ON FUZZY NUMBER CALCULUS

WITOLD KOSIŃSKI*,**

* Polish-Japanese Institute of Information Technology
Research Center
ul. Koszykowa 86, 02–008 Warsaw, Poland
e-mail: wkos@pjwstk.edu.pl

** Kazimierz Wielki University of Bydgoszcz
Institute of Environmental Mechanics and Applied Computer Science
ul. Chodkiewicza 30, 85–064 Bydgoszcz, Poland

In memory of the late Professor Ernest Czogała

Some generalizations of the concept of ordered fuzzy numbers (OFN) are defined to handle fuzzy inputs in a quantitative way, exactly as real numbers are handled. Additional two structures, an algebraic one and a normed (topological) one, are introduced to allow for counting with a more general type of membership relations.

Keywords: fuzzy numbers, bounded variation

1. Introduction

The commonly accepted theory of fuzzy numbers (Czogała and Pedrycz, 1985) is that set up by Dubois and Prade (1978), who proposed a restricted class of membership functions, called (L, R)–numbers with shape functions L and R. However, approximations of fuzzy functions and operations are needed if one wants to follow Zadeh's (Zadeh 1975; 1983) extension principle. It leads to some drawbacks that concern properties of fuzzy algebraic operations, as well as to unexpected and uncontrollable results of repeatedly applied operations (Wagenknecht, 2001; Wagenknecht *et al.*, 2001).

Classical fuzzy numbers (sets) are convenient as far as a simple interpretation in the set-theoretical language is concerned (Zadeh, 1965). However, we could ask: How can we imagine a fuzzy information, say X, in such a way that by adding it to fuzzy information (number) A another fuzzy number C will be obtained? In our previous papers (see (Kosiński *et al.*, 2003b) for references) we tried to answer that question in terms of the so-called ordered fuzzy numbers, which can be identified with pairs of continuous functions defined on the interval $[0, 1]$. In this paper we generalize the class of membership curves introduced earlier in order to make the algebra of ordered fuzzy numbers a more efficient tool in dealing with unprecise, fuzzy quantitative terms.

2. Ordered Fuzzy Numbers

In the series of papers (Kosiński *et al.*, 2001; 2002a; 2002b; 2003a; 2003b; Kosiński, 2004; Kosiński and Prokopowicz, 2004; Koleśnik *et al.*, 2004), we introduced and developed the main concepts of the space of ordered fuzzy numbers. In our approach the concept of membership functions (Czogała and Pedrycz, 1985) was weakened by requiring a mere *membership relation*. Consequently, a fuzzy number A was identified with an ordered pair of continuous real functions defined on the interval $[0, 1]$, i.e., $A = (f, g)$ with $f, g : [0, 1]\to\mathbb{R}$ as continuous functions. We call f and g the *up* and *down-parts* of the fuzzy number A, respectively. To be in agreement with the classical denotation of fuzzy sets (numbers), the independent variable of both functions f and g is denoted by y, and their values by x.

The continuity of both parts implies that their images are bounded intervals, say UP and $DOWN$, respectively (Fig. 1(a)). We used symbols to mark boundaries for $UP = [l_A, 1^-_A]$ and $DOWN = [1^-_A, p_A]$.

In general, the functions f and g need not be invertible as functions of $y \in [0, 1]$, and only continuity is required. If we assume, however, that they are monotonous, i.e., invertible, and add the constant function of x on the interval $[1^-_A, 1^+_A]$ with the value equal to 1, we

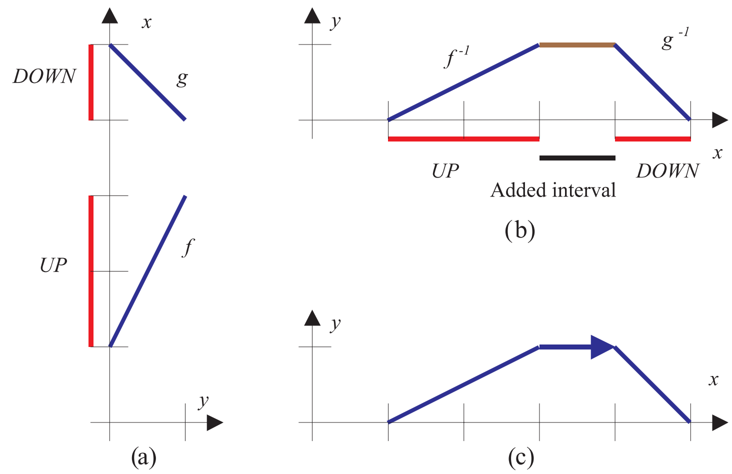

Fig. 1. Ordered fuzzy number (a), an ordered fuzzy number presented as a fuzzy number in classical meaning (b), and a simplified mark denoting the order of inverted functions (c).

might define the membership function

$$\mu(x) = \begin{cases} f^{-1}(x) & \text{if } x \in [f(0), f(1)] = [l_A, 1_A^-], \\ g^{-1}(x) & \text{if } x \in [g(1), g(0)] = [1_A^+, p_A], \\ 1 & \text{if } x \in [1_A^-, 1_A^+], \end{cases} \quad (1)$$

if

1. f is increasing and g is decreasing, and such that
2. $f \leq g$ (pointwise).

In this way, the obtained membership function $\mu(x), x \in \mathbb{R}$ represents a mathematical object which resembles a convex fuzzy number in the classical sense (Drewniak, 2001; Klir, 1997; Wagenknecht, 2001). Notice that for the representation of the membership function μ of the convex fuzzy number one can attach two monotonic functions $\mu_{up} := f^{-1}$ and $\mu_{down} := g^{-1}$ defined on the intervals $[f(0), f(1)]$ and $[g(1), g(0)]$, respectively.

In fact, in Fig. 1(c) to the ordered pair of two continuous functions (here just two affine functions) f and g there corresponds a membership function of a convex fuzzy number[1], with an extra arrow, which denotes the orientation of the closed curve formed of the graph of the function and the part of the x axis (which is just the part of the domain of the function on which its values are different from zero). On the other hand, the arrow underlines the fact that we are dealing with an ordered pair of functions. In this way, we appointed an extra feature to this object (as well as to its counterpart – the convex fuzzy number), named the *orientation*.

Notice that if some of the conditions formulated above are not satisfied, the construction of the classical membership function is not possible. However, in the $x - y$ plane the graphs of f and g (as functions of y) can be drawn together with the constant function of x on the interval $[f(1), g(1)]$, equal to 1. Consequently, the resulting graphs of three functions form together a curve which can be called the *membership curve of an ordered fuzzy number* (f, g) .

3. Operations

Now, in the most natural way, the operation of addition between two pairs of such functions is defined (cf. our main definition from (Kosiński *et al.*, 2003b)) as the pairwise addition of their elements, i.e., if (f_1, g_1) and (f_2, g_2) are two ordered fuzzy numbers, then $(f_1 + f_2, g_1 + g_2)$ will be just their sum. It is interesting to notice that as long as we are dealing with an ordered fuzzy number represented by pairs of affine functions of the variable $y \in [0, 1]$, its so-called classical counterpart, i.e., a membership function of the variable x is just a trapezoidal-type convex fuzzy number. One should notice, however, that a trapezoidal type membership function corresponds not to every pair of affine functions of y (cf. the requirement of the invertibility of f and g and conditions 1 and 2 formulated in Eqn. (1)); some of them are improper (as was noticed already in (Kosiński *et al.*, 2003b) like in Fig. 2.

If we want to add two pairs of affine functions (i.e., two particular types of ordered fuzzy numbers) defined on $[0, 1]$, the final result is easy to obtain, since interval calculus can then be used. Here a mnemotechnic method of adding (as well as subtracting and multiplying by a scalar, i.e., by a real, crisp number) of ordered fuzzy numbers represented by pairs of affine functions can be given. If for any pair of affine functions (f, g) of $y \in [0, 1]$ we form a quaternion (tetrad) of real numbers according

[1] As usual, the part of the graph representing vanishing values of the membership function is not presented here.

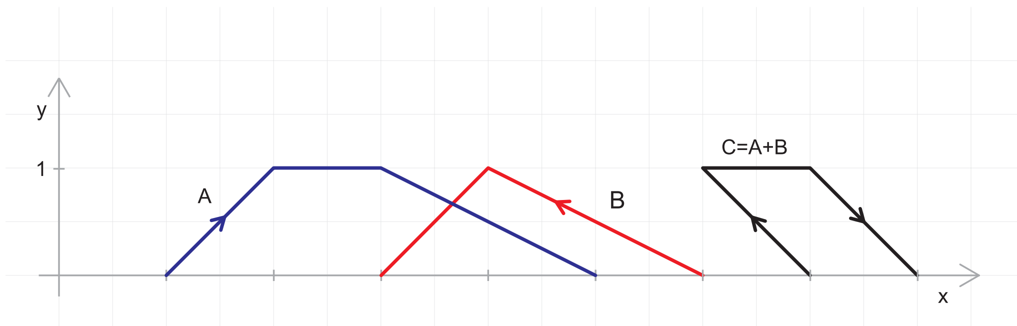

Fig. 2. Sum of two convex OFNs is an improper convex number.

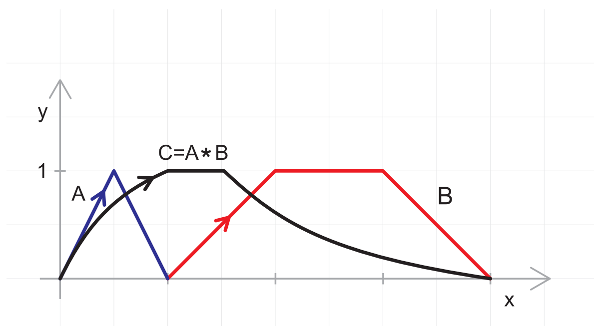

Fig. 3. Multiplication.

to the rule $[f(0), f(1), g(1), g(0)]$ (which correspond to the four numbers $[l_A, 1_A^+, 1_A^-, p_A]$) of Eqn. (1), then this tread uniquely determines[2] the ordered fuzzy number A. If $(e, h) =: B$ is another pairs of affine functions, then the sum $A + B = (f + e, g + h) =: C$ will be uniquely represented by the tread

$$\Big[f(0)+e(0), f(1)+e(1), g(1)+h(1), g(0)+h(0)\Big]. \quad (2)$$

In a similar way, if we want to multiply an OFN, say A, by a scalar $\lambda \in \mathbb{R}$, then the product λA will have its tread representation in the form

$$\lambda A \longleftrightarrow \Big[\lambda f(0), \lambda f(1), \lambda g(1), \lambda g(0)\Big]$$

$$\text{where} \quad A \longleftrightarrow \Big[f(0), f(1), g(1), g(0)\Big]. \quad (3)$$

In the assumed definitions (cf. Kosiński *et al.*, 2003a), the operation of subtraction is compatible with a linear structure of OFNs, i.e., $A - B := A + (-1)B$. The representations (2) and (3) are at our disposal to find the result of the subtraction $A - B$ in the form of the corresponding tread.

[2] Only one line segment can be drawn through two points in the plane.

If for $A = (f, g)$ we define its *complement* $\bar{A} = (-g, -f)$ (note that $\bar{A} \neq (-1) \cdot A$), then the sum $A + \bar{A}$ gives a fuzzy zero $0 = (f - g, -(f - g))$ in the sense of the classical fuzzy number calculus. If we attach to $A = (f, g)$ the corresponding number of the opposite orientation $A^\perp = (g, f)$, then we can see that the difference between them is a fuzzy zero, i.e.,

$$A - A^\perp = (f - g, -(f - g)) \quad (4)$$

like before. For a better presentation of the advantages of the new operations on OFN we add extra figures for the sum, the difference and the product of A by the inverse of B, i.e., the division A/B.

In Fig. 2 we can follow the operation of addition using the tread representation of two trapezoidal ordered fuzzy numbers. In fact, for the number A we have the tread $[1, 2, 3, 5]$, and for B (which has the opposite orientation to that of A) the corresponding tread is $[6, 4, 4, 3]$. Taking the sum of both treads (componentwise), we will get

$$[1, 2, 3, 5]+[6, 4, 4, 3] = [7, 6, 7, 8] \longleftrightarrow A+B = C, \quad (5)$$

which is the tread representation of the sum $C = A + B$.

For a better presentation of the advantages of the new operations on OFN, we add extra figures for the products of A by B and by the inverse of B, i.e., the division A/B. Notice that the inverse $1/B$ of an ordered fuzzy number B is defined as an ordered fuzzy number such that the product $B \cdot (1/B)$ gives a crisp one, i.e., an ordered fuzzy number represented by the pair of constant functions $(1^\dagger, 1^\dagger)$, where $1^\dagger(y) = 1$ for all $y \in [0, 1]$.

4. Generalization

However, there are some limitations if we pass from the concept of ordered fuzzy numbers (OFN) represented by ordered pairs of continuous functions (even those satisfying Conditions 1 and 2 above) to the theory of convex fuzzy numbers represented by their membership functions. This is because some membership functions already

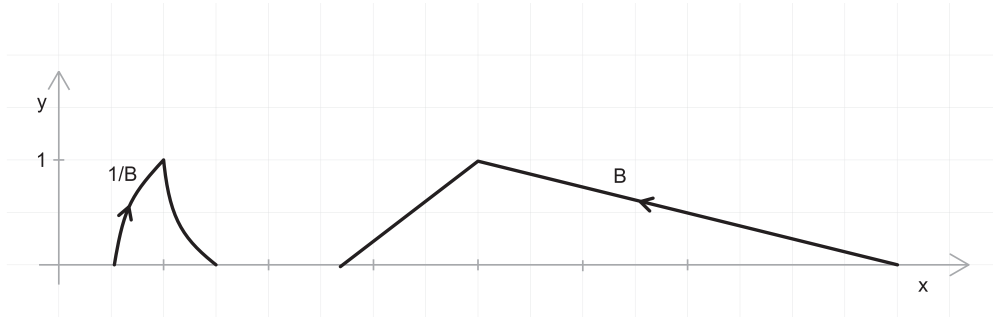

Fig. 4. Inverse of B.

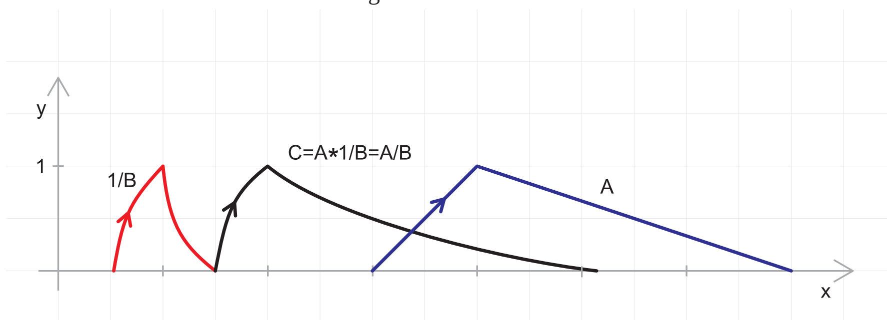

Fig. 5. Division A/B.

known in the classical theory of fuzzy numbers (cf. Czogała and Pedrycz, 1985; Guanrong and Tat, 2001; Łachwa, 2001; Piegat, 1999) cannot be obtained by taking inverses of continuous functions f and g in the process described above. We think here about such membership functions which are piecewise constant (cf. Fig. 7)), i.e., μ is one of them if its branches μ_{up} and μ_{down} are not strictly monotonous.

The lack of strict monotonicity of the branches μ_{up} and μ_{down} as functions of x and, consequently, the existence of constancy subintervals imply that the inverse functions to μ_{up} and μ_{down}, regarded as functions of y, do not exist in the classical sense. To solve this problem (in terms of the weaker concept of ordered fuzzy numbers, which is a membership relation) we may assume that for both functions μ_{up} and μ_{down} there exists a finite (or at most countable) number of such constancy subintervals, and then the inverse functions, say f and g, respectively, exist in a generalized sense, i.e., they are piecewise continuous and monotonous with a finite (or at most countable) number of discontinuity points. Those discontinuity points are *of the first order*, i.e., at each such point one-sided limits of the functions exist, which may be different. Then each jump of discontinuity in the y variable corresponds to a constancy subinterval in the x variable.

In this way we arrived at a class of functions larger than continuous ones from among which elements of pairs (f, g) are selected. This is the class of real-valued functions of *bounded* (finite) *variation* (Łojasiewicz, 1973). Now we are well prepared (cf. Appendix) to introduce a generalization of the original definition of ordered fuzzy numbers, cf. (Kosiński *et al.*, 2002a; 2002b; 2003a).

Definition 1. By an *ordered fuzzy number* A we mean an ordered pair (f, g) of functions such that $f, g : [0, 1]\to\mathbb{R}$ are of bounded variation.

Operations on new ordered fuzzy numbers are introduced in much the same way as in (Kosiński *et al.*, 2001; 2002a; 2002b; 2003a; 2003b; Kosiński, 2004; Kosiński and Prokopowicz, 2004; Koleśnik *et al.*, 2004). Notice, however, a minor difference in the definition of division.

Definition 2. Let $A = (f_A, g_A), B = (f_B, g_B)$ and $C = (f_C, g_C)$ be mathematical objects called ordered fuzzy numbers. The *sum* $C = A + B$, *subtraction* $C = A - B$, *product* $C = A \cdot B$, and *division* $C = A/B$ are defined by

$$f_C(y) = f_A(y) \star f_B(y), \quad g_C(y) = g_A(y) \star g_B(y), \quad (6)$$

where '$\star$' stands for '$+$', '$-$', '$\cdot$', and '$/$', respectively, and A/B is defined if the functions $|f_B|$ and $|g_B|$ are bounded from below by a positive number.

As was already noticed in the previous section, the subtraction of B is the same as the addition of the opposite of B, i.e., the number $(-1) \cdot B$.

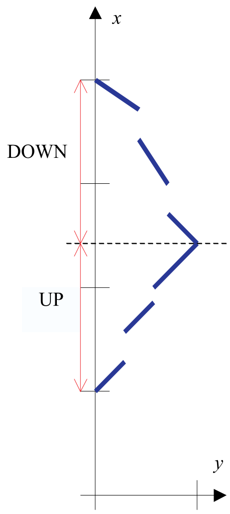

Fig. 6. Ordered fuzzy number as a pair of functions of bounded variation.

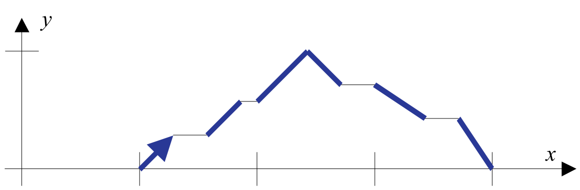

Fig. 7. Membership function of an ordered fuzzy number.

Additionally, the following, more set-theoretic operations can be defined:

Definition 3. Let $A = (f_A, g_A), B = (f_B, g_B)$ and $C = (f_C, g_C)$ be mathematical objects called *ordered fuzzy numbers*. The *maximum* $C = A \vee B$ and the *minimum* $C = A \wedge B$ are defined by

$$f_C(y) = \text{func}\Big\{f_A(y), f_B(y)\Big\},$$

$$g_C(y) = \text{func}\Big\{g_A(y), g_B(y)\Big\}, \tag{7}$$

where 'func' stands for '$\max$' and '$\min$', respectively.

Many operations can be defined in this way for pairs of functions. A `Fuzzy Calculator` was already created as a calculation tool by Roman Koleśnik (Koleśnik *et al.*, 2004). It facilitates an easy future use of all mathematical objects described as ordered fuzzy numbers.

This tool (a program called `zCalc`) was created with a graphical shell named `zWinCalc` and is not limited to piecewise linear parts (quasi-trapezoidal representations) only. It can run on any form of functions written in a symbolic way, i.e., by formulas, as well as given in a graphical way by points on the plane (a coordinate system). To create the main `zCalc` program, the following components were used:

(i) Visual Studio 6.0 – an environment for programming in the C++ language;

(ii) Bison-Flex – a generator of the language (a useful tool to build the syntax analyzer).

The tool `zCalc` was written as a component of the operating system Windows (9x/XP). To this end, a console interface which allows us to use the main module as a kind of the interpreter of a specifical simply language was added.

Algebraic operations on OFN offer a unique possibility to define new types of *compositional rules of fuzzy inference* which play a key role in approximate reasoning when conclusions from a set of fuzzy *if–then* rules are to be derived.

Examples of such compositional rules of inference were given based on the multiplication operator in which all fuzzy sets are OFNs, in the Ph.D. thesis (Prokopowicz, 2005). Moreover, to determine *activation levels* of multi-condition rules (or firing the strength of the fuzzy rule), new methods of aggregation of their premise parts were also proposed in (Prokopowicz, 2005). These aspects will be the subject of the next article.

The original case of OFNs with continuous elements (f, g) allows us to define a set of defuzzification operators thanks to the Riesz-Kakutami-Banach theorem.

5. Further Extensions

Pointwise multiplication by a scalar (crisp) number, together with addition, leads to a linear structure $\mathbb{R}$, which is isomorphic to the linear space of real 2D vector-valued functions defined on the unit interval $I = [0, 1]$.

Hence $\mathbb{R}$ can be identified with $BV([0,1]) \times BV([0,1])$, where $BV([0,1])$ is the space of real-valued functions of bounded variation defined on the interval $[0,1]$ (cf. Appendix). Since the space $BV([0,1])$ is a Banach space in the norm (12) (cf. Appendix), its Cartesian product can be equipped with the norm as follows:

$$\|(f, g)\| = \max(|f(0)| + \text{var}(f), |g(0)| + \text{var}(g))\ . \tag{8}$$

Finally, $\mathcal{R}$ is a Banach algebra with the unity $(1^\dagger, 1^\dagger)$. One should add that a Banach structure of an extension of convex fuzzy numbers was introduced by Goetschel and Voxman (1986). However, they were only interested in the linear structure of this extension.

A relation of *partial ordering* in $\mathcal{R}$ can be introduced by defining a subset of those ordered fuzzy numbers which are greater than or equal to zero. We say a fuzzy number $A = (f, g)$ is *no less than zero*, and write $A \geq 0$, iff $f \geq 0$ and $g \geq 0$. Hence for two ordered fuzzy numbers B, C the relation $B \geq C$ holds if $B - C \geq 0$. From this we see that $\mathcal{R}$ is a partially ordered ring to which the theory of such rings can be applied.

Acknowledgments

The research work reported in the paper was partially conducted in the framework of the project supported by the Ministry of Education and Science in Poland, no. 4 T11C 038 25. The author thanks Prof. Tadeusz Rzeżuchowski and Dr. Janusz Wąsowski from the Warsaw University of Technology for inspiring discussions.

References

Alexiewicz A. (1969): *Functional Analysis*. — Warsaw: Polish Scientific Publishers (in Polish).

Chen Guanrong and Pham Trung Tat (2001): *Fuzzy Sets, Fuzzy Logic, and Fuzzy Control Systems*. — Boca Raton, FL, CRS Press.

Czogała E. and Pedrycz W. (1985): *Elements and Methods of Fuzzy Set Theory*. — Warsaw: Polish Scientific Publishers (in Polish).

Drewniak J. (2001): Fuzzy numbers. In: Fuzzy Sets and their Applications (J. Chojcan, J. Łęski, Eds.). — Gliwice: Silesian University of Technology Press, pp. 103–129.

Dubois D. and Prade H. (1978): *Operations on fuzzy numbers*. — Int. J. Syst. Sci., Vol. 9, No. 6, pp. 613–626.

Goetschel R. Jr. and Voxman W. (1986): *Elementary fuzzy calculus*. — Fuzzy Sets Syst., Vol. 18, No. 1, pp. 31–43.

Klir G.J. (1997): *Fuzzy arithmetic with requisite constraints*. — Fuzzy Sets Syst., Vol. 91, No. 2, pp. 165–175.

Kosiński W., Piechór K., Prokopowicz K. and Tyburek K. (2001): *On algorithmic approach to opertions on fuzzy numbers*, In: Methods of Artificial Intelligence in Mechanics and Mechanical Engineering (T. Burczyński, W. Cholewa, Eds.). — Gliwice: PACM, pp. 95–98 (in Polish).

Kosiński W., P. Prokopowicz P. and Ślęzak D. (2002a): *Fuzzy numbers with algebraic operations: algorithmic approach*, In: Intelligent Information Systems 2002 (M. Klopotek, S.T. Wierzchoń, M. Michalewicz, Eds.). Proc.IIS'2002, Sopot, Poland — Heidelberg: Physica Verlag, pp. 311–320.

Kosiński W., Prokopowicz P. and Ślęzak D. (2002b): *Drawback of fuzzy arthmetics — New intutions and propositions*, In: Proc. Methods of Aritificial Intelligence (T. Burczyński, W. Cholewa, W. Moczulski, Eds.). — Gliwice: PACM, pp. 231–237.

Kosiński W., Prokopowicz P. and Ślęzak D. (2003a): *On algebraic operations on fuzzy numbers*, In: Intelligent Information Processing and Web Mining, Proc. Int. Symp. IIS: IIPWM'03, Zakopane, Poland, 2003 (M. Klopotek, S.T. Wierzchoń, K. Trojanowski, Eds.). — Heidelberg: Physica Verlag, pp. 353–362.

Kosiński W., Prokopowicz P. and Ślęzak D. (2003b): *Ordered fuzzy numbers*. — Bull. Polish Acad. Sci., Ser. Sci. Math., Vol. 51, No. 3, pp. 327–338.

Kosiński W. (2004): *On defuzzification of ordered fuzzy numbers*, In: Proc. ICAISC 2004, 7th Int. Conference, Zakopane, Poland, June 2004 (L. Rutkowski, Jörg Siekmann, R. Tadeusiewicz, Lofti A. Zadeh, Eds.), LNAI. — Berlin: Springer, Vol. 3070, pp. 326–331.

Kosiński W. and Prokopowicz P. (2004): *Algebra of fuzzy numbers*. — Matematyka Stosowana. Matematyka dla Społeczeństwa, Vol. 5, No. 46, pp. 37–63, (in Polish).

Koleśnik R., Prokopowicz P. and Kosiński W. (2004): *Fuzzy Calculator – useful tool for programming with fuzzy algebra*, In: Artficial Intelligence and Soft Computing – ICAISC 2004, 7th Int. Conference, Zakopane, Poland (L. Rutkowski, Jörg Siekmann, R. Tadeusiewicz, Lofti A. Zadeh, Eds.), Lecture Notes on Artificial Intelligence. — Berlin: Springer, Vol. 3070, pp. 320–325.

Łachwa A. (2001): *Fuzzy World of Sets, Numbers, Relations, Fazts, Rules and Decisions*. — Warsaw: EXIT, (in Polish).

Łojasiewicz S. (1973): *Introduction to the theory of real functions*. — Warsaw: Polish Scientific Publishers, (in Polish).

Martos B. (1983): *Nonlinear Programming – Theory and Methods*. — Warsaw: Polish Scientific Publishers, (in Polish).

Piegat A. (1999): *Fuzzy Modeling and Control*. — Warsaw: PLJ, (in Polish).

Prokopowicz P. (2005): *Algorithmic operations on fuzzy numbers and their applications*. — Ph. D. thesis, Institute of Fundamental Technological Research, Polish Acad. Sci., (in Polish).

Wagenknecht M. (2001): *On the approximate treatment of fuzzy arithmetics by inclusion, linear regression and information content estimation*, In: Fuzzy Sets and Their Applications (J. Chojcan, J. Łęski, Eds.). — Gliwice: Silesian University of Technology Press, pp. 291–310.

Wagenknecht M., Hampel R., Schneider V. (2001): *Computational aspects of fuzzy arithmetic based on Archimedean t-norms*. — Fuzzy Sets Syst., Vol. 123/1, pp. 49–62.

Zadeh L.A. (1965): *Fuzzy sets*. — Inf. Contr., Vol. 8, No. 3, pp. 338–353.

Zadeh L.A. (1975): *The concept of a linguistic variable and its application to approximate reasoning, Part I*. — Inf. Sci., Vol. 8, No. 3, pp. 199–249.

Zadeh L.A. (1983): *The role of fuzzy logic in the management of uncertainty in expert systems*. — Fuzzy Sets Syst., Vol. 11, No. 3, pp. 199–227.

Appendix

Here we give the most important facts concerning functions of bounded variations.

Each function h of bounded variation on $[0,1]$ possesses at most a countable number of *discontinuity points of the first order* and, moreover, each function can be represented as a sum of two functions $h(s) = h_{\mathbf{c}}(s) + h_{\mathbf{j}}(s),\ s \in [0,1]$, where $h_{\mathbf{c}}$ is a continuous function of bounded variation while $h_{\mathbf{j}}(s)$ is a function of jumps of h, called the *jump function*. In other words, if $\{s_k : k = 1, 2, \dots\}$ is the sequence (finite or infinite: however, with different terms) of all discontinuity points of the function h, then the jump function of h is $h_{\mathbf{j}}(s) = \sum_k u_k$, where for each k the value

$$u_k(s) = 0 \quad \text{for} \quad s < s_k$$

and

$$\begin{aligned} u_k(s_k) &= h(s_k) - h(s_k - 0), \\ u_k(s) &= h(s_k + 0) - h(s_k - 0) \quad \text{for} \quad s > s_k, \end{aligned} \tag{9}$$

where $h(s_k - 0), h(s_k + 0)$ are one-sided limits of h at s_k.

It is worthwhile to add that each function of bounded variation is a difference of two monotonous (exactly increasing) functions. Hence the function $h_{\mathbf{c}}$ in the above representation is a difference of two increasing continuous functions.

One needs to stress that not all ordered fuzzy numbers in the sense of our first definition (Kosiński *et al.*, 2003a) fulfil the present definition, since they are continuous functions which do not have bounded variation. For example, the function $h(s) = s\cos(\pi/2s)$ for $s \in (0,1]$ and $h(0) = 0$ is continuous in the whole interval $[0,1]$ while its variation, i.e., the upper limit of the sum

$$v(h) = \sum_{k=0}^{n-1} |h(s_{k+1}) - h(s_k)| \tag{10}$$

for an arbitrary partition $0 = s_0 < s_1 < s+2 < \cdots < s_{k-1} < s_n = 1$ of the interval $[0,1]$ is unbounded. To see this, it is enough to take for any n the partition

$$0 < \frac{1}{2n} < \frac{1}{2n-1} < \cdots < \frac{1}{3} < \frac{1}{2} < 1,$$

for which the sum $v(h)$ in (10) will be

$$v(h) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\ ,$$

and for a sufficiently large n the sum can be arbitrarily large. If, however, a function fulfils the Lipschitz condition with a constant M on the interval [0, 1], then its variation is bounded by M (Łojasiewicz, 1973).

On the other hand, each increasing (even discontinuous) function (or, more generally, monotonous) function on the interval $[0,1]$ is a function of bounded variation (which is rather obvious from the formula (10) in the case of an increasing function, since each component $h(s_{k+1}) - h(s_k)$ is positive and then the sum is equal to $h(1) - h(0)$).

In what follows, we will use the notation $\mathrm{var}(h)$ for the variation of the function f, i.e., the upper limit of the sum (10), for an arbitrary partition of the interval $[0,1]$,

$$\mathrm{var}(h) = \sup v(h), \tag{11}$$

where $v(h)$ is given by (10).

The facts quoted above are fundamental in the proof of the main proposition (Alexiewicz, 1969; Łojasiewicz, 1973).

Proposition 1. *Linear combinations and products of functions of bounded variation are functions of bounded variation. Moreover, a quotient of functions of bounded variation is a function of bounded variation if the absolute value of the divisor is bounded from below by a positive number.*

Moreover, on the set of functions of bounded variation one can introduce the norm by the relation

$$\|h\| = |h(0)| + \mathit{var}(h), \quad h : [0,1] \to \mathbb{R}, \tag{12}$$

and with this norm the space $BV([0,1])$ of all functions of bounded variation on $[0,1]$ with its linear structure defined by the pointwise addition of functions and multiplication by a scalar from $\mathbb{R}$ becomes a Banach space. This space is not separable.

Received: 7 November 2005
Revised: 4 February 2006