two-dimensional mathematical problem of the linearly elastic thin shell, in which the unknown is the field of the displacement of the shell middle surface. The proof of the existence and uniqueness of the solution of Koiter's model was first given by Bernardou and Ciarlet (1976). The most recent overview of shell theory can be found in a book by Ciarlet (2000). Blouza and Le Dret (1999) gave more elegant results with more relaxed assumptions than those of Bernardou and Ciarlet (they allow shells whose middle surface is parameterized by a function with discontinuous second derivative). The nonlinearity can be introduced to linear shell models in two ways. Firstly, one can consider nonlinear strain-displacement relationships. Such models are called geometrically nonlinear. They are widely discussed in (Ciarlet, 2000). Secondly, one can consider models physically nonlinear by using nonlinear stress-strain relationships (constitutive equations). This paper presents a generalization of the model presented by Blouza and Le Dret to shells governed by a family of nonlinear constitutive equations.

Physically nonlinear shells are used in technical models; however, for the justification of their use they need a rigorous mathematical statement. The linear shell model is, for instance, insufficient to express the behaviour of sophisticated biological materials like the tissue that constitutes the wall of an artery. We expect that the physically nonlinear shell presented here can model the control system that changes its elastic properties of the arterial wall with the changing rate of strain. The described nonlinear model has been used to model the wall of an artery in (Kalita, 2003).

The proof of the existence and uniqueness of the solution for the nonlinear shell problem is based on the theory of monotone operators presented in (Gajewski *et al.*, 1974) and (Ciarlet *et al.*, 1969). Moreover, the theory of monotone operators allows us to obtain the convergence of Galerkin approximations in finite-element spaces to the exact variational solution of the shell problem. We refer the reader to (Chapelle and Bathe, 1998) and (Kerdid and Mato Eiroa, 2000) for the finite-element approximation of the solution of the shell problem.

Monotone operators for the problems in thin domains were also considered in a different context in (Gaudiello *et al.*, 2002). The physical significance of monotonicity assumptions for constitutive equations in elasticity were verbosely discussed by Antman (1995).

In Sections 2 and 3 we recall some necessary facts from the theory of monotone operators and the theory of thin shells, respectively. The formulation of nonlinear shell problems together with the main results of this paper is given in Section 4. Section 5 delivers the physical discussion of the presented model and the numerical example comparing the behaviour of the shell in the linear and nonlinear cases. Proofs of some lemmas formulated in Section 5 are postponed to Appendix.

2. Strongly Monotone Operators

From now on we shall denote by V a reflexive real Banach space, by V^* the space of linear continuous functionals on V, and by $\langle\cdot,\cdot\rangle$ a duality pairing between V and V^*. By $\|\cdot\|$ we shall denote the norm in V and by $\|\cdot\|_*$ the norm in V^*. In the following definitions (see Definitions 1.1,1.2,1.3, Section III in (Gajewski *et al.*, 1974)) we will recall several properties of (not necessarily linear) mappings from V into V^*.

Definition 1. An operator $A : V \to V^*$ is *monotone* if

$$\forall\, y,z \in V: \quad \langle Ay - Az, y-z\rangle \geq 0. \tag{1}$$

Definition 2. An operator $A: V \to V^*$ is *strictly monotone* if

$$\forall\, y,z \in V: \quad y \neq z \Rightarrow \langle Ay - Az, y-z\rangle > 0. \tag{2}$$

Definition 3. An operator $A : V \to V^*$ is *strongly monotone* with a constant $\alpha \geq 0$ if

$$\forall\, y,z \in V: \quad \langle Ay - Az, y-z\rangle \geq \alpha\|y-z\|^2. \tag{3}$$

Of course, a mapping that is strongly monotone is strictly monotone. Furthermore, a mapping that is strictly monotone is monotone.

Definition 4. An operator $A : V \to V^*$ is *radially continuous* if for each $y,z \in V$ a mapping $s \to \langle A(y + sz), z\rangle$ is continuous on $[0,1]$.

Definition 5. $A : V \to V^*$ is *Lipschitz continuous on bounded sets* if

$$\forall\, r>0,\ \exists\, L>0,\ \forall\, y,z \in V:$$
$$\|y\| \leq r, \|z\| \leq r \Rightarrow \|Ay - Az\|_* \leq L\|y-z\|. \tag{4}$$

It is obvious that a mapping that is Lipschitz continuous on bounded sets is radially continuous.

Definition 6. An operator $A : V \to V^*$ is said to be *coercive* if there exists a function $\gamma : [0,\infty) \to \mathbb{R}$ such that

$$\lim_{t\to\infty} \gamma(t) = \infty, \tag{5}$$

$$\forall y \in V: \quad \langle Ay, y\rangle \geq \gamma(\|y\|)\|y\|. \tag{6}$$

A mapping that is strongly monotone is coercive (see Remark 1.4, Section III in (Gajewski *et al.*, 1974)).

Now let $A : V \to V^*$, and $f \in V^*$. By $\{V^k\}_{k=1}^{\infty}$ we denote a sequence of finite dimensional subspaces of V such that

$$\overline{\bigcup_{k=1}^{\infty} V^k} = V. \tag{7}$$

We consider the following problems:

(P) Find $u \in V$ *such that* $\langle Au, v\rangle = \langle f, v\rangle$ *for every* $v \in V$.

(P$_k$) Find $u^k \in V^k$ *such that* $\langle Au^k, v\rangle = \langle f, v\rangle$ *for every* $v \in V^k$.

We quote the following theorems (see Theorem 5.3.4 in (Ciarlet *et al.*, 1969) and Theorems 2.1,2.2,3.1,3.3, Section III in (Gajewski *et al.*, 1974)).

Theorem 1. *If* A *is radially continuous, monotone and coercive then the set of solutions of Problem (P) is nonempty, convex and weakly closed.*

Theorem 2. *If* A *is radially continuous, strictly monotone and coercive, then*

(i) Problem (P) has exactly one solution u*,*

(ii) Problem (P$_k$) has exactly one solution u^k*,*

(iii) the sequence u^k *converges to* u *weakly in* V.

Theorem 3. *If* A *is strongly monotone with a constant* $\alpha > 0$ *and Lipschitz continuous on bounded sets, then*

(i) Problem (P) has exactly one solution,

(ii) the mapping $A^{-1} : V^* \to V$ *is Lipschitz-continuous,*

(iii) Problem (P$_k$) has exactly one solution,

(iv) there exists a constant $K > 0$ *independent of the choice of* V^k *such that the following inequality is satisfied:*

$$\|u^k - u\| \leq K \inf\{\|v-u\|; v \in V^k\}.$$

The second part of the last theorem gives us the stability of Problem *(P)* with respect to the functional f. The last part (which is equivalent to the Cea lemma) gives us not only the convergence of the solutions u^k of the finite-dimensional problems *(P$_k$)* (which can be solved numerically) to the solution u of the infinite-dimensional problem *(P)*, but also the estimate of the error of the numerical method.

3. Linear Shell Problem

We will use the model of the linear elastic shell defined in (Blouza and Le Dret, 1999) that allows for a discontinuity of the curvature of the shell middle surface. From now on Greek indices and exponents will belong to the set $\{1,2\}$ while Latin indices and exponents will belong to $\{1,2,3\}$. We also use the summation convention. By $u\cdot v$ we denote the scalar product in $\mathbb{R}^3$, by $u\times v$ the vector product in $\mathbb{R}^3$ and by $|\cdot|$ the Euclidean norm in $\mathbb{R}^3$. Let ω denote an open, bounded, Lipschitz domain of $\mathbb{R}^2$ (such that the Sobolev Imbedding Theorem is satisfied). By φ we will denote an injective mapping which belongs to $W^{2,\infty}(\omega;\mathbb{R}^3)$ such that two vectors

$$a_\alpha(x) = \partial_\alpha \varphi(x)$$

are linearly independent at each $x \in \overline{\omega}$. Vectors $a_1(x)$ and $a_2(x)$ span the plane tangent to the middle surface of the shell. By

$$a_3(x) = \frac{a_1(x) \times a_2(x)}{|a_1(x) \times a_2(x)|}$$

we denote the normal versor on the midsurface at point x. Vectors $a_i(x)$ span the covariant basis at it. By $a^i(x)$ we denote the contravariant basis which is defined by the relations

$$a^i(x) \cdot a_j(x) = \delta^i_j,$$

where δ^i_j stands for the Kronecker symbol. Furthermore, we let

$$a(x) = |a_1(x) \times a_2(x)|^2,$$

so that $\sqrt{a}$ is the area element of the midsurface in the chart φ. Finally, by

$$\Gamma^\rho_{\alpha\beta} = a^\rho \cdot \partial_\beta a_\alpha$$

we denote the Christoffel symbols of the midsurface.

One can easily verify from the regularity of φ, ω and the linear independence of a_α that for each $x \in \overline{\omega}$ we have

$$a_i(x) \in W^{1,\infty}(\omega;\mathbb{R}^3), \tag{8}$$

$$a^i(x) \in W^{1,\infty}(\omega;\mathbb{R}^3), \tag{9}$$

$$a(x) \in W^{1,\infty}(\omega), \tag{10}$$

$$0 < C \leq a(x), \tag{11}$$

$$\Gamma^\rho_{\alpha\beta} \in L^\infty(\omega). \tag{12}$$

Now we define the space of admissible displacements for the shell problem

$$V = \{v \in H^1_0(\omega;\mathbb{R}^3) : \partial_{\alpha\beta} v \cdot a_3 \in L^2(\omega)\}. \tag{13}$$

The space V equipped with the norm

$$\|v\| = (\|v\|^2_{H^1(\omega;\mathbb{R}^3)} + \sum_{\alpha,\beta} \|\partial_{\alpha\beta} v \cdot a_3\|^2_{L^2(\omega)})^{\frac{1}{2}}$$

becomes a Hilbert space (Blouza and Le Dret, 1999).

Now we define the linearized strain tensor of the shell and the linearized change of curvature tensor of the shell. We assume the linear geometry of the shell, i.e. the displacement gradients are sufficiently small

$$\gamma_{\alpha\beta}(v) = \frac{1}{2}(\partial_\alpha v \cdot a_\beta + \partial_\beta v \cdot a_\alpha), \tag{14}$$

$$\Upsilon_{\alpha\beta}(v) = (\partial_{\alpha\beta} u - \Gamma^\rho_{\alpha\beta} \partial_\rho v) \cdot a_3. \tag{15}$$

One can easily see that $\gamma_{\alpha\beta}(v) \in L^2(\omega)$ and $\Upsilon_{\alpha\beta}(v) \in L^2(\omega)$ for $v \in V$.

Let us now formulate the linear shell problem. By $e(x) \in L^\infty(\omega)$ we denote the shell thickness such that

$$0 < C \leq e(x) \tag{16}$$

almost everywhere in ω with some constant C. By $a^{\alpha\beta\rho\sigma} \in L^\infty(\omega)$ we denote the constitutive tensor. We assume that it is symmetric ($a^{\alpha\beta\rho\sigma} = a^{\rho\sigma\alpha\beta}$) and coercive, i.e. there exists a positive constant C_1 such that for each symmetric tensor $\tau = (\tau_{\alpha\beta})$ and almost all $x \in \omega$ we have

$$a^{\alpha\beta\rho\sigma} \tau_{\alpha\beta} \tau_{\rho\sigma} \geq C_1(\tau_{\alpha\beta}\tau_{\alpha\beta}).$$

Finally, let $P \in L^2(\omega;\mathbb{R}^3)$ be an external load density. We define the bilinear form on $V \times V$ by

$$b(u,v) = \int_\omega \Big(e a^{\alpha\beta\rho\sigma}(\gamma_{\alpha\beta}(u)\gamma_{\rho\sigma}(v) + \frac{e^2}{12}\Upsilon_{\alpha\beta}(u)\Upsilon_{\rho\sigma}(u))\sqrt{a}\Big)\,\mathrm{d}x, \tag{17}$$

and a linear functional on V by

$$f(v) = \int_\omega P \cdot v \sqrt{a}\,\mathrm{d}x. \tag{18}$$

It can be easily seen that $f \in V^*$. The displacement of the shell is the solution to the following problem:

(LSP) Find $u \in V$ *such that* $b(u,v) = f(v)$ *for every* $v \in V$.

The proof of the existence and uniqueness of solutions to the above problem was given in (Blouza and Le Dret, 1999) and it is based on the following theorem:

Theorem 4. *Under the above hypotheses the expression*

$$|||v||| = \Big(\sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(v)\|^2_{L^2(\omega)} + \sum_{\alpha,\beta} \|\Upsilon_{\alpha\beta}(v)\|^2_{L^2(\omega)}\Big)^{\frac{1}{2}}$$

defines a norm on V *which is equivalent to* $\|\cdot\|$.

The above theorem implies the V-ellipticity of the form b defined by (17). The existence and uniqueness of solutions of Problem *(LSP)* follow from the Lax Milgram lemma (cf. e.g. (Gajewski *et al.*, 1974)). Moreover, by the Cea lemma we obtain the convergence of the solutions of appropriate finite-dimensional problems.

In the following part of the paper we will generalize the above result to a family of forms which are nonlinear with respect to their first argument.

4. Nonlinear Shell Problem

In this section we assume that V is defined by (13), $\gamma_{\alpha\beta}(v)$ by (14), $\Upsilon_{\alpha\beta}(v)$ by (15) and $a^{\alpha\beta\rho\sigma}$, e, P satisfy the assumptions of Section 3. By $\{V^k\}_{k=1}^{\infty}$ we denote the sequence of finite-dimensional subspaces of V that satisfy the condition (7). We shall introduce the following notation for the membrane energy density and flexural energy density:

$$|u|_\gamma = (a^{\alpha\beta\rho\sigma}\gamma_{\alpha\beta}(u)\gamma_{\rho\sigma}(u))^{\frac{1}{2}} \in L^2(\omega), \tag{19}$$

$$|u|_\Upsilon = (a^{\alpha\beta\rho\sigma}\Upsilon_{\alpha\beta}(u)\Upsilon_{\rho\sigma}(u))^{\frac{1}{2}} \in L^2(\omega). \tag{20}$$

We define the nonlinear constitutive operators according to the following formulae:

$$a_N^{\rho\sigma}\big(\gamma_{11}(u),\gamma_{12}(u),\gamma_{22}(u)\big) = \phi(\cdot,|u|_\gamma)a^{\alpha\beta\rho\sigma}\gamma_{\alpha\beta}(u), \tag{21}$$

$$A_N^{\rho\sigma}\big(\Upsilon_{11}(u),\Upsilon_{12}(u),\Upsilon_{22}(u)\big) = \psi(\cdot,|u|_\Upsilon)a^{\alpha\beta\rho\sigma}\Upsilon_{\alpha\beta}(u). \tag{22}$$

In the above formulae, $\phi : \omega\times[0,\infty)\to\mathbb{R}$ and $\psi : \omega\times[0,\infty)\to\mathbb{R}$ are functions which bring nonlinearity to our model. If $\phi\equiv 1$ and $\psi\equiv 1$, then the model simplifies to the linear one.

Further on, for simplicity, the operators in (21) and (22) are denoted by $a_N^{\rho\sigma}(\gamma_{\alpha\beta}(u))$ and $A_N^{\rho\sigma}(\Upsilon_{\alpha\beta}(u))$, respectively. Having defined the nonlinear operators, we introduce the following form on $V\times V$:

$$a_N(u,v) = \int_\omega \Big(e(a_N^{\rho\sigma}(\gamma_{\alpha\beta}(u))\gamma_{\rho\sigma}(v) + \frac{e^2}{12}A_N^{\rho\sigma}(\Upsilon_{\alpha\beta}(u))\Upsilon_{\rho\sigma}(u)\big)\sqrt{a}\Big)\,\mathrm{d}x. \tag{23}$$

The form a_N is linear with respect to the second variable, and, through the presence of functions ϕ and ψ, nonlinear with respect to the first variable.

Assuming that f is given by (18), we can formulate the nonlinear shell problem and the corresponding finite-dimensional problems:

(NLSP) Find $u\in V$ such that $a_N(u,v)=f(v)$ for every $v\in V$.

(NLSP$_k$) Find $u^k\in V^k$ such that $a_N(u^k,v)=f(v)$ for all $v\in V^k$.

The formulated problems are well defined due to the following two theorems:

Theorem 5. *If $\phi:\omega\times[0,\infty)\to\mathbb{R}$ and $\psi:\omega\times[0,\infty)\to\mathbb{R}$ satisfy the following assumptions:*

(i) $\phi(\cdot,t)$ and $\psi(\cdot,t)$ are Lebesgue measurable for all $t\in[0,\infty)$,

(ii) $\phi(x,\cdot)$ and $\psi(x,\cdot)$ are continuous for almost all $x\in\omega$,

(iii) there exist $M>0$ and $m>0$ such that for all $t\in[0,\infty)$ and almost all $x\in\omega$ we have

$$m\le\phi(x,t)\le M \quad \text{and} \quad m\le\psi(x,t)\le M, \tag{24}$$

(iv) for all $t\ge s\ge 0$ and almost all $x\in\omega$ we have

$$\phi(x,t)t-\phi(x,s)s\ge 0, \tag{25}$$

$$\psi(x,t)t-\psi(x,s)s\ge 0, \tag{26}$$

(v) for all $r>0$ there exists $l>0$ such that for any two real numbers $t\in[0,r]$ and $s\in[0,r]$ and almost all $x\in\omega$ we have

$$|\phi(x,t)t-\phi(x,s)s|\le l|t-s|, \tag{27}$$

$$|\psi(x,t)t-\psi(x,s)s|\le l|t-s|, \tag{28}$$

then the solution set of Problem (NLSP) *is nonempty, convex and weakly closed.*

Proof. In the course of the proof we will formulate several lemmas which will be proved in Appendix. First we show that the problem can be formulated in the dual space V^*. This is true due to the following result:

Lemma 1. *If the assumptions (i), (ii) and (iii) of Theorem 5 are satisfied, then for each $u,v\in V$ the form (23) is well defined. Furthermore, for a given $u\in V$ the mapping $v\to a_N(u,v)$ belongs to V^*.*

The above lemma implies that we can define the operator $A_N : V\to V^*$ such that

$$\langle A_N u, v\rangle = a_N(u,v).$$

It is now enough to show that the assumptions of Theorem 1 are satisfied. The following lemma gives the condition for the operator A_N to be coercive.

Lemma 2. *If there exists a real constant $m>0$ such that for every $t\ge 0$ and almost everywhere in ω we have*

$$\phi(x,t)\ge m, \tag{29}$$

$$\psi(x,t)\ge m, \tag{30}$$

then A_N is coercive.

The next lemma gives us the monotonicity of A_N.

Lemma 3. *If the assumption (iv) of Theorem 5 is satisfied, then A_N is monotone.*

The last property we need is the radial continuity. We prove the stronger property which is the Lipschitz continuity on bounded sets.

Lemma 4. *If the assumption (v) of Theorem 5 is satisfied, then A_N defined by (23) is Lipschitz continuous on bounded sets.*

We showed that all the assumptions of Theorem 1 are satisfied, which completes the proof. ■

Theorem 6. *If $\phi : \omega \times [0,\infty) \to \mathbb{R}$ and $\psi : \omega \times [0,\infty) \to \mathbb{R}$ satisfy the following assumptions:*

(i) $\phi(\cdot,t)$ and $\psi(\cdot,t)$ are Lebesgue measurable for all $t \in [0,\infty)$,

(ii) $\phi(x,\cdot)$ and $\psi(x,\cdot)$ are continuous for almost all $x \in \omega$,

(iii) there exists $M > 0$ such that for all $t \in [0,\infty)$ and almost all $x \in \omega$ we have

$$\phi(x,t) \le M \quad and \quad \psi(x,t) \le M, \tag{31}$$

(iv) there exists $m > 0$ such that for all $t \ge s \ge 0$ and almost all $x \in \omega$ we have

$$\phi(x,t)t - \phi(x,s)s \ge m(t-s), \tag{32}$$

$$\psi(x,t)t - \psi(x,s)s \ge m(t-s), \tag{33}$$

(v) for all $r > 0$ there exists $l > 0$ such that for any two real numbers $t \in [0,r]$ and $s \in [0,r]$ and almost all $x \in \omega$ we have

$$|\phi(x,t)t - \phi(x,s)s| \le l|t-s|, \tag{34}$$

$$|\psi(x,t)t - \psi(x,s)s| \le l|t-s|, \tag{35}$$

then

A. Problem (NLSP) *has exactly one solution u,*

B. Problem (NLSP) *is stable with respect to the shell load,*

C. Problem (NLSP$_k$) *has exactly one solution u^k,*

D. there exists a constant $K > 0$ independent of the choice of V^k such that the following inequality is satisfied:

$$\|u^k - u\| \le K \inf\{\|v-u\| : v \in V^k\}.$$

Proof. It is easy to see that the assumptions of Theorem 6 imply that the assumptions of Theorem 5 are also satisfied. We also notice that in the proof of Theorem 5 we showed that A_N is Lipschitz continuous on bounded sets. Due to Theorem 3, in order to obtain the thesis, it is therefore sufficient to show the strong monotonicity of A_N. This is true due to the following result:

Lemma 5. *If the assumption (iv) of Theorem 6 is satisfied, then A_N is strongly monotone.*

The proof of this lemma is postponed to Appendix.

Looking at the assumptions of Theorem 6 it is easy to see that the necessary condition for the functions ϕ and ψ to satisfy them is that there exist $m > 0$ and $M > 0$ such that for every positive $t \in [0,\infty)$

$$m \le \phi(x,t) \le M, \quad m \le \psi(x,t) \le M. \tag{36}$$

Now we give the sufficient condition for the functions ϕ and ψ to satisfy the assumptions of Theorem 6.

Corollary 1. *If $\phi : \omega \times [0,\infty) \to \mathbb{R}$ and $\psi : \omega \times [0,\infty) \to \mathbb{R}$ satisfy the following assumptions:*

(i) $\phi(\cdot,t)$ and $\psi(\cdot,t)$ are Lebesgue measurable for all $t \in [0,\infty)$,

(ii) $\phi(x,\cdot)$ and $\psi(x,\cdot)$ are $C^1[0,\infty)$ for almost all $x \in \omega$,

(iii) there exist $M > 0$ and $m > 0$ such that for all $t \in [0,\infty)$ and almost all $x \in \omega$ we have

$$m \le \phi(x,t) \le M \quad and \quad m \le \psi(x,t) \le M, \tag{37}$$

(iv) $\phi(x,\cdot)$ and $\psi(x,\cdot)$ are increasing,

then they also satisfy the assumptions of Theorem 6.

Proof. It is sufficient to prove the assumptions (iv) and (v) of Theorem 6. For the proof of the assumption (iv) let us take $0 \le s \le t$. We have

$$\begin{aligned}(t-s)m &\le \phi(x,s)(t-s) = \phi(x,s)t - \phi(x,s)s \\ &\le \phi(x,t)t - \phi(x,s)s.\end{aligned}$$

For the proof of the assumption (v) it suffices to notice that for $t \in [0,r]$ the first derivative of $\phi(x,t)$ (with respect to t) is bounded and therefore so is the first derivative of $\phi(x,t)t$. The mean value theorem completes the proof. The proof for ψ is analogous. ■

The last corollary allows us to give examples of functions that can be used in our constitutive equations. Such examples will be given in the next section.

In particular, the necessary condition (36) implies that the graphs of $\phi(x,t)t$ and $\psi(x,t)t$ should be included between two straight lines as depicted in Fig. 1.

5. Mechanical Aspects of the Presented Nonlinear Model

In the previous section we have suggested nonlinear three-dimensional stress-strain relationships ((21) and (22)) and we gave the conditions for the problem of finding the displacement of the shell governed by those relationships to

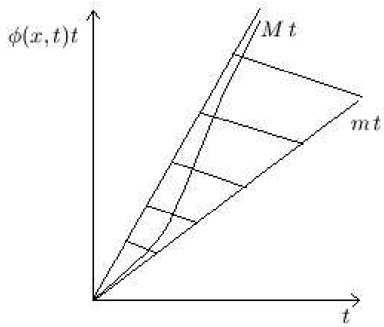


Fig. 1. Illustration to the necessary condition (36).

have only one solution which can be effectively approximated by the Galerkin method. Now we give some physical properties of the proposed equations:

(i) The equations satisfy the principle of determinism as the behaviour of each material point at time t is specified in terms of the behaviour of its arbitrarily small neighbourhood at the same time moment,

(ii) The tensorial form of expressions for $|\cdot|_\gamma$ and $|\cdot|_\Upsilon$ (see (19) and (20)) implies that they are invariant with respect to the rigid motion of spatial coordinates. Furthermore, the functions ϕ and ψ are also invariant with respect to the rigid motion of spatial coordinates and therefore the suggested equations satisfy the principle of material objectivity.

(iii) If we take ϕ and ψ independent of x, the equations can satisfy the principle of material isomorphism, e.g. they are invariant to a specific subgroup of the full orthogonal group of transformations of material coordinates (through the invariance of the tensor $a^{\alpha\beta\rho\sigma}$ with respect to this subgroup).

The fact that our constitutive equations satisfy the above principles implies that they are physically correct (Cemal Eringen, 1962; Noll and Truesdell, 1965).

Now we provide two examples of functions $\phi(x,t)$ that satisfy the assumptions of Corollary 1. The first is a piecewise polynomial (see Fig. 2 for the graph that explains the symbols used in the equation):

$$\phi(x,t)=\begin{cases} m & \text{for } t\in[0,t_0-\delta],\\ -\dfrac{(M-m)(t-t_0)^3}{4\delta^3}+\dfrac{3(M-m)(t-t_0)}{4\delta} & \\ \qquad +\dfrac{M+m}{2} & \text{for } t\in(t_0-\delta,t_0+\delta),\\ M & \text{for } t\in[t_0+\delta,\infty). \end{cases} \tag{38}$$

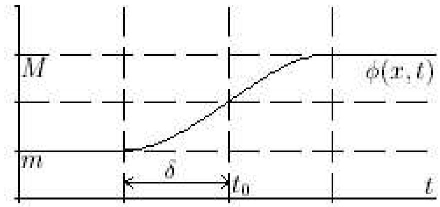


Fig. 2. Graph of a function of the type (38).

The second example concers a scaled and translated arcus tangent (see Fig. 3 for the graph that explains the symbols):

$$\phi(x,t)=\frac{M-m}{\pi}\arctan\frac{t-t_0}{\delta}+\frac{M+m}{2}. \tag{39}$$

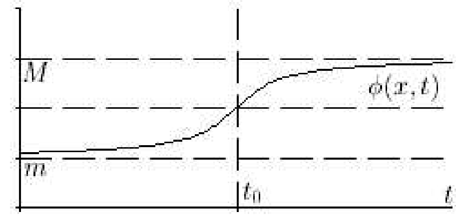


Fig. 3. Graph of a function of the type (39).

We remark here that any increasing $C^1[0,\infty)$ constitutive law of the type (21) and (22), e.g., such that the nonlinearity depends only on the energy density of the solution can be rendered to satisfy the assumption (37) by choosing an arbitrarily small $\epsilon>0$ and substituting $m\le\phi(x,t)$ and $m\le\psi(x,t)$ for $t\in[0,\epsilon)$ such that both functions remain C^1 with respect to t and, similarly, by choosing the large (nonphysical) energy density N and setting $\phi(x,t)=\text{const}=\phi(x,N)$ and $\psi(x,t)=const=\psi(x,N)$ for $t\ge N$. For a rigorous derivation of such a law see (Schaefer and Sędziwy, 1999).

The assumption (iv) of Corollary 1, e.g., the fact that ϕ and ψ are increasing functions of t means that the elastic modulus of the material increases with the increasing energy norm of strain. This means that with the growing strain the material strengthens itself. This is the case with the tissue constituting the walls of human (and mammalian) arteries. Nylon-like collagen fibres included in arteries cause a nonlinear passive response which can be interpreted using the presented formalism. For details of application of the presented model to the wall of human artery see (Kalita, 2003).

Now we present the benchmark for which we performed the finite-element simulation of the proposed equations. We used the setting suggested for shell benchmarks by Chapelle and Bathe (1998). The problem is

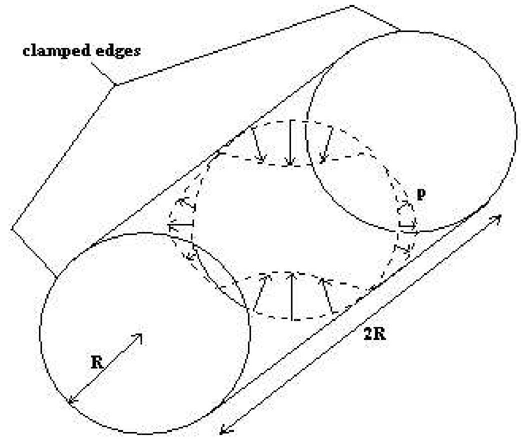
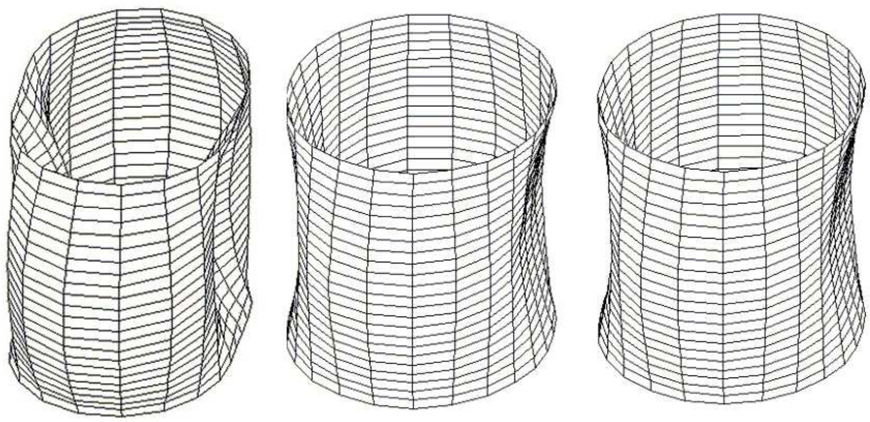
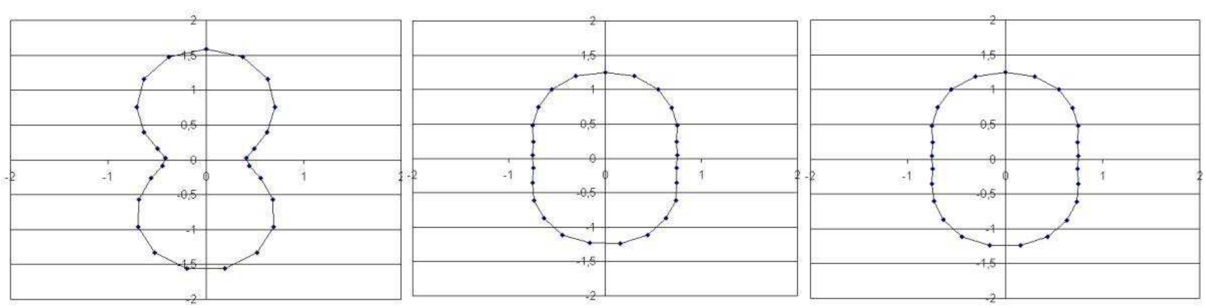


Fig. 4. Benchmark problem used as a numerical example.



(a) (b) (c)

Fig. 5. Displaced shells: linear case (a), nonlinearity of the type (38) (b), nonlinearity of the type (39) (c).



(a) (b) (c)

Fig. 6. Perpendicular cross sections of displaced shells: linear case (a), nonlinearity of the type (38) (b), nonlinearity of the type (39) (c).

shown in Fig. 4. The undeformed shell midsurface is a full cylinder with clamped edges (which corresponds to the homogenous Dirichlet boundary conditions enclosed in the definition of the space (13)). We note that for such geometry the shell problem is membrane dominated and therefore the influence of the bending term on the solution is neglectable. The cylinder is loaded by a periodic field of pressure $p = p_0 \cos(2\xi_2/R)$, where ξ_2 is the radial coordinate in the parameterization of the cylinder. The constants used were the following: $e = 0.1\,\mathrm{cm}$, $R = 1\,\mathrm{cm}$, $E = 500000\,\mathrm{Pa}$, $\nu = 0.4$, $p_0 = 10000\,\mathrm{Pa}$. Note that these constants correspond to the tissue of human arteries (Berne and Levy, 1983). The chosen type of elements were the simple P2 Lagrange triangles. The number of elements was 1250.

For the solution of the finite-dimensional nonlinear system resulting from nonlinear problems we used the quickly convergent Newton method (see (Kalita, 2003) for numeric details), with the Conjugate Gradient solver for the tangent linear system.

In the nonlinear simulations we used two different functions ϕ:

- the one described by the 'spline' formula (38) with the constants $\delta = 20$, $t_0 = 60$, $M = 3$, $m = 1$, and
- the one described by the 'arcus tangent' formula (39) with the constants $\delta = 1$, $t_0 = 60$, $M = 3$, $m = 1$.

In Fig. 5 we can see the deformed shell in the linear case, the nonlinear case with the 'arcus tangent' nonlinearity (middle) and 'spline' nonlinearity. In Fig. 6 we depict the middle cross section perpendicular to the axis of the shell.

Looking at the graphs, we can observe that for both nonlinear cases the results are similar, which means that in this specific case the nonlinear behaviour does not depend on the employed representation. Furthermore, we see that the nonlinearity made the material stronger, which inhibited the collapse of the shell under the same load — this fits the feature of the wall of an artery which is well protected against negative pressures which may occur, e.g., in the branching areas.

6. Conclusions

Here is a brief summary of the contributions provided by this paper:

- We restricted our study to a class of nonlinear constitutive formulae that are sufficiently flexible to model the behaviour of physically nonlinear shells composed of a wide range of materials.
- The presented model can be used for materials which strengthen themselves (e.g. their elastic modulus increases) with the increasing strain rate. As an example, we can give the tissue constituting the arterial wall in the circulatory system of mammals. A nonlinear passive behaviour of the arterial wall is due to the nylon-like fibres of collagen included in it.
- We gave a rigorous mathematical statement of the problem for the given class of materials. For that statement we proved the existence, uniqueness and stability of the solution, as well as the convergence of the numerical method. The given proofs verify the correctness of the models used in physics, engineeering and biomechanics, which can be included in the presented formalism.
- We showed that the constitutive equations of the presented type are physically correct, and presented an effective method of constructing them.
- We verified by simulation the strain strengthening the behaviour of the material due to the presented nonlinearity. We also showed that for the benchmark case the behaviour of the nonlinear material does not depend on the representation used as well as that the nonlinearity prevents the shell from collapsing. These results show that the proposed formulation may be useful for modelling arterial walls.

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Appendix: Proofs of the Lemmas

The following simple corollary gives some properties of $|\cdot|_\gamma$ and $|\cdot|_\Upsilon$.

Corollary 2. *For each $u, v \in V$ the following inequalities hold for almost all $x \in \omega$:*

$$a^{\alpha\beta\rho\sigma}\gamma_{\alpha\beta}(u)\gamma_{\rho\sigma}(v) \le |u|_\gamma |v|_\gamma, \tag{40}$$

$$a^{\alpha\beta\rho\sigma}\Upsilon_{\alpha\beta}(u)\Upsilon_{\rho\sigma}(v) \le |u|_\Upsilon |v|_\Upsilon, \tag{41}$$

$$|u+v|_\gamma \le |u|_\gamma + |v|_\gamma, \tag{42}$$

$$|u+v|_\Upsilon \le |u|_\Upsilon + |v|_\Upsilon. \tag{43}$$

Proof. Let us fix $x \in \omega$ such that $u(x)$, $v(x)$, $|u|_\gamma$, $|u|_\Upsilon$, $|v|_\gamma$ and $|v|_\Upsilon$ are defined. Such points constitute a set of full measure in ω. Write $\mathbb{R}^3 \ni \xi = (\xi_{11}, \xi_{12}, \xi_{22})$. We set $\xi_{21} = \xi_{12}$. Define $F(\xi, \zeta) = a^{\alpha\beta\rho\sigma}\xi_{\alpha\beta}\zeta_{\rho\sigma}$ which is a bilinear, symmetric, positive definite form on $\mathbb{R}^3$. Therefore if satisfies the Cauchy-Schwartz inequality $F(\xi,\zeta) \le \sqrt{F(\xi,\xi)}\sqrt{F(\zeta,\zeta)}$. Setting $\xi = (\gamma_{11}(u(x)), \gamma_{12}(u(x)), \gamma_{22}(u(x)))$ and $\zeta = (\gamma_{11}(v(x)), \gamma_{12}(v(x)), \gamma_{22}(v(x)))$, we get (40). Inequality (42) is a Minkowski inequality which follows directly from (40). For detailed proofs, see any textbook on functional analysis, e.g. (Rudin, 1973). The proofs of Inequalities (41) and (43) are analogous to those of (40) and (42). ■

Now we give proofs of Lemmas 1–4 and 5.

Lemma 6. *If*

(i) $\phi(\cdot, t)$ and $\psi(\cdot, t)$ are Lebesgue measurable for all $t \in [0, \infty)$,

(ii) $\phi(x, \cdot)$ and $\psi(x, \cdot)$ are continuous for almost all $x \in \omega$,

(iii) there exist $M > 0$ and $m > 0$ such that for all $t \in [0, \infty)$ and almost all $x \in \omega$ we have

$$m \le \phi(x,t) \le M \quad \text{and} \quad m \le \psi(x,t) \le M, \tag{44}$$

then for each $u, v \in V$ the form (23) is well defined. Furthermore, for a given $u \in V$ the mapping $v \to a_N(u, v)$ belongs to V^*.

Proof. For a given $u \in V$ the functions $\phi(x, |u|_\gamma)$ and $\psi(x, |u|_\Upsilon)$ are measurable and bounded almost everywhere. Therefore they belong to $L^\infty(\omega)$. From the formulae (21) and (22) one can see that $a_N^{\rho\sigma}(\gamma_{\alpha\beta}(u)) \in L^2(\omega)$ and $A_N^{\rho\sigma}(\Upsilon_{\alpha\beta}(u)) \in L^2(\omega)$. Therefore the form (23) is well defined. The Schwartz inequality for $L^2(\omega)$ and the application of the Theorem 4 complete the proof. ■

Lemma 7. *If there exists a real constant $m > 0$ such that for every $t \ge 0$ and almost everywhere in ω we have*

$$\phi(x,t) \ge m, \tag{45}$$

$$\psi(x,t) \ge m, \tag{46}$$

then A_N is coercive.

Proof. Let us fix $y \in V$. We have

$$\begin{aligned}\langle A_N y, y\rangle = \int_\omega e\sqrt{a}\Big[&\phi(x,|y|_\gamma)a^{\alpha\beta\rho\sigma}\gamma_{\alpha\beta}(y)\gamma_{\rho\sigma}(y) \\ &+ \frac{e^2}{12}\psi(x,|y|_\Upsilon)a^{\alpha\beta\rho\sigma}\Upsilon_{\alpha\beta}(y)\Upsilon_{\rho\sigma}(y)\Big]\,\mathrm{d}x \\ \ge \int_\omega e\sqrt{a}\Big[&m|y|_\gamma^2 + \frac{e^2}{12}m|y|_\Upsilon^2\Big]\,\mathrm{d}x \ge D|||y|||^2.\end{aligned}$$

The proof is thus complete. ■

Lemma 8. *If for every $t \ge s \ge 0$ and almost everywhere in ω we have*

$$\phi(x,t)t - \phi(x,s)s \ge 0, \tag{47}$$

$$\psi(x,t)t - \psi(x,s)s \ge 0, \tag{48}$$

then A_N is monotone.

Proof. (Cf. Lemma 1.6, Section III from (Gajewski *et al.*, 1974).) For every $y, z \in V$ we have the following inequalities:

$$\begin{aligned}
&\langle A_N y - A_N z, y - z\rangle \\
&= \int_\omega e\sqrt{a}\Big[\phi(x,|y|_\gamma)a^{\alpha\beta\rho\sigma}\gamma_{\alpha\beta}(y)(\gamma_{\rho\sigma}(y)-\gamma_{\rho\sigma}(z)) \\
&\quad + \frac{e^2}{12}\psi(x,|y|_\Upsilon)a^{\alpha\beta\rho\sigma}\Upsilon_{\alpha\beta}(y)(\Upsilon_{\rho\sigma}(y)-\Upsilon_{\rho\sigma}(z)) \\
&\quad - \phi(x,|z|_\gamma)a^{\alpha\beta\rho\sigma}\gamma_{\alpha\beta}(z)(\gamma_{\rho\sigma}(y)-\gamma_{\rho\sigma}(z)) \\
&\quad - \frac{e^2}{12}\psi(x,|z|_\Upsilon)a^{\alpha\beta\rho\sigma}\Upsilon_{\alpha\beta}(z)(\Upsilon_{\rho\sigma}(y)-\Upsilon_{\rho\sigma}(z))\Big]\,\mathrm{d}x \\
&\geq \int_\omega e\sqrt{a}\Big[\phi(x,|y|_\gamma)(|y|\gamma^2-|y|_\gamma|z|_\gamma) \\
&\quad + \frac{e^2}{12}\psi(x,|y|_\Upsilon)(|y|^2_\Upsilon-|y|_\Upsilon|z|_\Upsilon) \\
&\quad - \phi(x,|z|_\gamma)(|y|_\gamma|z|_\gamma-|z|\gamma^2) \\
&\quad - \frac{e^2}{12}\psi(x,|z|_\Upsilon)(|y|_\Upsilon|z|_\Upsilon-|z|^2_\Upsilon)\Big]\,\mathrm{d}x \\
&= \int_\omega e\sqrt{a}\Big[(\phi(x,|y|_\gamma)|y|_\gamma-\phi(x,|z|_\gamma)|z|_\gamma) \\
&\quad \times(|y|\gamma-|z|_\gamma) + \frac{e^2}{12}(\psi(x,|y|_\Upsilon)|y|_\Upsilon \\
&\quad - \psi(x,|z|_\Upsilon)|z|_\Upsilon)(|y|_\Upsilon-|z|_\Upsilon)\Big]\,\mathrm{d}x \geq 0.
\end{aligned}$$

During the derivation, we applied Corollary 2. ∎

Lemma 9. *If for every positive real constant r there exists a positive real constant l such that for any two real numbers t and s belonging to the interval $[0,r]$ and almost everywhere in ω we have*

$$|\phi(x,t)t - \phi(x,s)s| \leq l|t-s|, \tag{49}$$

$$|\psi(x,t)t - \psi(x,s)s| \leq l|t-s|, \tag{50}$$

then A_N defined by (23) is Lipschitz continuous on bounded sets.

Proof. (Cf. Lemma 1.9, Section III from (Gajewski *et al.*, 1974)) Let us fix $r > 0$. Then there exists l such that Inequalities (49) and (50) are satisfied. Let us first take $s = 0$ and $t \in [0,r]$. We have

$$|\phi(x,t)||t| \leq l|t|,$$

and

$$|\psi(x,t)||t| \leq l|t|.$$

Hence for $t \neq 0$ we have

$$|\phi(x,t)| \leq l, \tag{51}$$

and

$$|\psi(x,t)| \leq l. \tag{52}$$

The continuity of $\phi(x,\cdot)$ and $\psi(x,\cdot)$ implies that the above bounds are valid for $t = 0$ too. Let us further fix $y, z, v \in V$ such that $\|y\| \leq r$ and $\|z\| \leq r$. We have the following estimations:

$$\begin{aligned}
&\langle A_N y - A_N z, v\rangle \\
&= \int_\omega e\sqrt{a}\Big[\phi(x,|y|_\gamma)a^{\alpha\beta\rho\sigma}\gamma_{\alpha\beta}(y)\gamma_{\rho\sigma}(v) \\
&\quad - \phi(x,|z|_\gamma)a^{\alpha\beta\rho\sigma}\gamma_{\alpha\beta}(z)\gamma_{\rho\sigma}(v) \\
&\quad + \frac{e^2}{12}\big(\psi(x,|y|_\Upsilon)a^{\alpha\beta\rho\sigma}\Upsilon_{\alpha\beta}(y)\Upsilon_{\rho\sigma}(v) \\
&\quad - \psi(x,|z|_\Upsilon)a^{\alpha\beta\rho\sigma}\Upsilon_{\alpha\beta}(z)\Upsilon_{\rho\sigma}(v)\big)\Big]\,\mathrm{d}x \\
&= \int_\omega e\sqrt{a}\Big[\phi(x,|y|_\gamma)a^{\alpha\beta\rho\sigma}(\gamma_{\alpha\beta}(y)-\gamma_{\alpha\beta}(z))\gamma_{\rho\sigma}(v) \\
&\quad + (\phi(x,|y|_\gamma)-\phi(x,|z|_\gamma))a^{\alpha\beta\rho\sigma}\gamma_{\alpha\beta}(z)\gamma_{\rho\sigma}(v) \\
&\quad + \frac{e^2}{12}\big(\psi(x,|y|_\Upsilon)a^{\alpha\beta\rho\sigma}(\Upsilon_{\alpha\beta}(y)-\Upsilon_{\alpha\beta}(z))\Upsilon_{\rho\sigma}(v) \\
&\quad + (\psi(x,|y|_\Upsilon)-\psi(x,|z|_\Upsilon))a^{\alpha\beta\rho\sigma}\Upsilon_{\alpha\beta}(z)\Upsilon_{\rho\sigma}(v)\big)\Big]\,\mathrm{d}x.
\end{aligned}$$

We estimate the last integral using the linearity of γ and Υ and Corollary 2:

$$\begin{aligned}
&\langle A_N y - A_N z, v\rangle \leq \int_\omega e\sqrt{a}\Big[\phi(x,|y|_\gamma)|y-z|_\gamma|v|_\gamma \\
&\quad + (\phi(x,|y|_\gamma)-\phi(x,|z|_\gamma))|z|_\gamma|v|_\gamma \\
&\quad + \frac{e^2}{12}\big(\psi(x,|y|_\Upsilon)|y-z|_\Upsilon|v|_\Upsilon \\
&\quad + (\psi(x,|y|_\Upsilon)-\psi(x,|z|_\Upsilon))|z|_\Upsilon|v|_\Upsilon\big)\Big]\,\mathrm{d}x \\
&= \int_\omega e\sqrt{a}\Big[\big(\phi(x,|y|_\gamma)|y-z|_\gamma+\phi(x,|y|_\gamma)|y|_\gamma \\
&\quad - \phi(x,|y|_\gamma)|y|_\gamma+\phi(x,|y|_\gamma)|z|_\gamma-\phi(x,|z|_\gamma)|z|_\gamma\big)|v|_\gamma \\
&\quad + \frac{e^2}{12}\big(\psi(x,|y|_\Upsilon)|y-z|_\Upsilon+\psi(x,|y|_\Upsilon)|y|_\Upsilon \\
&\quad - \psi(x,|y|_\Upsilon)|y|_\Upsilon+\psi(x,|y|_\Upsilon)|z|_\Upsilon \\
&\quad - \psi(x,|z|_\Upsilon)|z|_\Upsilon\big)|v|_\Upsilon\Big]\,\mathrm{d}x.
\end{aligned}$$

For further estimations we use the bounds (51) and (52), the triangle inequality and the assumptions (49) and (50):

$$\begin{aligned}
&\langle A_N y - A_N z, v\rangle \\
&\le \int_\omega e\sqrt{a}\Big[\big(\phi(x,|y|_\gamma)|y-z|_\gamma + \phi(x,|y|_\gamma)\||z|_\gamma - |y|_\gamma| \\
&\quad + |\phi(x,|y|_\gamma)|y|_\gamma - \phi(x,|z|_\gamma)|z|_\gamma|\big)|v|_\gamma \\
&\quad + \frac{e^2}{12}\big(\psi(x,|y|_\Upsilon)|y-z|_\Upsilon + \psi(x,|y|_\Upsilon)\||z|_\Upsilon - |y|_\Upsilon| \\
&\quad + |\psi(x,|y|_\Upsilon)|y|_\Upsilon - \psi(x,|z|_\Upsilon)|z|_\Upsilon|\big)|v|_\Upsilon\Big]\,\mathrm{d}x \\
&\le \int_\omega e\sqrt{a}\Big[\big(2\phi(x,|y|_\gamma)|y-z|_\gamma + l\||y|_\gamma - |z|_\gamma|\big)|v|_\gamma \\
&\quad + \frac{e^2}{12}\big(2\psi(x,|y|_\Upsilon)|y-z|_\Upsilon + l\||y|_\Upsilon - |z|_\Upsilon|\big)|v|_\Upsilon\Big]\,\mathrm{d}x \\
&\le 3l\int_\omega e\sqrt{a}\Big[|y-z|_\gamma|v|_\gamma + \frac{e^2}{12}|y-z|_\Upsilon|v|_\Upsilon\Big]\,\mathrm{d}x.
\end{aligned}$$

Now we use the fact that for $v \in V$ we have $|v|_\gamma \in L^2(\omega)$, $|v|_\Upsilon \in L^2(\omega)$, $e(x) \in L^\infty(\omega)$ and $a(x) \in L^\infty(\omega)$. We have

$$\begin{aligned}
&\langle A_N y - A_N z, v\rangle \\
&\qquad \le 3l\Big(\int_\omega e\sqrt{a}[|y-z|_\gamma^2 + \frac{e^2}{12}|y-z|_\Upsilon^2]\,\mathrm{d}x\Big)^{\frac{1}{2}} \\
&\qquad\quad \times \Big(\int_\omega e\sqrt{a}[|v|_\gamma^2 + \frac{e^2}{12}|v|_\Upsilon^2]\,\mathrm{d}x\Big)^{\frac{1}{2}} \\
&\qquad \le 3lD|||y-z|||\,|||v|||,
\end{aligned}$$

where D is a positive constant independent of r.

Hence, using Theorem 4, we obtain

$$\|A_N y - A_N z\|_* \le l\overline{D}\|y - z\|,$$

where $\overline{D}$ is a positive constant independent of r. The proof is thus complete. ■

Lemma 10. *If there exists a real constant $m > 0$ such that for every $t \ge s \ge 0$ and almost everywhere in ω we have*

$$\phi(x,t)t - \phi(x,s)s \ge m(t-s), \tag{53}$$

$$\psi(x,t)t - \psi(x,s)s \ge m(t-s), \tag{54}$$

then A_N is strongly monotone.

Proof. (Cf. Lemma 1.9, Section III from (Gajewski *et al.*, 1974).) First we remark that the assumed inequalities are stronger then the assumptions (47) and (48) of Lemma 3. Let

$$\phi = \phi_1 + \phi_2, \quad \phi_1(x,t) = m,$$
$$\psi = \psi_1 + \psi_2, \quad \psi_1(x,t) = m.$$

Note that ϕ_2 is non-negative as substituting $s = 0$ in (53) we have $\phi(x,t) \ge m > 0$ for $t \ne 0$. From the continuity of ϕ this is also valid for $t = 0$. Further, as ϕ is bounded, both ϕ_1 and ϕ_2 are bounded. Moreover, we have

$$\begin{aligned}
\phi_2(x,t)t - \phi_2(x,s)s &= (\phi(x,t)-m)t - (\phi(x,s)-m)s \\
&= \phi(x,t)t - \phi(x,s)s + m(s-t) \\
&\ge m(t-s) + m(s-t) = 0.
\end{aligned}$$

An analogous estimate is satisfied for ψ_2. Therefore, if we define the nonlinear operators

$$\overline{a}_N^{\rho\sigma}(\gamma_{\alpha\beta}(u)) = \phi_2(x,|u|_\gamma)a^{\alpha\beta\rho\sigma}\gamma_{\alpha\beta}(u),$$
$$\overline{A}_N^{\rho\sigma}(\Upsilon_{\alpha\beta}(u)) = \phi_2(x,|u|_\Upsilon)a^{\alpha\beta\rho\sigma}\Upsilon_{\alpha\beta}(u),$$

then, by Lemma 3, the corresponding mapping $\overline{A}_N$ given by

$$\begin{aligned}
\langle \overline{A}_N u, v\rangle = &\int_\omega e(\overline{a}_N^{\rho\sigma}(\gamma_{\alpha\beta}(u))\gamma_{\rho\sigma}(v) \\
&+ \frac{e^2}{12}\overline{A}_N^{\rho\sigma}(\Upsilon_{\alpha\beta}(u))\Upsilon_{\rho\sigma}(u))\sqrt{a}\,\mathrm{d}x
\end{aligned}$$

is monotone. Now we will decompose the nonlinear operator $\overline{a}_N$ into the sum of the linear operator and a strongly monotone one:

$$\begin{aligned}
&\langle A_N y - A_N z, y - z\rangle \\
&= \int_\omega e\sqrt{a}\Big[(m + phi_2(x,|y|_\gamma))a^{\alpha\beta\rho\sigma} \\
&\quad \times \gamma_{\alpha\beta}(y)(\gamma_{\rho\sigma}(y) - \gamma_{\rho\sigma}(z)) + \frac{e^2}{12}(m + \psi_2(x,|y|_\Upsilon)) \\
&\quad \times a^{\alpha\beta\rho\sigma}\Upsilon_{\alpha\beta}(y)(\Upsilon_{\rho\sigma}(y) - \Upsilon_{\rho\sigma}(z)) \\
&\quad - (m + \phi_2(x,|z|_\gamma))a^{\alpha\beta\rho\sigma}\gamma_{\alpha\beta}(z)(\gamma_{\rho\sigma}(y) - \gamma_{\rho\sigma}(z)) \\
&\quad - \frac{e^2}{12}(m + \psi_2(x,|z|_\Upsilon))a^{\alpha\beta\rho\sigma} \\
&\quad \times \Upsilon_{\alpha\beta}(z)(\Upsilon_{\rho\sigma}(y) - \Upsilon_{\rho\sigma}(z))\Big]\,\mathrm{d}x \\
&= \langle \overline{A}_N y - \overline{A}_N z, y - z\rangle + mb(y - z, y - z).
\end{aligned}$$

In the last equation b is the bilinear form defined by (17). As it is V-elliptic (cf. Theorem 4) and the mapping $\overline{A}_N$ is monotone, we have

$$\langle A_N y - A_N z, y - z\rangle \ge \alpha\|y - z\|^2,$$

where α equals mK and K is the coercivity constant of the form b. ■

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