GRADIENT OBSERVABILITY FOR DIFFUSION SYSTEMS

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The aim of this paper is to study regional gradient observability for a diffusion system and the reconstruction of the state gradient without the knowledge of the state. First, we give definitions and characterizations of these new concepts and establish necessary conditions for the sensor structure in order to obtain regional gradient observability. We also explore an approach which allows for a regional gradient reconstruction. The developed method is original and leads to a numerical algorithm illustrated by simulations.

Keywords: diffusion system, observability, regional gradient observability, gradient strategic sensor, gradient reconstruction

1. Introduction

The analysis of distributed systems leads to a set of concepts, such as controllability, observability, stability, stabilisability, detectability, (Lions, 1988; Curtain and Zwart, 1995), which allow to understand them better and consequently enable us to steer them in a better way. This analysis can be made by means of the output and input parameter structures (El Jai and Pritchard, 1988). Recently, the concept of regional analysis was introduced in (Zerrik, 1993; El Jai *et al.*, 1994), which offers important tools for solving many real problems, particularly the concept of regional observability, which refers to problems in which the observed state of interest is not fully specified as a state, but concerns only a region ω, a portion of the spatial domain Ω on which the system is considered. It was extended by Zerrik *et al.* (1999) to the case where the subregion ω is a part of the boundary $\partial\Omega$ of Ω.

In this paper we introduce a new concept, i.e., regional gradient observability, where one is interested in the knowledge of the state gradient only in a critical subregion of the system domain without the knowledge of the state itself. The introduction of this concept is motivated by real situations. For example, this is the case of the determination of laminar boundary flux conditions developed in a steady state by a heated vertical plate. This problem consists in studying the thermal transfer by natural convection which is generated by a uniformly heated plate located in a small enclosure. Inside that enclosure, differences in wall surface temperatures produce natural convection movements.

The experimental device consists of an enclosure of $2\,\mathrm{m} \times 1.5\,\mathrm{m} \times 1\,\mathrm{m}$ with a specific wall and a temperature measurement system. The active wall with natural convection is successively composed (from the front to the back side) of a resin material ($4\,\mathrm{mm}$ thick), a copper plate ($5\,\mathrm{mm}$ thick) and a heat exchanger.

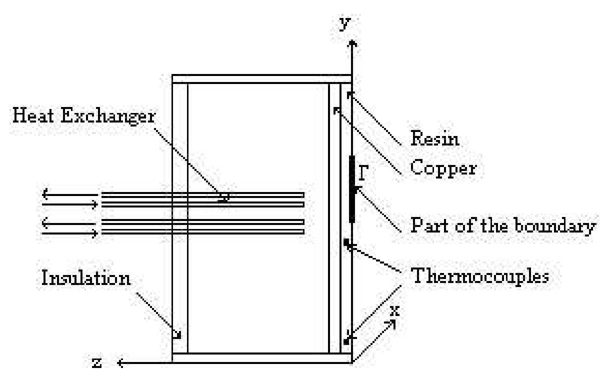

Fig. 1. Profile of the active plate.

In Fig. 1, we show the heat exchanger, which maintains a prescribed temperature on the back surface of the plate by means of hot water circulation. All the sides of this active wall are insulated (adiabatic conditions) except for the front surface. The objective is to find the unknown boundary convective condition on a part Γ of the front surface of the active plate using measurements given by internal thermocouples, cf. (Sparrow, 1963; Cuniasse-Langhans, 1998).

Considering directly this boundary problem seems to be difficult, but it can be solved by reconstructing the gradient in an internal subregion ω of the plate such that Γ is a part of the boundary of ω. More precisely, let (S) be a linear system evolving on Ω, with a suitable state space, and suppose that the initial state y_0 and its gradient ∇y_0 are unknowns. Assume that the measurements are given by means of an output function (depending on the number and the structure of the sensors). The problem concerns the reconstruction of the initial gradient ∇y_0 in a subregion ω located in the interior of the system domain Ω.

The paper is organized as follows: Section 2 is devoted to the presentation of the system considered, as well as to definitions and characterizations of this new concept. In the third section we establish the relation between regional gradient observability and sensors. Section 4 is focused on the regional reconstruction of the initial state gradient, and various types of sensors are discussed. In the last section we develop a numerical approach, which is illustrated by simulations that lead to some conjectures.

2. Regional Gradient Observability

2.1. Problem Statement

Let Ω be an open bounded subset of $\mathbb{R}^n (n = 1, 2, 3)$ with a smooth boundary $\partial\Omega$ and a subregion ω of Ω. For $T > 0$, we write $Q = \Omega\times]0, T[, \Sigma = \partial\Omega\times]0, T[$. Consider a parabolic system defined by

$$\begin{cases} \dfrac{\partial y}{\partial t}(x,t) = Ay(x,t) & \text{in } Q, \\ y(\xi,t) = 0 & \text{in } \Sigma, \\ y(x,0) = y_0(x) & \text{in } \Omega, \end{cases} \tag{1}$$

with the measurements given by the output function:

$$z(t) = Cy(t). \tag{2}$$

We have

$$A = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right)$$

with $a_{ij} \in \mathcal{D}(\bar{Q})$. Suppose that $-A$ is elliptic, i.e., there exists $\alpha > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq \alpha \sum_{j=1}^{n} |\xi_j|^2 \text{ a.e. on } Q$$

$$\forall\, \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n.$$

This operator is a second-order differential linear operator which generates a strongly continuous semigroup $(S(t))_{t\geq 0}$ on the Hilbert space $L^2(\Omega)$ endowed with the inner product

$$\langle u, v\rangle_{L^2(\Omega)} = \int_\Omega u(x)v(y)\,\mathrm{d}x\,\mathrm{d}y$$

and the norm

$$\|u\|^2_{L^2(\Omega)} = \int_\Omega \left|u(x)\right|^2 \mathrm{d}x.$$

The adjoint A^* is defined by

$$A^* = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ji}\frac{\partial}{\partial x_j}\right)$$

and C: $H_0^1(\Omega) \longrightarrow \mathbb{R}^q$ is linear and depends on the considered sensor structure. For the case of an unbounded C, care must be taken since $D(C) \subset H_0^1(\Omega)$ and $S(t)(D(C)) \subset D(C)$, $\forall\, t \geq 0$ (see El Jai and Pritchard, 1988, p. 43, Sect. 3.2).

We assume here that $y_0 \in H_0^1(\Omega)$. The system (1) is autonomous and can be interpreted in the mild sense with the solution $y(t) = S(t)y_0$, cf. (Curtain and Pritchard, 1978, p. 31, Def. 2.23; Curtain and Zwart, 1995, p. 104, Def. 3.1.4). The initial state y_0 and its gradient are supposed to be unknown.

For a region $\omega \subset \Omega$ assumed to be open, regular and of a positive Lebesgue measure, the problem of regional gradient observability consists in directly reconstructing the initial gradient ∇y_0 in the subregion ω with the knowledge of (1) and (2).

We first recall that a sensor is defined by the couple (D, f), where D is its spatial support represented by a nonempty part of $\bar{\Omega}$ and f represents the distribution of the sensing measurements on D. Depending on the nature of D and f, we could have various types of sensors. A sensor may be pointwise if $D = \{b\}$ with $b \in \overline{\Omega}$ and $f = \delta(\cdot - b)$, where δ is the Dirac mass concentrated at b. In this case the operator C is unbounded and the output function (2) can be written in the form

$$z(t) = y(b,t). \tag{3}$$

It may be zonal when $D \subset \overline{\Omega}$ and $f \in L^2(D)$. The output function (2) can be written in the form

$$z(t) = \int_D y(x,t)f(x)\,\mathrm{d}x. \tag{4}$$

For the definitions and properties of strategic and regional strategic sensors, we refer the reader to (El Jai and Pritchard, 1988; El Jai *et al.*, 1993; Zerrik, 1993; Zerrik *et al.*, 1999). The observation space is $\mathcal{O} = L^2(0, T; \mathbb{R}^q)$. The system (1) is autonomous and (2) allows us to write $z(t) = CS(t)y_0(x)$. We define by

$$K : \quad \begin{cases} X \longrightarrow \mathcal{O}, \\ h \longrightarrow CS(\cdot)h \end{cases}$$

the operator on $X = H_0^1(\Omega)$, which is linear and bounded in the zonal case, with the adjoint defined by

$$K^* : \begin{cases} \mathcal{O} \longrightarrow X, \\ z^* \longrightarrow \displaystyle\int_0^T S^*(\tau) C^* z^*(\tau) \, \mathrm{d}\tau. \end{cases}$$

Consider the operator

$$\nabla : \begin{cases} H_0^1(\Omega) \longrightarrow \big(L^2(\Omega)\big)^n, \\ y \longrightarrow \nabla y = \left(\dfrac{\partial y}{\partial x_1}, \dots, \dfrac{\partial y}{\partial x_n} \right). \end{cases}$$

Its adjoint ∇^* is given by

$$\nabla^* : \begin{cases} \big(L^2(\Omega)\big)^n \longrightarrow H_0^1(\Omega), \\ y \longrightarrow \nabla^* y = v, \end{cases}$$

where v is a solution of the Dirichlet problem

$$\begin{cases} \Delta v = -\mathrm{div}(y) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

For a nonempty subset ω of Ω, we consider the operators

$$\chi_\omega : \begin{cases} \big(L^2(\Omega)\big)^n \longrightarrow \big(L^2(\omega)\big)^n, \\ y \longrightarrow y_{|\omega} \end{cases}$$

and

$$\tilde{\chi}_\omega : \begin{cases} L^2(\Omega) \longrightarrow L^2(\omega), \\ y \longrightarrow y_{|\omega}. \end{cases}$$

Their adjoints are respectively denoted by χ_ω^* and $\tilde{\chi}_\omega^*$, and are defined by

$$\chi_\omega^* : \begin{cases} \big(L^2(\omega)\big)^n \longrightarrow \big(L^2(\Omega)\big)^n, \\ y \longrightarrow \chi_\omega^* y = \begin{cases} y_{|\omega} & \text{in } \omega, \\ 0 & \text{in } \Omega \setminus \omega, \end{cases} \end{cases}$$

and

$$\tilde{\chi}_\omega^* : \begin{cases} L^2(\omega) \longrightarrow L^2(\Omega), \\ y \longrightarrow \tilde{\chi}_\omega^* y = \begin{cases} y_{|\omega} & \text{in } \omega, \\ 0 & \text{in } \Omega \setminus \omega. \end{cases} \end{cases}$$

We recall that the system (1) is said to be exactly (resp. approximately) regionally observable if $\mathrm{Im}\,(\bar{\chi}_\omega K^*) = H^1(\omega)$ (resp. $\overline{\mathrm{Im}\,(\bar{\chi}_\omega K^*)} = H^1(\omega)$), where $\bar{\chi}_\omega$: $H^1(\Omega) \longrightarrow H^1(\omega)$ is the restriction to ω. For more details, we refer the reader to (El Jai *et al.*, 1993; Zerrik, 1993).

The idea is based on the existence of an operator H: $\mathcal{O} \longrightarrow (L^2(\omega))^n$ such that $Hz = \nabla y_0$. This is a natural extension of the observability concept (El Jai and Pritchard, 1988, p. 45). Then we introduce the operator $H = \chi_\omega \nabla K^*$ from $\mathcal{O}$ into $(L^2(\omega))^n$.

Definition 1. The system (1) together with the output (2) is said to be *approximately regionally gradient observable* in ω or *approximately G-observable* in ω if $\mathrm{Ker}\, H^* = \{0\}$.

We see that if a system is approximately G-observable, then there is a one-to-one relationship between the output and the initial gradient, viz. if z is given and y_0 satisfies $z = CS(\cdot)y_0$, then ∇y_0 is unique. In many physical problems this concept is not strong enough since if the output z is slightly perturbed, then the corresponding gradient of the initial state may considerably vary. We therefore introduce the following continuity hypothesis.

Definition 2. The system (1) together with the output (2) is said to be *exactly regionally gradient observable* in ω or *exactly G-observable* in ω if $\mathrm{Im}\,(H) = (L^2(\omega))^n$.

This problem is often encountred in many physical applications. This is the case of heat transfer (the Fourier law), the exchange concentration problem (the Fick law) and other problems, such as material deformation (or distortion), where one would like to know the gradient evolution in a subregion of the evolution domain.

It is clear that

- Approximate G-observability in ω amounts to the condition $[H^* z^* = 0 \Rightarrow z^* = 0]$, which is equivalent to the dual condition $\overline{\mathrm{Im}\, H} = (L^2(\omega))^n$.
- If a system is exactly (resp. approximately) regionally observable in ω (El Jai *et al.*, 1994), then it is exactly (resp. approximately) regionally G-observable in ω. Indeed, if the system (1) is observable in ω, then we can reconstruct the initial state y_0 using one of the approaches given in (Zerrik *et al.*, 1999) and then deduce its gradient ∇y_0 in ω.
- If a system is exactly (resp. approximately) G-observable in ω, it is exactly (resp. approximately) G-observable in ω_1 for all $\omega_1 \subset \omega$, but the following example outlines a system which may be regionally approximately G-observable in ω but not approximately G-observable in the whole domain Ω.

2.2. Counter-example

Let $\Omega =]0,1[\times]0,1[$. Consider the following two-dimensional system:

$$\begin{cases} \dfrac{\partial y}{\partial t}(x_1,x_2,t) = \dfrac{\partial^2 y}{\partial x_1^2}(x_1,x_2,t) \\ \qquad\qquad + \dfrac{\partial^2 y}{\partial x_2^2}(x_1,x_2,t) & \text{in } Q, \\ y(\zeta,\eta,t) = 0 & \text{on } \Sigma, \\ y(x_1,x_2,0) = y_0(x_1,x_2) & \text{in } \Omega. \end{cases} \tag{5}$$

The measurements are given by the output

$$z(t) = \int_D y(x_1,x_2,t) f(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2,$$

where $D = \{1/2\}\times]0,1[$ is the sensor support and $f(x_1,x_2) = \sin\pi x_2$ is the function measure.

Let $\omega =]0,1[\times]1/6,2/6[$ and $g(x_1,x_2) = (\cos\pi x_1 \sin 2\pi x_2, \sin\pi x_1\cos 2\pi x_2) \in (L^2(\Omega))^2$ be the gradient to be observed. The operator

$$A = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$

generates a semigroup $(S(t))_{t\geq 0}$ on $L^2(\Omega)$, given by

$$S(t)y = \sum_{i,j=1}^{\infty} e^{\lambda_{ij}t}\langle y,\ \varphi_{ij}\rangle_{L^2(\Omega)}\varphi_{ij},$$

where $\varphi_{ij}(x_1,x_2) = 2\sin(i\pi x_1)\sin(j\pi x_2)$ and $\lambda_{ij} = -(i^2+j^2)\pi^2$.

Proposition 1.
The gradient g is not approximately G-observable in the whole domain Ω. However, it is approximately G-observable in the subregion ω.

Proof. To prove that g is not approximately G-observable in Ω, we must show that $g \in \operatorname{Ker} K\nabla^*$. We have

$$\begin{aligned} K\nabla^*(g) &= \sum_{i,j=1}^{\infty} e^{\lambda_{ij}t}\langle \nabla^* g, \varphi_{ij}\rangle_{L^2(\Omega)}\langle \varphi_{ij}, f\rangle_{L^2(D)} \\ &= \sum_{i,j=1}^{\infty} e^{\lambda_{ij}t}\frac{6\pi}{\lambda_{ij}}\int_0^1 \sin(i\pi x_1)\sin(\pi x_1)\,\mathrm{d}x_1 \\ &\quad\times\int_0^1 \sin(j\pi x_2)\sin(2\pi x_2)\,\mathrm{d}x_2 \\ &\quad\times\sin\left(\frac{i\pi}{2}\right)\int_0^1 \sin(j\pi x_2)\sin(\pi x_2)\,\mathrm{d}x_2 = 0. \end{aligned}$$

Now, we show that the restriction of g to the subregion ω is approximately G-observable in ω. We get

$$\begin{aligned} &K\nabla^*\chi_\omega^*\chi_\omega(g) \\ &= \sum_{i,j=1}^{\infty} e^{\lambda_{ij}t}\langle \nabla^*\chi_\omega^*\chi_\omega(g), \varphi_{ij}\rangle_{L^2(\Omega)}\langle \varphi_{ij}, f\rangle_{L^2(D)} \\ &= \sum_{i,j=1}^{\infty} e^{\lambda_{ij}t}\frac{12\pi}{\lambda_{ij}}\int_0^1 \sin(i\pi x_1)\sin(\pi x_1)\,\mathrm{d}x_1 \\ &\quad\times\int_{\frac{1}{6}}^{\frac{2}{6}} \sin(j\pi x_2)\sin(2\pi x_2)\,\mathrm{d}x_2 \\ &\quad\times\sin\left(\frac{i\pi}{2}\right)\int_0^1 \sin(j\pi x_2)\sin(\pi x_2)\,\mathrm{d}x_2 \\ &= \frac{(2-3\sqrt{3})}{4\pi^2}e^{-2\pi^2 t} \neq 0 \end{aligned}$$

■

2.3. Characterizations

We can characterize G-observability in ω by the following results.

Proposition 2. 1. *The system (1) with the output (2) is exactly G-observable in ω if and only if, there exists $c > 0$ such that, for all $z^* \in (L^2(\omega))^n$,*

$$\|z^*\|_{(L^2(\omega))^n} \leq c\|K\nabla^*\chi_\omega^* z^*\|_{\mathcal{O}}.$$

2. *The system (1) together with the output (2) is approximately G-observable in ω if and only if the operator $N_\omega = HH^*$ is positive definite.*

Proof. We use standard functional analysis tools:

1. Let $h = \mathrm{Id}_{(L^2(\omega))^n}$ and $g = \chi_\omega\nabla K^*$. Since the system is exactly G-observable in ω, we have $\operatorname{Im} h \subset \operatorname{Im} g$, which is equivalent to the fact that there exists $c > 0$ such that

$$\|h^* z^*\|_{(L^2(\omega))^n} \leq c\|g^* z^*\|_{L^2(0,T;\mathbb{R}^q)}, \quad \forall\ z^* \in \left(L^2(\omega)\right)^n.$$

2. Let $z^* \in (L^2(\omega))^n$ such that $\langle N_\omega z^*, z^*\rangle_{(L^2(\omega))^n} = 0$. Then $\|H^* z^*\|_{\mathcal{O}} = 0$ and therefore $z^* \in \ker H^*$. Consequently, $z^* = 0$, i.e., N_ω is positive definite.

Conversely, let $z^* \in (L^2(\omega))^n$ such that $H^* z^* = 0$. Then $\langle H^* z^*, H^* z^*\rangle_{\mathcal{O}} = 0$ and thus $\langle N_\omega z^*, z^*\rangle_{(L^2(\omega))^n} = 0$. Hence $z^* = 0$, i.e., the system is approximately G-observable. ■

Now, assume that the system (1) is not G-observable in Ω and let $(\bar{\varphi}_i)_{i\in\mathbb{N}^n}$ be a basis in $(L^2(\Omega))^n$. Let $I \subset \mathbb{N}^n$ be such that $\ker K\nabla^* = \operatorname{span}\{(\bar{\varphi}_i)_{i\in I}\}$ and $J = \mathbb{N}^n\backslash I$.

Proposition 3. *The following properties are equivalent:*

1. *The system (1) is approximately G-observable in ω.*
2. $\overline{\operatorname{span}\{(\chi_\omega\bar{\varphi}_i)_{i\in J}\}} = (L^2(\omega))^n$.
3. *If $y \in (L^2(\omega))^n$ is such that $\langle y, \chi_\omega\bar{\varphi}_i\rangle_{(L^2(\omega))^n} = 0$ for all $i \in J$, then $y = 0$.*
4. *If $\sum_{i\in I} a_i\bar{\varphi}_i = 0$ in $\Omega\backslash\omega$, then $a_i = 0$ for all $i \in I$.*

Proof.

1⇒2 Let $y \in (L^2(\omega))^n$. Then for $\varepsilon > 0$ there exists $z \in \mathcal{O}$ such that $\|y - \chi_\omega\nabla K^* z\|_{(L^2(\omega))^n} \le \varepsilon$. But $\nabla K^* z = \sum_{i\in\mathbb{N}^n}\langle\nabla K^* z, \bar{\varphi}_i\rangle_{(L^2(\Omega))^n}\bar{\varphi}_i = \sum_{i\in J}\langle z, K\nabla^*\bar{\varphi}_i\rangle_{\mathcal{O}}\bar{\varphi}_i$ and thus $\chi_\omega\nabla K^* z = \sum_{i\in J}\langle z, K\nabla^*\bar{\varphi}_i\rangle_{\mathcal{O}}\chi_\omega\bar{\varphi}_i$. Then $\|y - \sum_{i\in J}\langle z, K\nabla^*\bar{\varphi}_i\rangle_{\mathcal{O}}\chi_\omega\bar{\varphi}_i\|_{(L^2(\omega))^n} \le \varepsilon$ and hence $y \in \overline{\{\chi_\omega\bar{\varphi}_i\}}_{i\in J}$.

2⇒3 Let $y \in (L^2(\omega))^n$. For any $\varepsilon > 0$, there exists $\alpha_j (j \in J)$ such that $\|y - \sum_{j\in J}\alpha_j\chi_\omega\bar{\varphi}_j\|^2_{(L^2(\omega))^n} < \varepsilon$ with $\langle y, \chi_\omega\bar{\varphi}_j\rangle_{(L^2(\omega))^n} = 0$, $\forall j \in J$. We deduce that $\|y\|^2_{(L^2(\omega))^n} < \varepsilon$ for all $\varepsilon > 0$. Thus $y = 0$.

3⇒4 Let $\sum_{i\in I} a_i\bar{\varphi}_i = 0$ in $\Omega\backslash\omega$. Consider $y = \chi_\omega(\sum_{i\in I} a_i\bar{\varphi}_i)$. For $j \in J$, we have $\langle y, \chi_\omega\bar{\varphi}_j\rangle_{(L^2(\omega))^n} = \sum_{i\in I} a_i\langle\bar{\varphi}_i, \bar{\varphi}_j\rangle_{(L^2(\Omega))^n} = 0$. Since $y = 0$, we get $\sum_{i\in I} a_i\bar{\varphi}_i = 0$ in Ω and $a_i = 0$, $\forall i \in I$.

4⇒1 Consider $y \in (L^2(\omega))^n$ such that $K\nabla^*\chi^*_\omega y = 0$. We have $\chi^*_\omega y \in (L^2(\Omega))^n$ and then $K\nabla^*\chi^*_\omega y = K\nabla^*(\sum_{i\in\mathbb{N}^n}\langle y, \chi_\omega\bar{\varphi}_i\rangle_{(L^2(\omega))^n}\bar{\varphi}_i) = K\nabla^*(\sum_{i\in J}\langle y, \chi_\omega\bar{\varphi}_i\rangle_{(L^2(\omega))^n}\bar{\varphi}_i) = 0$. Therefore $\sum_{j\in J}\langle y, \chi_\omega\bar{\varphi}_j\rangle_{(L^2(\omega))^n}\bar{\varphi}_j \in \operatorname{span}\{(\bar{\varphi}_i)_{i\in I}\}$ and thus $\langle y, \chi_\omega\bar{\varphi}_j\rangle_{(L^2(\omega))^n} = 0$, $\forall j \in J$. Then $\chi^*_\omega y = \sum_{i\in I}\langle y, \chi_\omega\bar{\varphi}_i\rangle_{(L^2(\omega))^n}\bar{\varphi}_i = 0$ in $\Omega\backslash\omega$. From the assumption we have $\langle y, \chi_\omega\bar{\varphi}_i\rangle_{(L^2(\omega))^n} = 0$, $\forall i \in I$. Hence $y = 0$. ■

Based on Property 4 of Proposition 2, it is easy to show the following result:

Corollary 1. *Under the hypotheses of Proposition 2, the system (1) is approximately G-observable in all $\omega \subset \Omega$ such that $\langle\bar{\varphi}_i, \bar{\varphi}_j\rangle_{(L^2(\omega))^n} = 0$, $\forall i, j \in I, i \ne j$.*

Proof. To deduce the result from Proposition 3, take $\sum_{i\in I} a_i\bar{\varphi}_i = 0$ in $\Omega\backslash\omega$. Then we only need to show that $a_i = 0$ for all $i \in I$. Let $y = \sum_{i\in I} a_i\bar{\varphi}_i$ in Ω and $i_0 \in I$. Then

$$\langle y, \bar{\varphi}_{i_0}\rangle_{(L^2(\Omega))^n} = \sum_{i\in I} a_i\langle\bar{\varphi}_i, \bar{\varphi}_{i_0}\rangle_{(L^2(\Omega))^n} = a_{i_0}. \tag{6}$$

Since $y = 0$ in $\Omega\backslash\omega$, under the assumptions of Corollary 1 we have

$$\begin{aligned}\langle y, \bar{\varphi}_{i_0}\rangle_{(L^2(\Omega))^n} &= \sum_{i\in I} a_i\langle\bar{\varphi}_i, \bar{\varphi}_{i_0}\rangle_{(L^2(\omega))^n}\\ &= a_{i_0}\|\bar{\varphi}_{i_0}\|^2_{(L^2(\omega))^n}.\end{aligned} \tag{7}$$

Combining (6) and (7), we obtain $a_i = 0$ for all $i \in I$. ■

3. Gradient Strategic Sensors

The aim of this section is to link the regional gradient observability with the sensors structure. Consider the system (1) observed by q sensors $(D_i, f_i)_{1\le i\le q}$, which may be pointwise or zonal.

Definition 3. A sensor (D, f) (or a sequence of sensors) is said to be *gradient strategic* in ω if the observed system is approximately G-observable in ω. Such a sensor will be called G-strategic in ω.

We assume that the operator A has a complete set of eigenfunctions in $H^1_0(\Omega)$, denoted by (φ_i), which is orthonormal in $L^2(\omega)$ and the associated eigenvalues λ_i are of multiplicities r_i. Assume also that $r = \sup_{i\in I} r_i$ is finite and A has constant coefficients.

Proposition 4. *If a sequence of sensors $(D_k, f_k)_{1\le k\le q}$ is G-strategic in ω, then $q \ge r$ and $\operatorname{rank} M_m = r_m$, where*

$$(M_m)_{i,j} = \begin{cases} \displaystyle\sum_{k=1}^{n}\frac{\partial\varphi_{m_j}}{\partial x_k}(b_i) & \textit{in the pointwise case},\\ \displaystyle\sum_{k=1}^{n}\left\langle\frac{\partial\varphi_{m_j}}{\partial x_k}, f_i\right\rangle_{L^2(D_i)} & \textit{in the zonal case}\end{cases}$$

for $1 \le i \le q$ and $1 \le j \le r_m$.

The proof is given in Appendix.

Remark 1. 1. Proposition 4 implies that the required number of sensors is greater than or equal to the largest multiplicity of the eigenvalues.

2. By infinitesimally deforming the domain, the multiplicity of the eigenvalues can be reduced to one (El Jai and El Yacoubi, 1993, p. 95, Sec. 3). Consequently, G-observability in ω can be achieved using only one sensor.

3. In the observability and regional observability cases, the above conditions are also sufficient, c.f. (Curtain and Zwart, 1995, p. 162, Thm. 4.2.1; El Jai *et al.*, 1993), but here we have only necessary conditions.

4. The above result can easily be extended to the case of pointwise sensors. ■

From Formula (A3) in Appendix, we deduce the result below.

Corollary 2. *In the one-dimensional case, the system (1) together with the output (2) is approximately G-observable if and only if $q \geq r$ and $\operatorname{rank} M_m = r_m$ where M_m is given in Proposition 4.*

Example 1. Consider the system described by the parabolic equation

$$\begin{cases} \dfrac{\partial y}{\partial t}(x,t) - \dfrac{\partial^2 y}{\partial x^2}(x,t) = 0 & \text{in }]0,1[\times]0,T[, \\ y(0,t) = y(1,t) = 0 & \text{on }]0,T[, \\ y(x,0) = y_0(x) & \text{in }]0,1[, \end{cases} \tag{8}$$

with the output $z(t) = y(b,t)$, $b \in]0,1[$ and $t \in]0,T[$. The system is observable on $]0,1[$ (El Jai and Pritchard, 1988, p. 108, Prop. 2.26) if and only if

$$b \notin S = \bigcup_{m=1}^{\infty} \left\{ \frac{k}{m} \mid 1 \leq k < m \right\}.$$

It is G-observable in $]0,1[$ if and only if

$$b \notin S_G = \bigcup_{m=1}^{\infty} \left\{ \frac{2k+1}{2m} \mid k \in [0, m-1] \cap \mathbb{N} \right\}.$$

We have $S_G \subset S$, which shows that there exist sensors which are G-strategic without being strategic. ♦

4. Regional Gradient Reconstruction

In this section we give an approach which allows us the reconstruction of the initial state gradient of the system (1) in ω. This approach is an extension of the H.U.M. method developed by Lions (Lions, 1988) and does not take into account what must be the residual initial gradient in the subregion $\Omega\backslash\omega$. Consider the set

$$\mathcal{F} = \left\{ h \in \left(L^2(\Omega)\right)^n \mid h = 0 \text{ in } \Omega \setminus \omega \right\} \cap \left\{ \nabla f \mid f \in H_0^1(\Omega) \right\}.$$

For $\phi_0 \in H_0^1(\Omega)$, consider the system

$$\begin{cases} \dfrac{\partial \phi}{\partial t}(x,t) = A\phi(x,t) & \text{in } Q, \\ \phi(\xi,t) = 0 & \text{on } \Sigma, \\ \phi(x,0) = \phi_0(x) & \text{in } \Omega, \end{cases} \tag{9}$$

which admits a unique solution $\phi \in L^2(0,T;H_0^1(\Omega)) \cap C(\Omega\times]0,T[)$, cf. (Lions and Magenes, 1968, Ex. 1, pp. 263–264).

4.1. Pointwise Sensors

We consider the system (1) with the output function (3). For $\widetilde{\phi}_0 \in \mathcal{F}$, there exists a unique $\phi_0 \in H_0^1(\Omega)$ such that $\widetilde{\phi}_0 = \nabla\phi_0$. Then we consider the semi-norm on $\mathcal{F}$ defined by

$$\widetilde{\phi}_0 \longrightarrow \|\widetilde{\phi}_0\|_{\mathcal{F}} = \left[\int_0^T \left(\sum_{k=1}^n \frac{\partial \phi}{\partial x_k}(b,t) \right)^2 \mathrm{d}t \right]^{\frac{1}{2}}, \tag{10}$$

where ϕ is the solution to (9). The solution ψ_1 of the equation

$$\begin{cases} \dfrac{\partial \psi_1}{\partial t}(x,t) = -A^*\psi_1(x,t) \\ \qquad\qquad - \displaystyle\sum_{k=1}^n \frac{\partial \phi}{\partial x_k}(b,t)\delta(x-b) & \text{in } Q, \\ \dfrac{\partial \psi_1}{\partial \nu_{A^*}}(\xi,t) = 0 & \text{on } \Sigma, \\ \psi_1(x,T) = 0 & \text{in } \Omega \end{cases} \tag{11}$$

is in $L^2(0,T;H_0^1(\Omega))$, cf. (Lions and Magenes, 1968, Ex. 2, p. 265).

When the seminorm (10) is a norm, we also denote by $\mathcal{F}$ the completion of $\mathcal{F}$ and consider the operator

$$\Lambda : \begin{cases} \mathcal{F} \longmapsto \mathcal{F}^*, \\ \widetilde{\phi}_0 \longmapsto P\big(\Psi_1(0)\big), \end{cases}$$

where $P = \chi_\omega^* \chi_\omega$ and $\Psi_1(0) = (\psi_1(0), \dots, \psi_1(0))$. Introduce the system

$$\begin{cases} \dfrac{\partial \psi}{\partial t}(x,t) = -A^*\psi(x,t) \\ \qquad\qquad - \displaystyle\sum_{k=1}^n \frac{\partial y}{\partial x_k}(b,t)\delta(x-b) & \text{in } Q, \\ \dfrac{\partial \psi}{\partial \nu_{A^*}}(\xi,t) = 0 & \text{on } \Sigma, \\ \psi(x,T) = 0 & \text{in } \Omega. \end{cases} \tag{12}$$

If $\widetilde{\phi}_0$ is chosen such that $\psi_1(0) = \psi(0)$ in ω, then the system (12) looks like the adjoint of the system (1), and the regional gradient observability amounts to the conditions for solving the equation

$$\Lambda\big(\widetilde{\phi}_0\big) = P\big(\Psi(0)\big), \tag{13}$$

where $\Psi(0) = (\psi(0), \dots, \psi(0))$ with ψ being the solution of (12).

Proposition 5. *If the sensor (b, δ_b) is G-strategic in ω, then Eqn. (13) has a unique solution $\widetilde{\phi}_0$, which is the initial state gradient to be observed in ω.*

Proof. (a) Let us show first that if the system (1) is approximately G-observable, then (10) defines a norm on $\mathcal{F}$. Consider a basis (φ_i) of the eigenfunctions of A. Without loss of generality, we suppose that the associated eigenvalues λ_i are of multiplicity one.

The mapping (10) defines a norm on $\mathcal{F}$. Indeed, $\|\widehat{\phi}_0\|_{\mathcal{F}} = 0$ gives

$$\sum_{i=1}^{\infty} e^{\lambda_i t} \langle \phi_0, \varphi_i \rangle_{L^2(\Omega)} \sum_{k=1}^{n} \frac{\partial \varphi_i}{\partial x_k}(b) = 0 \text{ a.e. on } (0, T).$$

Then

$$\langle \phi_0, \varphi_i \rangle_{L^2(\Omega)} \sum_{k=1}^{n} \frac{\partial \varphi_i}{\partial x_k}(b) = 0, \quad \forall\, i$$

and since the sensor (b, δ_b) is G-strategic in ω (cf. Proposition (4)), we obtain

$$\sum_{k=1}^{n} \frac{\partial \varphi_i}{\partial x_k}(b) \neq 0, \quad \forall\, i.$$

Then $\langle \phi_0, \varphi_i \rangle_{L^2(\Omega)} = 0$, $\forall\, i$. Consequently, $\phi_0 = 0$, and thus $\widetilde{\phi}_0 = 0$.

(b) Let us prove now that (13) has a unique solution. Equation (13) has a unique solution because the operator Λ is an isomorphism. Indeed, multiplying (11) by $\partial\phi/\partial x_k$ and integrating the result over Q, we obtain

$$\begin{aligned}
&\left\langle \frac{\partial \phi}{\partial x_k}(x,t), \frac{\partial \psi_1}{\partial t}(x,t) \right\rangle_{L^2(Q)} \\
&\quad = \left\langle \frac{\partial \phi}{\partial x_k}(x,t), -A^*\psi_1(x,t) \right\rangle_{L^2(Q)} \\
&\qquad - \left\langle \frac{\partial \phi}{\partial x_k}(x,t), \sum_{l=1}^{n} \frac{\partial \phi}{\partial x_l}(b,t)\delta(x-b) \right\rangle_{L^2(Q)}
\end{aligned}$$

which gives

$$\begin{aligned}
&\left[\left\langle \frac{\partial \phi}{\partial x_k}(x,t), \psi_1(x,t) \right\rangle_{L^2(\Omega)} \right]_0^T \\
&\qquad - \left\langle \frac{\partial}{\partial x_k}\Big(\frac{\partial \phi}{\partial t}(x,t)\Big), \psi_1(x,t) \right\rangle_{L^2(Q)} \\
&\quad = \left\langle \frac{\partial \phi}{\partial x_k}(x,t), -A^*\psi_1(x,t) \right\rangle_{L^2(Q)} \\
&\qquad - \int_0^T \frac{\partial \phi}{\partial x_k}(b,t) \sum_{l=1}^{n} \frac{\partial \phi}{\partial x_l}(b,t)\,\mathrm{d}t.
\end{aligned}$$

With the final condition, we obtain

$$\begin{aligned}
&- \left\langle \frac{\partial \phi}{\partial x_k}(x,0), \psi_1(x,0) \right\rangle_{L^2(\Omega)} \\
&\quad = \left\langle A\frac{\partial \phi}{\partial x_k}(x,t), \psi_1(x,t) \right\rangle_{L^2(Q)} \\
&\qquad - \left\langle \frac{\partial \phi}{\partial x_k}(x,t), A^*\psi_1(x,t) \right\rangle_{L^2(Q)} \\
&\qquad - \int_0^T \frac{\partial \phi}{\partial x_k}(b,t) \sum_{l=1}^{n} \frac{\partial \phi}{\partial x_l}(b,t)\,\mathrm{d}t.
\end{aligned}$$

Using the Green formula, we obtain

$$\begin{aligned}
&\left\langle \frac{\partial \phi}{\partial x_k}(x,0), \psi_1(x,0) \right\rangle_{L^2(\Omega)} \\
&\quad = \int_0^T \frac{\partial \phi}{\partial x_k}(b,t) \sum_{l=1}^{n} \frac{\partial \phi}{\partial x_l}(b,t)\,\mathrm{d}t.
\end{aligned}$$

Then

$$\begin{aligned}
&\sum_{k=1}^{n} \left\langle \frac{\partial \phi_0}{\partial x_k}(x), \psi_1(x,0) \right\rangle_{L^2(\Omega)} \\
&\quad = \sum_{k=1}^{n} \int_0^T \frac{\partial \phi}{\partial x_k}(b,t) \sum_{l=1}^{n} \frac{\partial \phi}{\partial x_l}(b,t)\,\mathrm{d}t.
\end{aligned}$$

Hence

$$\left\langle \widetilde{\phi}_0, \Lambda(\widetilde{\phi}_0) \right\rangle = \int_0^T \left(\sum_{l=1}^{n} \frac{\partial \phi}{\partial x_l}(b,t) \right)^2 \mathrm{d}t,$$

which proves that Λ is an isomorphism and, consequently, (13) has a unique solution which corresponds to the state gradient to be estimated in the subregion ω. ∎

4.2. Zonal Sensor

Here we consider the system (1) with the output function (4). For $\widetilde{\phi}_0$ in $\mathcal{F}$ the system (9) produces the solution ϕ. We consider the seminorm on $\mathcal{F}$ defined by

$$\widetilde{\phi}_0 \longrightarrow \|\widetilde{\phi}_0\|_{\mathcal{F}} = \left[\int_0^T \left(\sum_{k=1}^{n} \left\langle \frac{\partial \phi}{\partial x_k}(t), f \right\rangle_{L^2(D)} \right)^2 \mathrm{d}t \right]^{\frac{1}{2}} \tag{14}$$

and the system

$$\left\{ \begin{aligned}
\frac{\partial \psi_1}{\partial t}(x,t) &= -A^*\psi_1(x,t) - \sum_{k=1}^{n} \left\langle \frac{\partial \phi}{\partial x_k}(t), f \right\rangle_{L^2(D)} \\
&\qquad \times f(x)\chi_D \qquad \text{in } Q, \\
\frac{\partial \psi_1}{\partial \nu_{A^*}}(\xi,t) &= 0 \qquad \text{on } \Sigma, \\
\psi_1(x,T) &= 0 \qquad \text{in } \Omega.
\end{aligned} \right. \tag{15}$$

We consider the operator

$$\Lambda : \quad \begin{cases} \mathcal{F} \longmapsto \mathcal{F}^*, \\ \widetilde{\phi}_0 \longmapsto P\big(\Psi_1(0)\big), \end{cases}$$

and the system

$$\begin{cases} \dfrac{\partial \psi}{\partial t}(x,t) \;=\; -A^*\psi(x,t) - \displaystyle\sum_{k=1}^{n} \left\langle \frac{\partial y}{\partial x_k}(t), f \right\rangle_{L^2(D)} \\ \qquad\qquad \times f(x)\chi_D \qquad\qquad \text{in } Q, \\ \dfrac{\partial \psi}{\partial \nu_{A^*}}(\xi,t) \;=\; 0 \qquad\qquad \text{on } \Sigma, \\ \quad \psi(x,T) \;=\; 0 \qquad\qquad \text{in } \Omega. \end{cases} \tag{16}$$

Then regional gradient observability amounts to the solvability of the equation

$$\Lambda(\widetilde{\phi}_0) = P\big(\Psi(0)\big), \tag{17}$$

where $\Psi = (\psi(0), \dots, \psi(0))$ and ψ is the solution of the system (16).

Proposition 6. *If the sensor (D,f) is G-strategic in ω, then (17) has a unique solution which is the gradient of the initial state to be observed in ω.*

Proof. (Sketch) With some minor technical modifications, the proof is similar to the pointwise case. Indeed, it proceeds in two steps.

Step 1. $\|\widetilde{\phi}_0\|_{\mathcal{F}} = 0$ gives

$$\sum_{i=1}^{\infty} e^{\lambda_i t} \left\langle \phi_0, \varphi_i \right\rangle_{L^2(\Omega)} \times \sum_{k=1}^{n} \left\langle \frac{\partial \varphi_i}{\partial x_k}, f \right\rangle_{L^2(D)} = 0 \text{ a.e. on } (0,T).$$

Using

$$\sum_{k=1}^{n} \left\langle \frac{\partial \varphi_i}{\partial x_k}, f \right\rangle_{L^2(D)} \neq 0,$$

we obtain $\tilde{\phi}_0 = 0$.

Step 2. As in the pointwise case, we multiply the system (15) by $\partial\phi/\partial x_k$ and integrate the result over Q and then, using the Green formula, we obtain

$$\left\langle \widetilde{\phi}_0, \Lambda(\widetilde{\phi}_0) \right\rangle = \int_0^T \left(\sum_{l=1}^{n} \left\langle \frac{\partial \phi}{\partial x_l}(t), f \right\rangle_{L^2(D)} \right)^2 \mathrm{d}t,$$

which proves that Λ is an isomorphism. Consequently, Eqn. (17) has a unique solution which corresponds to the state gradient to be estimated in the subregion ω. ■

5. Numerical Approach

We consider the system (1) observed by a pointwise sensor located at $b \in \Omega$. In the previous section, it was shown that the regional observability of the initial gradient in ω amounts, in all cases, to the solvability of the equation

$$\Lambda\big(\widetilde{\phi}_0\big) = P\big(\Psi(0)\big). \tag{18}$$

The solution of this equation can be obtained very easily when we can calculate the components Λ_{ij} of Λ in a suitable basis $(\overline{\varphi}_i)_{i\in\mathbb{N}}$ of $(L^2(\Omega))^n$. The problem will then be approximated by the solution of the linear system

$$\sum_{j=1}^{M} \Lambda_{ij} \widetilde{\phi}_{0j} = \big(P(\Psi(0))\big)_i, \quad i = 1, \dots, M, \tag{19}$$

where $(P(\Psi(0)))_i$ are the components of $P(\Psi(0))$ in the basis considered.

Let $(\varphi_i)_{i\in\mathbb{N}}$ be a complete set of eigenfunctions of the operator A in $H_0^1(\Omega)$, which is orthonormal in $L^2(\Omega)$. We also consider a basis of $(L^2(\Omega))^n$ denoted by $(\overline{\varphi}_i)_{i\in\mathbb{N}}$. Then the components Λ_{ij} are solutions of the equation

$$\begin{cases} \displaystyle\sum_{i,j=1}^{\infty} \left\langle \left(\frac{\partial \varphi_k}{\partial x_1}, \dots, \frac{\partial \varphi_k}{\partial x_n} \right), \overline{\varphi}_i \right\rangle \\ \qquad \times \left\langle \left(\frac{\partial \varphi_l}{\partial x_1}, \dots, \frac{\partial \varphi_l}{\partial x_n} \right), \overline{\varphi}_j \right\rangle \Lambda_{ij} \\ \quad = \dfrac{e^{(\lambda_k+\lambda_l)T} - 1}{\lambda_k + \lambda_l} \displaystyle\sum_{m,p=1}^{n} \frac{\partial \varphi_k}{\partial x_m}(b) \frac{\partial \varphi_l}{\partial x_p}(b), \\ \qquad k, l = 1, \dots, \infty. \end{cases} \tag{20}$$

Indeed, in Section 4 it was demonstrated that

$$\left\langle \Lambda\widetilde{\phi}_0, \widetilde{\phi}_0 \right\rangle = \int_0^T \left(\sum_{j=1}^{n} \frac{\partial \phi}{\partial x_j}(b,t) \right)^2 \mathrm{d}t.$$

From

$$\phi(t) = \sum_{m=1}^{\infty} e^{\lambda_m t} \left\langle \phi_0, \varphi_m \right\rangle_{L^2(\Omega)} \varphi_m$$

we have

$$\sum_{j=1}^{n} \frac{\partial \phi}{\partial x_j}(b,t) = \sum_{j=1}^{n} \sum_{m=1}^{\infty} e^{\lambda_m t} \left\langle \phi_0, \varphi_m \right\rangle_{L^2(\Omega)} \frac{\partial \varphi_m}{\partial x_j}(b).$$

Then

$$\int_0^T \left(\sum_{j=1}^n \frac{\partial \phi}{\partial x_j}(b,t)\right)^2 \mathrm{d}t$$

$$= \sum_{p,m=1}^{\infty} \langle \phi_0, \varphi_p \rangle_{L^2(\Omega)} \langle \phi_0, \varphi_m \rangle_{L^2(\Omega)}$$

$$\times \sum_{j,l=1}^{n} \frac{\partial \varphi_p}{\partial x_l}(b) \frac{\partial \varphi_m}{\partial x_j}(b) \frac{e^{(\lambda_m+\lambda_p)T}-1}{\lambda_m+\lambda_p}. \quad (21)$$

On the other hand, since $\partial \phi_0 / \partial x_j \in L^2(\Omega)$, we can express $\widetilde{\phi}_0$ in the basis $(\overline{\varphi}_i)$:

$$\widetilde{\phi}_0 = \sum_{i=1}^{\infty} \left\langle \left(\frac{\partial \phi_0}{\partial x_1}, \dots, \frac{\partial \phi_0}{\partial x_n} \right), \overline{\varphi}_i \right\rangle_{(L^2(\Omega))^n} \overline{\varphi}_i.$$

Therefore

$$\widetilde{\phi}_0 = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \langle \phi_0, \varphi_k \rangle \left\langle \left(\frac{\partial \varphi_k}{\partial x_1}, \dots, \frac{\partial \varphi_k}{\partial x_n} \right), \overline{\varphi}_i \right\rangle_{(L^2(\Omega))^n} \overline{\varphi}_i$$

$$= \sum_{k=1}^{\infty} \langle \phi_0, \varphi_k \rangle \sum_{i=1}^{\infty} \left\langle \left(\frac{\partial \varphi_k}{\partial x_1}, \dots, \frac{\partial \varphi_k}{\partial x_n} \right), \overline{\varphi}_i \right\rangle_{(L^2(\Omega))^n} \overline{\varphi}_i.$$

Thus

$$\left\langle \Lambda \widetilde{\phi}_0, \widetilde{\phi}_0 \right\rangle = \sum_{m,p=1}^{\infty} \langle \phi_0, \varphi_m \rangle_{L^2(\Omega)} \langle \phi_0, \varphi_p \rangle_{L^2(\Omega)}$$

$$\times \Bigg(\sum_{i,j=1}^{\infty} \left\langle \left(\frac{\partial \varphi_m}{\partial x_1}, \dots, \frac{\partial \varphi_m}{\partial x_n} \right), \overline{\varphi}_i \right\rangle_{(L^2(\Omega))^n}$$

$$\times \left\langle \left(\frac{\partial \varphi_p}{\partial x_1}, \dots, \frac{\partial \varphi_p}{\partial x_n} \right), \overline{\varphi}_j \right\rangle_{(L^2(\Omega))^n}$$

$$\times \langle \Lambda \overline{\varphi}_i, \overline{\varphi}_j \rangle_{(L^2(\Omega))^n} \Bigg). \quad (22)$$

Let $\Lambda_{ij} = \langle \Lambda \overline{\varphi}_i, \overline{\varphi}_j \rangle_{(L^2(\Omega))^n}$. From (21) and (22), we obtain (20). ■

Remark 2. 1. In the case of a zonal sensor (D, f), the same derivations give

$$\begin{cases} \displaystyle\sum_{i,j=1}^{\infty} \left\langle \left(\frac{\partial \varphi_m}{\partial x_1}, \dots, \frac{\partial \varphi_m}{\partial x_n} \right), \overline{\varphi}_i \right\rangle_{(L^2(\Omega))^n} \\ \displaystyle\quad \times \left\langle \left(\frac{\partial \varphi_p}{\partial x_1}, \dots, \frac{\partial \varphi_p}{\partial x_n} \right), \overline{\varphi}_j \right\rangle_{(L^2(\Omega))^n} \Lambda_{ij} \\ \displaystyle\quad = \frac{e^{(\lambda_m+\lambda_p)T}-1}{\lambda_m+\lambda_p} \sum_{k,l=1}^{n} \left\langle \frac{\partial \varphi_p}{\partial x_k}, f \right\rangle_{L^2(D)} \\ \displaystyle\quad \times \left\langle \frac{\partial \varphi_m}{\partial x_l}, f \right\rangle_{L^2(D)} \\ \text{for } m, p = 1, \dots, \infty. \end{cases}$$

2. In the case of many pointwise sensors $(b_s, \delta(\cdot - b_s))_{s=1,\dots,q}$, the components of Λ are solutions of the equation

$$\begin{cases} \displaystyle\sum_{i,j=1}^{\infty} \left\langle \left(\frac{\partial \varphi_m}{\partial x_1}, \dots, \frac{\partial \varphi_m}{\partial x_n} \right), \overline{\varphi}_i \right\rangle_{(L^2(\Omega))^n} \\ \displaystyle\quad \times \left\langle \left(\frac{\partial \varphi_p}{\partial x_1}, \dots, \frac{\partial \varphi_p}{\partial x_n} \right), \overline{\varphi}_j \right\rangle_{(L^2(\Omega))^n} \Lambda_{ij} \\ \displaystyle\quad = \frac{e^{(\lambda_m+\lambda_p)T}-1}{\lambda_m+\lambda_p} \\ \displaystyle\quad \times \sum_{s=1}^{q} \sum_{k,l=1}^{n} \frac{\partial \varphi_m}{\partial x_k}(b_s) \frac{\partial \varphi_p}{\partial x_l}(b_s) \\ \text{for } m, p = 1, \dots, \infty. \end{cases}$$

Summing up, regional gradient reconstruction is obtained via the following simplified steps:

1. The solution of (12) gives $\psi(x, 0)$.
2. The components of the operator Λ constitute the solution of (20).
3. With Steps 1 and 2, the system (19) gives $\tilde{\phi}_0$, which corresponds to the initial gradient to be observed in ω.

6. Simulations

In this section we present a numerical example that leads to some conjectures related to the best sensor location, the results being related to the choice of the subregion and the gradient to be observed.

Let $\Omega =]0,1[$ and consider the system

$$\begin{cases} \dfrac{\partial y}{\partial t}(x,t) - 0.01 \dfrac{\partial^2 y}{\partial x^2}(x,t) = 0 & \text{in } Q, \\ y(0,t) = y(1,t) = 0 & \text{on }]0,T[, \\ y(x,0) = y_0(x) & \text{in } \Omega. \end{cases} \quad (23)$$

The output is given by means of a pointwise sensor $z(t) = y(b,t)$ with $b = 0.91$ and $T = 2$.

Let $\omega = [0.15, 0.85]$ and $g(x) = 2x(x-1)(2x-1)$ be the initial gradient to be observed in ω. Figure 2 shows how close the estimated gradient $\tilde{g}$ is to the initial gradient g which is estimated with the reconstruction error $\|\tilde{g} - g\|^2_{L^2(\omega)} = 1.398 \times 10^{-4}$. The resulting numerical method is efficient provided that the truncation order is small.

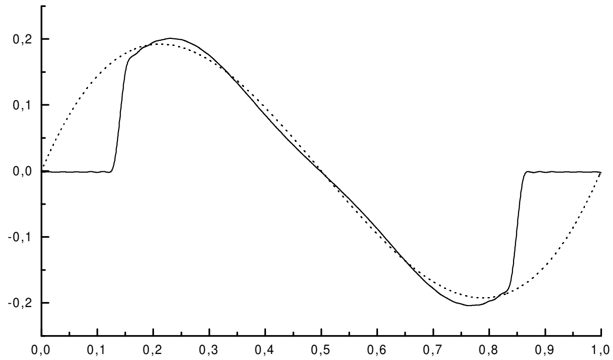

Fig. 2. Actual initial gradient g (dashed line) and the estimated gradient $\tilde{g}$ (continuous line) in ω.

6.1. Relation between the Estimated Gradient Error and the Pointwise Sensor Location

In this subsection we study numerically the evolution of the observer gradient error with respect to the sensor location.

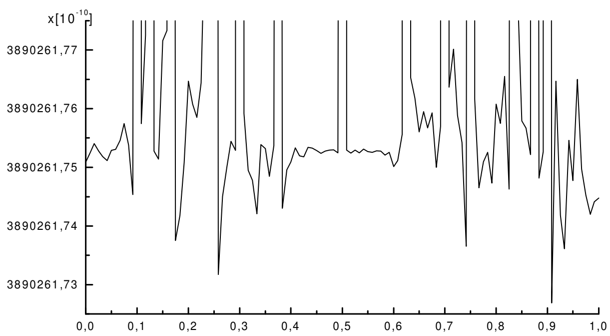

Fig. 3. Evolution of the estimated gradient error with respect to the sensor location.

The following conclusions can be drawn from Figure 3:

- For a given subregion and an initial gradient, there is an optimal sensor location (optimal in the sense that it leads to a very good estimate of the initial gradient).
- When the sensor is located sufficiently far from the subregion ω, the gradient error is constant for any location.
- The worse observed locations correspond to non G-strategic sensors on $\Omega =]0,1[$, where
$$b \in \left\{ \frac{2k+1}{2m} \mid k \in [0, m-1] \cap \mathbb{N}, \quad 1 \leq m \leq 5 \right\}$$
(the order of the approximation of the system is five).

6.2. Relation between the Subregion Area and the Estimated Gradient Error

Here we study numerically the dependence of the gradient reconstruction error on the subregion area. Table 1 shows that both the error and the subregion area increase or decrease. This means that the larger the region, the greater the error. G-observability is realized by means of one pointwise sensor located at $b = 0.91$. The results are similar for other types of sensors.

Table 1. Relation between the reconstruction gradient error and the subregion area.

Subregion ω	$\|\tilde{g} - g\|^2_{L^2(\omega)}$
$]1/3, 2/3[$	6.862×10^{-5}
$]0.32, 0.67[$	7.231×10^{-5}
$]0.29, 0.72[$	9.290×10^{-5}
$]0.1, 0.9[$	1.581×10^{-4}
$]0, 1[$	3.890×10^{-4}

7. Conclusion

This paper concerns the study of the gradient observability concept, which is motivated by many real applications where the objective is to obtain information about the state gradient in a given subregion. Moreover, we explore an approach which allows us to reconstruct the gradient. Interesting results on regional gradient controllability were developed by Zerrik *et al.* (1999). Various open questions are still under consideration. For example, this is the case of the problem where the subregion ω is a part of the boundary of the system domain. This case is presently being studied and the results will appear in a separate paper. The problem of optimal sensor location in order to achieve regional gradient observability is also of great interest.

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Appendix

The proof of Proposition 4 is presented for the case of zonal sensors located inside the domain Ω. We recall that G-observability in ω is equivalent to $[K\nabla^*\chi_\omega^* z = 0 \Rightarrow z = 0]$, which allows us to say that the sequence of sensors $(D_k, f_k)_{1\le k\le q}$ is G-strategic in ω if and only if

$$\big\{z \in \big(\mathcal{L}^2(\omega)\big)^n \mid \langle Hu, z\rangle_{(L^2(\omega))^n} = 0, \quad \forall\, u \in \mathcal{O}\big\} \Rightarrow z = 0.$$

Suppose that the sequence of sensors $(D_k, f_k)_{1\le k\le q}$ is G-strategic in ω but for some $m \in \mathbb{N}$ we have rank $M_m \neq r_m$. Then there exists $z_m = (z_{m_1}, \dots, z_{m_{r_m}})^t \neq 0$ such that $M_m z_m = 0$.

Let

$$z_0 = \sum_{j=1}^{r_m} z_{m_j}\varphi_{m_j}, z_p = \Big(\langle z_0, \varphi_{p_1}\rangle, \dots, \big\langle z_0, \varphi_{p_{r_p}}\big\rangle\Big)^t$$

and

$$Z_0 = (z_0, \dots, z_0).$$

Then

$$\begin{aligned}\langle Hu, Z_0\rangle_{(L^2(\omega))^n} &= \sum_{k=1}^{n}\Big\langle \tilde\chi_\omega \frac{\partial}{\partial x_k}(K^*u), z_0\Big\rangle_{L^2(\omega)}\\ &= \sum_{k=1}^{n}\Big\langle \frac{\partial}{\partial x_k}(K^*u), \tilde\chi_\omega^* z_0\Big\rangle_{L^2(\Omega)}\\ &= \sum_{k=1}^{n}\Big\langle \frac{\partial}{\partial x_k}(\tilde y(T)), \tilde\chi_\omega^* z_0\Big\rangle_{L^2(\Omega)},\end{aligned}$$

where $\tilde y$ is the solution of the system

$$\begin{cases}\dfrac{\partial\tilde y}{\partial t}(x,t) = A^*\tilde y(x,t) \\ \qquad\qquad + \displaystyle\sum_{i=1}^{q} f_i\chi_{D_i}u_i(T-t) & \text{in } Q,\\ \tilde y(\xi,t) = 0 & \text{on } \Sigma,\\ \tilde y(x,0) = 0 & \text{in } \Omega,\end{cases} \tag{A1}$$

which is in $L^2(0,T;H_0^2(\Omega))$ (Lions and Magenes, 1968, Ex. 1, p. 263–264).

Consider the system

$$\begin{cases}\dfrac{\partial\varphi}{\partial t}(x,t) = -A\varphi(x,t) & \text{in } Q,\\ \varphi(\xi,t) = 0 & \text{on } \Sigma,\\ \varphi(x,T) = \tilde\chi_\omega^* z_0 & \text{in } \Omega.\end{cases} \tag{A2}$$

The solution φ is in $L^2(0,T;H_0^1(\Omega)) \cap C(\Omega\times]0,T[)$ (Lions and Magenes, 1968, Ex. 2, p. 265). Multiplying (A1) by $\partial\varphi/\partial x_k$ and integrating the result over Q, we obtain

$$\begin{aligned}&\int_Q \frac{\partial\varphi}{\partial x_k}(x,t)\frac{\partial\tilde y}{\partial t}(x,t)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad= \int_Q A^*\tilde y(x,t)\frac{\partial\varphi}{\partial x_k}(x,t)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+ \int_Q \sum_{i=1}^{q} f_i\chi_{D_i}u_i(T-t)\frac{\partial\varphi}{\partial x_k}(x,t)\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

but we have

$$\begin{aligned}&\int_Q \frac{\partial\varphi}{\partial x_k}(x,t)\frac{\partial\tilde y}{\partial t}(x,t)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad= \int_\Omega \Big[\frac{\partial\varphi}{\partial x_k}(x,t)\tilde y(x,t)\Big]_0^T\,\mathrm{d}x\\ &\quad+ \int_Q A\Big(\frac{\partial\varphi}{\partial x_k}\Big)(x,t)\tilde y(x,t)\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

$$= \int_\Omega \frac{\partial \varphi}{\partial x_k}(x,T)\tilde{y}(x,T)\,\mathrm{d}x + \int_Q A\left(\frac{\partial \varphi}{\partial x_k}(x,t)\right)\tilde{y}(x,t)\,\mathrm{d}x\,\mathrm{d}t.$$

Then

$$\begin{aligned}\int_\Omega \frac{\partial \varphi}{\partial x_k}(x,T)\tilde{y}(x,T)\,\mathrm{d}x &= -\int_Q A\left(\frac{\partial \varphi}{\partial x_k}(x,t)\right)\tilde{y}(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_Q A^*\tilde{y}(x,t)\frac{\partial \varphi}{\partial x_k}(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_Q \left(\sum_{i=1}^q f_i\chi_{D_i}u_i(T-t)\right)\frac{\partial \varphi}{\partial x_k}(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_\Sigma \frac{\partial \tilde{y}}{\partial \nu_{A^*}}(\zeta,t)\frac{\partial \varphi}{\partial x_k}(\zeta,t)\,\mathrm{d}\zeta\,\mathrm{d}t \\ &\quad + \int_\Sigma \frac{\partial}{\partial \nu_A}\left(\frac{\partial \varphi}{\partial x_k}(\zeta,t)\right)\tilde{y}(\zeta,t)\,\mathrm{d}\zeta\,\mathrm{d}t \\ &\quad + \int_Q \left(\sum_{i=1}^q f_i\chi_{D_i}u_i(T-t)\right)\frac{\partial \varphi}{\partial x_k}(x,t)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Since $\tilde{y}(\zeta,t)\in L^2(0,T;H_0^2(\Omega))$, we have

$$\int_\Omega \frac{\partial \varphi}{\partial x_k}(x,T)\tilde{y}(x,T)\,\mathrm{d}x = \int_Q (\sum_{i=1}^q f_i\chi_{D_i}u_i(T-t))\frac{\partial \varphi}{\partial x_k}(x,t)\,\mathrm{d}x\,\mathrm{d}t.$$

Thus

$$\int_\Omega \varphi(x,T)\frac{\partial \tilde{y}}{\partial x_k}(x,T)\,\mathrm{d}x = -\sum_{i=1}^q\int_0^T \left\langle f_i,\frac{\partial \varphi}{\partial x_k}\right\rangle_{L^2(D_i)} u_i(T-t)\,\mathrm{d}t,$$

and we have

$$\begin{aligned}\langle \chi_\omega \nabla K^*u, Z_0\rangle_{(L^2(\Omega))^n} &= \sum_{k=1}^n \int_\Omega \frac{\partial \tilde{y}}{\partial x_k}(x,T)\varphi(x,T)\,\mathrm{d}x \\ &= -\sum_{i=1}^q\int_0^T \sum_{k=1}^n \left\langle f_i,\frac{\partial \varphi}{\partial x_k}\right\rangle_{L^2(D_i)} u_i(T-t)\,\mathrm{d}t.\end{aligned}$$

But

$$\varphi(x,t) = \sum_{p=1}^\infty e^{-\lambda_p(t-T)}\sum_{j=1}^{r_p}\left\langle z_0,\varphi_{p_j}\right\rangle_{L^2(\omega)}\varphi_{p_j}.$$

Then

$$\begin{aligned}\sum_{k=1}^n \left\langle f_i,\frac{\partial \varphi}{\partial x_k}(t)\right\rangle_{L^2(D_i)} &= \sum_{p=1}^\infty e^{\lambda_p(T-t)}\sum_{j=1}^{r_p}\left\langle z_0,\varphi_{p_j}\right\rangle_{L^2(\omega)} \\ &\quad \times \sum_{k=1}^n \left\langle \frac{\partial \varphi_{p_j}}{\partial x_k}, f_i\right\rangle_{L^2(D_i)} \\ &= \sum_{p=1}^\infty e^{\lambda_p(T-t)}(M_pz_p)_i.\end{aligned}$$

Therefore

$$\langle \chi_\omega \nabla K^*u, Z_0\rangle_{(L^2(\omega))^n} = -\sum_{i=1}^q\int_0^T\sum_{p=1}^\infty e^{\lambda_p(T-t)}(M_pz_p)_iu_i(T-t)\,\mathrm{d}t. \quad \text{(A3)}$$

Thus

$$\langle \chi_\omega \nabla K^*u, Z_0\rangle_{(L^2(\omega))^n} = -\sum_{i=1}^q\int_0^T e^{\lambda m(T-t)}(M_mz_m)_iu_i(T-t)\,\mathrm{d}t = 0.$$

This is true for all $u\in L^2(0,T;\mathbb{R}^q)$. Hence $Z_0\in \mathrm{Ker}(H^*)$, which contradicts the assumption that the sequence of sensors is G-strategic. ■

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