

## OPTIMAL ERASURES IN DECISION–FEEDBACK EQUALIZATION FOR THE GAUSSIAN NOISE

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A new method of optimizing decision feedback parameters for intersymbol interference equalizers is described. The coefficient existing in the decision feedback loop depends on risk qualification of the received decision. We prove that bit error probability can be decreased with this method for any channel with a single interference sample and small Gaussian noise. Experimental results are presented for selected channels. The dependences of optimal feedback parameters on channel interference samples and noise power are presented, too.

**Keywords:** data communication, decision feedback equalizer, intersymbol interference, error rate minimization

### 1. Introduction

The intersymbol interference equalization problem with the use of decision feedback has been described in many papers. It is a decision problem concerned with receiving incoming data. This is done by taking the recognized binary data to the tapped-delay line of the transversal filter and by using these data for incoming interference erasure, as shown in Fig. 1. If the discrete transfer function of the channel is

$$Y(z) = y_{-h} + y_{1-h}z^{-1} + \dots + y_{-1}z^{-(h-1)} \\ + y_0z^{-h} + y_1z^{-(h+1)} + \dots + y_gz^{-(h+g)},$$

then the sample  $v_k$  of the equalizer input at the time  $t_k$  is

$$v_k = a_{k+h}y_{-h} + \dots + a_{k+1}y_{-1} + a_k y_0 + a_{k-1}y_1 \\ + \dots + a_{k-g}y_g + z_k,$$

where  $a_{k+h}, \dots, a_{k-g}$  are amplitudes of pulses representing the data being transmitted and  $z_k$  is the noise sample. Using  $v_k$ , we try to recover  $a_k$ . The part of  $v_k$  which depends on data, except for those represented by  $a_k$ , is called the *intersymbol interference* and has to be minimized. This problem is described in the books (Benedetto *et al.*, 1987; Clark, 1976; Dąbrowski, 1982; Dąbrowski and Dymarski, 2004).

To use the decision-feedback equalizer, the channel must be preequalized so that  $y_{-h}, y_{1-h}, \dots, y_{-1}$  is close to zero, and hence we can write  $v_k = a_k y_0 +$

$a_{k-1}y_1 + a_{k-2}y_2 + \dots + a_{k-g}y_g + z_k$ . Data represented by  $a_{k-1}, a_{k-2}, \dots, a_{k-g}$  were received before  $a_k$ , so at the time  $t_k$  they are recognized as data represented by  $d_{k-1}, d_{k-2}, \dots, d_{k-g}$ . Since the data symbol is recognized, its interference with the incoming next datum becomes known and can be subtracted from the incoming signal. The sample  $e_k$  at the decision unit input is

$$e_k = a_k y_0 + (a_{k-1} - d_{k-1})y_1 + (a_{k-2} - d_{k-2})y_2 \\ + \dots + a_{k+g} + (a_{k-g} - d_{k-g})y_g + z_k.$$

The decision feedback equalizer works properly if the decisions in its delay line are correct, i.e., when  $a_{k-1} = d_{k-1}, a_{k-2} = d_{k-2}, \dots, a_{k-g} = d_{k-g}$ , which means  $e_k = a_k y_0 + z_k$ . In the case of a received data error, the wrong data are taken to the feedback and the calculated incoming interference differs from the real one. For binary data, the existing interference is rather duplicated instead of being cancelled, so it increases the error probability for  $a_k$ . In this case no cancellation or partial cancellation are proposed, by subtracting a smaller value than the one calculated from the signal.

Partial cancellation is proposed only when the decision is risky, i.e., when the probability of the error is high. We qualify the decision to be risky if the signal value  $e_k$  at the decision unit input is close to the decision level  $S_*$ , i.e., if  $|e_k - S_*| < \beta$ . If the probabilities of sending 0 and 1 are the same and the signal mean value is equal to zero, the optimal decision level is  $S_* = 0$ .

We assume that the transmission path is composed of a partially equalized channel and a decision feedback

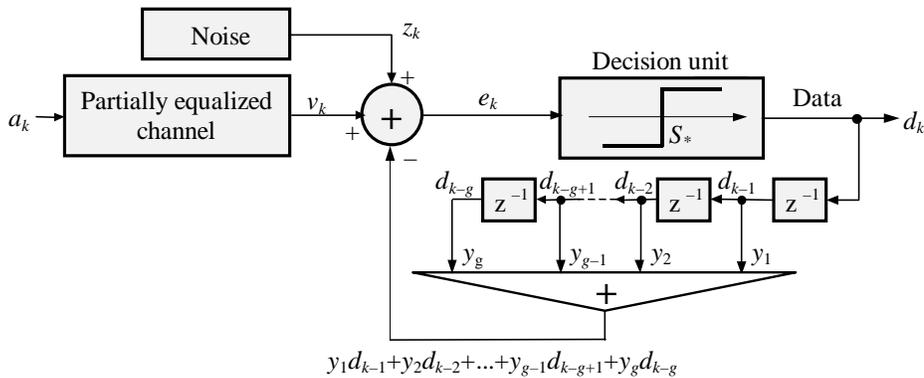


Fig. 1. Decision feedback and the equalizer.

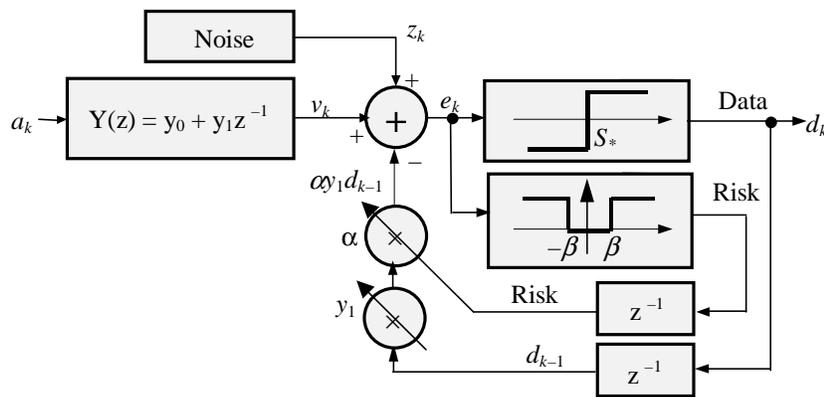


Fig. 2. Assumed channel and the equalizer.

equalizer, as shown in Fig. 2. The discrete transfer function of the channel is  $Y(z) = y_0 + y_1 z^{-1}$  with the parameters  $|y_0| > |y_1|$ .

Decision feedback equalization with decision risk analysis is described in (Bergmans *et al.*, 1997; Chiani, 1997; Hacıoglu and Amca, 1999). Similar methods, which do not include in equalization interference from the received signal when the received data were riskily detected, were proposed by Chiani (1997) and Hacıoglu (1999). These methods do not subtract the calculated interference from the existing one if risky data have been used in calculations. This reduces the error extension phenomena described in (Altekar and Beaulieu, 1993; Choy and Beaulieu, 1997; Hacıoglu and Amca, 1999; Labat and Laot, 2001). Chiani (1997) examined the dependence of the bit error rate (BER) on the risk threshold  $\beta$ , searching for the best values of  $\beta$  for chosen channels to obtain the minimum BER.

If the decision  $d_k$  is risky, the algorithm presented

in this paper multiplies interference calculated from  $d_k$  by the factor of  $\alpha$  and then it subtracts the result from the signal. In other words, if the value  $e_k$  at the decision unit input differs from the decision level  $S_* = 0$  by less than  $\beta$ , i.e.,  $|e_k| < \beta$ , the incoming interference calculated from  $d_k$  is multiplied by  $\alpha$  (where  $0 < \alpha < 1$ ) and the result is subtracted from the signal. In the case of an error ( $d_k \neq a_k$ ), interference at the decision unit input is multiplied by  $1 + \alpha$ , which is less than 2. The decision feedback contains an extra delay line which remembers risk qualifications of the decisions existing in a normal delay line of the feedback.

In this paper we will prove that it is possible to decrease the BER by properly choosing the parameters  $\alpha$  and  $\beta$ . The optimal value of the risk level  $\beta$  as an analytic function of the factor  $\alpha$  is given for channels with a single interference sample. The dependences of optimal  $\alpha$  and  $\beta$  on the parameters  $y_0$ ,  $y_1$  and on the noise power  $\sigma^2$  are presented graphically.

## 2. Calculation of the Risk Level

Assume that the noise probability density function is even and the probabilities of binary data values in the transmitting sequences are equal to 0.5. Let the bit 0 be represented by a negative pulse ( $a_k = -1$ ) and the bit 1 be represented by a positive pulse ( $a_k = 1$ ).

The first step is to find the dependence of bit error probability on the parameters  $\alpha$  and  $\beta$  for the system shown in Fig. 2 with the discrete transfer function given by

$$Y(z) = y_0 + y_1 z^{-1}. \quad (1)$$

The sample of the channel response to the input signal at the time  $t = kT$  is equal to  $v_k = a_k y_0 + a_{k-1} y_1 + z_k$ .

If the tap gain of the digital delay line is  $y_1$ , the value  $e_k$  at the decision module input is (Grzybowski and Kisilewicz, 1998):

$$e_k = a_k y_0 + (a_{k-1} - d_{k-1}) y_1 + z_k \quad (2)$$

for  $\alpha = 1$ , or

$$e_k = a_k y_0 + (a_{k-1} - \alpha d_{k-1}) y_1 + z_k \quad (3)$$

for  $0 < \alpha < 1$ .

In the case of  $\alpha = 1$  and when the decisions (received data) are correct, ( $d_{k-1} = a_{k-1}$ ), (2) gives

$$e_k = a_k y_0 + z_k = \begin{cases} y_0 + z_k & \text{if } a_k = 1, \\ -y_0 + z_k & \text{if } a_k = -1. \end{cases} \quad (4)$$

The sign of  $e_k$  determines the decision  $d_k$ . If  $e_k > 0$ , then  $d_k = 1$ , otherwise  $d_k = -1$ . If  $|e_k| < \beta$ , i.e., if  $e_k$  is close to zero, the decision  $d_k$  is assumed to be risky and in the next time period the calculated interference is compensated carefully, using (3) with a coefficient  $0 < \alpha < 1$ .

Let  $f_z(z)$  be the probability density function of the noise  $z$ . We make the following assumptions:

- $f_z(z)$  is even, i.e.,  $f_z(-z) = f_z(z)$ ,
- the probabilities of  $a_k = 1$  and  $a_k = -1$  are equal.

Therefore, we will obtain the probabilities of the error in the decision  $d_k$  ( $d_k \neq a_k$ ) in case the decision  $d_{k-1}$  was assumed risky ( $|e_{k-1}| < \beta$ ) or not ( $|e_{k-1}| > \beta$ ).

In further investigations, we assume that  $y_0$  and  $y_1$  are positive. This assumption does not limit the generality of our deliberations. For negative  $y_0$  and  $y_1$ , the appropriate signs preceding the variables  $y_0$  and  $y_1$  in the following equations will reverse all other signs, giving the same final result.

If the decision  $d_{k-1}$  was not risky ( $|e_{k-1}| > \beta$ ), then the probability of a wrong decision  $d_k$  is

$$p_{e1} = \int_{-\infty}^{-y_0} f_z(x) dx \quad (5)$$

if  $d_{k-1} = a_{k-1}$ , and

$$p_{e2} = \frac{1}{2} \int_{-\infty}^{-y_0 - 2y_1} f_z(x) dx + \frac{1}{2} \int_{-\infty}^{-y_0 + 2y_1} f_z(x) dx \quad (6)$$

if  $d_{k-1} \neq a_{k-1}$ .

If the correct decision  $d_{k-1} = a_{k-1}$  was risky ( $|e_{k-1}| < \beta$ ), then (3) gives

$$e_k = \begin{cases} y_0 \pm (1 - \alpha) y_1 + z_k & \text{if } a_k = 1, \\ -y_0 \pm (1 - \alpha) y_1 + z_k & \text{if } a_k = -1, \end{cases} \quad (7)$$

and the probability of the binary error is

$$p_{e3} = \frac{1}{2} \int_{-\infty}^{-y_0 - (1 - \alpha) y_1} f_z(x) dx + \frac{1}{2} \int_{-\infty}^{-y_0 + (1 - \alpha) y_1} f_z(x) dx. \quad (8)$$

If the decision  $d_{k-1}$  was false ( $d_{k-1} \neq a_{k-1}$ ) and risky ( $|e_{k-1}| < \beta$ ), then (3) gives

$$e_k = \begin{cases} y_0 \mp (1 + \alpha) y_1 + z_k & \text{if } a_k = 1, \\ -y_0 \mp (1 + \alpha) y_1 + z_k & \text{if } a_k = -1, \end{cases} \quad (9)$$

and the probability of the binary error is

$$p_{e4} = \frac{1}{2} \int_{-\infty}^{-y_0 - (1 + \alpha) y_1} f_z(x) dx + \frac{1}{2} \int_{-\infty}^{-y_0 + (1 + \alpha) y_1} f_z(x) dx. \quad (10)$$

Suppose that the correct decision  $d_{k-2}$  was not risky. Now we calculate the following probabilities:

- (a) the probability of the correct and non-risky decision  $d_{k-1}$ :

$$q_1 = \int_{-y_0 + \beta}^{\infty} f_z(x) dx = \int_{-\infty}^{y_0 - \beta} f_z(x) dx, \quad (11)$$

- (b) the probability of the false and non-risky decision  $d_{k-1}$ :

$$q_2 = \int_{-\infty}^{-y_0 - \beta} f_z(x) dx, \quad (12)$$

(c) the probability of the correct and risky decision  $d_{k-1}$ :

$$q_3 = \int_{-y_0}^{-y_0+\beta} f_z(x) dx, \quad (13)$$

(d) the probability of the false and risky decision  $d_{k-1}$ :

$$q_4 = \int_{-y_0-\beta}^{-y_0} f_z(x) dx. \quad (14)$$

From (5), (6), (8), (10) and (11)–(14), we obtain the probability of getting a false decision  $d_k$  in the case of a correct and non-risky decision  $d_{k-2}$ :

$$P_e = q_1 p_{e1} + q_2 p_{e2} + q_3 p_{e3} + q_4 p_{e4}. \quad (15)$$

For the assumed value of the parameter  $\alpha$ , the optimal  $\beta$  satisfies

$$\frac{dP_e}{d\beta} = 0. \quad (16)$$

Since  $p_{e1}$ ,  $p_{e2}$ ,  $p_{e3}$  and  $p_{e4}$  are not dependent on  $\beta$ , and the parameter  $\beta$  appears in (11)–(14) only in upper or lower limits of integrals, we have

$$\frac{dP_e}{d\beta} = (p_{e3} - p_{e1})f_z(-y_0 + \beta) + (p_{e4} - p_{e2})f_z(-y_0 - \beta), \quad (17)$$

when

$$p_{e3} - p_{e1} = \frac{1}{2} \left[ \int_{-y_0}^{-y_0+(1+\alpha)y_1} f_z(x) dx - \int_{-y_0-(1-\alpha)y_1}^{-y_0} f_z(x) dx \right],$$

$$p_{e4} - p_{e2} = \frac{1}{2} \left[ \int_{-y_0-2y_1}^{-y_0-(1+\alpha)y_1} f_z(x) dx - \int_{-y_0+(1+\alpha)y_1}^{-y_0+2y_1} f_z(x) dx \right].$$

Substituting the above expressions into (16), we obtain

$$\begin{aligned} & \frac{f_z(-y_0 + \beta)}{f_z(-y_0 - \beta)} \\ &= \frac{\int_{-y_0+(1+\alpha)y_1}^{-y_0+2y_1} f_z(x) dx - \int_{-y_0-(1-\alpha)y_1}^{-y_0-(1+\alpha)y_1} f_z(x) dx}{\int_{-y_0+(1-\alpha)y_1}^{-y_0+(1+\alpha)y_1} f_z(x) dx - \int_{-y_0-(1-\alpha)y_1}^{-y_0} f_z(x) dx} \\ &= C(\alpha). \end{aligned} \quad (18)$$

Assuming the white Gaussian noise  $N(0, \sigma)$ , i.e.,

$$f_z(x) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{x^2}{2\sigma^2}},$$

we obtain

$$\frac{f_z(-y_0 + \beta)}{f_z(-y_0 - \beta)} = e^{\frac{(\beta+y_0)^2 - (\beta-y_0)^2}{2\sigma^2}} = e^{\frac{2\beta y_0}{\sigma^2}}$$

and, finally, the optimal  $\beta$  is given by

$$\beta = \frac{\sigma^2}{2y_0} \ln C(\alpha). \quad (19)$$

**Lemma 1.** *If  $0 < y_1 < y_0$  and  $0 < \alpha < 1$ , then  $\beta$  given by (19) minimizes the error probability  $P_e$ .*

*Proof.* The minimization of  $P_e$  is implied by the positive-ness of the second derivative  $d^2 P_e / d\beta^2$ . Following (16) and (17) for the optimal  $\beta$ , i.e., for

$$(p_{e2} - p_{e4})f_z(-y_0 - \beta) = (p_{e3} - p_{e1})f_z(-y_0 + \beta),$$

we get

$$\frac{d^2 P_e}{d\beta^2} = (p_{e2} - p_{e4}) \frac{2y_0}{\sigma^3 \sqrt{2\pi}} e^{-\frac{(y_0+\beta)^2}{2\sigma^2}}. \quad (20)$$

The second derivative is positive if so is the difference  $p_{e2} - p_{e4}$ . It is easy to prove that this difference is positive for the white Gaussian noise  $N(0, \sigma)$  and for  $0 < y_1 < y_0$  and  $0 < \alpha < 1$ . From (17) we get

$$2(p_{e2} - p_{e4}) = \int_{-y_0+(1+\alpha)y_1}^{-y_0+2y_1} f_z(x) dx - \int_{-y_0-2y_1}^{-y_0-(1+\alpha)y_1} f_z(z) dz.$$

The ranges of changes in the variables  $x$  and  $z$  are the same and equal to  $(1 + \alpha)y_1$ . But for  $0 < y_1 < y_0$  and  $0 < \alpha < 1$ , we have  $|x| \leq y_0$  and  $|z| \geq y_0$ , and therefore  $|x| \leq |z|$ . Since  $f_z(x)$  is a Gaussian function, we obtain  $f_z(x) \geq f_z(z)$ . This gives the same inequality for the foregoing integrals, so  $p_{e2} > p_{e4}$ . This proves that the second derivative (20) is positive and the minimum of  $P_e$  exists for  $\beta$  which satisfies (19). ■

The proposed method of optimizing the decision feedback can be useful for a large number of channels. Next we will prove that is possible to decrease error probability by this method for the white Gaussian noise if the noise power is sufficiently low.

**Lemma 2.** *For any channel with the samples  $y_0$  and  $y_1$  satisfying  $0 < y_1 < y_0$ , if the white Gaussian noise power is less than some positive value, there exist  $\beta > 0$  and  $0 < \alpha < 1$  that minimize the bit error probability  $P_e$ .*

*Proof.* From Lemma 1 we get the parameter  $\beta$  that minimizes the bit error probability  $P_e$  for fixed  $0 < \alpha < 1$ . We will prove Lemma 2 if we prove that the threshold  $\beta$  is positive ( $\beta > 0$ ) for low noise ( $\sigma \rightarrow 0$ ). In this case

we prove that the threshold  $\beta$  which minimizes  $P_e$  for the low noise is physically realizable.

To get positive  $\beta$  from (19),  $C(\alpha)$  has to be greater than 1. From (18) we get

$$\begin{aligned} & \int_{-y_0-(1-\alpha)y_1}^{-y_0} f_z(x) dx + \int_{-y_0+(1+\alpha)y_1}^{-y_0+2y_1} f_z(x) dx \\ & > \int_{-y_0-2y_1}^{-y_0-(1+\alpha)y_1} f_z(x) dx + \int_{-y_0}^{-y_0+(1-\alpha)y_1} f_z(x) dx, \quad (21) \end{aligned}$$

where

$$f_z(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}.$$

From  $0 < \alpha < 1$  and  $0 < y_1 < y_0$  it is clear that the three integrals in (21) have negative limits. For such arguments the Gaussian function  $f_z(x)$  is positive and increasing. Therefore

$$\begin{aligned} & \int_{-y_0-2y_1}^{-y_0-(1+\alpha)y_1} f_z(x) dx < \Delta_x f_z(-y_0-(1+\alpha)y_1), \\ & \int_{-y_0}^{-y_0+(1-\alpha)y_1} f_z(x) dx < \Delta_x f_z(-y_0+(1-\alpha)y_1), \\ & \int_{-y_0-(1-\alpha)y_1}^{-y_0} f_z(x) dx > \Delta_x f_z(-y_0-(1-\alpha)y_1), \end{aligned}$$

where  $\Delta_x = (1-\alpha)y_1$ .

The estimation of the fourth integral is dependent on the absolute values of its limits. Therefore

$$\int_{-y_0+(1+\alpha)y_1}^{-y_0+2y_1} f_z(x) dx > \Delta_x f_z(-y_0+(1+\alpha)y_1),$$

when  $|-y_0+(1+\alpha)y_1| > |-y_0+2y_1|$ , i.e., when

$$0 < y_1 < \frac{2}{3+\alpha}y_0$$

or

$$\int_{-y_0+(1+\alpha)y_1}^{-y_0+2y_1} f_z(x) dx > \Delta_x f_z(-y_0+2y_1),$$

when  $|-y_0+(1+\alpha)y_1| \leq |-y_0+2y_1|$ , i.e., when  $\frac{2}{3+\alpha}y_0 \leq y_1 < y_0$ .

The inequality (21) will be true if the following inequalities are true:

$$\begin{aligned} & f_z(-y_0-(1-\alpha)y_1) + f_z(-y_0+(1+\alpha)y_1) \\ & > f_z(-y_0-(1+\alpha)y_1) + f_z(-y_0+(1-\alpha)y_1) \quad (22a) \end{aligned}$$

for  $0 < y_1 < \frac{2}{3+\alpha}y_0$ , and

$$\begin{aligned} & f_z(-y_0-(1-\alpha)y_1) + f_z(-y_0+2y_1) \\ & > f_z(-y_0-(1+\alpha)y_1) + f_z(-y_0+(1-\alpha)y_1) \quad (22b) \end{aligned}$$

for  $\frac{2}{3+\alpha}y_0 \leq y_1 < y_0$ .

Lemma 2 will be proved if we prove (22) for  $\sigma \rightarrow 0$  and for the Gaussian function  $f_z(x)$ . Replacing  $f_z(x)$  by  $N(0, \sigma)$  in (22), from (22a) we have that

$$\begin{aligned} & 1 + \exp\left(\frac{2y_1(\alpha y_1 - y_0)}{\sigma^2}\right) \\ & > \exp\left(\frac{-2(1+\alpha)y_1 y_0}{\sigma^2}\right) + \exp\left(\frac{2\alpha y_1(y_1 - y_0)}{\sigma^2}\right) \quad (23a) \end{aligned}$$

and, from (22b),

$$\begin{aligned} & 1 + \exp\left(\frac{y_1[4(y_1 - y_0) - 2(1-\alpha)y_0 - (1-\alpha)^2 y_1]}{2\sigma^2}\right) \\ & > \exp\left(\frac{y_1[4(y_1 - y_0) - 2(1+\alpha)y_0 - (1+\alpha)^2 y_1]}{2\sigma^2}\right) \\ & \quad + \exp\left(\frac{y_1[4(y_1 - y_0) + 2(1-\alpha)y_0 - (1-\alpha)^2 y_1]}{2\sigma^2}\right). \quad (23b) \end{aligned}$$

The left-hand sides of (23a) and (23b) are greater than one. For  $0 < y_1 < y_0$  both exponents appearing on the right-hand side of (23a) are negative and they tend to  $-\infty$  as  $\sigma \rightarrow 0$ . The right-hand side of the inequality decreases to zero. So (22a) is true for small  $\sigma$ .

The first exponent on the right-hand side of (23b) is negative. The second exponent is negative if  $y_1 < \frac{2}{3+\alpha}y_0$ , and for  $\alpha = 1$ , it is negative if  $y_1 < y_0$ . In any case, it is possible to choose  $\alpha$  to get a negative value of this exponent. In this case (23b) is true. Accordingly, it is possible to find  $\alpha$  making (21) true for low noise. ■

It is easy to notice that Lemma 2 can be also generalized for the negative values of  $y_0$  or  $y_1$ .

**Lemma 3.** For any channel with the samples  $y_0$  and  $y_1$  satisfying  $0 < |y_1| < |y_0|$ , if the power of the white Gaussian noise is less than some positive value, there exist  $\beta > 0$  and  $0 < \alpha < 1$  that minimize the bit error probability  $P_e$ .

*Proof.* Lemma 3 is nothing but Lemma 2 with the assumption that  $|y_1| < |y_0|$  instead of  $0 < y_1 < y_0$ . Given bit error probability, (15) includes the probabilities  $p_{e1}$ ,  $p_{e2}$ ,  $p_{e3}$  and  $p_{e4}$ . They are expressed by the integrals (5), (6), (8) and (10) with the positive values  $0 < y_1 < y_0$ . If  $y_1 < 0$ , the replacement of  $y_1$  by  $-y_1$  does not change these equations, but the new  $y_1$  will be positive. For negative  $y_0$ , the original equations (5), (6), (8), (10) and (11)–(14) will use negative  $y_0$  in place of  $-y_0$  and hence the same limits of integration can be used. The replacement of  $y_0$  by  $-y_0$ , where the new  $y_0$  is positive, converts the above equations to the same formulas which were used in (5), (6), (8) (10) and (11)–(14). Because the new positive quantities  $y_1$  and  $y_0$  satisfy the assumption of Lemma 2 and the equation for  $P_e$  is not changed, Lemma 3 is true. ■

Lemma 3 gives information as to when decision feedback optimization by decision risk qualification can decrease bit error probability. We proved that this method can be used successfully for any channel (1) with  $|y_1| < |y_0|$  if the noise power is sufficiently low.

The scale of decreasing the bit error probability  $P_e$  using the best values  $\alpha$  and  $\beta$  depends on the noise power  $\sigma^2$ , and on  $y_0$  and  $y_1$ . This will be illustrated with computational results for three selected channels.

### 3. Simulation Experiment

The aim of the experiment is to find out how much the optimization of decision feedback can decrease bit error probability and to find the optimal values of  $\alpha$  and  $\beta$  as functions of channel parameters.

The channel chosen in the experiment was described by two pulse response samples, and therefore its transfer function is given by (1). These samples are normalized for their energy equal to one, so that  $y_0^2 + y_1^2 = 1$ . The sample  $y_1$  is the interference sample and the sample  $y_0$  is the main sample. If  $|y_1| < |y_0|$ , then  $|y_1| = \sqrt{1 - y_0^2}$ , where  $1/\sqrt{2} < y_0 \leq 1$ . First, the value of  $y_0$  was increased by some constant and then  $y_1$  was calculated.

For such channels, the assumed noise power  $\sigma^2$  and the assumed values  $\alpha$  ( $0 \leq \alpha < 1$ ), the optimal risk qualification threshold  $\beta$  was calculated to minimize the bit error probability  $P_e$ . Optimal  $\alpha^*$  and  $\beta^*$  were found by changing  $\alpha$  and calculating optimal  $\beta$  and  $P_e$  for  $\alpha$ . The main result of this experiment is to find optimal  $\alpha^*$  and  $\beta^*$  as functions of the quotient  $y_1/y_0$  and the noise power  $\sigma^2$ ,  $\alpha^*(y_1/y_0, \sigma)$  and  $\beta^*(y_1/y_0, \sigma)$ .

### 4. Results and Conclusion

Three channels were chosen for experiments. The parameter  $\alpha$  was increased from 0.1 to 0.9 for each channel.

For each  $\alpha$  the optimal level  $\beta$  was calculated using (19) and (20), and then the bit error rate  $P_e$  was obtained from (15). For each channel the bit error rate  $P_{e0}$  was calculated using (15) with the assumption that  $\beta = 0$  and  $\alpha = 1$ . Bit error rate reduction in the effect of decision feedback optimization is given by the quotient  $P_e/P_{e0}$ . These experiments were repeated for several values of the noise power  $\sigma^2$ . The resulting bit error rate reductions are shown in Figs. 3–5 for three selected channels.

The minimum of this dependence represents the best reduction and gives the optimal point  $(\alpha^*, \beta^*)$ . The minimum  $P_e$  is reached at this point. These approximate points  $(\alpha^*, \beta^*)$  are shown as crosses, triangles, squares or circles.

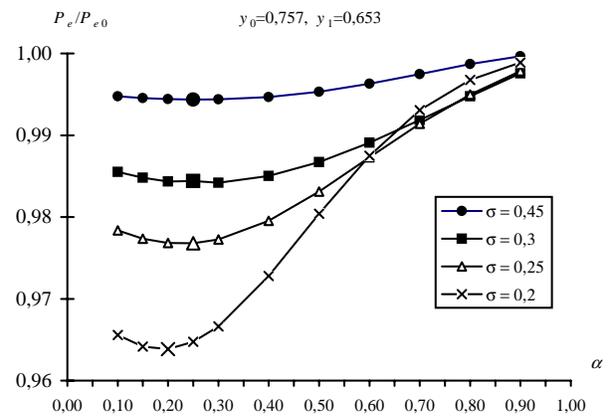


Fig. 3. BER reduction for the channel with  $y_0 = 0,757$ ,  $y_1 = 0,653$ .

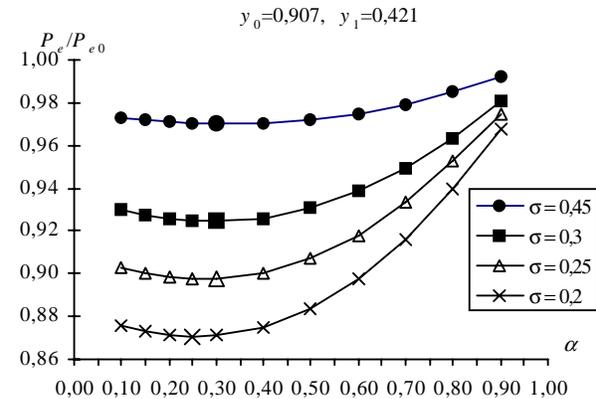


Fig. 4. BER reduction for the channel with  $y_0 = 0,907$ ,  $y_1 = 0,421$ .

Optimal  $\alpha^*$  depends on the noise power only to some extent, as seen in Figs. 3–5. The results for particular channels are as follows:

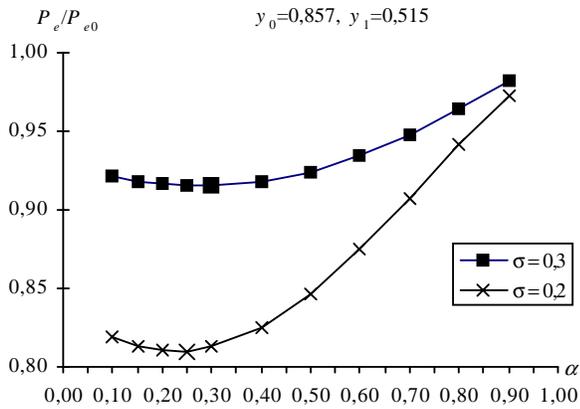


Fig. 5. BER reduction for the channel with  $y_0 = 0,857, y_1 = 0,515$ .

- For Channel 1 ( $y_0 = 0.757, y_1 = 0.653$ ) we obtain  $0.20 \leq \alpha^* \leq 0.25$ ,
- For Channel 2 ( $y_0 = 0.907, y_1 = 0.421$ ) we obtain  $0.25 \leq \alpha^* \leq 0.30$ ,
- For Channel 3 ( $y_0 = 0.857, y_1 = 0.515$ ) we obtain  $0.25 \leq \alpha^* \leq 0.30$ .

The functions  $\alpha^*(\sigma)$  and  $\beta^*(\sigma)$  are shown in Figs. 6 and 7 for all three channels. The optimal  $\alpha^*$  and  $\beta^*$  depend on the channel and increase if  $\sigma^2$  increases for small  $\sigma$ . The functions  $\alpha^*(y_1/y_0)$  and  $\beta^*(y_1/y_0)$  are shown in Figs. 8–9 for some  $\sigma$ . Usually, optimal  $\alpha^*$  and  $\beta^*$  are smaller for larger interference. These diagrams can be helpful in the optimization of decision feedback given a channel with known noise.

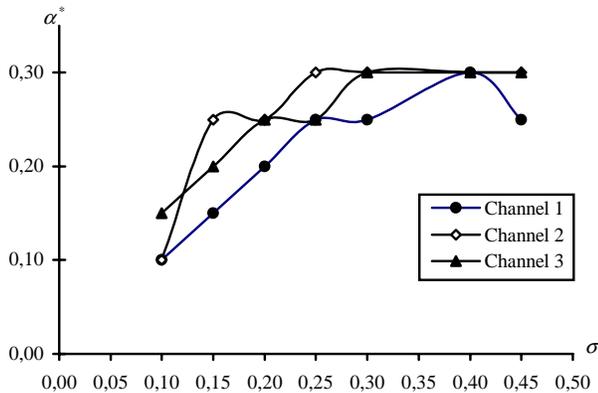


Fig. 6. Dependence of the parameter  $\alpha^*$  on  $\sigma$ .

### 5. Summary

The presence of errors in decision feedback of an equalizer involves a high probability of new errors. This is

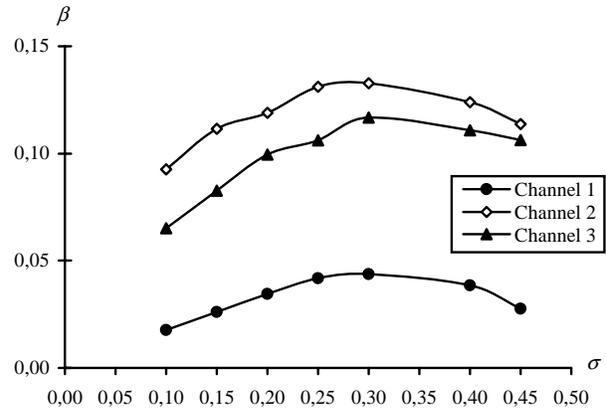


Fig. 7. Dependence of the parameter  $\beta^*$  on  $\sigma$ .

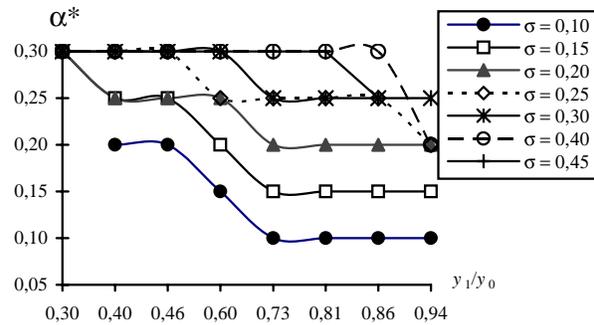


Fig. 8. Optimal values of the parameter  $\alpha$ .

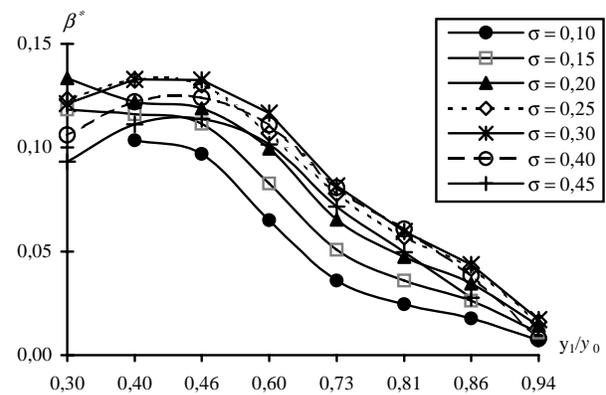


Fig. 9. Optimal values of the parameter  $\beta$ .

called the error extension. The reduction of *error extension* is still an open problem. The method presented in this paper allows us to reduce error extension for some kind of channels. It was proved in an analytical way that the presented method has an optimal solution. It was also proved for channels with discrete transfer functions  $Y(z) = y_0 + y_1z^{-1}$ , i.e., with one interference sample  $y_1$  ( $|y_1| < |y_0|$ ) and with the white Gaussian noise, that the optimization of decision feedback using the risk

threshold  $\beta$  and the erasure factor  $\alpha$  decreases the bit error rate if the noise power is suitably low. Under these assumptions, the optimal values for  $\alpha$  and  $\beta$  were found numerically. The results are presented in Figs. 8 and 9.

BER reduction depends on channel parameters and the signal-to-noise ratio. The proposed optimization of decision feedback decreases the bit error rate by 1% to 19% depending on the channel ( $y_1/y_0$ ) and the noise power. For lower noise (a higher signal-to-noise ratio), the bit error rate decreases more than for higher noise. In the presented experiments, the BER decreases from 3.6% to 19.0% for  $\sigma = 0.2$  (SNR = 5), and from 1.6% to 8.5% for  $\sigma = 0.3$  (SNR = 3.3).

The presented theoretical deliberations were performed for the optimization of the second decision following error absence in decision feedback of the equalizer. They have to be generalized for any decision assuming the Markov chain model for this decision process.

Another problem that has to be solved is the implementation or extension of the presented results for channels with more than one interference sample, i.e., for longer decision feedback. We can test BER reduction for separate optimization of each tap of the feedback using more than one decision, as in Fig. 1. Next we can test simultaneous optimization of all parameters of feedback with many taps.

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Received: 11 April 2005  
Revised: 17 October 2005