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A FUNCTORIAL APPROACH TO THE BEHAVIOUR OF MULTIDIMENSIONAL CONTROL SYSTEMS

JEAN-FRANÇOIS POMMARET*, ALBAN QUADRAT**

* CERMICS, Ecole Nationale des Ponts et Chaussées,
6 et 8 Avenue Blaise Pascal,
77455 Marne-La-Vallée Cedex 02, France,
e-mail: pommaret@cermics.enpc.fr

** INRIA, CAFE project, 2004 Route des Lucioles,
BP 93, 06902 Sophia Antipolis cedex, France,
e-mail: Alban.Quadrat@sophia.inria.fr

We show how to use the extension and torsion functors in order to compute the torsion submodule of a differential module associated with a multidimensional control system. In particular, we show that the concept of the weak primeness of matrices corresponds to the torsion-freeness of a certain module.

Keywords: multidimensional systems, weak primeness, controllability, algebraic analysis, torsion and extension functors, rings of differential operators

1. Introduction

It is well known that the controllability of a linear multidimensional control system depends on an algebraic property (namely, the *torsion-freeness*) of a certain module M associated with the system (Oberst, 1990; Pillai and Shankar, 1999; Pommaret and Quadrat, 1999a; 1999b). The recent survey (Wood, 2000) gives different equivalent formulations of controllability and, in particular, the equivalence obtained in (Pillai and Shankar, 1999) between the torsion-freeness of M and the definition of controllability given by Willems (1991).

In this paper, we show how to use the powerful tools of homological algebra in order to compute the torsion submodule $t(M)$ of an A-module M, i.e. the non-controllable part of a control system (Pommaret and Quadrat, 1999a; 1999b; Willems, 1991). The obtained results show the link existing between the concept of the weak primeness of a matrix R with entries in A and the torsion part $t(M)$ of the module M associated with R. Finally, we prove the isomorphisms $t(M) \cong \mathrm{tor}_1^A(K/A, M) \cong \mathrm{ext}_A^1(N, A)$, where N is the *transposed* A*-module* of M, A is any *Ore Noetherian ring* and $K = Q(A)$ stands for the *field of fractions* of A. This result generalizes the previous results obtained in (Pommaret and Quadrat, 1999a; 1999b) for the ring D of differential operators.

2. Torsion Functor and Weak Primeness

In the remainder of the paper, we shall denote by A a Noetherian integral domain which is supposed to be either a commutative ring or a *left Ore domain*, namely a domain such that, for any couple $(a, b) \in A^2$, there exists a non-trivial couple $(u, v) \in A^2$ such that $u\,a = v\,b$. Moreover, let $K = Q(A)$ be the quotient field of A. If A and B are two integral domains, then we shall denote by ${}_B M_A$ a module M with the structure of a left B-module and of a right A-module (see (Kashiwara, 1970; Pommaret and Quadrat, 1999a) for more details for the non-commutative case).

In the literature on internal stabilization, the concept of *weak primeness* is useful (Quadrat, 2003; Smith, 1989).

Definition 1. Let R be an $l \times m$ matrix with entries in A. R is said to be *weakly left-prime* if

$$K^l R \cap A^m = A^l R,$$

where the vectors of A^l, A^m and K^l are row vectors.

Remark 1. If R is a full row rank $l \times m$ matrix ($l \leq m$) with entries in A, then R is weakly left-prime iff

$$\forall\, z \in K^l :\; z\,R \in A^m \Rightarrow z \in A^l.$$

Example 1. Let $D = \mathbb{R}(x_1, x_2)[d_1, d_2]$ be the ring of differential operators in d_i with rational coefficients in x_i, and let R be the full row rank matrix defined by

$$R = (x_2^3 d_1^2 - 2x_2^2 d_1 + x_2 : x_2^2 d_1 d_2 - x_2 d_2 - 1) \in D^{1\times 2}.$$

R is not weakly left-prime because we have

$$d_2 R = (x_2 d_1 - 1)R', \tag{1}$$

and thus

$$(x_2 d_1 - 1)^{-1} d_2 R = R',$$

where $(x_2 d_1 - 1)^{-1}\, d_2 \in K = Q(D)$ and

$$R' = (x_2^2 d_1 d_2 + 3x_2 d_1 - x_2 d_2 - 1 : x_2 d_2^2 + 2d_2) \in D^{1\times 2}.$$

♦

Let us interpret weak primeness in terms of modules. We need several definitions (Rotman, 1979).

Definition 2. The A-module defined by

$$t(M)\big\{m \in M \mid \exists\, 0 \neq a \in A:\ am = 0\big\}$$

is called the *torsion submodule* of an A-module M. An A-module M is *torsion-free* if $t(M) = 0$ and *torsion* if $t(M) = M$. We have the following exact sequence:

$$0 \longrightarrow t(M) \longrightarrow M \longrightarrow M/t(M) \longrightarrow 0, \tag{2}$$

where $M/t(M)$ is a torsion-free A-module. A module M_A (resp. ${}_B M_A$) is *flat* if for every short exact sequence of left A-modules

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0,$$

we have the following exact sequence of Abelian groups (resp. left B-modules):

$$0 \longrightarrow M \otimes_A N' \longrightarrow M \otimes_A N \longrightarrow M \otimes_A N'' \longrightarrow 0.$$

K is a flat A-module, i.e. the tensor product by K transforms exact sequences of left A-modules into exact sequences of left K-vector spaces, and we have the exact sequence of left A-modules (Rotman, 1979):

$$0 \longrightarrow t(M) \longrightarrow M \xrightarrow{i_K} K \otimes_A M \longrightarrow (K/A) \otimes_A M \longrightarrow 0. \tag{3}$$

Theorem 1. (Quadrat, 2003) *Let R be an $l \times m$ matrix with entries in A and $M = A^m / A^l\, R$, where the vectors of A^l and A^m are row vectors. The following assertions are equivalent:*

1. *R is weakly left-prime, i.e. $K^l\ R \cap A^m = A^l\, R$,*
2. *$t(M) = 0$, i.e. M is a torsion-free A-module.*

Proof. It is well known that K is a flat A-module (Rotman, 1979). Thus, tensoring by K the exact sequence

$$A^l \xrightarrow{.R} A^m \longrightarrow M \longrightarrow 0, \tag{4}$$

we obtain the exact sequence

$$K^l \xrightarrow{\mathrm{id}_K \otimes .R} K^m \longrightarrow K \otimes_A M \longrightarrow 0, \tag{5}$$

where $(\mathrm{id}_K \otimes .R)(z) = z\, R,\ \forall\, z \in K^l$. Hence we have the commutative exact diagram

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
0 & & 0 & & t(M) & & \\
\downarrow & & \downarrow & & \downarrow & & \\
A^l & \xrightarrow{.R} & A^m & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow i_K & & \\
K^l & \xrightarrow{\mathrm{id}_K \otimes .R} & K^m & \xrightarrow{i_K \otimes \pi} & K \otimes_A M & \longrightarrow & 0.
\end{array} \tag{6}$$

$1 \Rightarrow 2$. Let us take $m \in t(M)$, i.e. $i_K(m) = 0$. Using the fact that π is a surjective A-morphism, there exists $\lambda \in A^m$ such that $m = \pi(\lambda)$. Thus, we have $(i_K \otimes \pi)(\lambda) = i_K(\pi(\lambda)) = 0$, and, since (5) is an exact sequence, there exists $\mu \in K^l$ such that $\mu\, R = \lambda$. By hypothesis, R is weakly left-prime, and thus there exists $\nu \in A^l$ such that $\nu\, R = \lambda \Rightarrow m = \pi(\lambda) = \pi(\nu\, R) = 0$, i.e. $t(M) = 0$.

$2 \Rightarrow 1$. Let $\mu \in K^l$ such that $\mu\, R = \lambda \in A^m$. If we write $m = \pi(\lambda)$, then we have

$$i_K(m) = i_K\big(\pi(\lambda)\big) = (i_K \otimes \pi)(\lambda) = (i_K \otimes \pi)(\mu\, R) = 0,$$

and thus $i_K(m) = 0$. By hypothesis, we have $t(M) = 0$, i.e. $m = \pi(\lambda) = 0$. Since (4) is an exact sequence, there exists $\nu \in A^l$ such that $\nu\, R = \lambda$, and thus, we have $\mu\, R = \nu\, R$, i.e. $K^l\, R \cap A^m \subseteq A^l\, R$. Finally, using the trivial fact that $A^l\, R \subseteq K^l\, R \cap A^m$, we obtain $K^l\, R \cap A^m = A^l\, R$. ■

Example 2. Let us reconsider the matrix R defined in Example 1. We proved that R is not weakly left-prime. Therefore, by Theorem 1, we know that the D-module $M = D^2 / D\, R$ is not torsion-free. Let us denote by y (resp. u) the class of the first vector $e_1 = (1 : 0)$ (resp. $e_2 = (0 : 1)$) of the canonical basis of D^2 in M. Hence, with the notation $d_i\, d_j\, y = y_{ij}$, we find that M is defined by the following equation:

$$x_2^3\, y_{11} - 2\, x_2^2\, y_1 + x_2\, y + x_2^2\, u_{12} - x_2\, u_2 - u = 0,$$

as well as all its D-linear combinations. Using (1), we find that the class of the vector R' is in M, i.e. the element

$$z = x_2^2\, y_{12} + 3\, x_2\, y_1 - x_2\, y_2 - y + x_2\, u_{22} + 2\, u_2 \in M,$$

satisfies the equation $(x_2\, d_1 - 1)\, z = x_2\, z_1 - z = 0$, which means that z is a torsion element of the D-module M. ♦

Corollary 1. (Quadrat, 2003) *Let R be an $l \times m$ matrix with entries in A and the A-module $M = A^m/A^l R$. Then we have*

$$\begin{cases} t(M) = (K^l\, R \cap A^m)/A^l\, R, \\ M/t(M) = A^m/(K^l\, R \cap A^m). \end{cases} \tag{7}$$

Proof. We have the commutative exact diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 \longrightarrow & & A^l R & \longrightarrow & A^m \longrightarrow & M & \longrightarrow 0 \\ & & \downarrow & & \| & \downarrow & \\ 0 \longrightarrow & & K^l R \cap A^m & \longrightarrow & A^m \longrightarrow & A^m/(K^l R \cap A^m) & \longrightarrow 0 \\ & & \downarrow & & \downarrow & \downarrow & \\ & & (K^l R \cap A^m)/A^l R & & 0 & 0 & \\ & & \downarrow & & & & \\ & & 0, & & & & \end{array} \tag{8}$$

from which we deduce the following exact sequence:

$$0 \to (K^l\, R \cap A^m)/A^l\, R \to M \to A^m/(K^l\, R \cap A^m) \to 0.$$

We have $K \otimes_A ((K^l R \cap A^m)/A^l R) = K^l R/K^l R = 0$, i.e. the A-module $(K^l R \cap A^m)/A^l R$ is a torsion module. We have $K \otimes_A (K^l R \cap A^m) \cap A^m = K^l R \cap A^m$, which shows that the A-module $A^m/(K^l R \cap A^m)$ is torsion-free. From the exact sequence (2), we obtain (7). ■

Remark 2. Smith (1989) shows that if A is a commutative integral domain, then weak left-primeness implies minor left-primeness and the two concepts are equivalent if A is a *greatest common divisor domain*. See (Quadrat, 2003; Smith, 1989) for counter-examples of the fact that minor left-primeness does not generally imply weak left-primeness. We thank both the anonymous referee and J. Wood for pointing out to us that the concept of weak left-primeness is also equivalent to the generalized factor left-primeness in the case of a polynomial ring $A = k[\chi_1, \dots, \chi_n]$ with coefficients in a field k (Fornasini and Valcher, 1997; Wood *et al.*, 1998).

To study the defect of exactness that the tensor product by K/A introduces in exact sequences, we shall need the definition of the *torsion functor*.

Definition 3. (Rotman, 1979) Let $\cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\pi} M \longrightarrow 0$ be a free resolution of the left A-module M and a module N_A (resp. ${}_B N_A$). Then the Abelian groups (resp. left B-modules) of homology of the complex

$$\dots \xrightarrow{\mathrm{id}_N \otimes d_3} N \otimes_A F_2 \xrightarrow{\mathrm{id}_N \otimes d_2} N \otimes_A F_1 \xrightarrow{\mathrm{id}_N \otimes d_1} N \otimes_A F_0 \longrightarrow 0$$

do not depend on the choice of the free resolution of M and they are called $\mathrm{tor}_i^A(N, M)$. We have

$$\begin{cases} \mathrm{tor}_0^A(N, M) = N \otimes_A M, \\ \mathrm{tor}_i^A(N, M) = \ker(\mathrm{id}_N \otimes d_i)/\mathrm{im}(\mathrm{id}_N \otimes d_{i+1}),\ \forall\, i \geq 1. \end{cases}$$

Proposition 1. (Rotman, 1979) *If $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an exact sequence of left A-modules and a module N_A (resp. ${}_B N_A$), then we have the following exact sequence of Abelian groups (resp. left B-modules):*

$$\begin{array}{lll} & \cdots & \longrightarrow \mathrm{tor}_2^A(N, M'') \longrightarrow \\ \mathrm{tor}_1^A(N, M') \longrightarrow & \mathrm{tor}_1^A(N, M) \longrightarrow & \mathrm{tor}_1^A(N, M'') \longrightarrow \\ N \otimes_A M' & \longrightarrow N \otimes_A M & \longrightarrow N \otimes_A M'' \longrightarrow 0. \end{array}$$

In particular, a module N_A is flat *iff for any left A-module M we have $\mathrm{tor}_i^A(N, M) = 0$, $\forall\, i \geq 1$.*

Proposition 2. *Let M be a left A-module. Then*

$$t(M) \cong \mathrm{tor}_1^A(K/A, M).$$

Proof. Applying Proposition 1 to the exact sequence $0 \longrightarrow A \longrightarrow K \longrightarrow K/A \longrightarrow 0$, we obtain the following exact sequence:

$$\mathrm{tor}_1^A(K, M) \longrightarrow \mathrm{tor}_1^A(K/A, M) \longrightarrow M \xrightarrow{i_K} K \otimes_A M \longrightarrow (K/A) \otimes_A M \longrightarrow 0.$$

Since K is a flat A-module, we have $\mathrm{tor}_1^A(K, M) = 0$, and thus, $\ker i_K = \mathrm{tor}_1^A(K/A, M)$. Finally, using the exact sequence (3), we obtain $t(M) = \ker i_K = \mathrm{tor}_1^A(K/A, M)$. ■

Example 3. Consider the ring $D = \mathbb{R}[d_1, d_2, d_3]$ of the differential operators in d_i with coefficients in $\mathbb{R}$ and the multidimensional system defined by means of the gradient operator in $\mathbb{R}^3$:

$$\begin{cases} d_1\, z = 0, \\ d_2\, z = 0, \\ d_3\, z = 0. \end{cases} \tag{9}$$

We have the following free resolution of the D-module $M = D/D^3\,(d_1 : d_2 : d_3)^T$ corresponding to (9):

$$0 \longrightarrow D \xrightarrow{.R_3} D^3 \xrightarrow{.R_2} D^3 \xrightarrow{.R_1} D \xrightarrow{\pi} M \longrightarrow 0,$$

where

$$R_1 = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & -d_3 & d_2 \\ d_3 & 0 & -d_1 \\ -d_2 & d_1 & 0 \end{pmatrix}, \quad R_3 = R_1^T.$$

We have the following complex:

$$(K/A)^3 \overset{\mathrm{id}_{K/A}\otimes .R_2}{\longrightarrow} (K/A)^3 \overset{\mathrm{id}_{K/A}\otimes .R_1}{\longrightarrow} K/A,$$

and $\mathrm{tor}_1^A(K/A, M) = \ker(\mathrm{id}_{K/A} \otimes .R_1)/\mathrm{im}(\mathrm{id}_{K/A} \otimes .R_2)$. Let y be the class of the vector $(d_1^{-1}\ 0\ 0) \in K^3$ in $(K/A)^3$. Here y belongs to $\ker(\mathrm{id}_{K/A} \otimes .R_1)$ because we have

$$(d_1^{-1} : 0 : 0)\, R_1 = 1 \in A. \tag{10}$$

$y \in \mathrm{im}\,(\mathrm{id}_{K/A} \otimes .R_2)$ iff there exist $(t_1 : t_2 : t_3) \in K^3$ and $(s_1 : s_2 : s_3) \in A^3$ which satisfy the system

$$(t_1 : t_2 : t_3)\, R_2 + (s_1 : s_2 : s_3) = (d_1^{-1} : 0 : 0). \tag{11}$$

If (11) has a solution, then necessarily, by applying R_1 on the right of (11) and using (10), we must have

$$(s_1 : s_2 : s_3)\, R_1 = 1 \Leftrightarrow s_1 d_1 + s_2 d_2 + s_3 d_3 = 1, \quad s_i \in A.$$

We easily check that the last equality is never satisfied, and thus R_1 is not weakly left-prime. ♦

3. Extension Functor and Behaviour

Definition 4. Let $\cdots \overset{d_2}{\longrightarrow} F_1 \overset{d_1}{\longrightarrow} F_0 \overset{\pi}{\longrightarrow} M \longrightarrow 0$ be a free resolution of the left A-module M and ${}_AS$ (resp. ${}_AS_B$) a module. Then the Abelian groups (resp. the right B-modules) of cohomology of the complex

$$\cdots \overset{d_2^\star}{\longleftarrow} \mathrm{hom}_A(F_1, S) \overset{d_1^\star}{\longleftarrow} \mathrm{hom}_A(F_0, S) \longleftarrow 0,$$

where $d_i^\star(f) = f \circ d_i,\ \forall\, f \in \mathrm{hom}_A(F_{i-1}, S)$, do not depend on the choice of the free resolution of M and they are called $\mathrm{ext}_A^i(M, S)$. Thus we have

$$\begin{cases} \mathrm{ext}_A^0(M,S) = \mathrm{hom}_A(M,S), \\ \mathrm{ext}_A^i(M,S) = \ker d_{i+1}^\star / \mathrm{im}\, d_i^\star, \quad \forall\, i \geq 1. \end{cases}$$

A module ${}_AS$ is called *injective* if for every exact sequence of left A-modules

$$0 \longrightarrow M' \overset{f}{\longrightarrow} M \overset{g}{\longrightarrow} M'' \longrightarrow 0,$$

we have the exact sequence

$$0 \longleftarrow \mathrm{hom}_A(M', S) \overset{f^\star}{\longleftarrow} \mathrm{hom}_A(M, S) \overset{g^\star}{\longleftarrow} \mathrm{hom}_A(M'', S) \longleftarrow 0.$$

Example 4. If M is a finitely generated left A-module, then a free resolution of M can be written in the form

$$\cdots \longrightarrow A^{r_2} \overset{.R_2}{\longrightarrow} A^{r_1} \overset{.R_1}{\longrightarrow} A^{r_0} \overset{\pi}{\longrightarrow} M \longrightarrow 0,$$

where R_i is an $r_i \times r_{i-1}$ matrix with entries in A and $.R_i$ is the A-morphism defined by multiplying a row vector of length r_i on the left of R_i to obtain a row vector of length r_{i-1}. Hence the extension functor gives the defects of exactness of the following sequence:

$$\cdots \longleftarrow S^{r_2} \overset{R_2 .}{\longleftarrow} S^{r_1} \overset{R_1 .}{\longleftarrow} S^{r_0} \longleftarrow 0,$$

where $R_i.$ is the A-morphism defined by multiplying a column vector of length r_{i-1} on the right of R_i to obtain a column vector of length r_i. In particular, $\mathrm{ext}_A^0(M,S) = \mathrm{hom}_A(M,S)$ represents the solution $y \in S^{r_0}$ of the system $R_1\, y = 0$, whereas $\mathrm{ext}_A^1(M,S)$ is the obstruction for $z \in S^{r_1}$ satisfying $R_2\, z = 0$ to be of the form $z = R_1\, y$ with $y \in S^{r_0}$. If A is the ring D of differential operators with constant coefficients and Ω is an open convex of $\mathbb{R}^n$, then $C^\infty(\Omega), \mathcal{D}'(\Omega)$ and $\mathcal{S}'(\Omega)$ are examples of injective D-modules (Malgrange, 1966; Oberst, 1990; Shankar, 2001; Wood, 2000). In particular, if R_1 is a matrix which defines a multidimensional control system, then $\mathrm{ext}_A^0(M,S) = \mathrm{hom}_A(M,S)$ corresponds to the *behaviour* of the system with respect to the *signal module* S (Willems, 1991). ♦

Proposition 3. (Rotman, 1979) *If* $0 \longrightarrow M' \overset{f}{\longrightarrow} M \overset{g}{\longrightarrow} M'' \longrightarrow 0$ *is an exact sequence of left* A*-modules and* ${}_AS$ *(resp.* ${}_AS_B$*) is a module, then we have the following exact sequence of Abelian groups (resp. right* B*-modules):*

$$\begin{aligned} 0 &\longrightarrow \mathrm{hom}_A(M'', S) \overset{g^\star}{\longrightarrow} \mathrm{hom}_A(M, S) \overset{f^\star}{\longrightarrow} \mathrm{hom}_A(M', S) \\ &\longrightarrow \mathrm{ext}_A^1(M'', S) \longrightarrow \mathrm{ext}_A^1(M, S) \longrightarrow \mathrm{ext}_A^1(M', S) \\ &\longrightarrow \mathrm{ext}_A^2(M'', S) \longrightarrow \cdots \end{aligned}$$

In particular, a module ${}_AS$ *is injective iff, for any left* A*-module* M*, we have* $\mathrm{ext}_A^i(M,S) = 0,\ i \geq 1$.

If M is a left A-module defined by a finite-free presentation $F_1 \overset{d_1}{\longrightarrow} F_0 \overset{\pi}{\longrightarrow} M \longrightarrow 0$, then we can define the right A-module N by

$$0 \longleftarrow N \longleftarrow F_1^\star \overset{d_1^\star}{\longleftarrow} F_0^\star \longleftarrow M^\star \longleftarrow 0. \tag{12}$$

Two different free resolutions of M give two different right A-modules N and N', and thus the right A-module N is not uniquely defined by M. But in (Pommaret and Quadrat, 2000; Quadrat, 1999) it is shown that N is uniquely defined up to a *projective equivalence* (Rotman, 1979), a fact which implies $\mathrm{ext}_A^i(N,S) \cong \mathrm{ext}_A^i(N',S)$ and $\mathrm{tor}_i^A(N,S) \cong \mathrm{tor}_i^A(N',S)$ for $i \geq 1$ and any left

A-module S. The module N plays a crucial role in the study of algebraic properties of the left A-module M and their correspondence to the different types of primeness (Oberst, 1990; Pommaret and Quadrat, 1999b; 2000). An interesting application of this result is the following. Let us take a free resolution of N of the form

$$0 \longleftarrow N \longleftarrow A^{r_1} \xleftarrow{R_1 .} A^{r_0} \xleftarrow{R_0 .} A^{r_{-1}} \longleftarrow \cdots .$$

Deleting N and taking the tensor product by a left A-module S, we obtain the following sequence:

$$0 \longleftarrow S^{r_1} \xleftarrow{R_1 .} S^{r_0} \xleftarrow{R_0 .} S^{r_{-1}} \longleftarrow \cdots .$$

The defects of exactness are given by $\mathrm{tor}_i^A(N,S)$. Thus, if S is a flat left A-module, we have parametrized the solution $y \in S^{r_0}$ of the system $R_1\, y = 0$ by $y = R_0 z$, with $z \in S^{r_{-1}}$ and so on. For example, if A is the ring D of differential operators with constant coefficients and Ω is an open convex of $\mathbb{R}^n$, then $\mathcal{S}(\Omega), \mathcal{D}(\Omega)$ and $\mathcal{E}'(\Omega)$ are flat D-modules (Malgrange, 1966; Shankar, 2001).

4. Duality between Extension and Torsion Functors

Theorem 2. (Quadrat, 1999) *Let M be a left A-module defined by the finite presentation $F_1 \xrightarrow{d_1} F_0 \xrightarrow{\pi} M \longrightarrow 0$ and an exact sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ of right A-modules. Then we have the commutative exact diagram (13), where we denote by N the right A-module defined by (12). There exist two connecting maps*

$$\begin{cases} \delta : \ \hom_A(N,Z) \longrightarrow X \otimes_A M, \\ \kappa : \ \mathrm{tor}_1^A(Z,M) \longrightarrow \mathrm{ext}_A^1(N,X), \end{cases}$$

such that the following two sequences are exact:

$$\begin{array}{l} 0 \longrightarrow \hom_A(N,X) \longrightarrow \hom_A(N,Y) \longrightarrow \hom_A(N,Z) \\ \xrightarrow{\delta} X \otimes_A M \longrightarrow Y \otimes_A M \longrightarrow Z \otimes_A M \longrightarrow 0, \end{array}$$

$$\begin{array}{lll} & \cdots & \longrightarrow \mathrm{tor}_2^A(Z,M) \\ \longrightarrow \mathrm{tor}_1^A(X,M) \longrightarrow & \mathrm{tor}_1^A(Y,M) & \longrightarrow \mathrm{tor}_1^A(Z,M) \\ \xrightarrow{\kappa} \mathrm{ext}_A^1(N,X) \longrightarrow & \mathrm{ext}_A^1(N,Y) & \longrightarrow \mathrm{ext}_A^1(N,Z) \\ \longrightarrow \mathrm{ext}_A^2(N,X) \longrightarrow & \cdots . & \end{array}$$

Proof. First of all, let us notice that if F is a finitely generated projective (free) left A-module and X is a right A-module, then we have $\hom_A(F^\star, X) \cong X \otimes_A F$, where $F^\star = \hom_A(F,A)$ (Rotman, 1979). By taking the tensor product of the finite presentation of M with respect to X, we obtain the following exact sequence:

$$X \otimes_A F_1 \xrightarrow{\mathrm{id}_X \otimes_A d_1} X \otimes_A F_0 \xrightarrow{\mathrm{id}_X \otimes_A \pi} X \otimes_A M \longrightarrow 0,$$

whereas, by taking the A-morphisms of the exact sequence (12) in X, we obtain the following exact sequence:

$$\begin{array}{ccccc} 0 \longrightarrow \hom_A(N,X) \longrightarrow & \hom_A(F_1^\star, X) & \longrightarrow & \hom_A(F_0^\star, X). \\ & \| & & \| \\ & X \otimes_A F_1 & \longrightarrow & X \otimes_A F_0 \end{array}$$

Therefore, we have the exact sequence

$$\begin{array}{l} 0 \longrightarrow \hom_A(N,X) \longrightarrow X \otimes_A F_1 \xrightarrow{\mathrm{id}_X \otimes_A d_1} X \otimes_A F_0 \\ \qquad \xrightarrow{\mathrm{id}_X \otimes_A \pi} X \otimes_A M \longrightarrow 0. \end{array}$$

Finally, we obtain (13) if we notice that we have the following short exact sequence:

$$0 \longrightarrow X \otimes_A F_i \longrightarrow Y \otimes_A F_i \longrightarrow Z \otimes_A F_i \longrightarrow 0,$$

because F_i is a free, and thus, a flat A-module (Rotman, 1979) for $i = 0, 1$. Then two chases in (13) prove the theorem. ■

Lemma 1. *If M is a left A-module, then $\mathrm{ext}_A^i(M,A)$ is a finitely generated torsion right A-module for $i \geq 1$.*

Proof. The fact that $\mathrm{ext}_A^i(M,A)$ is a finitely generated right A-module for all $i \geq 1$ can be easily proved (Rotman, 1979). Now, let F be a maximal free A-module included in M. Then we have the exact sequence

$$0 \longrightarrow F \longrightarrow M \longrightarrow T \longrightarrow 0,$$

where $T = M/F$ is a torsion left A-module. Then we can apply Proposition 3 to the previous exact sequence to obtain the following exact sequences:

$$\begin{array}{l} 0 = \hom_A(T,A) \longrightarrow \hom_A(M,A) \longrightarrow \hom_A(F,A) \\ \longrightarrow \mathrm{ext}_A^1(T,A) \longrightarrow \mathrm{ext}_A^1(M,A) \longrightarrow \mathrm{ext}_A^1(F,A) = 0, \end{array}$$

$$\begin{array}{l} 0 = \mathrm{ext}_A^{i-1}(F,A) \longrightarrow \mathrm{ext}_A^i(T,A) \longrightarrow \mathrm{ext}_A^i(M,A) \\ \qquad \longrightarrow \mathrm{ext}_A^i(F,A) = 0, \ \ \forall\, i \geq 2. \end{array}$$

From the second exact sequence, we deduce that $\mathrm{ext}_A^i(M,A) \cong \mathrm{ext}_A^i(T,A)$, $\forall\, i \geq 2$. Now, using the fact that K is a flat A-module, we have $\forall\, i \geq 1$, $K \otimes_A \mathrm{ext}_A^i(T,A) \cong \mathrm{ext}_K^i(K \otimes_A T, K) = 0$, because T is a torsion left A-module (Rotman, 1979). Therefore we have $K \otimes_A \mathrm{ext}_A^i(M,A) = 0$, $\forall\, i \geq 2$, i.e. $\mathrm{ext}_A^i(M,A)$ is a torsion right A-module for all $i \geq 2$. Finally, if we take the tensor product of the first exact sequence with respect to K, then we obtain $K \otimes_A \mathrm{ext}_A^1(M,A) = 0$, i.e. $\mathrm{ext}_A^1(M,A)$ is a torsion right A-module. ■

$$
\begin{array}{ccccccccccc}
 & & & & & & & & \vdots & & \\
 & & & & & & & & \downarrow & & \\
 & & & & & & & & \mathrm{tor}_1^A(X,M) & & \\
 & & & & & & & & \downarrow & & \\
 & & & & & & & & \mathrm{tor}_1^A(Y,M) & & \\
 & & & & & & & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & \mathrm{tor}_1^A(Z,M) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \hom_A(N,X) & \longrightarrow & X\otimes_A F_1 & \longrightarrow & X\otimes_A F_0 & \longrightarrow & X\otimes_A M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \hom_A(N,Y) & \longrightarrow & Y\otimes_A F_1 & \longrightarrow & Y\otimes_A F_0 & \longrightarrow & Y\otimes_A M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \hom_A(N,Z) & \longrightarrow & Z\otimes_A F_1 & \longrightarrow & Z\otimes_A F_0 & \longrightarrow & Z\otimes_A M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathrm{ext}_A^1(N,X) & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & & & & & & & \\
 & & \mathrm{ext}_A^1(N,Y) & & & & & & & & \\
 & & \downarrow & & & & & & & & \\
 & & \mathrm{ext}_A^1(N,Z) & & & & & & & & \\
 & & \downarrow & & & & & & & & \\
 & & \vdots & & & & & & & &
\end{array}
\qquad (13)
$$

$$
\begin{array}{ccccccccccc}
 & & & & & & & & 0 & & \\
 & & & & & & & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & \mathrm{tor}_1^A(K/A,M) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \hom_A(N,A) & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \hom_A(N,K) & \longrightarrow & K\otimes_A F_1 & \longrightarrow & K\otimes_A F_0 & \longrightarrow & K\otimes_A M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \hom_A(N,K/A) & \longrightarrow & (K/A)\otimes_A F_1 & \longrightarrow & (K/A)\otimes_A F_0 & \longrightarrow & (K/A)\otimes_A M & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathrm{ext}_A^1(N,A) & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & & & & & & & \\
 & & 0 & & & & & & & &
\end{array}
\qquad (14)
$$

Corollary 2. (Quadrat, 1999) *We have the commutative exact diagram (14) of left A-modules and the following exact sequence of left A-modules:*

$$
\begin{aligned}
0 \longrightarrow \hom_A(N,A) \longrightarrow \hom_A(N,K) \longrightarrow \hom_A(N,K/A)\\
\xrightarrow{\delta} M \xrightarrow{i_K} K\otimes_A M \longrightarrow (K/A)\otimes_A M \longrightarrow 0.
\end{aligned}
$$

Moreover, we have the following isomorphisms:

$$
t(M) \cong \mathrm{ext}_A^1(N,A) \cong \mathrm{tor}_1^A(K/A,M). \qquad (15)
$$

Proof. Using the fact that K is a flat A-module and $\mathrm{ext}_A^1(N,A)$ is a torsion left A-module (see Lemma 1), we obtain $\mathrm{ext}_A^1(N,K) \cong K\otimes_A \mathrm{ext}_A^1(N,A) = 0$ and $\mathrm{tor}_1^A(K,M) = 0$. Now, applying Theorem 2 to the exact sequence $0 \longrightarrow A \longrightarrow K \longrightarrow K/A \longrightarrow 0$ of both left and right A-modules, we obtain the commutative exact diagram (14). Finally, the snake lemma gives the isomorphisms (15). ■

Remark 3. K is an injective module over A as it is a torsion-free and *divisible* A-module (Rotman, 1979). Therefore, we could have obtained directly:

$$
\mathrm{ext}_A^i(N,K) = 0, \quad \forall\, i \geq 1.
$$

Example 5. Let us reconsider Example 3. The D-module N corresponding to M is defined by the following exact sequence: $0 \longleftarrow N \longleftarrow D^3 \stackrel{R_1.}{\longleftarrow} D \longleftarrow 0$. Therefore, dualizing this exact sequence, we have the sequence $0 \longrightarrow N^\star \longrightarrow D^3 \stackrel{.R_1}{\longrightarrow} D \longrightarrow 0$ and we obtain $\mathrm{ext}^1_D(N, D) = M$. Finally, we have $t(M) = M$ and M is a torsion D-module.

If $D = K[d_1, d_2, \ldots, d_n]$ is the ring of differential operators with entries in a differential field K, i.e. a field K endowed with n derivations ∂_i which satisfy

$$\begin{cases} \partial_i(a+b) = \partial_i\, a + \partial_i\, b, \\ \partial_i\,(a\, b) = (\partial_i\, a)\, b + a\,(\partial_i\, b), \end{cases}$$

$\forall\, a,\, b \in K$, then, using the *adjoint functor* which transforms a right D-module $N = D^l / R\, D^m$ into a left D-module defined by $\tilde{N} = D^l / D^m\, \tilde{R}$, where $\tilde{R}$ is the *formal adjoint* of R, we can effectively compute $\mathrm{ext}^1_D(N, D) \cong \mathrm{ext}^1_D(\tilde{N}, D)$ using the algorithms developed in (Pommaret, 2001; Pommaret and Quadrat, 1999a; 1999b). We refer the reader to (Pommaret and Quadrat, 1999a; 1999b) for more information and examples.

5. Conclusion

We hope to have convinced the reader that homological tools such as extension and torsion functors are very useful and powerful in the study of multidimensional control systems. They allowed us to show the link existing between the concept of weak primeness and the concept of torsion-freeness in module theory. Moreover, we gave a purely algebraic proof of the isomorphism existing between $t(M)$ and $\mathrm{ext}^1_A(N, A)$, for any Noetherian left Ore integral domain A and any finitely generated A-module M. This result generalizes those obtained in (Pommaret and Quadrat, 1999b) for rings of differential operators.

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