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GREEN'S FUNCTION IN FREE AXISYMMETRIC VIBRATION ANALYSIS OF ANNULAR THIN PLATES WITH DIFFERENT BOUNDARY CONDITIONS

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Free vibration analysis of homogeneous and isotropic annular thin plates by using Green's functions is considered. The formula of the influence function for uniform thin circular and annular plates is presented in closed-form. The limited independent solutions of differential Euler equation were expanded in the Neumann power series based on properties of integral equations. The analytical frequency equations as power series were obtained using the method of successive approximations. The natural axisymmetric frequencies for singularities when the core radius approaches zero are calculated. The results are compared with selected results presented in the literature.

Key words: annular plates, Green's function, singularities.

1. Introduction

The study of the vibration of a thin annular plate is basic in structural mechanics because it has many applications in civil and mechanical engineering. Circular and annular plates are the most critical structural elements in high speed rotating engineering systems. The natural frequencies of the circular and annular plates have been studied extensively for more than a century. Since the frequency of external load matches the natural frequency of the plate, destruction may occur.

The free vibration of annular plates has received considerable attention in the literature. The vibration of annular plates has been discussed by many authors (Vogel and Skinner, 1965; Ramaiah, 1980; Vera *et al*., 1998; Gabrielson, 1999). The work of Leissa (1969) is an excellent source of information about methods used for free vibration analysis of plates. Free vibration analysis has been carried out by using a variety of weighting function methods (Leissa, 1969) such as the Ritz method, the Galerkin method or the finite element method. Kim and Dickinson (1989) made studied lateral vibration of a thin annular plate subject to certain complicating effects. Wang *et al*. (1993) free vibration analysis of thin annular plates using the differential quadrature method (DQM). Liu *et al*. (2000) studied the effect of satisfying stress boundary conditions in the axisymmetric vibration analysis of circular and annular plates. Wang and Wang (2005) analyzed the fundamental frequencies of annular plates with small core. Kukla and Szewczyk (2006) analyzed axisymmetric natural frequencies of annular plates with elastic concentric supports using Green's functions. Taher *et al*. (2006) studied free vibration of circular and annular plates with variable thickness and different combinations of boundary conditions. Zhou *et al*. (2011) analyzed natural vibration of thin circular and annular plates by Hamiltonian approach. Wang (2014) studied vibration modes of concentrically supported free circular and annular plates.

In the present study Green's functions (GF) are used to obtain axisymmetric natural frequencies of annular plates. The formula of influence function for uniform annular (circular) plates is obtained in closed-form. The characteristic equations for different boundary conditions such as free, clamped, simply supported, sliding for inner and outer edges of plates are obtained for different values of core radius. The numerical results of the investigation are compared with selected results presented in open literature.

2. Statement of the problem

Consider an isotropic, homogeneous annular thin plate of constant thickness h in cylindrical coordinate (r, θ, z) with the z-axis along the longitudinal direction. The geometry and coordinate system of the plate is shown in Fig.1. The partial differential equation for free vibration of thin uniform annular (circular) plates has the following form

$$\nabla^4 W + \frac{\rho h}{D}\frac{\partial^2 W}{\partial t^2} = 0 \tag{2.1}$$

where ρ is the mass density, $D = Eh^3/12\left(1-\nu^2\right)$ is the flexural rigidity, E is Young's modulus, ν is the Poisson ratio, $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$ is the Laplacian and $W(r,\theta,t)$ is the small deflection compared with the thickness h of the plate. For free axisymmetric vibration of annular plates $\Theta = 0$.

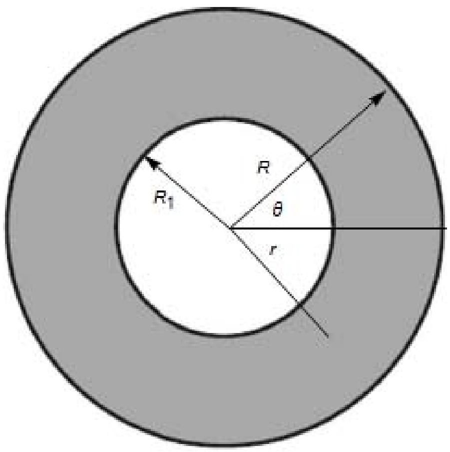

Fig.1. Geometry and coordinate system of the annular plate.

The axisymmetric deflection of an annular plate may be expressed as follows

$$W = w(r)e^{i\omega t} \tag{2.2}$$

where $w(r)$ is the radial mode function, ω is natural frequency, and $i^2 = -1$. Substituting Eq.(2.2) into Eq.(2.1) using the dimensionless coordinate $\xi = r/R$, the governing differential equation of the annular (circular) plate becomes

$$L(w) - \lambda^2 w = 0 \tag{2.3}$$

where

$$L(w) \equiv \frac{d^4 w}{d\xi^4} + \frac{2}{\xi}\frac{d^3 w}{d\xi^3} - \frac{1}{\xi^2}\frac{d^2 w}{d\xi^2} + \frac{1}{\xi^3}\frac{dw}{d\xi}, \tag{2.4}$$

is the differential operator and

$$\lambda = \omega R^2 \sqrt{\rho h / D}, \tag{2.5}$$

is the dimensionless frequency of vibration.

The boundary conditions at the inner edge $(\xi = R_1 / R = \xi_1)$ of the annular plate may be one of the following: clamped, simply supported, free and sliding supports. These conditions may be written in terms of the radial mode function $w(\xi)$ in the following form:

Clamped

$$w(\xi)|_{\xi=\xi_1} = 0, \tag{2.6a}$$

$$\frac{dw}{d\xi}|_{\xi=\xi_1} = 0. \tag{2.6b}$$

Simply supported

$$w(\xi)|_{\xi=\xi_1} = 0, \tag{2.7a}$$

$$M(w)|_{\xi=\xi_1} = \left(\frac{d^2 w}{d\xi^2} + \frac{\nu}{\xi}\frac{dw}{d\xi}\right)_{\xi=\xi_1} = 0. \tag{2.7b}$$

Free

$$M(w)|_{\xi=\xi_1} = 0, \tag{2.8a}$$

$$V(w)|_{\xi=\xi_1} = \left(\frac{d^3 w}{d\xi^3} + \frac{1}{\xi}\frac{d^2 w}{d\xi^2} - \frac{1}{\xi^2}\frac{dw}{d\xi}\right)_{\xi=\xi_1} = 0. \tag{2.8b}$$

Sliding (vertical) supports

$$\frac{dw}{d\xi}|_{\xi=\xi_1} = 0, \tag{2.9a}$$

$$V(w)|_{\xi=\xi_1} = 0. \tag{2.9b}$$

$M(w)$ and $V(w)$ are the normalized radial bending moment and the normalized effective shear force, respectively. Similar boundary conditions may be defined for the outer edge $(\xi = 1)$ depending on the type of support of annular plates.

3. Finding Green's function

The characteristic equation of homogeneous differential Euler equation for thin annular (circular) plates

$$L(w) \equiv \frac{d^4 w}{d\xi^4} + \frac{2}{\xi}\frac{d^3 w}{d\xi^3} - \frac{1}{\xi^2}\frac{d^2 w}{d\xi^2} + \frac{1}{\xi^3}\frac{dw}{d\xi} = 0\,, \tag{3.1}$$

has the following form

$$s^4 - 4s^3 + 4s^2 = 0. \tag{3.2}$$

The roots of Eq.(3.2) have the form

$$s_1 = s_2 = 0; \qquad s_3 = s_4 = 2\,. \tag{3.3}$$

Based on properties of Euler differential equations, the linear independent solutions of Eq.(3.1) have the form

$$w_1(\xi) = 1; \quad w_2(\xi) = \xi^2; \quad w_3(\xi) = \xi ln\xi; \quad w_4(\xi) = \xi^2 ln\xi\,. \tag{3.4}$$

Green's function (general solution of Eq.(3.1)) may be received from the formula presented in the following form (Tricomi, 1957)

$$K(\xi,\alpha) = \frac{|A|}{W(\alpha)p_0(\alpha)} \tag{3.5}$$

where $p_0(\alpha) = 1$ is the coefficient placed before the highest order of derivative of Euler differential Eq.(3.1),

$$|A| = \begin{vmatrix} 1 & \alpha^2 & \alpha ln\alpha & \alpha^2 ln\alpha \\ 0 & \frac{d(\alpha^2)}{d\alpha} & \frac{d(\alpha ln\alpha)}{d\alpha} & \frac{d(\alpha^2 ln\alpha)}{d\alpha} \\ 0 & \frac{d^2(\alpha^2)}{d\alpha^2} & \frac{d^2(\alpha ln\alpha)}{d\alpha^2} & \frac{d^2(\alpha^2 ln\alpha)}{d\alpha^2} \\ 1 & \xi^2 & \xi ln\xi & \xi^2 ln\xi \end{vmatrix}, \tag{3.6}$$

$$W(\alpha)=\begin{vmatrix} 1 & \alpha^2 & \alpha ln\alpha & \alpha^2 ln\alpha \\ 0 & \frac{d(\alpha^2)}{d\alpha} & \frac{d(\alpha ln\alpha)}{d\alpha} & \frac{d(\alpha^2 ln\alpha)}{d\alpha} \\ 0 & \frac{d^2(\alpha^2)}{d\alpha^2} & \frac{d^2(\alpha ln\alpha)}{d\alpha^2} & \frac{d^2(\alpha^2 ln\alpha)}{d\alpha^2} \\ 0 & \frac{d^3(\alpha^2)}{d\alpha^3} & \frac{d^3(\alpha ln\alpha)}{d\alpha^3} & \frac{d^3(\alpha^2 ln\alpha)}{d\alpha^3} \end{vmatrix}. \tag{3.7}$$

Functions 1, α^2, $\alpha ln\alpha$, $\alpha^2 ln\alpha$ are linear independent solutions, then the Wronskian satisfies the condition $W(\alpha)\neq 0$.

After calculations, Green's function (GF) has the following form

$$K(\xi,\alpha)=\frac{\alpha}{4}\left[\alpha^2-\xi^2+\left(\xi^2+\alpha^2\right)\left(ln\xi-ln\alpha\right)\right], \tag{3.8}$$

and satisfies conditions

$$K(\alpha,\alpha)=\frac{\partial K(\xi,\alpha)}{\partial\xi}\Big|_{\xi=\alpha}=\frac{\partial^2 K(\xi,\alpha)}{\partial\xi^2}\Big|_{\xi=\alpha}=0, \tag{3.9a}$$

$$\frac{\partial^3 K(\xi,\alpha)}{\partial\xi^3}\Big|_{\xi=\alpha}=1, \tag{3.9b}$$

according to the properties of influence functions (Kukla, 2009).

4. Solution of the problem

Based on the method of successive approximations and properties of integral equations, the limited solutions were expanded in the Neumann power series (Tricomi, 1957) in the following form

$$K(\xi)_u=K_0(\xi)_u+\sum_{i=1}^{\eta}K_i(\xi)_u\cdot\lambda^{2i},\qquad \lambda\in\Re^+, \tag{4.1a}$$

$$K(\xi)_v=K_0(\xi)_v+\sum_{i=1}^{\eta}K_i(\xi)_v\cdot\lambda^{2i}, \tag{4.1b}$$

$$K(\xi)_{u_1}=K_0(\xi)_{u_1}+\sum_{i=1}^{\eta}K_i(\xi)_{u_1}\cdot\lambda^{2i}, \tag{4.1c}$$

$$K(\xi)_{v_1}=K_0(\xi)_{v_1}+\sum_{i=1}^{\eta}K_i(\xi)_{v_1}\cdot\lambda^{2i} \tag{4.1d}$$

where $K_i(\xi)_u$, $K_i(\xi)_v$, $K_i(\xi)_{u_1}$ and $K_i(\xi)_{v_1}$ are iterated kernels defined in the following form

$$K_i(\xi)_u=\int_0^{\xi}K(\xi,\alpha)K_{i-1}(\alpha)_u\,d\alpha,\qquad K_0(\alpha)_u=1, \tag{4.2a}$$

$$K_i(\xi)_\nu = \int_0^\xi K(\xi,\alpha) K_{i-1}(\alpha)_\nu \, d\alpha, \qquad K_0(\alpha)_\nu = \alpha^2 \,, \tag{4.2b}$$

$$K_i(\xi)_{u_1} = \int_0^\xi K(\xi,\alpha) K_{i-1}(\alpha)_{u_1} \, d\alpha, \qquad K_0(\alpha)_{u_1} = \alpha \ln \alpha \,, \tag{4.2c}$$

$$K_i(\xi)_{\nu_1} = \int_0^\xi K(\xi,\alpha) K_{i-1}(\alpha)_{\nu_1} \, d\alpha, \qquad K_0(\alpha)_{\nu_1} = \alpha^2 \ln \alpha \,. \tag{4.2d}$$

Using condition for non-zero values for integral constants, the characteristic equations for different boundary conditions for considering annular plates have the following form:

Clamped inner edge and free outer edge

$$\Delta(\lambda) \equiv \begin{vmatrix} M\left[K(\xi)_u\right]|_{\xi=1} & M\left[K(\xi)_v\right]|_{\xi=1} & M\left[K(\xi)_{u_1}\right]|_{\xi=1} & M\left[K(\xi)_{v_1}\right]|_{\xi=1} \\ V\left[K(\xi)_u\right]|_{\xi=1} & V\left[K(\xi)_v\right]|_{\xi=1} & V\left[K(\xi)_{u_1}\right]|_{\xi=1} & V\left[K(\xi)_{v_1}\right]|_{\xi=1} \\ K(\xi)_u|_{\xi=\xi_1} & K(\xi)_v|_{\xi=\xi_1} & K(\xi)_{u_1}|_{\xi=\xi_1} & K(\xi)_{v_1}|_{\xi=\xi_1} \\ \dfrac{dK(\xi)_u}{d\xi}|_{\xi=\xi_1} & \dfrac{dK(\xi)_v}{d\xi}|_{\xi=\xi_1} & \dfrac{dK(\xi)_{u_1}}{d\xi}|_{\xi=\xi_1} & \dfrac{dK(\xi)_{v_1}}{d\xi}|_{\xi=\xi_1} \end{vmatrix} = \Lambda_0 = 0; \tag{4.3}$$

Simply supported inner edge and free outer edge

$$\Delta(\lambda) \equiv \begin{vmatrix} M\left[K(\xi)_u\right]|_{\xi=1} & M\left[K(\xi)_v\right]|_{\xi=1} & M\left[K(\xi)_{u_1}\right]|_{\xi=1} & M\left[K(\xi)_{v_1}\right]|_{\xi=1} \\ V\left[K(\xi)_u\right]|_{\xi=1} & V\left[K(\xi)_v\right]|_{\xi=1} & V\left[K(\xi)_{u_1}\right]|_{\xi=1} & V\left[K(\xi)_{v_1}\right]|_{\xi=1} \\ K(\xi)_u|_{\xi=\xi_1} & K(\xi)_v|_{\xi=\xi_1} & K(\xi)_{u_1}|_{\xi=\xi_1} & K(\xi)_{v_1}|_{\xi=\xi_1} \\ M\left[K(\xi)_u\right]|_{\xi=\xi_1} & M\left[K(\xi)_v\right]|_{\xi=\xi_1} & M\left[K(\xi)_{u_1}\right]|_{\xi=\xi_1} & M\left[K(\xi)_{v_1}\right]|_{\xi=\xi_1} \end{vmatrix} = \Lambda_0 = 0; \tag{4.4}$$

Sliding inner edge and free outer edge

$$\Delta(\lambda) \equiv \begin{vmatrix} M\left[K(\xi)_u\right]|_{\xi=1} & M\left[K(\xi)_v\right]|_{\xi=1} & M\left[K(\xi)_{u_1}\right]|_{\xi=1} & M\left[K(\xi)_{v_1}\right]|_{\xi=1} \\ V\left[K(\xi)_u\right]|_{\xi=1} & V\left[K(\xi)_v\right]|_{\xi=1} & V\left[K(\xi)_{u_1}\right]|_{\xi=1} & V\left[K(\xi)_{v_1}\right]|_{\xi=1} \\ \dfrac{dK(\xi)_u}{d\xi}|_{\xi=\xi_1} & \dfrac{dK(\xi)_v}{d\xi}|_{\xi=\xi_1} & \dfrac{dK(\xi)_{u_1}}{d\xi}|_{\xi=\xi_1} & \dfrac{dK(\xi)_{v_1}}{d\xi}|_{\xi=\xi_1} \\ V\left[K(\xi)_u\right]|_{\xi=\xi_1} & V\left[K(\xi)_v\right]|_{\xi=\xi_1} & V\left[K(\xi)_{u_1}\right]|_{\xi=\xi_1} & V\left[K(\xi)_{v_1}\right]|_{\xi=\xi_1} \end{vmatrix} = \Lambda_0 = 0\text{ł} \tag{4.5}$$

Free inner edge and clamped outer edge

$$\Delta(\lambda) \equiv \begin{vmatrix} M\left[K(\xi)_u\right]|_{\xi=\xi_1} & M\left[K(\xi)_v\right]|_{\xi=\xi_1} & M\left[K(\xi)_{u_1}\right]|_{\xi=\xi_1} & M\left[K(\xi)_{v_1}\right]|_{\xi=\xi_1} \\ V\left[K(\xi)_u\right]|_{\xi=\xi_1} & V\left[K(\xi)_v\right]|_{\xi=\xi_1} & V\left[K(\xi)_{u_1}\right]|_{\xi=\xi_1} & V\left[K(\xi)_{v_1}\right]|_{\xi=\xi_1} \\ K(\xi)_u|_{\xi=1} & K(\xi)_v|_{\xi=1} & K(\xi)_{u_1}|_{\xi=1} & K(\xi)_{v_1}|_{\xi=1} \\ \frac{dK(\xi)_u}{d\xi}|_{\xi=1} & \frac{dK(\xi)_v}{d\xi}|_{\xi=1} & \frac{dK(\xi)_{u_1}}{d\xi}|_{\xi=1} & \frac{dK(\xi)_{v_1}}{d\xi}|_{\xi=1} \end{vmatrix} = \Lambda_0 = 0; \quad (4.6)$$

Free inner edge and simply supported outer edge

$$\Delta(\lambda) \equiv \begin{vmatrix} M\left[K(\xi)_u\right]|_{\xi=\xi_1} & M\left[K(\xi)_v\right]|_{\xi=\xi_1} & M\left[K(\xi)_{u_1}\right]|_{\xi=\xi_1} & M\left[K(\xi)_{v_1}\right]|_{\xi=\xi_1} \\ V\left[K(\xi)_u\right]|_{\xi=\xi_1} & V\left[K(\xi)_v\right]|_{\xi=\xi_1} & V\left[K(\xi)_{u_1}\right]|_{\xi=\xi_1} & V\left[K(\xi)_{v_1}\right]|_{\xi=\xi_1} \\ K(\xi)_u|_{\xi=1} & K(\xi)_v|_{\xi=1} & K(\xi)_{u_1}|_{\xi=1} & K(\xi)_{v_1}|_{\xi=1} \\ M\left[K(\xi)_u\right]|_{\xi=1} & M\left[K(\xi)_v\right]|_{\xi=1} & M\left[K(\xi)_{u_1}\right]|_{\xi=1} & M\left[K(\xi)_{v_1}\right]|_{\xi=1} \end{vmatrix} = \Lambda_0 = 0; \quad (4.7)$$

Free inner edge and sliding outer edge

$$\Delta(\lambda) \equiv \begin{vmatrix} M\left[K(\xi)_u\right]|_{\xi=\xi_1} & M\left[K(\xi)_v\right]|_{\xi=\xi_1} & M\left[K(\xi)_{u_1}\right]|_{\xi=\xi_1} & M\left[K(\xi)_{v_1}\right]|_{\xi=\xi_1} \\ V\left[K(\xi)_u\right]|_{\xi=\xi_1} & V\left[K(\xi)_v\right]|_{\xi=\xi_1} & V\left[K(\xi)_{u_1}\right]|_{\xi=\xi_1} & V\left[K(\xi)_{v_1}\right]|_{\xi=\xi_1} \\ \frac{dK(\xi)_u}{d\xi}|_{\xi=1} & \frac{dK(\xi)_v}{d\xi}|_{\xi=1} & \frac{dK(\xi)_{u_1}}{d\xi}|_{\xi=1} & \frac{dK(\xi)_{v_1}}{d\xi}|_{\xi=1} \\ V\left[K(\xi)_u\right]|_{\xi=1} & V\left[K(\xi)_v\right]|_{\xi=1} & V\left[K(\xi)_{u_1}\right]|_{\xi=1} & V\left[K(\xi)_{v_1}\right]|_{\xi=1} \end{vmatrix} = \Lambda_0 = 0; \quad (4.8)$$

Free inner and outer edge

$$\Delta(\lambda) \equiv \begin{vmatrix} M\left[K(\xi)_u\right]|_{\xi=1} & M\left[K(\xi)_v\right]|_{\xi=1} & M\left[K(\xi)_{u_1}\right]|_{\xi=1} & M\left[K(\xi)_{v_1}\right]|_{\xi=1} \\ V\left[K(\xi)_u\right]|_{\xi=1} & V\left[K(\xi)_v\right]|_{\xi=1} & V\left[K(\xi)_{u_1}\right]|_{\xi=1} & V\left[K(\xi)_{v_1}\right]|_{\xi=1} \\ M\left[K(\xi)_u\right]|_{\xi=\xi_1} & M\left[K(\xi)_v\right]|_{\xi=\xi_1} & M\left[K(\xi)_{u_1}\right]|_{\xi=\xi_1} & M\left[K(\xi)_{v_1}\right]|_{\xi=\xi_1} \\ V\left[K(\xi)_u\right]|_{\xi=\xi_1} & V\left[K(\xi)_v\right]|_{\xi=\xi_1} & V\left[K(\xi)_{u_1}\right]|_{\xi=\xi_1} & V\left[K(\xi)_{v_1}\right]|_{\xi=\xi_1} \end{vmatrix} = \Lambda_0 = 0; \quad (4.9)$$

Clamped inner and outer edge

$$\Delta(\lambda) \equiv \begin{vmatrix} K(\xi)_u|_{\xi=1} & K(\xi)_v|_{\xi=1} & K(\xi)_{u_1}|_{\xi=1} & K(\xi)_{v_1}|_{\xi=1} \\ \frac{dK(\xi)_u}{d\xi}|_{\xi=1} & \frac{dK(\xi)_v}{d\xi}|_{\xi=1} & \frac{dK(\xi)_{u_1}}{d\xi}|_{\xi=1} & \frac{dK(\xi)_{v_1}}{d\xi}|_{\xi=1} \\ K(\xi)_u|_{\xi=\xi_1} & K(\xi)_v|_{\xi=\xi_1} & K(\xi)_{u_1}|_{\xi=\xi_1} & K(\xi)_{v_1}|_{\xi=\xi_1} \\ \frac{dK(\xi)_u}{d\xi}|_{\xi=\xi_1} & \frac{dK(\xi)_v}{d\xi}|_{\xi=\xi_1} & \frac{dK(\xi)_{u_1}}{d\xi}|_{\xi=\xi_1} & \frac{dK(\xi)_{v_1}}{d\xi}|_{\xi=\xi_1} \end{vmatrix} = \Lambda_0 = 0; \tag{4.10}$$

Simply supported inner and outer edge

$$\Delta(\lambda) \equiv \begin{vmatrix} K(\xi)_u|_{\xi=1} & K(\xi)_v|_{\xi=1} & K(\xi)_{u_1}|_{\xi=1} & K(\xi)_{v_1}|_{\xi=1} \\ M\left[K(\xi)_u\right]|_{\xi=1} & M\left[K(\xi)_v\right]|_{\xi=1} & M\left[K(\xi)_{u_1}\right]|_{\xi=1} & M\left[K(\xi)_{v_1}\right]|_{\xi=1} \\ K(\xi)_u|_{\xi=\xi_1} & K(\xi)_v|_{\xi=\xi_1} & K(\xi)_{u_1}|_{\xi=\xi_1} & K(\xi)_{v_1}|_{\xi=\xi_1} \\ M\left[K(\xi)_u\right]|_{\xi=\xi_1} & M\left[K(\xi)_v\right]|_{\xi=\xi_1} & M\left[K(\xi)_{u_1}\right]|_{\xi=\xi_1} & M\left[K(\xi)_{v_1}\right]|_{\xi=\xi_1} \end{vmatrix} = \Lambda_0 = 0. \tag{4.11}$$

Sliding both edges

$$\Delta(\lambda) \equiv \begin{vmatrix} \frac{dK(\xi)_u}{d\xi}|_{\xi=\xi_1} & \frac{dK(\xi)_v}{d\xi}|_{\xi=\xi_1} & \frac{dK(\xi)_{u_1}}{d\xi}|_{\xi=\xi_1} & \frac{dK(\xi)_{v_1}}{d\xi}|_{\xi=\xi_1} \\ V\left[K(\xi)_u\right]|_{\xi=\xi_1} & V\left[K(\xi)_v\right]|_{\xi=\xi_1} & V\left[K(\xi)_{u_1}\right]|_{\xi=\xi_1} & V\left[K(\xi)_{v_1}\right]|_{\xi=\xi_1} \\ \frac{dK(\xi)_u}{d\xi}|_{\xi=1} & \frac{dK(\xi)_v}{d\xi}|_{\xi=1} & \frac{dK(\xi)_{u_1}}{d\xi}|_{\xi=1} & \frac{dK(\xi)_{v_1}}{d\xi}|_{\xi=1} \\ V\left[K(\xi)_u\right]|_{\xi=1} & V\left[K(\xi)_v\right]|_{\xi=1} & V\left[K(\xi)_{u_1}\right]|_{\xi=1} & V\left[K(\xi)_{v_1}\right]|_{\xi=1} \end{vmatrix} = \Lambda_0 = 0. \tag{4.12}$$

For all boundary conditions formula of Λ_0 has the following form

$$\Lambda_0 = a_0 + \sum_{i=1}^{\pi}(-1)^i a_i \lambda^{2i} \tag{4.13}$$

where $a_0, a_1, \ldots, a_\eta$ are coefficients of characteristic equations dependent on boundary conditions and parameter *m*.

5. Results

The numerical results for axisymmetric dimensionless frequencies of annular plates with different boundary conditions are presented in Tabs 1-3. The Neumann power series Eq.(4.1) were expanded for the degree of approximation $\eta = 30$ for all cases under consideration. The numerical results for free annular plates with different boundary conditions on the inner edge are shown in Tab.1 in comparison to Vera *et al*. (1998), Vogel and Skinner (1965). The dimensionless frequencies of annular plates with different boundary conditions on the outer edge and free inner edge are presented in Tab.2 in comparison to Zhou *et al*. (2011). The numerical results for annular plates with the same boundary conditions on both edges are shown in Tab.3 in comparison with Zhou *et al*. (2011).

Table 1. Axisymmetric dimensionless frequencies $\lambda = \omega R^2 \sqrt{\rho h_R / D_R}$ for free outer edge of annular plate with different conditions on the inner edge $(\eta = 30, \nu = 0.3)$.

ξ_1	Dimensionless frequency, λ		Boundary conditions at the inner edge		
			Clamped	Simply supported	Sliding
0	λ_0	GF	*3.7519*	*3.7519*	*9.003*
		Ref.[13]	*3.7520*	*3.7519*	*9.003*
	λ_1	GF	*20.908*	*20.908*	*38.443*
	λ_2	GF	*60.645*	*60.644*	*87.750*
0.1	λ_0	GF	*4.237*	*3.449*	*9.426*
		Ref.[13]	*4.237*	*3.449*	*9.425*
		Ref.[14]	*4.237*	*3.449*	-
	λ_1	GF	*25.262*	*20.889*	*41.684*
		Ref.[13]	*25.262*	*20.899*	-
	λ_2	GF	*73.900*	*64.202*	*97.480*
0.2	λ_0	GF	*5.181*	*3.339*	*10.675*
		Ref.[13]	*5.181*	*3.339*	*10.676*
	λ_1	GF	*32.291*	*24.797*	*49.955*
		Ref.[13]	*32.291*	*24.797*	-
	λ_2	GF	*94.084*	*79.073*	*119.61*
0.3	λ_0	GF	*6.660*	*3.422*	*12.919*
	λ_1	GF	*42.614*	*31.603*	*63.519*
	λ_2	GF	*123.46*	*102.31*	*154.23*
0.4	λ_0	GF	*9.020*	*3.672*	*16.685*
		Ref.[13]	*9.020*	*3.672*	*16.685*
		Ref.[14]	*9.020*	*3.672*	-
	λ_1	GF	*58.549*	*42.556*	*85.206*
	λ_2	GF	*168.69*	*138.68*	*208.58*
0.5	λ_0	GF	*13.024*	*4.120*	*23.189*
		Ref.[13]	*13.024*	*4.120*	*23.189*
	λ_1	GF	*85.032*	*61.009*	*121.69*
	λ_2	GF	*243.71*	*199.34*	*298.45*
0.7	λ_0	GF	*36.953*	*6.187*	*62.152*
		Ref.[13]	*36.953*	*6.186*	-
		Ref.[14]	*36.953*	*6.186*	-
	λ_1	GF	*756.74*	*169.49*	*1449.1*
	λ_2	GF	*2059.05*	*730.13*	*6541.1*
0.8	λ_0	GF	*84.499*	*8.892*	*139.08*
		Ref.[13]	*84.500*	*8.892*	-
	λ_1	GF	*494.49*	*205.74*	*246.57*
	λ_2	GF	*789.61*	*492.53*	*291.72*
0.9	λ_0	GF	*167.38*	*17.113*	*168.41*
	λ_1	GF	*394.20*	*166.81*	*594.57*
	λ_2	GF	*739.59*	*393.63*	*794.96*

Table 2. Axisymmetric dimensionless frequencies $\lambda = \omega R^2 \sqrt{\rho h_R / D_R}$ for free inner edge of annular plate with different boundary conditions on the outer edge $(\eta = 30, \nu = 0.33)$.

ξ_1	Dimensionless frequency, λ		Boundary conditions at the outer edge		
			Clamped	Simply supported	Sliding
0	λ_0	GF	*10.215*	*4.983*	*14.682*
	λ_1	GF	*39.771*	*29.758*	*49.218*
	λ_2	GF	*89.104*	*74.192*	*103.50*
0.1	λ_0	GF	*10.134*	*4.890*	*14.434*
	λ_1	GF	*39.378*	*29.370*	*48.959*
	λ_2	GF	*90.161*	*74.610*	*105.32*
0.2	λ_0	GF	*10.347*	*4.732*	*14.652*
		Ref.[15]	*10.347*	*4.732*	-
	λ_1	GF	*42.799*	*31.262*	*54.322*
	λ_2	GF	*104.71*	*85.546*	*123.68*
0.3	λ_0	GF	*11.338*	*4.659*	*16.153*
	λ_1	GF	*51.513*	*36.882*	*66.642*
	λ_2	GF	*132.14*	*107.25*	*151.09*
0.4	λ_0	GF	*13.500*	*4.743*	*19.486*
		Ref.[15]	*13.500*	*4.743*	-
	λ_1	GF	*66.925*	*47.303*	*87.817*
	λ_2	GF	*176.77*	*143.06*	*211.02*
0.5	λ_0	GF	*17.598*	*5.043*	*25.809*
	λ_1	GF	*93.605*	*65.680*	*124.16*
	λ_2	GF	*251.94*	*203.66*	*301.76*
0.6	λ_0	GF	*25.540*	*5.663*	*38.098*
		Ref.[15]	*25.541*	*5.663*	-
	λ_1	GF	*143.39*	*100.24*	*191.85*
	λ_2	GF	*327.34*	*297.61*	*341.79*
0.7	λ_0	GF	*42.980*	*6.864*	*65.209*
	λ_1	GF	*432.03*	*175.39*	*248.33*
	λ_2	GF	*1014.9*	*265.76*	*631.14*
0.8	λ_0	GF	*92.816*	*9.455*	*143.13*
		Ref.[15]	*92.816*	*9.454*	-
	λ_1	GF	*202.62*	*336.95*	*211.30*
	λ_2	GF	*244.31*	*587.30*	*296.38*
0.9	λ_0	GF	*181.95*	*17.510*	*286.85*
	λ_1	GF	*361.31*	*443.94*	*363.73*
	λ_2	GF	*768.66*	*1391.4*	*1200.4*

Table 3. Axisymmetric dimensionless frequencies $\lambda = \omega R^2 \sqrt{\rho h_R / D_R}$ for annular plate with the same boundary conditions on both the edges $\left(\eta = 30, \nu = 0.33\right)$.

ξ_1	Dimensionless frequency, λ		Boundary conditions at the both edges: Clamped	Simply supported	Free	Sliding
0	λ_0	GF	*22.736*	*14.859*	*9.076*	*14.682*
	λ_1	GF	*61.951*	*49.513*	*38.514*	*49.218*
	λ_2	GF	*121.17*	*103.80*	*87.820*	*103.50*
0.1	λ_0	GF	*27.280*	*14.438*	*8.818*	*15.531*
	λ_1	GF	*75.366*	*51.654*	*38.169*	*53.737*
	λ_2	GF	*148.21*	*112.81*	*88.815*	*115.52*
0.2	λ_0	GF	*34.609*	*16.730*	*8.441*	*17.941*
		Ref.[15]	*34.609*	*16.733*	*8.441*	-
	λ_1	GF	*95.740*	*63.275*	*41.570*	*64.888*
	λ_2	GF	*188.15*	*140.48*	*103.14*	*142.24*
0.3	λ_0	GF	*45.346*	*21.035*	*8.316*	*22.144*
	λ_1	GF	*125.36*	*81.662*	*50.222*	*82.887*
	λ_2	GF	*246.15*	*182.44*	*130.30*	*183.68*
0.4	λ_0	GF	*61.872*	*28.090*	*8.551*	*29.064*
		Ref.[15]	*61.872*	*28.183*	*8.551*	-
	λ_1	GF	*170.90*	*110.45*	*65.567*	*111.46*
	λ_2	GF	*335.37*	*247.63*	*174.58*	*248.58*
0.5	λ_0	GF	*89.250*	*40.011*	*9.228*	*40.872*
	λ_1	GF	*246.35*	*158.60*	*92.208*	*159.38*
	λ_2	GF	*450.24*	*354.56*	*249.27*	*359.09*
0.6	λ_0	GF	*139.61*	*62.126*	*10.546*	*62.886*
		Ref.[15]	*139.61*	*62.122*	*10.548*	-
	λ_1	GF	*325.93*	*247.28*	*142.08*	*248.26*
	λ_2	GF	*866.48*	*456.30*	*331.73*	*323.67*
0.7	λ_0	GF	*402.92*	*110.04*	*13.018*	*110.71*
	λ_1	GF	*855.34*	*250.46*	*1104.7*	*304.01*
	λ_2	GF	*8088.1*	*614.32*	*1837.2*	*1096.8*
0.8	λ_0	GF	*559.17*	*247.17*	*18.262*	*277.12*
		Ref.[15]	*559.16*	*247.07*	*18.262*	-
	λ_1	GF	*1129.1*	*1026.5*	*209.37*	*735.44*
	λ_2	GF	*7018.8*	*2079.1*	*543.15*	*2511.4*
0.9	λ_0	GF	*347.08*	*172.18*	*34.428*	*171.64*
	λ_1	GF	*1996.4*	*328.80*	*162.00*	*232.28*
	λ_2	GF	-	*391.45*	*351.50*	*769.86*

6. Conclusions

In this paper, Green's function has been employed to solve natural vibration of annular thin plates with different combinations of boundary conditions on both the edges. Green's function for annular (circular) plates is obtained in closed form. The limited independent solutions of Euler equation were expanded in the Neumann power series rapidly convergent to exact values. Exact solutions were obtained for a low value of degree of approximation $(\eta = 30)$. The characteristic equations were obtained for all possible values of core radius. It was possible to obtain the exact eigenvalues using properties of the Volterra integral equations of second kind and calculation software without using the Bessel functions and bisection method (or Newton-Raphson method) presented in other works (Wang and Wang, 2005; Wang, 2014). The natural frequencies for singularities when the core radius approaches zero are calculated. The results of the investigation for annular plates are in good agreement with results obtained by other methods presented in open literature. The results can be used to validate the accuracy of other numerical methods as benchmark values. The calculations were made with the help of Mathematica v10, which is a symbolic calculation software.

Nomenclature

D- – flexural rigidity
E – Young modulus
$K(\xi,\alpha)$ – Green's function
$L(w)$- – differential operator
$W(\alpha)$ – Wronskian
$w(r)$ - – radial mode function
$w_i(\xi)$ - – fundamental solution
$\Delta(\lambda)$ – characteristic equation
λ - – dimensionless frequency
ν – Poisson ratio
$\xi = r/R$ - – dimensionless radial coordinate
$\xi_I = R_I/R$ - – dimensionless radial coordinate of the hole
ρ – mass density

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