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RESPONSE OF NON-LINEAR SHOCK ABSORBERS-BOUNDARY VALUE PROBLEM ANALYSIS

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A nonlinear boundary value problem of two degrees-of-freedom (DOF) untuned vibration damper systems using nonlinear springs and dampers has been numerically studied. As far as untuned damper is concerned, sixteen different combinations of linear and nonlinear springs and dampers have been comprehensively analyzed taking into account transient terms. For different cases, a comparative study is made for response versus time for different spring and damper types at three important frequency ratios: one at $r = 1$, one at $r > 1$ and one at $r < 1$. The response of the system is changed because of the spring and damper nonlinearities; the change is different for different cases. Accordingly, an initially stable absorber may become unstable with time and vice versa. The analysis also shows that higher nonlinearity terms make the system more unstable. Numerical simulation includes transient vibrations. Although problems are much more complicated compared to those for a tuned absorber, a comparison of the results generated by the present numerical scheme with the exact one shows quite a reasonable agreement.

Key words: shock absorber, untuned vibration damper, frequency ratios, non-linear springs, non-linear dampers, stability, boundary value problem, multisegment method of integration.

1. Introduction

Shock absorbers are basically untuned viscous vibration dampers designed to smooth out or damp shock impulse and dissipate kinetic energy. A tuned vibration absorber (having ideally no damping) is the most effective only at the tuned frequency and its usefulness is narrowly limited in the region of the tuned frequency (Rahman and Ahmed, 2009; Thomson, 1981). In contrast, when the forcing frequency varies over a wide range, an untuned viscous damper (also called a shock absorber) becomes very useful (Ahmed, 2009; Thomson, 1981). As far as vibration control is concerned, the boundary value problem analysis of shock absorbers will be important from practical engineering point of view.

Practically vibration problem becomes quite complicated as springs and dampers do not behave linearly in actual engineering applications (Mikhlin and Reshetnikova, 2005; Zhu *et al*., 2004; Kalnins and Dym, 1976; Rahman and Ahmed, 2009). But nonlinear problems, usually having no closed form solutions, are always challenges for practicing engineers. The superposition principle cannot be applied and therefore different mathematical techniques are developed and used to solve such problems.

Boundary value problems have been solved by researchers by different mathematical techniques, for example, the Runge-Kutta integration method and bifurcation method (Asfar and Akour, 2005; Chatterjee, 2007), univariate search optimization method (Asfar and Akour, 2005), method of multiple scales (Plaut and Limam, 1991), singular perturbation method of first order (Dohnal, 2007). These methods have been used to solve self-excited vibration problems.

However, the present paper considers a multisegment method of integration which was originally developed for solving thin shell equations by Kalnins and Lestingi (1967). To the best of the authors' knowledge this method has not been tried for solving nonlinear shock absorber's stability problems. It has

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been found that the method used in this paper makes the computation faster and yields reliable results under any practically possible boundary condition for a tuned absorber (Rahman and Ahmed, 2009).

A few studies of the TDOFS involving nonlinearities in springs and dampers and somewhat related to the present study, are briefly described below. Recently, Alexander *et al*. (2009) explored the performance of a nonlinear tuned mass damper (NTMD), which is modeled as a TDOFS with a cubic nonlinearity. This nonlinearity is physically derived from a geometric configuration of two pairs of springs. The springs in one pair rotate as they extend, which results in a hardening spring stiffness. The other pair provides a linear stiffness term.

Rahman and Ahmed (2009) investigated stability of tuned vibration absorbers having nonlinear springs involving the boundary value problem and demonstrated that under certain conditions an initially stable system can become unstable and vice versa.

Mikhlin and Reshetnikova (2005) studied the nonlinear 2DOF system having a linear oscillator with a relatively big mass which is an approximation of some continuous elastic system, and an essentially nonlinear oscillator with a relatively small mass which is an absorber of the main linear system vibrations. Zhu *et al*. (2004) investigated the nonlinear response of a two degrees of freedom (2DOF) vibration system with nonlinear damping and nonlinear springs. In papers by Manevitch *et al*. (2002) and McFarland *et al*. (2002), theoretical investigation and some experimental verification on the use of nonlinear localization for reducing the transmitted vibrations in structures subjected to transient base motions have been presented. In particular, the experimental assembly, containing the main linear subsystem and the nonlinear absorber, is described by Vakakis *et al*. (2002). Nakhaie *et al.* (2003) used the root mean square of absolute acceleration and relative displacement to find the optimal damping ratio and natural frequency of the isolator.

Vakakis and Paipetis (1986) investigated the effect of a viscously damped dynamic absorber on an undamped multiple degrees of freedom system (MDOFS). Soom and Lee (1983) and Jordanov (1988) studied the optimal parameter design of linear and non-linear dynamic vibration absorbers for damped primary systems. The presence of the non-linearities introduces dangerous instabilities, which in some cases may result in amplification rather than reduction of the vibration amplitudes (Rice,1986 and Shaw *et al*., 1989). Natsiavas (1992) applied the method of averaging to investigate the steady state oscillations and stability of non-linear dynamic vibration absorbers. Oueini *et al.* (1998) exploited the saturation phenomenon in devising an active vibration suppression technique.

Shekhar *et al*. (1998) considered a single stage shock isolator comprising a parallel combination of a cubic nonlinearity in spring and damper. Combining the straightforward perturbation method and Laplace transform they determined the transient response of the system. Three types of input base excitations were considered: the rounded step, the rounded pulse and the oscillatory step. Although the solution is valid for a limited range of nonlinearities, it was shown that the nonlinearity in the damping rather than in the stiffness has a more pronounced effect on the performance of a shock isolator. The presence of a nonlinear velocity dependent damping term with a positive coefficient was found to have detrimental effects on the isolator performance. Shekhar *et al.* (1999) considered different alternatives to improve the performance of an isolator having a nonlinear cubic damping over and above the usual viscous damping.

As can be verified from literature, most numerical studies regarding nonlinear vibration of structures, particularly involving multi degrees of freedom system (MDOFS), have been carried out in the form of initial value problems: all the boundary conditions, termed as system's responses (displacement, velocity etc.), were specified at an initial time reference, followed by numerical integration of the governing differential equation. Such type of analysis involves simultaneous solution of a system of nonlinear equations where the number of equations to be solved is determined by the order of the governing equations. For example, for a 2nd order equation, each time a *2x 2* system of equation needs to be solved and the frequently used method is Newton-Raphson. Obviously, the problem becomes much more complex if all the boundary conditions are not specified at the same initial time reference, that is, some conditions are also specified at the final time reference. This type of problem needs to simultaneously solve a large number of nonlinear equations that depends on the number of intermediate grid points in between the two time references.

Though, Newton-Raphson method can be used to solve that large number of equations, there are chances of non-convergence of solutions.

The present work aims to solve both boundary and initial value problems for an untuned vibration damper system. Generally, stability of the MDOFS system is studied by the method of perturbation. But a simple and direct method, like that of the multisegment integration technique (Kalnins and Lestingi, 1967), that helps to directly visualize the system's response with time, would be very useful, in particular for the present study, when a boundary value problem is dealt with. Therefore, the objectives of this study can be described as below:

(a.) At first to develop a generalized computer code (using Turbo C) for nonlinear vibration analysis of a TDOFS having nonlinear springs and dampers.
(b.) To study the system's response and stability for different mass ratios and the nonlinearity of the springs and dampers.
(c.) Soundness of the code can be checked by comparing the available standard result, for the case of a linear viscous damper, given in Thomson (1981).

Thus results have been obtained for different untuned vibration dampers (Cases 1–16 in Tab.1) for chosen boundary conditions and different parameters of interest (Tabs 2–4 and Figs 1–2). Ideally, an untuned viscous damper is basically a 2DOFS with a very small mass ratio, having zero damping for the main mass (m_1) and zero spring force for the absorber. Further, both the main mass spring and absorber's damper are linear. Such an ideal case has been called Case 1 in this paper. However only to simulate practical situations, a small damping for the main mass (m_1) and a small spring force for the absorber are assumed. It is also assumed that these two nonzero forces can be nonlinear as well (Cases 2-16).

In Tab.2 data set #1 and data set #2 consist of same masses and main spring stiffness. But changes are made in damping constants and 2nd mass spring stiffness. For set #1, the damping constant equals the optimum damping ratio (ζ_o) but for set #2, the chosen damping constant is larger than that of data set #1.

For nonlinearity indices (both springs and dampers), values chosen for data set #1 are much smaller than that of data set #2. These data are so selected to demonstrate the fact that untuned viscous vibration dampers become highly unstable because of increased nonlinearity.

As in our previous work on tuned absorbers (Rahman and Ahmed, 2009) no damping effect was studied this present study includes both linear and non-linear damping terms as well as the optimum damping ratio term to extensively investigate the shock absorber's response.

2. Mathematical models and governing equations

Figure 1 shows the proposed model for the TDOFS while Tab.1 shows the different cases. Following Fig.1,

Spring force for the 1st spring= $k_1 x_1 + k_1' x_1^3$ (2.1)

Spring force for the 2nd spring= $k_2 \left(x_1 - x_2\right) + k_2' (x_1 - x_2)^3$ (2.2)

Damping force for the 1st damper= $c_1 \dot{x}_1 + c_1' \dot{x}_1 x_1^2$ (2.3)

Damping force for the 2nd damper= $c_2 \left(\dot{x}_1 - \dot{x}_2\right) + c_2' \left(\dot{x}_1 - \dot{x}_2\right)\left(x_1 - x_2\right)^2$ (2.4)

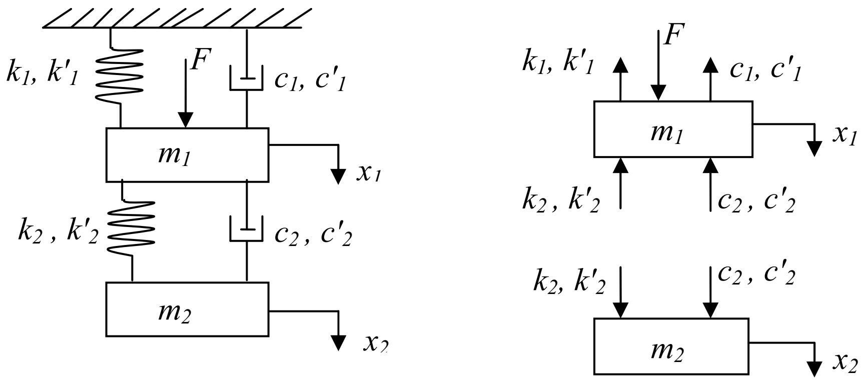

Fig.1. Arrangement of masses, springs and dampers for TDOF non-linear shock absorbers.

Table 1. Different cases of shock absorbers.

Case	Combinations of Springs	Combinations of Dampers	Summary of the Systems
Case 1	Linear spring with main mass only	Linear damper with second mass only	$c_1=0.0$, $k_2=0.0$ $c'_1=0.0=k'_2$
Case 2	Both springs linear	Linear dampers	Both linear and nonlinear spring combinations with linear dampers, Cases 2 – 6
Case 3	Both springs hard		
Case 4	1st spring: hard 2nd spring: soft		
Case 5	1st spring: soft 2nd spring: hard		
Case 6	Both springs soft		
Case 7	Both springs linear	Hard dampers	Both linear and nonlinear spring combinations with hard dampers, Cases 7 – 11
Case 8	Both springs hard		
Case 9	1st spring: hard 2nd spring: soft		
Case 10	1st spring: soft 2nd spring: hard		
Case 11	Both springs soft		
Case 12	Both springs linear	Soft dampers	Both linear and nonlinear spring combinations with soft dampers, Cases 12 – 16
Case 13	Both springs hard		
Case 14	1st spring: hard 2nd spring: soft		
Case 15	1st spring: soft 2nd spring: hard		
Case 16	Both springs soft		

The equations of motion are as follows for the main mass and the damper mass, respectively,

$$\begin{aligned} & m_1\ddot{x}_1+\left(k_1x_1+k'_1x_1^3\right)+\left(c_1\dot{x}_1+c'_1\dot{x}_1x_1^2\right)+\left\{k_2\left(x_1-x_2\right)+k'_2(x_1-x_2)^3\right\}+ \\ & +\left\{c_2\left(\dot{x}_1-\dot{x}_2\right)+c'_2\left(\dot{x}_1-\dot{x}_2\right)\left(x_1-x_2\right)^2\right\}=F, \end{aligned} \tag{2.5}$$

$$m_2\ddot{x}_2-\left\{k_2\left(x_1-x_2\right)+k_2'(x_1-x_2)^3\right\}-\left\{c_2\left(\dot{x}_1-\dot{x}_2\right)+c_2'\left(\dot{x}_1-\dot{x}_2\right)\left(x_1-x_2\right)^2\right\}=0. \tag{2.6}$$

For transformations, let $x_1=y_1$, $x_2=y_3$, $\frac{dx_1}{dt}=\dot{x}_1=y_2$ and $\frac{dx_2}{dt}=\dot{x}_2=y_4$

With those transformations, Eqs (2.5) and (2.6) become,

$$\begin{aligned}&m_1\frac{dy_2}{dt}+\left(k_1y_1+k_1'y_1^3\right)+\left(c_1y_2+c_1'y_2y_1^2\right)+\left\{k_2\left(y_1-y_3\right)+k_2'\left(y_1-y_3\right)^3\right\}+\\&+\left\{c_2\left(y_2-y_4\right)+c_2'\left(y_2-y_4\right)\left(y_1-y_3\right)^2\right\}=F,\end{aligned} \tag{2.7}$$

$$\begin{aligned}&m_2\frac{dy_4}{dt}-\left\{k_2\left(y_1-y_3\right)+k_2'\left(y_1-y_3\right)^3\right\}+\\&-\left\{c_2\left(y_2-y_4\right)+c_2'\left(y_2-y_4\right)\left(y_1-y_3\right)^2\right\}=0.\end{aligned} \tag{2.8}$$

Rearrangement of Eqs (2.7) and (2.8) gives,

$$\frac{dy_2}{dt}=\frac{1}{m_1}\left[\begin{aligned}&F-\left(k_1y_1+k_1'y_1^3\right)-\left(c_1y_2+c_1'y_2y_1^2\right)-\left\{k_2\left(y_1-y_3\right)+k_2'\left(y_1-y_3\right)^3\right\}+\\&-\left\{c_2\left(y_2-y_4\right)+c_2'\left(y_2-y_4\right)\left(y_1-y_3\right)^2\right\}\end{aligned}\right], \tag{2.9}$$

$$\frac{dy_4}{dt}=\frac{1}{m_2}\left[\left\{k_2\left(y_1-y_3\right)+k_2'\left(y_1-y_3\right)^3\right\}+\left\{c_2\left(y_2-y_4\right)+c_2'\left(y_2-y_4\right)\left(y_1-y_3\right)^2\right\}\right]. \tag{2.10}$$

The governing Eqs (2.9) and (2.10) can now be rewritten as a set of four nonlinear first order ordinary differential equations (ODE) as follows

$$\frac{dy_1}{dt}=y_2, \tag{2.11}$$

$$\frac{dy_2}{dt}=\frac{1}{m_1}\left[\begin{aligned}&F-\left(k_1y_1+k_1'y_1^3\right)-\left(c_1y_2+c_1'y_2y_1^2\right)-\left\{k_2\left(y_1-y_3\right)+k_2'\left(y_1-y_3\right)^3\right\}+\\&-\left\{c_2\left(y_2-y_4\right)+c_2'\left(y_2-y_4\right)\left(y_1-y_3\right)^2\right\}\end{aligned}\right], \tag{2.12}$$

$$\frac{dy_3}{dt}=y_4, \tag{2.13}$$

$$\frac{dy_4}{dt}=\frac{1}{m_2}\left[\left\{k_2\left(y_1-y_3\right)+k_2'\left(y_1-y_3\right)^3\right\}+\left\{c_2\left(y_2-y_4\right)+c_2'\left(y_2-y_4\right)\left(y_1-y_3\right)^2\right\}\right]. \tag{2.14}$$

The additional field equations, needed for the multisegment method of integration are derived now from Eqs (2.11)-(2.14). This is done by differentiating both sides of Eqs (2.11)-(2.14), partially with respect to y (a). For example, at first $\dfrac{\partial}{\partial y_1(a)}$ of Eq.(2.11) gives

$$\frac{\partial}{\partial y_1(a)}\left\{\frac{dy_1}{dt}\right\}=\frac{\partial}{\partial y_1(a)}\{y_2\} \qquad \text{or} \qquad \frac{d}{dt}\left\{\frac{\partial y_1(t)}{\partial y_1(a)}\right\}=\frac{\partial y_2(t)}{\partial y_1(a)}.$$

Finally, using the symbol Y for the partial derivative term of y (t) w.r.t. y (a), we get the additional field equations that are actually the governing differential Equations of Y, as follows

$$\frac{d}{dt}\left(Y_{11}\right)=Y_{21}. \tag{2.15}$$

Now $\dfrac{\partial}{\partial y_1(a)}$ of Eq.(2.12)

$$\frac{\partial}{\partial y_1(a)}\left\{\frac{dy_2}{dt}\right\}=\frac{\partial}{\partial y_1(a)}\left\{\frac{1}{m_1}\left[\begin{array}{l}F-\left(k_1y_1+k_1'y_1^3\right)-\left(c_1y_2+c_1'y_2y_1^2\right)+\\ -\left\{k_2\left(y_1-y_3\right)+k_2'\left(y_1-y_3\right)^3\right\}+\\ -\left\{c_2\left(y_2-y_4\right)+c_2'\left(y_2-y_4\right)\left(y_1-y_3\right)^2\right\}\end{array}\right]\right\}, \quad \text{or}$$

$$\frac{d\left(Y_{21}\right)}{dt}=-\frac{1}{m_1}\left[\begin{array}{l}\left(k_1Y_{11}+3k_1'y_1^2Y_{11}\right)+\left(c_1Y_{21}+c_1'y_1^2Y_{21}+2c_1'y_1y_2Y_{11}\right)+\\ +\left\{k_2\left(Y_{11}-Y_{31}\right)+3k_2'\left(y_1-y_3\right)^2\left(Y_{11}-Y_{31}\right)\right\}+\\ +\left\{c_2\left(Y_{21}-Y_{41}\right)+c_2'\left(y_1-y_3\right)^2\left(Y_{21}-Y_{41}\right)+2c_2'\left(y_1-y_3\right)\left(y_2-y_4\right)\left(Y_{11}-Y_{31}\right)\right\}\end{array}\right]. \tag{2.16}$$

Next $\dfrac{\partial}{\partial y_1(a)}$ of Eq.(2.13)

$$\frac{\partial}{\partial y_1(a)}\left\{\frac{dy_3}{dt}\right\}=\frac{\partial}{\partial y_1(a)}\{y_4\} \qquad \text{or,} \qquad \frac{d}{dt}\left\{\frac{\partial y_3(t)}{\partial y_1(a)}\right\}=\frac{\partial y_4(t)}{\partial y_1(a)},$$

or,

$$\frac{d}{dt}\left(Y_{31}\right)=Y_{41}. \tag{2.17}$$

Finally, $\dfrac{\partial}{\partial y_1(a)}$ of Eq.(2.14)

$$\frac{\partial}{\partial y_1(a)}\left\{\frac{dy_4}{dt}\right\}=\frac{\partial}{\partial y_1(a)}\left\{\frac{1}{m_2}\begin{bmatrix}\left\{k_2\left(y_1-y_3\right)+k_2'\left(y_1-y_3\right)^3\right\}+\\+\left\{c_2\left(y_2-y_4\right)+c_2'\left(y_2-y_4\right)\left(y_1-y_3\right)^2\right\}\end{bmatrix}\right\}\text{ or,}$$

$$\frac{d\left(Y_{41}\right)}{dt}=\frac{1}{m_2}\begin{bmatrix}\left\{k_2\left(Y_{11}-Y_{31}\right)+3k_2'\left(y_1-y_3\right)^2\left(Y_{11}-Y_{31}\right)\right\}+\\+\left\{\begin{matrix}c_2\left(Y_{21}-Y_{41}\right)+c_2'\left(y_1-y_3\right)^2\left(Y_{21}-Y_{41}\right)+\\+2c_2'\left(y_1-y_3\right)\left(y_2-y_4\right)\left(Y_{11}-Y_{31}\right)\end{matrix}\right\}\end{bmatrix}. \tag{2.18}$$

Similarly partial differentiations of Eqs (2.11), (2.12), (2.13), (2.14) with respect to $y(a)$ give similar differential equations of Y for the other three columns of Y. In all of these equations of Y, only coefficients change keeping a similar form. These governing equations can be solved when the boundary conditions are specified. More detail can be found in Kalnins and Lestingi (1967) or recent work of Ahmed (2009).

Boundary conditions

For different cases and chosen parameters (Tabs 1-2) prescribed boundary conditions are as given in Tabs 3 and 4. According to the multisegment method of integration, the boundary conditions for any boundary value problem are arranged in the following matrix form,

$$Ay(a)+By(b)=C, \tag{2.19}$$

that means, for the prescribed boundary conditions in Tab.3,

$$\begin{bmatrix}y_1(a)\\y_2(b)\\y_3(a)\\y_4(b)\end{bmatrix}=\left[C\right]=\begin{bmatrix}0.05\\0.06\\-0.07\\-0.06\end{bmatrix},$$

and, matrices of Eq.(2.19) are then as follows

$$\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}y_1(a)\\y_2(a)\\y_3(a)\\y_4(a)\end{bmatrix}+\begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}y_1(b)\\y_2(b)\\y_3(b)\\y_4(b)\end{bmatrix}=\begin{bmatrix}y_1(a)\\y_2(b)\\y_3(a)\\y_4(b)\end{bmatrix}$$

where, $\left[C\right]$ determines the arrangements of elements in matrices A and B. After deriving the governing equations and the boundary condition Eq.(2.19) the multisegment integration technique is used to solve those equations as a boundary value problem.

Solutions of the boundary value problem with any arbitrarily chosen boundary conditions are possible by the present method. For example, in Tab.3 it could be all zeros in 2nd–3rd columns. These results are not shown in this study for brevity.

Table 2. Chosen parameters of shock absorbers.

Parameters	Solution as Initial Value Problem	Solution as Boundary Value Problem.	
		Data set #1	Data set #2
m_1 (kg)	100.0	100.0	100.0
m_2 (kg)	100.0	1.0	1.0
k_1 (N/m)	1000.0	1000.0	1000.0
k'_1 (N/m^3)	0.0	±0.5	±0.5
k_2 (N/m)	0.0	10.0	10.0
k'_2 (N/m^3)	0.0	±0.005	±0.5
c_1 (Ns/m)	0.0	0.03139	1.0
c'_1 (Ns/m^3)	0.0	±0.003	±0.01
c_2 (Ns/m)	63.246, 182.174, 316.23, 632.46	3.139	10.0
c'_2 (Ns/m^3)	0.0	±0.03	±0.1
$\sqrt{\frac{k_1}{m_1}}$	3.162	3.162	3.162
μ	1.0	0.01	0.01
ζ	0.1, ζ_o =0.288, 0.5, 1.0	ζ_o=0.00496	0.0158
f (N)	20.0	20.0	20.0
ω_1 (rad/s)	3.162	3.006	3.006
ω_2 (rad/s)	-	3.326	3.326

Damping ratio, $\zeta = \frac{c_1}{2\sqrt{m_1 k_1}}$.

Optimum damping ratio $\zeta_o = \frac{\mu}{\sqrt{2(1+\mu)(2+\mu)}}$.

Table 3. Chosen boundary conditions of boundary value problem.

Parameters	Values for shock absorbers for Data set #1 and Data set #2
$y_1(t=0.0s)$ (m)	0.05
$y_2(t=50.0s)$ (m/s)	0.06
$y_3(t=0.0s)$ (m)	-0.07
$y_4(t=50.0s)$ (m/s)	-0.06

Table 4. Chosen boundary conditions of initial value problem for shock absorbers.

Parameters	Values
$y_1(t=0.0s)$ (m)	0.0
$y_2(t=0.0s)$ (m/s)	0.0
$y_3(t=0.0s)$ (m)	0.0
$y_4(t=0.0s)$ (m/s)	0.0

3. Method of solution

3.1. Multisegment integration technique

This method originally developed by Kalnins and Lestingi (1967) is briefly described below and interested readers may refer to the original paper or Rahman and Ahmed (2009) for more detail.

At first the m^{th} order ordinary differential equation (ODE) is reduced to 'm' first order ODEs

$$\frac{dy(x)}{dx} = F\left(x, y^{1}(x), y^{2}(x), \ldots\ldots\ldots\ldots, y^{m}(x)\right). \tag{3.1}$$

Then the scheme of the multisegment method of integration of a system of m first order ordinary differential equations, in the interval ($x_1 < x < x_{M+1}$), is as follows

(a.) the division of the given interval into M segments;

(b.) ($m+1$) initial value integrations over each segment;

(c.) solution of a system of matrix equations to ensure continuity of the dependent variables at the nodal points;

(d.) repetition of (b) and (c) until continuity of the independent variables at the nodal points is achieved.

In this study 4th order accurate Runge-Kutta formula has been used for initial value integration (in step b) as described above. It is worth mentioning that the number of segment, that is, M has been kept to one to get the results and these are also reliable as discussed in the next chapter.

4. Results and discussion

First, solutions for both initial and boundary value problems are shown and discussed for a so called ideal untuned viscous damper, Case 1 in section 4.1. The validity of the code is also demonstrated for Case 1 (Figs 3-5).

Next, results for data set #1 and data set #2 are shown and discussed in sections 4.2 and 4.3, respectively for three different frequency ratios. These frequency ratios are $r = 0.3162$, 1.0 and 4.744. At $r=1.0$ system vibrates with high amplitude. It is found that in some cases the system is unstable at $r=1.0$, $r=0.3162$ represents the response of the system at lower forcing frequency, whereas $r = 4.744$ represents the system response at a higher forcing frequency ratio. The results are described now sequentially.

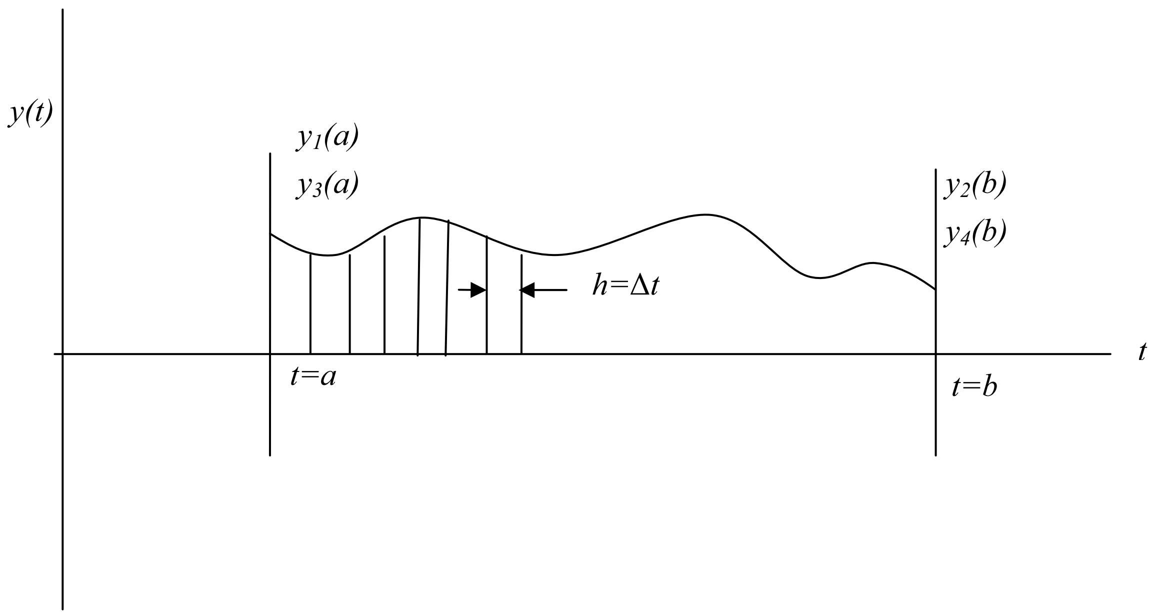

Fig.2.Prescribed boundary conditions at $t(a)$ and $t(b)$.

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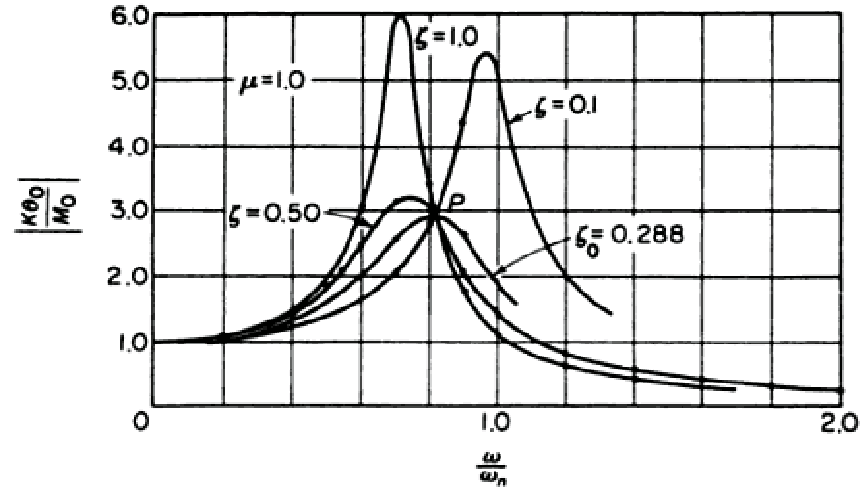

Figure 5.8-3. Response of an untuned viscous damper (all curves pass through P).

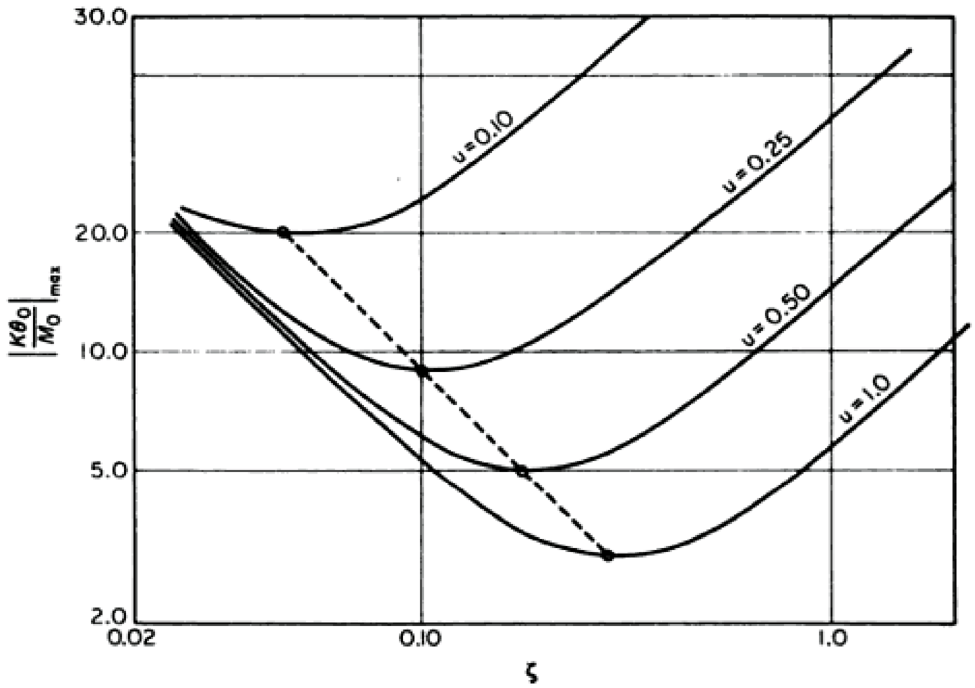

Figure 5.8-4.

Fig.3. Exact solution given in Thomson (1981) for untuned vibration dampers.

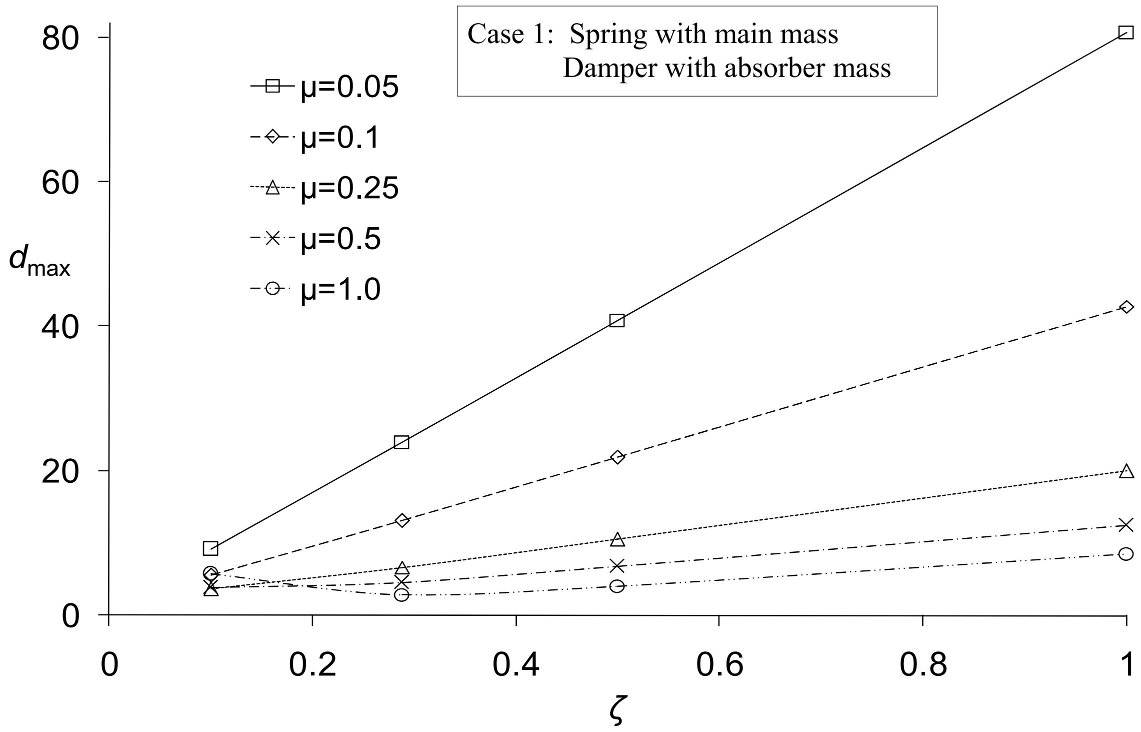

Fig.4. d_{max} - ζ with varying mass ratio (μ) for Case 1 and solved as boundary value problem at *t=20s*.

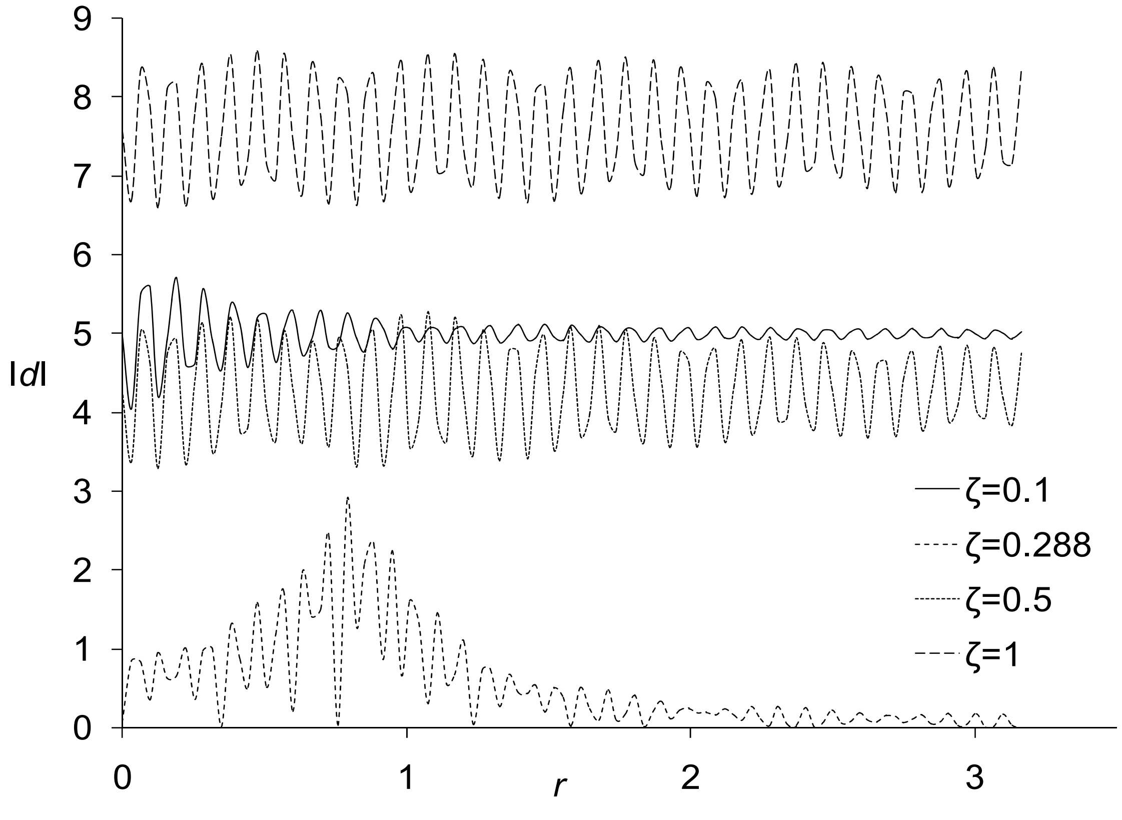

Fig.5. $|d| - r$ for Case 1 (spring with main mass and damper with absorber mass) at *t=20s* having μ=*1.0* and solved as boundary value problem.

4.1. Validity of the code

At first, untuned vibration dampers (Case 1) has been treated as an initial value problem using boundary conditions in Tab.4 to validate the code developed. Later in this section, results for the boundary value analysis are also discussed.

Although Case 1 (spring with main mass and damper mass only) is a two degrees of freedom system, it actually responds like a single degree of freedom system (SDOF) (Thomson, 1981). Similarly, all of Cases 2-16 are basically 2DOFS but non-dimensional amplitude is a function of 3 parameters : mass ratio (μ), damping ratio(ζ), and frequency ratio (*r*). Therefore, if μ is held constant the non-dimensional amplitude versus the frequency ratio curve for any ζ appears somewhat similar to that of a SDOFS with a single peak. Consequently, $r = 1$ appears to be a resonant frequency when damping ratio is zero (Thomson 1981).
If ζ is infinitely large, the damper mass and the spring mass will move together as a single mass, and again we have an undamped SDOFS, but with lower natural frequency of

$$\sqrt{\frac{k_1}{m_1 + m_2}} \quad \text{(Thomson 1981).}$$

Thomson (1981) evidently concluded that both rotary and linear viscous dampers (TDOFS) show similar responses. Ahmed (2009) demonstrated that results very similar as it may seen in Fig.3. Further, instability and jump phenomenon (Thomson, 1981) can also be predicted by the same code. Figure 4 is shown here to compare the response with the damping ratio curve of Fig.3.

d_{max}–ζ relation for varying mass ratio is shown in Fig.4. Each curve in this figure was drawn by taking the peak amplitudes of a particular mass ratio while varying the damping ratio. From Fig.5 the line of a particular μ becomes steeper with the decrease of its value. So the system with lower mass ratio but with the fixed damping ratio gives larger vibration. For cross-check it can be readily seen that for $\mu = 1.0$ and $\zeta = 0.288$, d_{max} is minimum as discussed earlier.

Next, *d*–*r* curves for Case 1 having $\mu = 1.0$ and solved as boundary value problem are shown in Fig.5. The boundary conditions given in Tab.3 were used for this observation. As the boundary conditions were fixing final velocity of the system and damping force is proportional to velocity, eventually the damping force became fixed at that particular boundary ($t = b$). The damper in this case has little effect on the system's final displacement unless optimum $\zeta = \zeta_o$ is used. As seen from the figure, for $\zeta = 1.0$, the amplitude (*d*) range varies from *6.60–8.59*, for $\zeta = 0.1$ range of *d* varies from *5.0-5.7* and for $\zeta = 0.5$ *d* varies from *3.295* to *5.28*. But, for the optimum damping ratio ($\zeta = 0.288$) the system shows similar deflection to that of the initial value problem. This also indicates that the effect of the optimum damping ratio (ζ_o) on the system's response is independent of boundary conditions. This figure also proves the fact that ζ_o gives the minimum *d*.

4.2. Stability analysis in terms of boundary value problem for data set # 1

For data set #1, the system response for Cases 2–16 are illustrated in Figs 6–14. A summary of stability of the systems for data set #1 is given in Tab.5. Figure 6 shows absolute non-dimensional displacement versus the frequency ratio for Case 2 at $t = 50s$. At $r \cong 1$ resonance occurs as the system behaves as SDOFS. Due to imposed velocity of the system at final condition the damper force also becomes fixed at the end. As seen from the figure, the peak amplitude occurs at the frequency ratio near unity and the peak value is *3.37*. Cases 3–16 show similar type of deflection at $t = 50s$.

Figure 7 shows the *d*–*t* curve for Case 2 at $r = 1.0$. As seen from the figure, the system vibrates with very high amplitude initially and with time amplitude decreases to a small value. For the entire *50s* period, amplitude (*d*) varies from *102.09* to *1.60*. As seen from the figure, the system is initially unstable but approaches stability with time.

Table 5. Characteristics of the responses of shock absorbers for Data set #1.

<table>
<tr><td rowspan="2">Cases</td><td colspan="3">Characteristics of the Responses at</td></tr>
<tr><td>r= 0.3162</td><td>r= 1.0</td><td>r= 4.744</td></tr>
<tr><td>2</td><td rowspan="15">approaches farther stability from an initially stable state</td><td>Approaches stability from an initially unstable state</td><td rowspan="15">approaches farther stability from an initially stable state</td></tr>
<tr><td>3</td><td>Remains unstable with continuous high amplitude</td></tr>
<tr><td>4</td><td>Unstable from the very beginning if r=1.0. But approaches stability from an initially unstable state at r=1.012</td></tr>
<tr><td>5</td><td>Approaches stability from an initially unstable state</td></tr>
<tr><td>6</td><td>Unstable from the very beginning if r=1.0. But approaches stability from an initially unstable state at r=1.012</td></tr>
<tr><td>7</td><td rowspan="5">Remains unstable with increasing amplitude</td></tr>
<tr><td>8</td></tr>
<tr><td>9</td></tr>
<tr><td>10</td></tr>
<tr><td>11</td></tr>
<tr><td>12</td><td rowspan="5">Unstable from the very beginning if r=1.0. But approaches stability from an initially unstable state at r=1.012</td></tr>
<tr><td>13</td></tr>
<tr><td>14</td></tr>
<tr><td>15</td></tr>
<tr><td>16</td></tr>
</table>

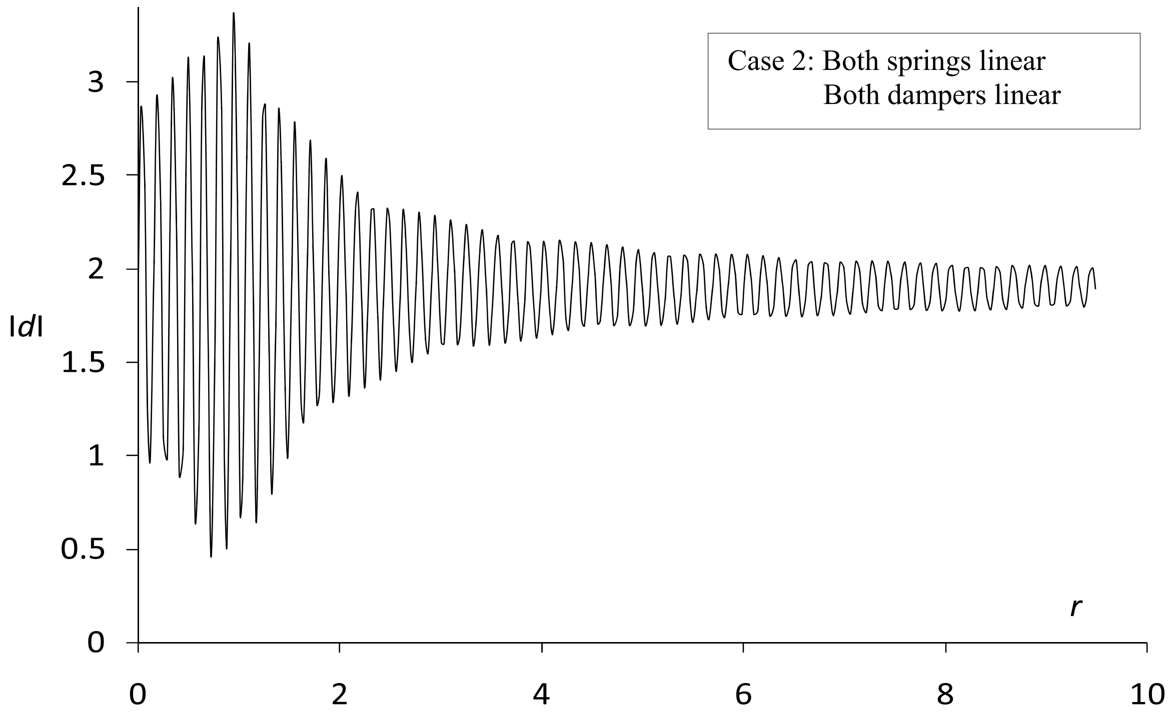

Fig.6. $|d| - r$ for Case 2 having $\mu = 0.01$ and $\zeta = 0.00496$ at $t = 50s$ and solved as boundary value problem (data set #1).

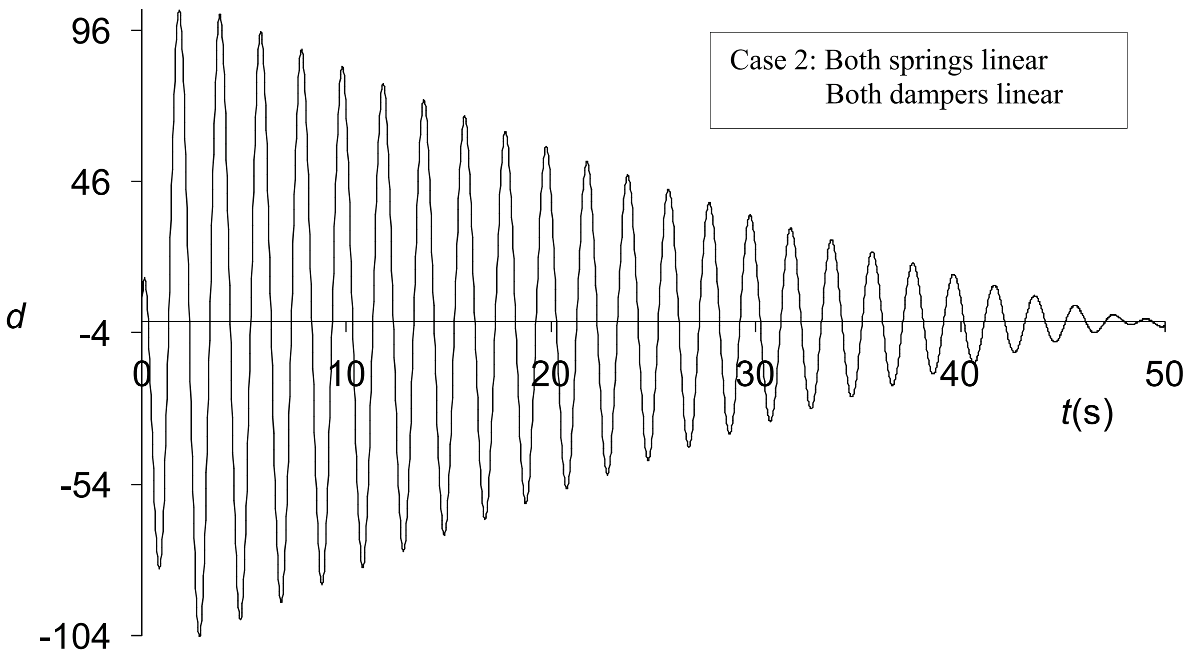

Fig.7. $d - t$ for Case 2 with μ=*0.01*, ζ *=0.00496* and *r=1.0.* (Data set #1).

Figure 8 shows *d–t* curve for Case 2 at *r=4. 744*. Compared to Fig.7, at high forcing frequency ratio, the system vibrates with lower amplitude somewhat similar to the response of a vibration isolator (Thomson, 1981), and the amplitude decreases continuously ranging from *3.955–1.866* (farther stability is achieved). Other linear and nonlinear combinations of springs and dampers (Cases 3–16) show a similar type of response at *r=4.744* as presented in Fig 8 but the amplitude range varies with the cases.

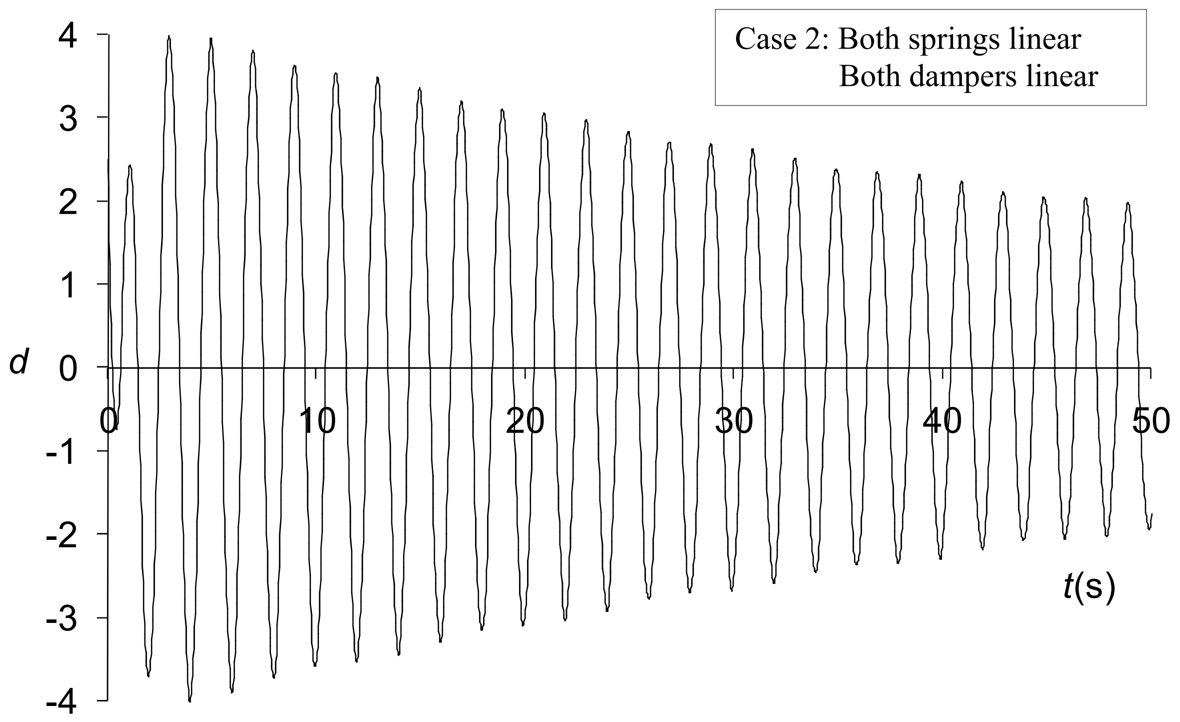

Fig.8. $d - t$ for Case 2 with μ=0.01, ζ *=0.00496* and *r=4.744*. (Data set #1).

In Figure 9, the $d-t$ curve is shown for Case 3 at $r = 1.0$. With this combination the spring - damper system vibrates with continuous high amplitude. As seen from the figure, d varies in the range from *64.04* to *54.9*.

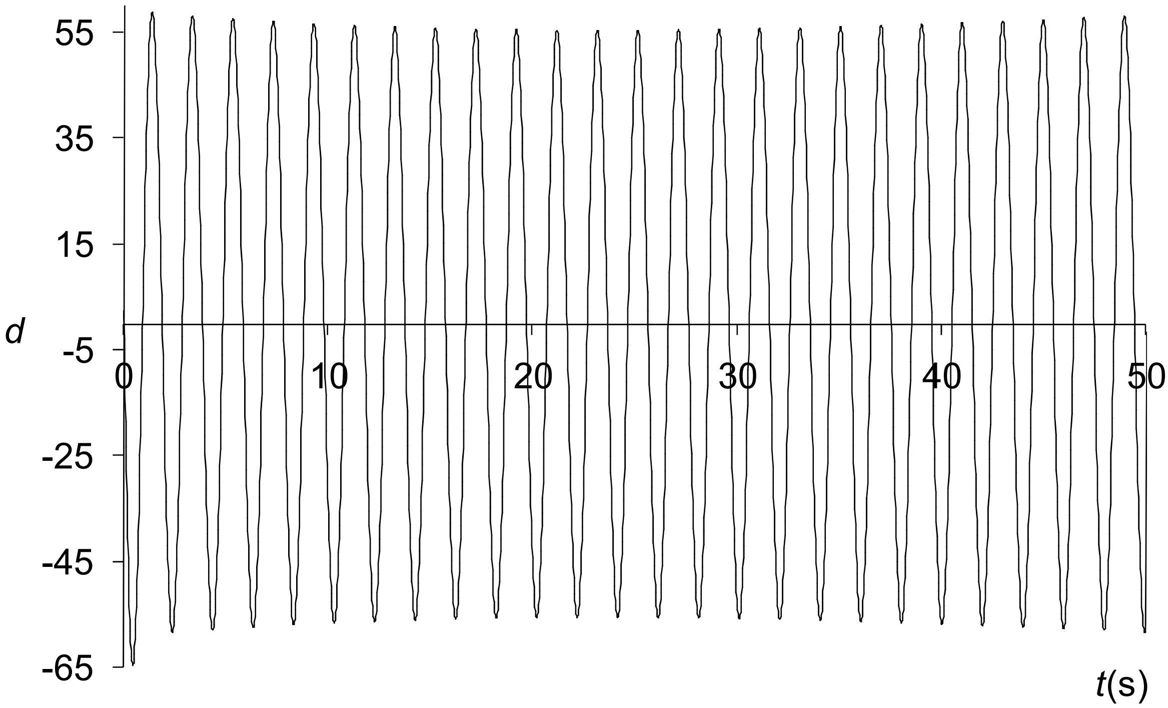

Fig.9. $d-t$ for Case 3 (Both springs hard and Linear dampers) with $\mu=0.01$, $\zeta=0.00496$ and $r=1.0$. (Data set #1).

d–t curve for Case 5 at $r=1.0$ is shown in Fig.10. Here the system vibrates with decreasing amplitude from an initial unstable state, as in Case 2 at $r=1.0$ (Fig.7) but its approach to stability is much slower than that of Case 2. For the entire *50s* period of vibration, the amplitude range of the system is *104.5–40.33*.

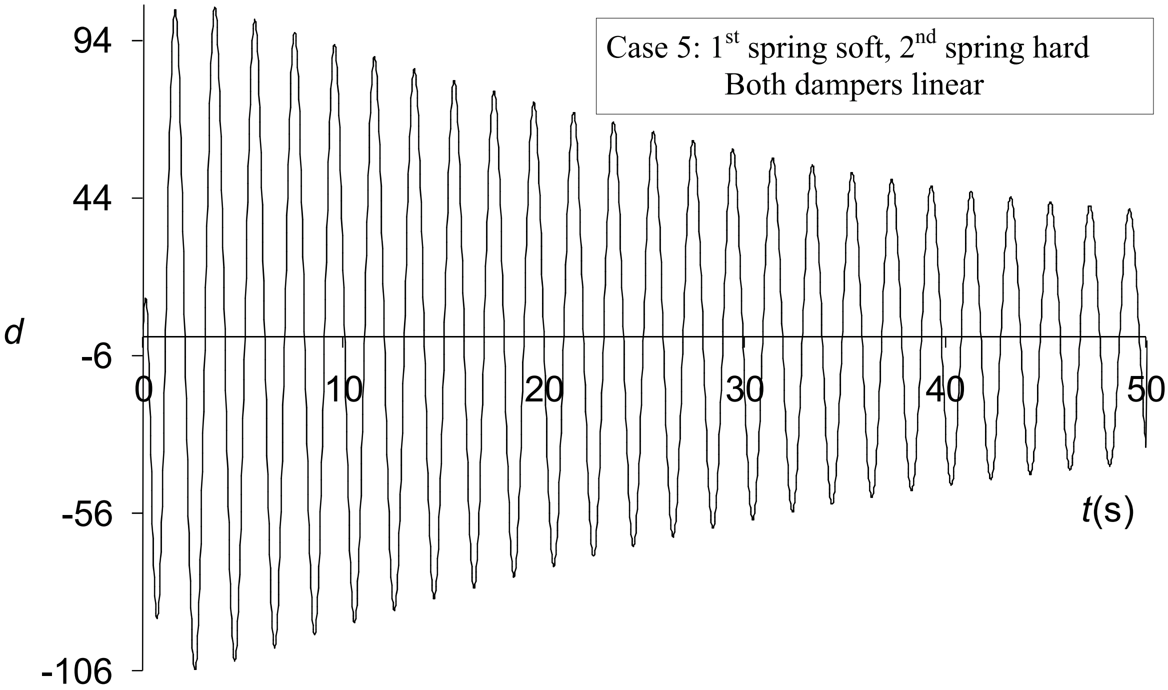

Fig.10. $d-t$ for Case 5 with $\mu=0.01$, $\zeta=0.00496$ and $r=1.0$. (Data set #1).

Figure 11 shows the *d–t* curves for case 6 at $r = 1.012$. Case 4 also shows a similar type of response at $r=1.012$. As seen from the figure, amplitude d ranges from $86.53–2.09$. Here again the system approaches stability with time.

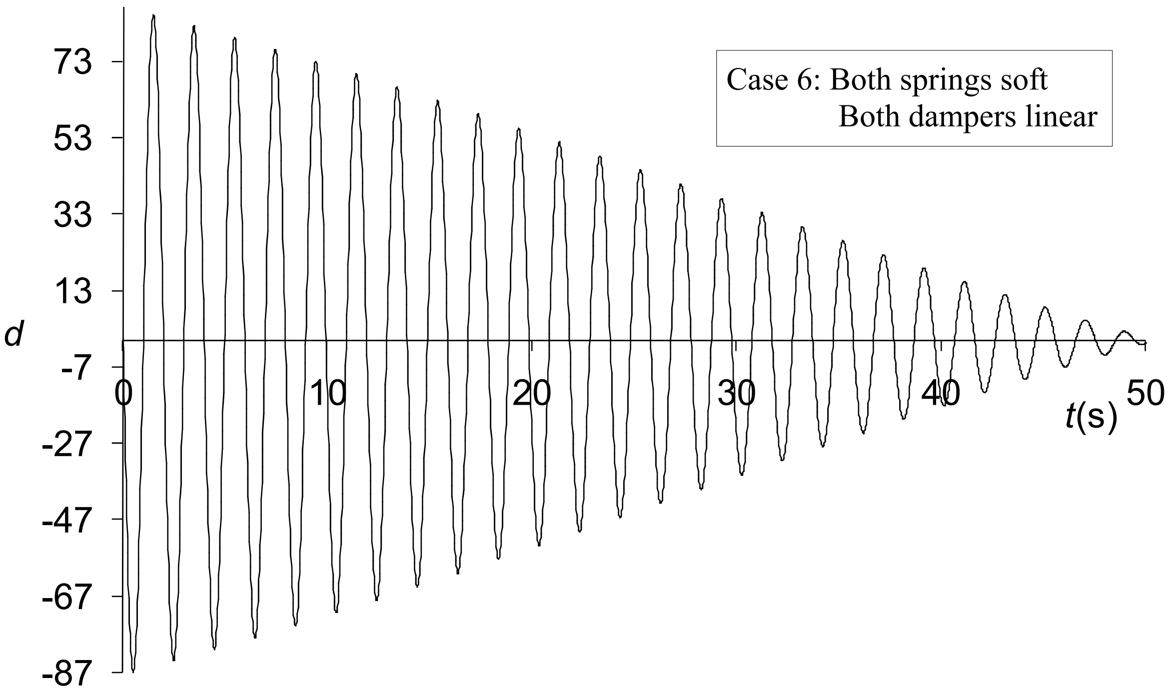

Fig.11. $d - t$ for Case 6 with $\mu=0.01$, $\zeta=0.00496$ and $r=1.012$. (Data set #1).

The *d–t* curve for Case 7 at $r=1.0$ is shown in Fig.12. From this figure, it can be said that in the case of hard damper as in Fig.12, the system never reaches its stability, rather vibrates with high amplitude. The range of d varies from 49.91 to 30.214. Cases 7–11 give similar types of response at $r=1.0$.

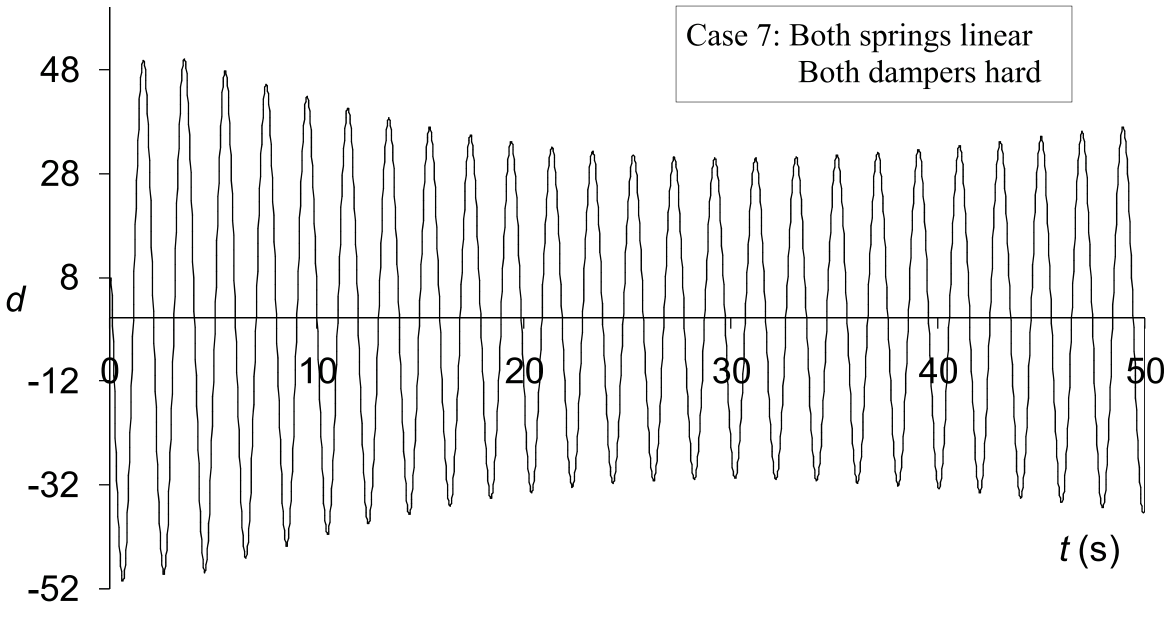

Fig.12. $d - t$ for Case 7 with $\mu=0.01$, $\zeta=0.00496$ and $r=1.0$. (Data set #1).

In Figure 13, the *d–t* curve for Case 11 at $r=1.0$ is shown. Amplitude ranges from *52.07–32.28*. Cases 12–16 at $r=1.0$ show a similar response.

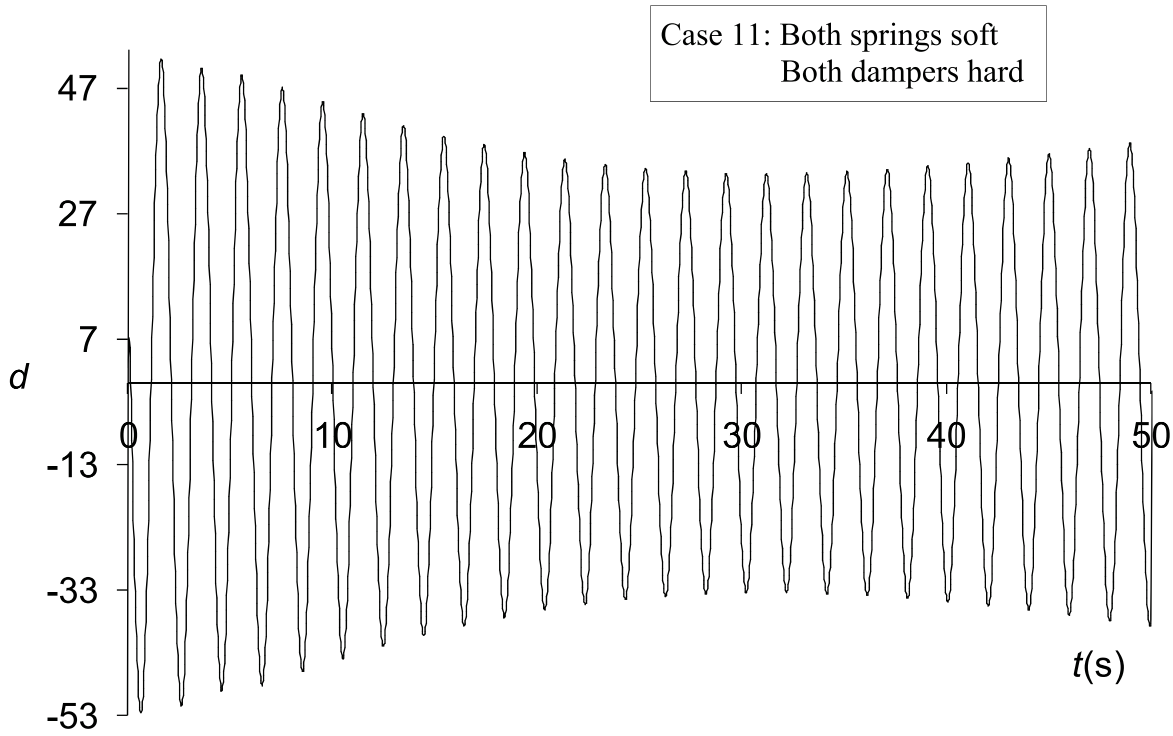

Fig.13. $d-t$ for Case 11 with $\mu=0.01$, $\zeta=0.00496$ and $r=1.0$. (Data set #1).

Figure 14 shows the *d–t* relation for Case 13 at $r=1.012$. As seen from the figure, the range of *d* varies from *86.1–2.09* approaching to stability with time.

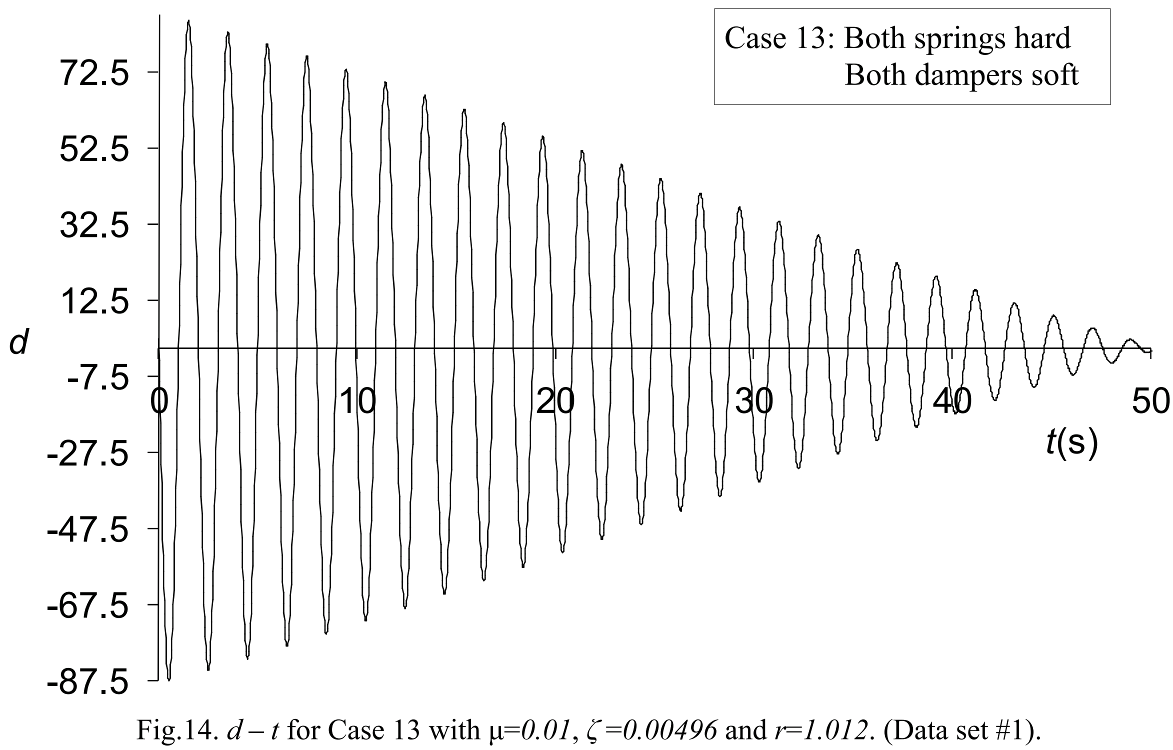

Fig.14. $d-t$ for Case 13 with $\mu=0.01$, $\zeta=0.00496$ and $r=1.012$. (Data set #1).

Table 5 summarizes the effect of optimum damping terms together with different nonlinear terms as mentioned in Data set #1. As solutions of the system comprising soft dampers (Cases 12–16) do not converge, the system is unstable at $r=1.0$. Cases 12 and 14–16 show a similar type of response at $r=1.012$ but with a varying range of amplitude.

4.3. Stability analysis of untuned vibration damper solved as boundary value problem for data set # 2

Table 6 shows a summary of different cases that can be compared to Tab.5 to see the effect of increasing the values of damping terms for the nonlinear shock absorbers on stability. For Cases 4, 6, 9, and 11–16 the numerical solution does not converge due to larger spring and damper indices and as a result, these cases do not show any convergence even at a lower or higher speed ratio. Results of all the Cases 2–16 with data set #1 have already been discussed previously. Interestingly, as found in this study case 1 appears to be stable for any value of the damping ratio but other Cases 2-16 become unstable with an increase of any nonlinearity index (Tabs 5 and 6). A few causes of instability can be attributed to the jump phenomenon and negative damping for the present study. The amplitude suddenly jumps discontinuously near the resonant frequency making the unstable. The instability region near the resonant frequency depends on the amount of damping and rate of change of forcing frequency among others (Thomson, 1981).

Table 6. Characteristics of the responses of shock absorbers for Data set #2.

<table>
<tr><td rowspan="2">Cases</td><td colspan="3">Characteristics of the Responses at</td></tr>
<tr><td>$r=0.3162$</td><td>$r=1.0$</td><td>$r=4.744$</td></tr>
<tr><td>2</td><td rowspan="2">Stable</td><td>Approaches stability from an initially unstable state</td><td rowspan="2">approaches farther stability from an initially stable state</td></tr>
<tr><td>3</td><td>Remains unstable with increasing amplitude</td></tr>
<tr><td>4</td><td colspan="3">Unstable from the very beginning</td></tr>
<tr><td>5</td><td>Stable</td><td>Remains unstable with increasing amplitude</td><td>approaches farther stability from an initially stable state</td></tr>
<tr><td>6</td><td colspan="3">Unstable from the very beginning</td></tr>
<tr><td>7</td><td rowspan="2">Stable</td><td rowspan="2">Unstable from the very beginning if $r=1.0$. But approaches stability from an initially unstable state at r=1.012</td><td rowspan="2">Stable</td></tr>
<tr><td>8</td></tr>
<tr><td>9</td><td colspan="3">Unstable from the very beginning</td></tr>
<tr><td>10</td><td>Stable</td><td>Unstable from the very beginning if $r=1.0$. But approaches stability from an initially unstable state at $r=1.012$</td><td>Stable</td></tr>
<tr><td>11</td><td rowspan="6" colspan="3">Unstable from the very beginning</td></tr>
<tr><td>12</td></tr>
<tr><td>13</td></tr>
<tr><td>14</td></tr>
<tr><td>15</td></tr>
<tr><td>16</td></tr>
</table>

Negative damping because of nonlinear damping force can also lead to instability as the system's response starts to increase with time (Thomson 1981). All these facts appear to be applicable for the present study for cases 2-16 as discussed in details below.

The cases not discussed in detail here can be found in Ahmed (2009). Before concluding it is again mentioned that both steady state and transient terms are taken into account while predicting the response of the dynamic system and the term instability refers to increase of response with time.

5. Conclusions

As found in this study Case 1 appears to be stable for any value of the damping ratio but other Cases 2-16 become unstable with an increase of any nonlinearity index. A few causes of instability can be attributed to the jump phenomenon and negative damping for the present study. Soundness of the code has been demonstrated comparing a few results of the present analysis with available exact results. The following conclusions can be drawn from the present study:

The effect of optimum damping ratio (ζ_o) on the system's response is independent of boundary conditions. In case of optimum damping, the untuned vibration damper vibrates with minimum amplitude. This happens for both initial value and boundary value problems. For example, Case 1 with the optimum damping ratio ($\zeta_o=0.288$) shows similar deflections for initial value and boundary value problems. The mass ratio (μ) plays a significant role in the maximum deflection of the untuned vibration damper. For the same damping ratio, the system with larger mass ratio shows a smaller peak amplitude. But all the systems with a particular mass ratio show a minimum peak at optimum damping ratio.

An unstable system at $r=1.0$ is observed due to soft springs in (Case 4 and Case 6). Spring force of 2nd mass becomes negative due to the negative index and this force cannot be diminished by the linear or soft damper. This problem is eliminated when a hard damper is used. Another way to solve this type of problem is to reduce the value of the spring index of the second mass.

Increased nonlinearity in the spring and damper makes the system more unstable. For example, systems with data set #1 show more stability than those for data set #2 (having higher nonlinearity indexes compared to data set #1) for different cases at $r=1.0$.

The untuned system with different Cases (2-16) shows similar responses for both lower and higher forcing frequency ratios. For example, at forcing frequency ratio $r=0.3162$ and 4.744 all the systems with data set #1 show similar responses. For data set #2, systems that converge at $r=0.3162$ and 4.744 show a similar response too.

We can conclude from this study, that practical springs and dampers should contain smaller nonlinearity indices as systems with lower nonlinearity indexes approached more stability. Responses using data set #1 are stable as discussed.

Practically no spring or damper remains linear with ever-increasing response. Therefore, this study is particularly useful for cars' shock absorber design and application as stability of a shock absorber (which can be considered an untuned vibration damper) can easily change because of damper and spring nonlinearities.

In future, the developed computer code can be used to study the response of a similar system having a higher DOF. The effect of self-excited force on the system can also be studied. Self-excited vibration forces with and without nonlinearity can be added to this study. Experimental studies can be carried out to verify the results obtained for the untuned vibration dampers.

Practically, this method of vibration analysis will provide handy information for active/passive vibration control of structures.

Nomenclature

a	– initial time reference
b	– final time reference
c	– damping coefficient (n-s/m)
c'	– damping nonlinearity index (n-s/m^3)
c_1, c_2	– main mass and absorber mass damping coefficients (n-s/m)
c'_1, c'_2	– main mass and absorber mass damping nonlinearity indexes (n-s/m^3)
d	– k_1*x_1/f: nondimensional displacement for main mass
e	– k_1*x_2/f: nondimensional displacement for absorber mass
F	– $f \sin(\omega t)$ (n)
f	– amplitude of the applied force (n)
k	– spring constant (n/m)
k_1, k_2	– main mass and absorber mass spring constants (n/m)
k'	– spring nonlinearity index (n/m^3)
k'_1, k'_2	– main mass and absorber mass spring nonlinearity indexes (n/m^3)
m_1, m_2	– main mass and absorber mass (kg)
r	– frequency ratio: $\dfrac{\omega}{\sqrt{\dfrac{k_1}{m_1}}}$
r_1	– value of r at first natural frequency of the system
r_2	– value of r at second natural frequency of the system
t	– time (s)
x_1	– x
x_1, x_2	– main mass and absorber mass deflections (m)
$\dot{x}_1$, $\dot{x}_2$	– main mass and absorber mass velocities (m/s)
y_1, y_3	– x_1, x_2 (m)
y_2, y_4	– $\dot{x}_1$, $\dot{x}_2$ (m/s)
ζ	– damping ratio: $\dfrac{c_2}{2\sqrt{m_1 k_1}}$
ζ_o	– optimum damping ratio : $\dfrac{\mu}{\sqrt{2(1+\mu)(2+\mu)}}$
μ	– mass ratio: m_2/m_1
ω	– forcing frequency (rad/s)
ω_1,ω_2	– natural frequencies of the main mass and absorber masses
DOF	– degrees of freedom
DOFS	– degrees of freedom system
MDOF	– multiple degrees of freedom
SDOF	– single degree of freedom
Tuned Absorber	– 2 dofs without damping
Shock Absorber	– 2 dofs with damping
Spring force	– $kx \pm k'x^3$
Damping force	– $c\dot{x} \pm c'\dot{x}x^2$
Hard spring	– nonlinearity index (k') is positive
Soft spring	– nonlinearity index (k') is negative
Hard damper	– nonlinearity index (c') is positive
Soft damper	– nonlinearity index (c') is negative
Linear spring	– nonlinearity index $k' = 0.0$
Linear damper	– nonlinearity index $c' = 0.0$

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