# Robust PD-type iterative learning control design for uncertain batch processes subject to nonrepetitive disturbances

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**Abstract**

This paper develops PD-type iterative learning control schemes for a class of uncertain batch processes subject to nonrepetitive disturbances. By means of two-dimensional/repetitive setting the conditions for batch-to-bach error convergence and $\mathcal{H}_\infty$ disturbance attenuation are formulated and analyzed. Subsequently, the procedure for computing the desired control law matrices is formulated in terms of solvability of linear matrix inequalities. The proposed control law is able to fulfil the imposed design specifications, i.e., they are suitable for the batch processes with time-varying uncertainties as well as non-repetitive disturbances. An illustrative example is used to validate the proposed control scheme and demonstrates a possible applicability of the developed results.

## 1 Introduction

In the recent years, increasing research effort has been directed at the development of ILC schemes, which are one of the most popular feedforward control scenarios for improving tracking response in systems that perform a given task repeatedly. Each repetition of a given task is known as a trial, or pass, and when a trial is complete, the system resets to the same initial conditions and the next trial can begin. This allows to use the information collected from the previous trials, such as control input and error signals, to modify the current control input signals, aiming to track the desired trajectories of the controlled plant and hence the desired trial-to-trial performance level is reached. Specifically, ILC aims to construct the control input signals such that the output tracks

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the reference as accurately as possible. Hence, the basic ILC problem is to design both feedback and learning controllers which produce such control signals to ensure that the error sequence generated over the trials converges to the prescribed value.

A detailed overview of main developments in ILC research can be found in, e.g., the survey papers [1, 3, 14], where the last of these has a special focus on run-to-run control as found in the chemical process industries. Anyway, a great benefit of using ILC has been demonstrated in industrial robotics, see, e.g., [10], where the pick and place operation common in many mass manufacturing processes is an immediate fit to ILC, and wafer stage motion systems, see, for example, [6].

As known, the primary challenge to ILC is to obtain trial-to-trial error convergence of the resulting system. However, the error convergence should be achieved even if there is uncertainty in the plant model which additionally is affected by non-repetitive disturbances. Clearly, these factors make the problem more complex and a robust ILC scheme should be designed to keep the tracking performance within the acceptable range. A possible approach, see, for example, [1, 3] as starting points for the literature, to ILC design for nominal and uncertain plants is to first apply a feedback control law to stabilize and/or produce acceptable along the trial dynamics and then apply ILC to force trial-to-trial error convergence of the resulting system. Hence, this classical approach includes separate procedures for designing feedback and learning controllers where, for example, the ILC learning update (control law) is calculated as the inverse of the dynamics resulting from the feedback controller design. However, due to the finite nature of the time axis, the properties along the trial are not particularly addressed by many papers. The main reason behind this issue is the fact that the response of the linear system can never become unbounded in a finite time. Therefore, the main drawback of classical approaches to design ILC schemes is that the tracking error or control signal may become unacceptably large before convergence finally occurs. Thus, it is evident that application of two-dimensional/repetitive setting, see [13], allows us to study the convergence properties of ILC in two directions: the time direction ($p$-direction) and the trial-to-trial direction ($k$-direction). This means that both transient response and trial-to-trial convergence goals can be simultaneously satisfied and hence a great performance superiority is reached. Often these requirements are defined over the entire frequency spectrum but most of design requirements and specifications are defined for limited frequency ranges of relevance. Specifically, the possibility to specify different performance specifications has considerable practical significance since common performance issues occur over different and limited frequency ranges. For example, trial-to-trial error convergence rate is in the 'low' frequency range whereas low sensitivity to non-repetitive disturbances and sensor noise is rather in 'high' frequency range.

The contribution of this paper is to provide new insights into the currently known ILC design procedures with the two-dimensional/repetitive setting. Specifically, systematic guidelines for design of robust ILC laws are proposed for plants with time-varying uncertainties. Additionally, the conditions for attenuating non-repetitive disturbances are included. The generalized version of Kalman-Yakubovich-Popov (KYP) lemma [8] is extensively used to attenuate disturbances at specific frequencies, whereas the most of currently known results cannot impose such performance specifications. Moreover, the proposed approach leads to design based on LMI computations. In particular, sufficient LMI-based conditions for the existence of a robust ILC updating law are derived together with the design algorithms for the associated controller matrices.

Throughout this paper, the null and identity matrices with the required dimensions are denoted by 0 and $I$, respectively, and the notation $Y \prec 0$ (respectively $Y \succ 0$) means that the matrix $Y$

is negative definite (respectively, positive definite). The notation $(\star)$ represents the transposed elements in some symmetric matrices and sym{M} is a shorthand notation for $M + M^T$.

The following well-known results are adopted from the literature and used in developing the new results in this paper.

**Lemma 1** *[4] Given a symmetric matrix $\Upsilon$ and two matrices $\Lambda$, $\Sigma$ of compatible dimensions, there exist a matrix $\mathcal{W}$ satisfying*

$$\Upsilon + \mathrm{sym}\left\{\Lambda^{\mathrm{T}}\mathcal{W}\Sigma\right\} \prec 0, \tag{1}$$

*if, and only if the following projection inequalities hold*

$$\Lambda^{\perp\mathrm{T}}\Upsilon\Lambda^{\perp} \prec 0,\ \Sigma^{\perp\mathrm{T}}\Upsilon\Sigma^{\perp} \prec 0, \tag{2}$$

*where $\Lambda^{\perp}$ and $\Sigma^{\perp}$ are arbitrary matrices whose columns form a basis of null spaces of $\Lambda$ and $\Sigma$, respectively. Clearly, this means that $\Lambda\Lambda^{\perp} = 0$ and $\Sigma\Sigma^{\perp} = 0$.*

**Lemma 2** *[12] Given matrices $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z} = \mathcal{Z}^T$, $\Delta$ of compatible dimensions, then*

$$\mathcal{Z} + \mathrm{sym}\{\mathcal{X}\Delta\mathcal{Y}\} \prec 0,$$

*for all $\Delta$ satisfying $\Delta^T\Delta \preceq I$ if, and only if, there exists $\varepsilon > 0$ such that*

$$\mathcal{Z} + \epsilon\mathcal{X}\mathcal{X}^T + \epsilon^{-1}\mathcal{Y}^T\mathcal{Y} \prec 0.$$

# 2 Problem statement and preliminaries

Let $p \in [0, \alpha - 1]$ be the discrete-time index where $\alpha$ is the number of time steps for each batch and $k \geq 0$ be the batch index.

In this paper, we consider the following model of a batch process over a finite time interval $p \in [0, \alpha - 1]$,

$$\begin{cases} zx(p,k) = [A + \Delta A(p,k)]x(p,k) \\ \qquad\qquad + [B + \Delta B(p,k)]u(p,k) + w(p,k), \\ y(p,k) = Cx(p,k), \end{cases} \tag{3}$$

where $x(p,k) \in \mathbb{R}^{n_x}$ ,$u(p,k) \in \mathbb{R}^{n_u}$, $y(p,k) \in \mathbb{R}^{n_y}$ and $w(p,k) \in \mathbb{R}^{n_x}$ represent, respectively, the process state, input, output, and external disturbance vectors at time instant $p$ and batch number $k$. Furthermore, $z$ is the forward shift operator along the time ($p$) axis, i.e., $zx(p,k) = x(p+1,k)$. The given matrices $A$, $B$, $C$ are supposed to be real with compatible dimensions. Furthermore, $\Delta A(p,k)$ and $\Delta B(p,k)$ represent admissible time-varying uncertainties (modelled as additive perturbations to the nominal model matrices), that may be not repetitive from batch to batch, specified as

$$\Delta A(p,k) = H\Delta(p,k)E_A,\ \Delta B(p,k) = H\Delta(p,k)E_B, \tag{4}$$

where $H$, $E_A$ and $E_B$ are known constant matrices and $\Delta(p,k)$ is unknown time-varying matrix with Lebesgue measurable elements bounded by

$$\Delta^{\mathrm{T}}(p,k)\Delta(p,k) \preceq I. \tag{5}$$

The problem can now be formulated as follows. A given batch process of (3) is supposed to execute repetitive tasks of tracking the desired reference signal $\{Y_r(p), p = 1, 2, \ldots, \alpha\}$ with the initial resetting condition $x_0$ , i.e. $x(0, k) = x_0, \ \ \forall k \geq 0$. Also, the output tracking error is defined as

$$e(p, k) = Y_r(p) - y(p, k) \tag{6}$$

and the output tracking error and inputs are recorded for each batch. Then the primary task is to use these recorded data to generate an appropriate control input in the next batch. In other words, the objective is to construct a sequence of ILC inputs $\{u(\cdot, k)\}_{k\geq 0}$ such that the tracking error converges to zero or is below the prescribed level and hence the performance is improved in the batch-to-batch domain. Therefore, the convergence condition on the input and error can be defined as

$$\lim_{k\to\infty} \|e(\cdot, k)\| = 0, \lim_{k\to\infty} \|u(\cdot, k) - u(\cdot, \infty)\| = 0,$$

where $\|\cdot\|$ denotes the norm on the underlying function and $u(\cdot, \infty)$ is termed the learned control.

According to the aforementioned issues, this paper proposes analysis and design over a repetitive process setting since the dynamics of repetitive processes evolve in two independent directions and information in the temporal domain is limited to a finite duration. As the result this setting gives a systematic way to simultaneously consider behaviour along the time axis and from batch-to-batch. Therefore, the next section provides the adaptation of repetitive processes setting for ILC design of uncertain processes affected by disturbances.

## 2.1 Repetitive process model for ILC schemes

To formulate the ILC design problem over the repetitive process setting, let

$$\begin{aligned}
\delta x(p, k{+}1) &= x(p, k{+}1){-}x(p, k),\\
\delta u(p, k{+}1) &= u(p, k{+}1){-}u(p, k),\\
\delta e(p, k{+}1) &= e(p, k{+}1){-}e(p, k),\\
\delta w(p, k{+}1) &= w(p, k{+}1){-}w(p, k).
\end{aligned} \tag{7}$$

Then, according to (3), (6) and (7), we have

$$\begin{aligned}
e(p, k{+}1) =& e(p, k) - C\delta x(p, k + 1),\\
z\delta x(p, k{+}1) =& \mathfrak{A}\delta x(p, k{+}1){+}\mathfrak{B}\delta u(p, k{+}1)\\
&+\varpi(p, k{+}1),
\end{aligned} \tag{8}$$

where

$$\begin{aligned}
\mathfrak{A} =& A{+}\Delta A(p, k{+}1), \mathfrak{B} = B{+}\Delta B(p, k{+}1),\\
\varpi(p, k{+}1) =& [\Delta A(p, k{+}1){-}\Delta A(p, k)]x(p, k)\\
&+[\Delta B(p, k{+}1){-}\Delta B(p, k)]u(p, k)\\
&+\delta w(p, k{+}1).
\end{aligned}$$

It deserves to point out that $\varpi(p, k + 1) \neq 0$ for any non-repeatable uncertainties and/or external disturbances. Next, consider a classical formula for learning process of updating control input sequence as

$$u(p, k + 1) = u(p, k) + r(p, k + 1), \tag{9}$$

where $r(p,k+1)$ is the modification of the control input. Then it is found from (7) and the above formula that

$$e(p,k+1)-e(p,k)=-C\mathfrak{A}\delta x(p-1,k+1)-C\mathfrak{B}\delta u(p-1,k+1) \\ -C\varpi(p-1,k+1).$$

Next, define the vectors

$$\begin{aligned}\overline{x}(p,k)=&\delta x(p-1,k+1),\ \overline{u}(p,k)=\delta u(p-1,k+1),\\ \overline{\varpi}(p,k)=&\varpi(p-1,k+1),\end{aligned} \tag{10}$$

to write

$$\overline{x}(p+1,k) = \mathfrak{A}\overline{x}(p,k)+\mathfrak{B}(p,k)+\overline{\varpi}(p,k). \tag{11}$$

A widely used, ILC law to determine an input at the subsequent batches takes the form (9) where

$$\begin{aligned}r(p,k+1)=&K_1\delta x(p,k+1)+K_2e(p+1,k)\\ &-K_3(e(p+1,k)-e(p,k)).\end{aligned} \tag{12}$$

In the above formula $K_1$, $K_2$ and $K_3$ are compatibly dimensioned control law matrices to be found. Application of (12) results in the controlled dynamics written as

$$\begin{aligned}\begin{bmatrix} z\overline{x}(p,k) \\ ze(p-1,k)\end{bmatrix}=&\mathcal{A}\begin{bmatrix}\overline{x}(p,k)\\ e(p-1,k)\end{bmatrix}+\mathcal{B}_0e(p,k)+\mathcal{B}_1\overline{\varpi}(p,k),\\ e(p,k+1)=&\mathcal{C}\begin{bmatrix}\overline{x}(p,k)\\ e(p-1,k)\end{bmatrix}+\mathcal{D}_0e(p,k)+\mathcal{D}_1\overline{\varpi}(p,k),\end{aligned} \tag{13}$$

where

$$\begin{aligned}&\mathcal{A}=\begin{bmatrix}\mathfrak{A}+\mathfrak{B}K_1 & \mathfrak{B}K_3\\ 0 & 0\end{bmatrix},\ \mathcal{B}_0=\begin{bmatrix}\mathfrak{B}(K_2-K_3)\\ I\end{bmatrix},\ \mathcal{B}_1=\begin{bmatrix}I\\0\end{bmatrix},\\ &\mathcal{C}=\begin{bmatrix}-C(\mathfrak{A}+\mathfrak{B}K_1) & -C\mathfrak{B}K_3\end{bmatrix},\ \mathcal{D}_0=I-C\mathfrak{B}(K_2-K_3),\\ &\mathcal{D}_1=-C.\end{aligned}$$

**Remark 1** *Having the repetitive model of* (13) *we can address 3 different forms of ILC laws. In particular, when $K_2 \neq K_3$ the PD-type ILC law is addressed. On the other side, when $K_3 = 0$ then the P-type ILC law is addressed. Finally, set $K_2 = K_3$ to address the D-type ILC law.*

# 3 Robust PD-type ILC scheme design

Given repetitive process model (13), the problem of selecting of $K_1$, $K_2$, $K_3$ in (12) can be formulated by means of LMI-based stability condition for these processes. In a next step, this result will be extended to propose the design method for the robust ILC scheme with the requirement for $\mathcal{H}_\infty$ non-repetitive disturbance attenuation - see [11] for more details on $\mathcal{H}_\infty$ disturbance attenuation for discrete repetitive processes. To proceed, introduce the matrices

$$\Pi_1 = \begin{bmatrix}1 & 0\\ 0 & -\gamma_1^2\end{bmatrix}, \Phi = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \tag{14}$$

where a scalar $\gamma_1$ is a priori known. The following lemma gives an LMI-based sufficient condition for robust stability along the pass of discrete repetitive processes described by (13) and this result is adopted from [2].

**Lemma 3** *Let $\gamma_1$ be a positive scalar satisfying $0 < \gamma_1 \leq 1$. Then a linear repetitive process described by (13) is stable along the pass if there exist compatibly dimensioned $P_1 \succ 0$, $P_2 \succ 0$ such that*

$$\begin{bmatrix} \mathcal{A} & I \\ \mathcal{C} & 0 \end{bmatrix} (\Phi \otimes P_1) \begin{bmatrix} \mathcal{A} & I \\ \mathcal{C} & 0 \end{bmatrix}^T + \begin{bmatrix} \mathcal{B}_0 & 0 \\ \mathcal{D}_0 & I \end{bmatrix} (\Pi_1 \otimes P_2) \begin{bmatrix} \mathcal{B}_0 & 0 \\ \mathcal{D}_0 & I \end{bmatrix}^T \prec 0 \tag{15}$$

*is feasible.*

Clearly, the result of Lemma 3 cannot be directly applied for the considered ILC design since there exist product terms between control law matrices and matrix variables $P_1$ and $P_2$. Additionally, the unknown matrix $\Delta(p,k)$ is coupled too. To convert it into LMI problem and reduce the some conservatism imposed by uncertainty we firstly write

$$\begin{aligned} \left[\begin{array}{c|c} \mathcal{A} & \mathcal{B}_0 \\ \hline \mathcal{C} & \mathcal{D}_0 \end{array}\right] &= \left( \left[\begin{array}{cc|c} A & 0 & 0 \\ 0 & 0 & I \\ \hline -CA & 0 & I \end{array}\right] + \left[\begin{array}{cc|c} \Delta A(p,k+1) & 0 & 0 \\ 0 & 0 & 0 \\ \hline -C\Delta A(p,k+1) & 0 & 0 \end{array}\right] \right) \\ &+ \left( \left[\begin{array}{c} B \\ 0 \\ \hline -CB \end{array}\right] + \left[\begin{array}{c} \Delta B(p,k+1) \\ 0 \\ \hline -C\Delta B(p,k+1) \end{array}\right] \right) \left[\begin{array}{cc|c} K_1 & K_3 & N \end{array}\right] \\ &= (\overline{\mathcal{A}} + \Delta\overline{\mathcal{A}}) + (\overline{\mathcal{B}} + \Delta\overline{\mathcal{B}})\, K, \end{aligned}$$

where $N = K_2 - K_3$. Additionally, let us introduce the below matrices

$$\overline{H} = \left[\begin{array}{c} H \\ 0 \\ \hline -CH \end{array}\right], \ \overline{E}_A = \left[\begin{array}{cc|c} E_A & 0 & 0 \end{array}\right]$$

to rewrite $\Delta\overline{\mathcal{A}}$ and $\Delta\overline{\mathcal{B}}$ as

$$\Delta\overline{\mathcal{A}} = \overline{H}\Delta(p,k)\overline{E}_A, \ \Delta\overline{\mathcal{B}} = \overline{H}\Delta(p,k)E_B.$$

Therefore, the following result can be established.

**Theorem 1** *Let $\gamma_1$ be a positive scalar satisfying $0 < \gamma_1 \leq 1$. Suppose also that an ILC law (12) is applied to an uncertain batch process (3). Then the resulting ILC scheme described as a discrete linear repetitive process of the form (13) is robustly stable along the pass, and hence batch-to-batch error convergence occurs, if there exist compatibly dimensioned matrices $P_1 \succ 0$, $P_2 \succ 0$, $W$, $F_1$, $F_2$, $F_3$, $Y$ and scalars $\beta \in (-1, 1)$ and $\epsilon_1 > 0$ such that the following LMI is feasible*

$$\begin{bmatrix} \Upsilon_1 - \mathrm{sym}\{W\} & (\star) & (\star) & (\star) & (\star) \\ \Upsilon_3 + \overline{\mathcal{A}}W^T + \overline{\mathcal{B}}Y - \beta W & \Upsilon_2 + \beta\mathrm{sym}\left\{\overline{\mathcal{A}}W^T + \overline{\mathcal{B}}Y\right\} & (\star) & (\star) & (\star) \\ F_b - [0\ I]W & -F_a^T + [0\ I](\overline{\mathcal{A}}W^T + \overline{\mathcal{B}}Y)^T & P_2 - \mathrm{sym}\{F_3\} & (\star) & (\star) \\ 0 & \epsilon_1 \overline{H}^T & 0 & -\epsilon_1 I & (\star) \\ \overline{E}_A W + E_B Y & \overline{E}_A W + E_B Y & (\overline{E}_A W + E_B Y)\begin{bmatrix} 0 \\ I \end{bmatrix} & 0 & -\epsilon_1 I \end{bmatrix} \prec 0, \tag{16}$$

*where*

$$\begin{aligned}\Upsilon_1=&\begin{bmatrix}P_1 & 0\\ 0 & 0\end{bmatrix},\Upsilon_2=\begin{bmatrix}-P_1 & 0\\ 0 & -\gamma_1^2 P_2\end{bmatrix},\\ \Upsilon_3=&\begin{bmatrix}0 & F_1\\ 0 & F_2\end{bmatrix},F_a=\begin{bmatrix}F_1\\ F_2\end{bmatrix},F_b=\begin{bmatrix}0 & F_3\end{bmatrix}.\end{aligned} \tag{17}$$

*Also, if this last LMI is feasible then the corresponding ILC law matrices of* (12) *are given by*

$$\begin{bmatrix}K_1 \; K_3 \,|\, N\end{bmatrix} = YW^{-T},\; K_2 = N + K_3. \tag{18}$$

**Proof 1** *Assume that there exist* $P_1 \succ 0$*,* $P_2 \succ 0$*,* $W$*,* $F_1$*,* $F_2$*,* $F_3$*,* $Y$ *and scalars* $\beta \in (-1, 1)$ *and* $\epsilon_1 > 0$*, such that the LMI of* (16) *is feasible. Then application of the Schur's complement formula to* (16) *gives*

$$\Gamma_1 + \epsilon_1 \mathcal{H}_1 \mathcal{H}_1{}^T + \epsilon_1{}^{-1} \mathcal{E}_1{}^T \mathcal{E}_1 \prec 0,$$

*where*

$$\Gamma_1=\begin{bmatrix}\Upsilon_1-\mathrm{sym}\{W\} & (\star) & (\star)\\ \Upsilon_3+\overline{\mathcal{A}}W^T+\overline{\mathcal{B}}Y-\beta W & \Upsilon_2+\beta\mathrm{sym}\left\{\overline{\mathcal{A}}W^T+\overline{\mathcal{B}}Y\right\} & (\star)\\ F_b-[0\; I]W & -F_a^T+[0\; I](\overline{\mathcal{A}}W^T+\overline{\mathcal{B}}Y)^T & P_2-\mathrm{sym}\{F_3\}\end{bmatrix},\; \mathcal{H}_1=\begin{bmatrix}0\\ \overline{H}\\ 0\end{bmatrix},$$

$$\mathcal{E}_1=\begin{bmatrix}\overline{E}_A W+E_B Y & \overline{E}_A W+E_B Y & (\overline{E}_A W+E_B Y)\begin{bmatrix}0\\ I\end{bmatrix}\end{bmatrix}.$$

*Next, assign* $\mathcal{Z} \leftarrow \Gamma_1$*,* $\mathcal{X} \leftarrow \mathcal{H}_1$*,* $\mathcal{Y} \leftarrow \mathcal{E}_1$*,* $\Delta \leftarrow \Delta(p, k)$*and by Lemma 2 the last inequality is feasible if and only if*

$$\Gamma_1 + sym\{\mathcal{H}_1 \Delta(p, k) \mathcal{E}_1\} \prec 0.$$

*Introducing* $M = \left(\overline{\mathcal{A}}+\Delta\overline{\mathcal{A}}\right) + \left(\overline{\mathcal{B}}+\Delta\overline{\mathcal{B}}\right) K$ *the last inequality can be is transformed to the form of* (1) *with*

$$\begin{aligned}\Gamma=&\begin{bmatrix}\Upsilon_1 & \Upsilon_3^T & F_b^T\\ \Upsilon_3 & \Upsilon_2 & -F_a\\ F_b & -F_a^T & P_2-sym\{F_3\}\end{bmatrix},\Lambda^T=\begin{bmatrix}I & 0\\ \beta I & 0\\ 0 & I\end{bmatrix},\\ \mathcal{W}=&\begin{bmatrix}W\\ [0\; I]W\end{bmatrix},\Sigma=\begin{bmatrix}-I & M^T & 0\end{bmatrix}.\end{aligned} \tag{19}$$

*Next, by Lemma 1, the inequality* (16) *is feasible if and only if the inequality* (1) *holds for matrices chosen as in* (19)*. Also, by construction, the matrices* $\Sigma^\perp$ *and* $\Lambda^\perp$ *are*

$$\Sigma^\perp = \begin{bmatrix}M^T & 0\\ I & 0\\ 0 & I\end{bmatrix},\; \Lambda^\perp = \begin{bmatrix}\beta I\\ -I\\ 0\end{bmatrix}.$$

*Also, it follows immediately that the first inequality in* (2) *for this case is*

$$\begin{bmatrix} (\beta^2-1)P_1 & 0 \\ 0 & -\gamma^2 P_2 \end{bmatrix} + sym\left\{ \begin{bmatrix} 0 \\ -\beta I \end{bmatrix} \begin{bmatrix} F_1 \; F_2 \end{bmatrix} \begin{bmatrix} I \; 0 \\ 0 \; I \end{bmatrix} \right\} \prec 0. \tag{20}$$

*Therefore, by assigning*

$$\Gamma \leftarrow \begin{bmatrix} (\beta^2-1)P_1 & 0 \\ 0 & -\gamma^2 P_2 \end{bmatrix}, \Lambda^T \leftarrow \begin{bmatrix} 0 \\ -\beta I \end{bmatrix}, \mathcal{W} \leftarrow \begin{bmatrix} F_1 \; F_2 \end{bmatrix}, \Sigma \leftarrow \begin{bmatrix} I \; 0 \\ 0 \; I \end{bmatrix}$$

*and in virtue of Lemma 1 with* $\Lambda^\perp = [I \; 0]^T$ *and noting that* ${\Sigma^\perp}^T \Gamma \Sigma^\perp$ *vanishes,* (20) *is feasible if and only if* $(\beta^2-1)P_1 \prec 0$*. This meas that one can require* $\beta \in (-1,1)$ *and* $P_1 \succ 0$ *to satisfy* (20)*. Additionally, using the notation in* (19) *the second inequality in* (2) *yields*

$$\begin{bmatrix} \mathcal{A}P_1\mathcal{A}^T - P_1 & \mathcal{A}P_1\mathcal{C}^T & 0 \\ \mathcal{C}P_1\mathcal{A}^T & \mathcal{C}P_1\mathcal{C}^T - \gamma_1^2 P_2 & 0 \\ 0 & 0 & P_2 \end{bmatrix} + sym\left\{ \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \begin{bmatrix} \mathcal{B}_0^T & \mathcal{D}_0^T & -I \end{bmatrix} \right\} \prec 0. \tag{21}$$

*Finally, assign*

$$\Gamma \leftarrow \begin{bmatrix} \mathcal{A}P_1\mathcal{A}^T - P_1 & \mathcal{A}P_1\mathcal{C}^T & 0 \\ \mathcal{C}P_1\mathcal{A}^T & \mathcal{C}P_1\mathcal{C}^T - \gamma_1^2 P_2 & 0 \\ 0 & 0 & P_2 \end{bmatrix}, \mathcal{W} \leftarrow \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}, \Lambda^T \leftarrow I, \; \Sigma \leftarrow \begin{bmatrix} \mathcal{B}_0^T & \mathcal{D}_0^T & -I \end{bmatrix}$$

*and by Lemma 1, the inequality* (21) *can be directly transformed into* (15)*. This implies robust stability of the resulting repetitive process* (13) *and hence and hence batch-to-batch error convergence occurs. This completes the proof.*

# 4 Nonrepetitive disturbances attenuation

In this subsection, we address the problem of nonrepetitive disturbance attenuation. Specifically, we are interested in imposing additional constraint that allows the following frequency specification

$$\sup_{\theta \in \Omega} \| G(e^{j\theta}) \|_\infty < \gamma_2, \tag{22}$$

be satisfied for the prescribed value of $\gamma_2 > 0$, where $G(e^{j\theta}) = \mathbb{C}(e^{j\theta} I - \mathbb{A})^{-1}\mathbb{B}$ and

$$\mathbb{A} = \left(\overline{\mathcal{A}} + \Delta\overline{\mathcal{A}}\right) + \left(\overline{\mathcal{B}} + \Delta\overline{\mathcal{B}}\right) K, \; \mathbb{B} = \begin{bmatrix} \mathcal{B}_1 \\ \mathcal{D}_1 \end{bmatrix}, \; \mathbb{C} = [0 \, I].$$

As seen, the specification (22) imposes the desired small gain (or $\mathcal{H}_\infty$ norm) specification over finite frequency domain $\Omega$. In other words, (22) captures the desired attenuation of nonrepetitive disturbances since $G(e^{\mathrm{j}\theta})$ denotes the frequency response matrix from $\varpi(p,k)$ to $e(p,k)$. Also, to include the finite frequency domain specifications, $\Omega$ is the finite frequency range of interest as shown in Table 1, and LF, MF, and HF stand for low, middle and high frequency ranges respectively. As known, the generalized KYP (gKYP) lemma has been proven to be an efficient

Table 1: Frequency ranges of interest

| | LF | MF | HF |
|---|---|---|---|
| $\Omega$ | $\lvert\theta\rvert < \theta_l$ | $\theta_1 \leq \theta \leq \theta_2$ | $\lvert\theta\rvert > \theta_h$ |

tool for a given transfer function to satisfy a frequency domain specification over a limited frequency range. Moreover, it results in conditions over LMIs. Thus it can be directly applied to address the considered attenuation problem. The required gKYP lemma (its dual version) is given below for convenience.

**Lemma 4** *[8] For a given linear discrete time-invariant system with the transfer function matrix $G(z)$ and the frequency response matrix $G(e^{\mathrm{j}\theta}) = \mathbb{C}(e^{\mathrm{j}\theta}I - \mathbb{A})^{-1}\mathbb{B}$, the following statements are equivalent*

*i) The frequency domain inequality*

$$\begin{bmatrix} (e^{\mathrm{j}\theta}\omega I - \mathbb{A}^T)^{-1}\mathbb{C}^T \\ I \end{bmatrix}^* \Theta \begin{bmatrix} (e^{\mathrm{j}\theta}\omega I - \mathbb{A}^T)^{-1}\mathbb{C}^T \\ I \end{bmatrix} \prec 0 \tag{23}$$

*holds $\forall\theta \in \Omega$ where $\Omega$ is the frequency range, i.e. $\theta$ belongs to a subset of real numbers denoted by $\Omega$ and specified as in Table 1.*

*ii) There exist matrices $Q \succ 0$ and a symmetric matrix $P_3$ such that*

$$\begin{bmatrix} \mathbb{A} & I \\ \mathbb{C} & 0 \end{bmatrix} (\Psi^* \otimes Q + \Phi^* \otimes P_3) \begin{bmatrix} \mathbb{A} & I \\ \mathbb{C} & 0 \end{bmatrix}^T + \Theta \prec 0, \tag{24}$$

*where Phi is as in (14) and $\Phi$ is defined with the reference to specified choices of frequency ranges given below in Table 2.*

Table 2: The values of $\Psi$ over $\Omega$

| | LF | MF | HF |
|---|---|---|---|
| $\Theta$ | $\lvert\theta\rvert < \theta_l$ | $\theta_1 \leq \lvert\theta\rvert \leq \theta_2$ | $\lvert\theta\rvert > \theta_h$ |
| $\Psi$ | $\begin{bmatrix} 0 & 1 \\ 1 & -2\cos(\theta_l) \end{bmatrix}$ | $\begin{bmatrix} 0 & e^{j\theta_c} \\ e^{-j\theta_c} & -2\cos(\theta_d) \end{bmatrix}$ | $\begin{bmatrix} 0 & -1 \\ -1 & 2\cos(\theta_h) \end{bmatrix}$ |

where

$$\theta_d = \frac{\theta_2 - \theta_1}{2}, \quad \theta_c = \frac{\theta_1 + \theta_2}{2}.$$

Importantly, to satisfy (22), i.e. $\mathcal{H}_\infty$ disturbance attenuation over finite frequency domain, the matrix $\Theta$ is fixed as

$$\Theta = \begin{bmatrix} \mathbb{B} & 0 \\ 0 & I \end{bmatrix} (\Pi_2 \otimes I) \begin{bmatrix} \mathbb{B} & 0 \\ 0 & I \end{bmatrix}^T, \tag{25}$$

where $\Pi_2 = \text{diag}\{1, -\gamma_2\}$. In view of the above lemma, the following result can be obtained.

**Theorem 2** *Let $\gamma_1$, $\gamma_2$ be given positive scalars. Suppose also that an ILC law* (12) *is applied to an uncertain batch process* (3)*. Then the resulting ILC scheme described as a discrete linear repetitive process of the form* (13) *is robustly stable along the pass and the finite frequency $\mathcal{H}_\infty$ performance in* (22) *is satisfied for all $\theta \in \Omega$ and hence batch-to-batch error convergence occurs if there exist compatibly dimensioned matrices $P_1 \succ 0$, $P_2 \succ 0$, $P_3 \succ 0$, $Q \succ 0$, $W$, $F_1$, $F_2$, $F_3$, $Y$ and scalars $\beta \in (-1, 1)$ and $\epsilon_2 > 0$ such that the LMIs* (16) *and*

$$\begin{aligned} &\begin{bmatrix} \Psi^* \otimes Q + \Phi^* \otimes P_3 & 0 & \begin{bmatrix} 0 \\ \mathbb{B} \end{bmatrix} & \begin{bmatrix} 0 & \Upsilon_4^T \\ \epsilon_2 \overline{H} & \Upsilon_4^T \end{bmatrix} \\ 0 & -I & 0 & \begin{bmatrix} 0 & I \end{bmatrix} \Upsilon_4^T \\ \begin{bmatrix} 0 & \mathbb{B}^T \end{bmatrix} & 0 & -\gamma_2^2 I & 0 \\ \begin{bmatrix} 0 & \epsilon_2 \overline{H}^T \\ \Upsilon_4 & \Upsilon_4 \end{bmatrix} & \Upsilon_4 \begin{bmatrix} 0 \\ I \end{bmatrix} & 0 & -\epsilon_2 I \end{bmatrix} \\ &+ sym \left\{ \begin{bmatrix} \begin{bmatrix} I \\ I \end{bmatrix} \\ \begin{bmatrix} 0 & I \end{bmatrix} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -[W \; W\overline{\mathcal{A}}^T + Y^T \overline{\mathcal{B}}^T] & W\mathbb{C}^T & 0 & 0 \end{bmatrix} \right\} \prec 0, \end{aligned} \tag{26}$$

*hold and where $\Upsilon_4 = \overline{E}_A W + E_B Y$. Moreover, if LMIs in* (16) *and* (26) *are feasible then the ILC law matrices of* (12) *given by* (18) *ensure that the resulting ILC scheme is robustly convergent from batch to batch and keep the $\mathcal{H}_\infty$ disturbance attenuation below $\gamma_2$.*

**Proof 2** *Assume that (16) and* (26) *hold. Then feasibility of (26) implies that $W$ is non-singular and hence invertible. Clearly, the feasibility* (16) *ensures the batch-to-batch error convergence for uncertain processes. Next, apply the Schur's complement formula to* (26) *and employ the similar lines to that of the proofs of Theorem 1. Subsequently, it can be observed that it is possible to rewrite the inequality* (24) *as*

$$\begin{bmatrix} \mathbb{A}^T & \mathbb{C}^T \\ I & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} \Upsilon_5 & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \end{bmatrix} & -\gamma_2^2 I \end{bmatrix} \begin{bmatrix} \mathbb{A}^T & \mathbb{C}^T \\ I & 0 \\ 0 & I \end{bmatrix} \prec 0, \tag{27}$$

*where*

$$\Upsilon_5 = (\Psi^* \otimes Q + \Phi^* \otimes P_3) + \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{B}\mathbb{B}^T \end{bmatrix}.$$

*Furthermore, by means of Lemma 1 (or the Finsler Lemma which is a specialized version of Lemma 1) the LMI (2) can be obtained, and the proof is complete.*

# 5 Illustration

To illustrate effectiveness of the developed approach, the ILC algorithm design problem for the linearized dynamics of injection molding process is considered. Following studies in [9, 7, 5], the nozzle pressure response is formulated as the following state-space model,

$$\begin{aligned} x(p{+}1,k) &= \left( \begin{bmatrix} 1.607 & 1 \\ -0.6086 & 0 \end{bmatrix} + \Delta A(p,k) \right) x(p,k) \\ &\quad + \left( \begin{bmatrix} 1.239 \\ -0.9282 \end{bmatrix} + \Delta B(p,k) \right) u(p,k) + \omega(p,k), \\ y(p,k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(p,k) \end{aligned}$$

where the time-varying uncertainties $\Delta A(p,k)$, $\Delta B(p,k)$ are of the form (4) with

$$H = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix},\ E_{\mathrm{A}} = \begin{bmatrix} 0.0804 & 0 \\ -0.0304 & 0 \end{bmatrix},\ E_{\mathrm{B}} = \begin{bmatrix} 0.062 \\ -0.0464 \end{bmatrix}.$$

For illustration, the target profile takes the following form:

$$Y_{\mathrm{r}}(p) = \begin{cases} 200, & 1 \le p < 100; \\ 200 + 5(p - 100), & 100 \le p < 120; \\ 300, & 120 \le p \le T_{\mathrm{p}} = 200. \end{cases}$$

For practical implementation the initial part of $Y_{\mathrm{r}}(p)$ is pre-filtered by $G_f = (z^{-1} + z^{-2})/(3 - z^{-1})$. Also assume that $|\Delta(p,k)| \le 1$, together with non-repetitive disturbance $w(p,k) = 5\sin(0.1\delta_1 p + 0.2\delta_2 k)$, where $\delta_1$ and $\delta_2$ are randomly selected from interval $[0,1]$. To demonstrate the effectiveness of the proposed results, the design procedure given in Theorem 2 is executed for $\gamma_1 = 0.5$, $\beta = 0.1$, $\gamma_2 = 10$ and the low frequency range ($\theta_l = 0.3$). The obtained solution gives the following controller matrices

$$K_1 = [-1.2948 - 0.8127],\ K_2 = 0.8473,\ K_3 = 0.0073$$

The resulting controlled system is stable along the trial and hence trial-to-trial error convergence occurs. This can be verified in Figure 1 where the tracking error in the form of the RMSE (Root Mean Squared Error) is shown and compared to the previously presented results in [7, 5]. From comparison in Figure 1, the RMSE of the developed method is significantly lower when comparing with the method of [7, 5]. Obviously, the effectiveness of the presented ILC design is apparent.

# 6 Conclusions

In this paper we have developed new results on the iterative learning tracking control problem for a class of uncertain batch processes with nonrepetitive disturbances. A robust PD-type ILC law based on the repetitive process theory and $\mathcal{H}_\infty$ disturbance attenuation over finite frequency has been developed. Sufficient conditions for the existence of a monotonically convergent ILC law have been obtained in terms of the corresponding LMIs. In order to reduce the conservatism of the control problem, additional slack matrix variables have been introduced. A simulation study based on the nozzle velocity control system of injection molding process verifies the effectiveness of this design method. Topics for future research include a detailed investigation into continuous-time processes and more complex control laws which use only measured outputs.

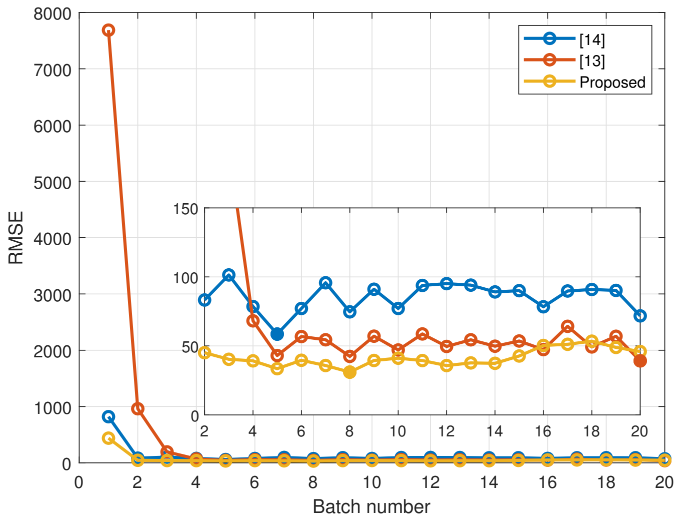


Figure 1: RMSE values obtained in the simulation study.

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