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CONSTRAINED CONTROLLABILITY OF NONLINEAR STOCHASTIC IMPULSIVE SYSTEMS

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This paper is concerned with complete controllability of a class of nonlinear stochastic systems involving impulsive effects in a finite time interval by means of controls whose initial and final values can be assigned in advance. The result is achieved by using a fixed-point argument.

Keywords: complete controllability, nonlinear stochastic system, impulsive effect, Banach contraction principle.

1. Introduction

There are many real-world systems and natural processes which display some kind of dynamic behavior in a style of both continuous and discrete characteristics. For instance, many evolutionary processes, particularly some biological systems such as biological neural networks and bursting rhythm models in pathology, as well as optimal control models in economics, frequency-modulated signal processing systems, flying object motions, and the like are characterized by abrupt changes in states at certain time instants (Gelig and Churilov, 1998; Lakshmikantham *et al.*, 1989). This is the familiar impulsive phenomenon. Often, sudden and sharp changes occur instantaneously, in the form of impulses, which cannot be well described using purely continuous or purely discrete models. On the other hand, stochastic modelling has come to play an important role in many branches of science and industry because any real world system and natural process may be disturbed by many stochastic factors. Therefore, stochastic impulsive systems arise naturally from a wide variety of applications and can be used as an appropriate description of these phenomena of abrupt qualitative dynamical changes of essentially continuous time systems which are disturbed by stochastic factors.

Control systems are often subject to constraints on their manipulated inputs and state variables. Input constraints arise as a manifestation of the physical limitations inherent in the capacity of control actuators (e.g., bounds on the magnitude of valve opening) and are enforced at all times (hard constraints). State constraints, on the other hand, arise either due to the necessity to keep the state variables within acceptable ranges, to avoid, for example, runaway reactions (in which case they need to be enforced at all times and treated as hard constraints) or due to the desire to maintain them within desirable bounds dictated by performance considerations (in which case they may be relaxed and treated as soft constraints). Neglecting such constraints in controller design and implementation can drastically degrade system performance or, worse, lead to catastrophic failures (Gilbert, 1992).

It has been found that in some control system operations it is necessary to change operational limits. For example, the need for such a control system occurs in industrial electric motor control for motors which are comprised of stator and rotor assemblies. It is frequently desirable to limit both synchronous frequency and slip frequency of such motors within prescribed limits. Also, it is desirable to limit the supply voltage to the stator as a function of both synchronous and slip frequency so that the airgap flux between the stator and rotor of the motor may never exceed the saturation limit of the rotor core. Any control system design methodology must include these properties as objectives in the design procedure. For more applications on constrained controls in industrial plants one can refer to the works of Alotabi *et al*. (2004), Respondek (2007) or Semino and Ray (1995). This problem is important and challenging in both theory and practice, which has motivated the present study.

The theory of controllability of nonlinear deterministic systems is well developed (Balachandran and Dauer, 1987; Klamka, 2000b). Many important results for controllability of linear as well as nonlinear stochastic systems have also been established (Balachandran and Karthikeyan, 2007; Balachandran *et al.*, 2009; Klamka, 2007a; Mahmudov, 2001; Mahmudov and Zorlu, 2003; Zabczyk, 1981). When the control is constrained, the major global results are those by Conti (1976). Benzaid (1988) studied global null controllability with bounded controls of perturbed linear systems in $\mathbb{R}^n$.

The theory of constrained controllability of linear and nonlinear systems in finite dimensional space has been extensively studied (Chukwu, 1992; Klamka, 1991; 1993; Sikora, 2003). Klamka (1996; 1999; 2001) formulated sufficient conditions for exact and approximated constrained controllability assuming that the values of controls are in a convex and closed cone with the vertex at zero. Respondek (2008) generalized earlier results to a system of an arbitrary, n-th order system with respect to time, with possible delays in controls and with consideration of arbitrary multiplicities of its characteristic equation eigenvalues.

Klamka (2000a) and Respondek (2004) established necessary and sufficient conditions for constrained approximate controllability for linear dynamical systems described by abstract differential equations with an unbounded control operator. However, such a type of control constraints models only non-negative controls and is thus of minor industrial importance. Much better control constraints are the so-called compact constraints, which can consider both the lower and upper limitations of the control. Schmitendorf (1981) and Respondek (2010) investigated controllability with compact control constraints for ordinary differential equations and partial differential equations, respectively.

Generally, the control may be any element of the control space U, but sometimes some constraints are imposed on the control function u. Concerning the concept of controllability with prescribed controls, Anichini (1980; 1983) discussed complete controllability of the nonlinear boundary-value problem with boundary conditions on the control and used a fixed-point argument. A similar approach can be found in the work of Lukes (1972) for nonlinear differential systems which arise when a linear system is perturbed. Controllability for nonlinear Volterra integro-differential systems with prescribed controls was studied by Balachandran and Lalitha (1992) as well as Sivasundaram and Uvah (2008). Recently, Balachandran and Karthikeyan (2010) studied controllability of stochastic integrodifferential systems with prescribed controls. However, it should be emphasized that most of the works in this direction are mainly concerned with deterministic controllability problems and there have been no attempts made to study constrained controllability of stochastic impulsive systems. In order to fill this gap, the present paper studies the complete controllability problem for a class of nonlinear stochastic impulsive systems with prescribed controls (that is, a controllability condition for which the initial and the final value of the control are given *a priori*).

In this article we obtain sufficient controllability conditions for the nonlinear stochastic impulsive system

$$\begin{aligned} \mathrm{d}x(t) &= \Big[A(t)x(t) + B(t)u(t) + f(t,x(t))\Big]\mathrm{d}t \\ &\quad + \sigma(t,x(t))\,\mathrm{d}w(t), \quad t \neq t_k, \\ \Delta x(t_k) &= I_k(x(t_k^-)), \quad t = t_k, \quad k = 1,2,\dots,\rho, \\ x(0) &= x_0, \quad x(T) = x_T, \\ u(0) &= u_0, \quad u(T) = u_T, \end{aligned} \tag{1}$$

by means of controls whose initial and final values can be prescribed in advance. That is, we want to establish conditions on $A(t)$, $B(t)$, $f(t,x(t))$ and $\sigma(t,x(t))$ which ensure that, for $x_0, x_T \in \mathbb{R}^n$, there exists a control $u \in L_2([t_0,T];\mathbb{R}^m)$ with $u(0) = u_0, u(T) = u_T$ which produces a response $x(t;u)$ satisfying the boundary conditions $x(0;u) = x_0$ and $x(T;u) = x_T$. Further, we show complete controllability of the nonlinear stochastic impulsive system under the natural assumption that the associated linear stochastic impulsive system is completely controllable.

2. Preliminaries

Consider the linear stochastic impulsive system represented by the Itô equation of the form

$$\begin{aligned} \mathrm{d}x(t) &= \Big[A(t)x(t) + B(t)u(t)\Big]\,\mathrm{d}t \\ &\quad + \tilde{\sigma}(t)\,\mathrm{d}w(t), \quad t \neq t_k, \\ \Delta x(t_k) &= I_k(x(t_k^-)), \quad t = t_k, \quad k = 1,2,\dots,\rho, \\ x(t_0) &= x_0, \quad t_0 \geq 0, \end{aligned} \tag{2}$$

where $A(t)$ and $B(t)$ are known $n \times n$ and $n \times m$ continuous matrices, respectively, $x(t) \in \mathbb{R}^n$ is the vector describing the instantaneous state of the stochastic system, $u(t) \in \mathbb{R}^m$ is a control input to the stochastic dynamical system, w is an n-dimensional Wiener process, $\tilde{\sigma} : [t_0,T] \to \mathbb{R}^{n\times n}$, $I_k : [t_0,T] \to \mathbb{R}^n$, $\Delta x(t) = x(t^+) - x(t^-)$, where

$$\lim_{h\to 0^+} x(t+h) = x(t^+), \quad \lim_{h\to 0^+} x(t-h) = x(t^-)$$

and

$$0 = t_0 < t_1 < t_2 < \cdots < t_\rho < t_{\rho+1} = T,$$

$$I_k(x(t_k^-)) = \big(I_{1k}(x(t_k^-)), \dots, I_{nk}(x(t_k^-))\big)^{\mathsf{T}}$$

represents an impulsive perturbation of x at time t_k and $x(t_k^-) = x(t_k), k = 1, 2, \dots, \rho$, which implies that the solution of system (2) is left continuous at t_k.

Consider the following ordinary differential system corresponding to the stochastic impulsive system (2):

$$x'(t) = A(t)x(t), x(0) = x_0. \tag{3}$$

Suppose that $\Phi(t, t_0)$ is the fundamental solution matrix of (3). Then $\Phi(t, s) = \Phi(t)\Phi^{-1}(s)$, $t, s \in [t_0, T]$, is the transition matrix associated with matrix $A(t)$. It is easy to see that, for any $t, s, \tau \in [t_0, T], \Phi(t, t) = I$, the identity matrix of order n, $\Phi(t, \tau)\Phi(\tau, s) = \Phi(t, s)$, and $\Phi(t, s) = \Phi^{-1}(s, t)$.

Lemma 1. *For any $t \in (t_{k-1}, t_k]$, $k = 1, 2, \dots, \rho$, the general solution of the system (2) is given by*

$$\begin{aligned} x(t) = \Phi(t, t_0)x_0 &+ \int_{t_0}^t \Phi(t, s)B(s)u(s)\,\mathrm{d}s \\ &+ \int_{t_0}^t \Phi(t, s)\tilde{\sigma}(s)\,\mathrm{d}w(s) \\ &+ \sum_{i=1}^k \Phi(t, t_i)I_i(x(t_i^-)), \end{aligned} \tag{4}$$

where $\Phi(t, s)$ is the transition matrix of the system (3).

Proof. The proof is quite similar to that by in Karthikeyan and Balachandran (2009). ∎

For convenience, we define some notation that will be used throughout this paper. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a probability measure $\mathbb{P}$ on Ω and $w(t) = (w_1(t), w_2(t), \dots, w_n(t))^T$ be an n-dimensional Wiener process defined on this probability space. Let $\{\mathcal{F}_t | t \in [t_0, T]\}$ be the filtration generated by $\{w(s) : 0 \le s \le t\}$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $L_2(\Omega, \mathcal{F}_t, \mathbb{R}^n)$ denote the Hilbert space of all $\mathcal{F}_t$-measurable square integrable random variables with values in $\mathbb{R}^n$. Let $L_2^{\mathcal{F}}([t_0, T], \mathbb{R}^n)$ be the Hilbert space of all square-integrable and $\mathcal{F}_t$-measurable processes with values in $\mathbb{R}^n$. Let $PC([t_0, T], \mathbb{R}^n) = \{x : x$ is a function from $[t_0, T]$ into $\mathbb{R}^n$ such that $x(t)$ is continuous at $t \ne t_k$ and left continuous at $t = t_k$ and the right limit $x(t_k^+)$ exists for $k = 1, 2, \dots, \rho\}$. Let $\mathcal{B}_2$ denote the Banach space $PC^b_{\mathcal{F}_t}([t_0, T], L_2(\Omega, \mathcal{F}_t, \mathbb{R}^n))$, the family of all bounded $\mathcal{F}_t$-measurable, $PC([t_0, T], \mathbb{R}^n)$-valued random variables φ, satisfying

$$\|\varphi\|^2_{L_2} = \sup_{t \in [t_0, T]} \mathbb{E}\|\varphi(t)\|^2,$$

where $\mathbb{E}$ denotes the mathematical expectation operator of a stochastic process with respect to the given probability measure $\mathbb{P}$. Let $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ be the space of all linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$.

In the sequel, for simplicity, we shall assume that the set of admissible controls is $\mathcal{U}_{ad} := L_2^{\mathcal{F}}([0, T], \mathbb{R}^m)$.

For brevity, we set

$$P(t; \theta) = \int_0^t \Phi(\theta, \theta - s)B(\theta - s)\,\mathrm{d}s,$$

$$\bar{C}(t; T) = \int_{T-t}^T P^*(s; T)\,\mathrm{d}s - \frac{t}{T}\int_0^T P^*(s; T)\,\mathrm{d}s,$$

$$S(t; T) = \int_0^t \Phi(t, s)B(s)\bar{C}(s; T)\,\mathrm{d}s$$

and define

$$M(0, t) = \int_0^t B(s)B^*(s)\,\mathrm{d}s,$$

$$\begin{aligned} \bar{S}(T) = &\int_0^T P(s; \theta)P^*(s; \theta)\,\mathrm{d}s \\ &- \frac{1}{T}\left[\int_0^T P(s; \theta)\,\mathrm{d}s\right]\left[\int_0^T P^*(s; \theta)\,\mathrm{d}s\right], \end{aligned}$$

where the star denotes the matrix transpose. We observe that $P(t; \theta)$, $\bar{C}(t; T)$, and $S(t; T)$ are continuous.

The set of all states attainable from x_0 in time $t > 0$ is given by

$$\mathcal{R}_t(x_0) = \{x(t; x_0, u) : u(\cdot) \in U_{ad}\},$$

where $x(t; x_0, u)$ is the solution to (1) corresponding to $x_0 \in \mathbb{R}^n$, $u(\cdot) \in U_{ad}$.

Definition 1. The stochastic impulsive system (1) is said to be *controllable on* $[t_0, T]$ if, for given any initial state $x_0 \in \mathbb{R}^n$ and $x_T \in \mathbb{R}^n$, there exists a piecewise continuous input signal $u(t) : [t_0, T] \to \mathbb{R}^m$ such that the corresponding solution of (1) satisfies $x(T) = x_T$.

Since for the stochastic dynamical system (1) the state space $L_2(\Omega, \mathcal{F}_t, \mathbb{R}^n)$ is, in fact, an infinite-dimensional space, we distinguish exact or complete controllability and approximate controllability. Using the notation given above for the stochastic dynamical system (1) we define the following complete and approximate controllability concepts for nonlinear stochastic systems.

Definition 2. The stochastic impulsive system (1) is *completely controllable* on $[t_0, T]$ if

$$\mathcal{R}_T(x_0) = L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n),$$

that is, all the points in $L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n)$ can be exactly reached from the arbitrary initial condition arrived at from an arbitrary initial $x_0 \in L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n)$ at time T.

Definition 3. The stochastic impulsive system (1) is *approximately controllable* on $[t_0, T]$ if

$$\overline{\mathcal{R}_T(x_0)} = L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n),$$

that is, if all the points in $L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n)$ can be approximately reached from an arbitrary initial condition $x_0 \in L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n)$ at time T.

Consider the deterministic dynamical system of the following form:

$$\dot{z}(t) = A(t)z(t) + B(t)v(t), \tag{5}$$

where the admissible controls $v \in L_2([t_0, t_1], \mathbb{R}^m)$.

Lemma 2. *The following conditions are equivalent:*

(i) *The deterministic system* (5) *is controllable on* $[t_0, T]$.

(ii) *The stochastic system* (2) *is completely controllable on* $[t_0, T]$.

(iii) *The stochastic system* (2) *is approximately controllable on* $[t_0, T]$.

Proof. The proof is quite similar to that by Klamka (2007b). ■

The solution of the linear stochastic system (2) can be written as follows:

$$\begin{aligned} x(t) = \Phi(t, t_0)x_0 &+ \int_{t_0}^{t} \Phi(t, s)B(s)u(s)\,\mathrm{d}s \\ &+ \int_{t_0}^{t} \Phi(t, s)\tilde{\sigma}(s)\,\mathrm{d}w(s) \\ &+ \sum_{i=1}^{k} \Phi(t, t_i)I_i(x(t_i^-)). \end{aligned} \tag{6}$$

Proposition 1. *For all* $u \in \mathbb{R}^m$, *we have*

$$\begin{aligned} &\int_0^t \Phi(t, s)B(s)u(s)\,\mathrm{d}s \\ &\quad = P(t; t)u_0 + \frac{1}{T}(u_T - u_0)\int_0^t P(s; t)\,\mathrm{d}s \\ &\quad + S(t; T)y(T) \end{aligned} \tag{7}$$

and $S(T; T) = \bar{S}(T)$.

Proof. The proof is quite similar to that by Sivasundaram and Uvah (2008). ■

By restricting our attention to systems with a controllable linear part, we are able to obtain global results for systems in which the control can enter in a nonlinear fashion. The results cover linear systems as a simple special case and, moreover, show that the steering can be accomplished using continuous controls with arbitrarily prescribed initial and final values. The following lemma gives a formula for a minimum energy control steering the linear stochastic system (2) from the state x_0 to an arbitrary point x_T with prescribed controls.

Lemma 3. *Assume that the matrix* $M(0, T)$ *is invertible. Then, for an arbitrary* $x_T \in L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n)$ *and* $\tilde{\sigma}(\cdot) \in L_2^{\mathcal{F}}([0, T], \mathbb{R}^{n\times n})$*, the control*

$$u^0(t) = \left(1 - \frac{t}{T}\right)u_0 + \frac{t}{T}u_T + \bar{C}(t; T)y(T), \tag{8}$$

where

$$\begin{aligned} y(T) = \mathbf{E}\Big\{[\bar{S}(T)]^{-1}\Big(&x_T - \Phi(T, 0)x_0 \\ &- P(T; T)u_0 - \frac{1}{T}(u_T - u_0)\int_0^T P(s; t)\,\mathrm{d}s \\ &- \int_0^T \Phi(T, s)\tilde{\sigma}(s)\,\mathrm{d}w(s) \\ &- \sum_{i=1}^{k} \Phi(T, t_i)I_i(x(t_i^-))\Big)\Big|\mathcal{F}_t\Big\}, \end{aligned}$$

transfers the system

$$\begin{aligned} x(t) = \Phi(t, 0)x_0 &+ P(t; t)u_0 \\ &+ \frac{1}{T}(u_T - u_0)\int_0^t P(s; t)\,\mathrm{d}s \\ &+ S(t; T)y(T) + \int_0^t \Phi(t, s)\tilde{\sigma}(s)\,\mathrm{d}w(s) \\ &+ \sum_{i=1}^{k} \Phi(t, t_i)I_i(x(t_i^-)) \end{aligned} \tag{9}$$

from $x_0 \in \mathbb{R}^n$ *to* x_T *at time* T *with* $u(0) = u_0$ *and* $u(T) = u_T$.

Moreover, among all the admissible controls $u(t)$ *transferring the initial state* x_0 *to the final state* x_1 *at time* $T > 0$, *the control* $u^0(t)$ *minimizes the integral performance index*

$$\mathcal{J}(u) = \mathbb{E}\int_0^T \|u(t)\|^2\,\mathrm{d}t.$$

Proof. If the matrix $M(0, T)$ is invertible, then the impulsive system (2) is controllable on $[0, T]$. Moreover, the inverse $[\bar{S}(T)]^{-1}$ exists (Anichini, 1980). Thus the pair $(x(t), u^0(t))$ defined in (8) and (9) is well defined.

Now, by Proposition 1, we have

$$\begin{aligned} x(t) = \Phi(t, 0)x_0 &+ \int_0^t \Phi(t, s)B(s)u^0(s)\,\mathrm{d}s \\ &+ \int_0^t \Phi(t, s)\tilde{\sigma}(s)\,\mathrm{d}w(s) + \sum_{i=1}^{k} \Phi(t, t_i)I_i(x(t_i^-)). \end{aligned}$$

From (7) and (9) we have

$$\begin{aligned}
x(T) &= \Phi(T,0)x_0 + P(T;T)u_0 + S(T;T)y(T) \\
&\quad + \frac{1}{T}(u_T - u_0)\int_0^T P(s;T)\,\mathrm{d}s \\
&\quad + \int_0^T \Phi(T,s)\tilde{\sigma}(t)(s)\,\mathrm{d}w(s) \\
&\quad + \sum_{i=1}^{k}\Phi(T,t_i)I_i(x(t_i^-)) \\
&= \Phi(T,0)x_0 + P(T;T)u_0 \\
&\quad + \frac{1}{T}(u_T - u_0)\int_0^T P(s;T)\,\mathrm{d}s \\
&\quad + S(T;T)\bar{S}(T)^{-1}\Big[x_T - \Phi(T,0)x_0 \\
&\quad - P(T;T)u_0 - \frac{1}{T}(u_T - u_0)\int_0^T P(s;t)\,\mathrm{d}s \\
&\quad - \int_0^T \Phi(T,s)\tilde{\sigma}(s)\,\mathrm{d}w(s) \\
&\quad - \sum_{i=1}^{k}\Phi(T,t_i)I_i(x(t_i^-))\Big] \\
&\quad + \int_0^T \Phi(T,s)\tilde{\sigma}(s)\,\mathrm{d}w(s) \\
&\quad + \sum_{i=1}^{k}\Phi(T,t_i)I_i(x(t_i^-)) = x_T
\end{aligned}$$

and $x(0) = x_0,\ \ u^0(0) = u_0,\ \ u^0(T) = u_T$. The second part of the proof is similar to that of Theorem 2 by Klamka (2007a). ■

3. Controllability results

In this section, we investigate the possibility of designing a nonlinear controller which conforms to the prescribed control and derive controllability conditions for the nonlinear stochastic impulsive system (3) by using the contraction mapping principle. Here we prove complete controllability of the nonlinear stochastic impulsive system under the natural assumption that the associated linear stochastic impulsive control system is completely controllable.

Consider the nonlinear stochastic impulsive system

$$\begin{aligned}
\mathrm{d}x(t) &= \Big[A(t)x(t) + B(t)u(t) + f(t,x(t))\Big]\,\mathrm{d}t \\
&\quad + \sigma(t,x(t))\,\mathrm{d}w(t), \qquad t \neq t_k, \\
\Delta x(t_k) &= I_k(t_k, x(t_k^-)), \quad t = t_k, \quad k = 1,2,\dots,\rho, \\
x(0) &= x_0, \quad x(T) = x_T, \\
u(0) &= u_0, \quad u(T) = u_T,
\end{aligned} \tag{10}$$

with $f : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n,\ \sigma : [0,T]\times\mathbb{R}^n \to \mathbb{R}^{n\times n}$, $I_k : \Omega \to \mathbb{R}^n, \Omega \subset [t_0,T]\times\mathbb{R}^n$, $\Delta x(t) = x(t^+) - x(t^-)$, where

$$\lim_{h\to 0^+} x(t+h) = x(t^+), \qquad \lim_{h\to 0^+} x(t-h) = x(t^-)$$

and w is an n-dimensional Wiener process.

For the study of this problem we impose the following hypotheses on the problem data:

($\mathbf{H_1}$) The functions $f,\ I_k$ and σ satisfy the following Lipschitz condition: There exist constants L_1 and $\alpha_k > 0, k = 1,2,\dots,\rho$ for $x,y \in \mathbb{R}^n$ and $t_0 \le t \le T$ such that

$$\begin{aligned}
\|f(t,x) - f(t,y)\|^2 + \|\sigma(t,x) - \sigma(t,y)\|^2 \\
\le L_1\|x-y\|^2, \\
\|I_k(t,x) - I_k(t,y)\|^2 \le \alpha_k\|x-y\|^2.
\end{aligned}$$

($\mathbf{H_2}$) The functions $f,\ I_k$ and σ are continuous and satisfies the usual linear growth condition, i.e., there exist a constants K_1 and $\beta_k > 0,\ k = 1,2,\dots,\rho$ for $x \in \mathbb{R}^n$ and $t_0 \le t \le T$ such that

$$\begin{aligned}
\|f(t,x)\|^2 + \|\sigma(t,x)\|^2 \le K_1(1 + \|x\|^2), \\
\|I_k(t,x)\|^2 \le \beta_k(1 + \|x\|^2),
\end{aligned}$$

By a solution of the system (10), we mean a solution of the nonlinear integral equation

$$\begin{aligned}
x(t) &= \Phi(t,0)x_0 + \int_0^t \Phi(t,s)B(s)u(s)\,\mathrm{d}s \\
&\quad + \int_0^t \Phi(t,s)f(s,x(s))\,\mathrm{d}s \\
&\quad + \int_0^t \Phi(t,s)\sigma(s,x(s))\,\mathrm{d}w(s) \\
&\quad + \sum_{k=1}^{\rho}\Phi(t,t_k)I_k(t_k, x(t_k^-)).
\end{aligned} \tag{11}$$

It is obvious that, under the conditions (H_1) and (H_2), for every $u(\cdot) \in U_{ad}$ the integral equation (11) has a unique solution in $\mathcal{B}_2$.

For $x \in \mathbb{R}^n$, consider

$$\begin{aligned}
x(t) &= \Phi(t,0)x_0 + P(t;t)u_0 + S(t;T)y(T) \\
&\quad + \frac{1}{T}(u_T - u_0)\int_0^t P(s;t)\,\mathrm{d}s \\
&\quad + \int_0^t \Phi(t,s)f(s,x(s))\,\mathrm{d}s \\
&\quad + \int_0^t \Phi(t,s)\sigma(s,x(s))\,\mathrm{d}w(s)
\end{aligned}$$

$$+\sum_{k=1}^{\rho}\Phi(t,t_k)I_k(t_k,x(t_k^-)),$$

$$u(t)=\left(1-\frac{t}{T}\right)u_0+\frac{t}{T}u_T+\bar{C}(t;T)y(T), \quad (12)$$

where

$$\begin{aligned}y(T)=\mathbf{E}\Big\{&[\bar{S}(T)]^{-1}\Big(x_T-\Phi(T,0)x_0-P(T;T)u_0\\&-\frac{1}{T}(u_T-u_0)\int_0^T P(s;t)\,\mathrm{d}s\\&-\int_0^T\Phi(T,s)f(s,x(s))\,\mathrm{d}s\\&-\int_0^T\Phi(T,s)\sigma(s,x(s))\,\mathrm{d}w(s)\\&-\sum_{k=1}^{\rho}\Phi(T,t_k)I_k(t_k,x(t_k^-))\Big)\Big|\mathcal{F}_t\Big\}.\end{aligned}$$

To apply the contraction mapping principle, we define the nonlinear operator $\mathcal{Q}$ from $\mathcal{B}_2$ to $\mathcal{B}_2$ as follows:

$$\begin{aligned}(\mathcal{Q}x)(t)=\Phi(t,0)x_0&+P(t;t)u_0+S(t;T)y(T)\\&+\frac{1}{T}(u_T-u_0)\int_0^t P(s;t)\,\mathrm{d}s\\&+\int_0^t\Phi(t,s)f(s,x(s))\,\mathrm{d}s\\&+\int_0^t\Phi(t,s)\sigma(s,x(s))\,\mathrm{d}w(s)\\&+\sum_{k=1}^{\rho}\Phi(t,t_k)I_k(t_k,x(t_k^-)).\end{aligned}$$

From Lemma 3, if the operator $\mathcal{Q}$ has a fixed point, then the system (10) has a solution $x(t,u)$ with respect to $u(\cdot)$. Clearly, $x(t_0,u)=x_0$, $x(T,u)=x_1$. Then the system (10) is controllable by $u(\cdot)$. Thus the problem of discussing the controllability of the system (10) can be reduced into that of the existence of a fixed point of $\mathcal{Q}$.

Note that if the linear stochastic system (2) is completely controllable, then there exists a positive constant m_1 such that, for $t_0<s<t\le T$ (Mahmudov, 2001),

$$\|\Phi(s,t)\|^2\le m_1.$$

Now, for convenience, let us introduce the following notation:

$$\begin{aligned}m_2&=\max\{\|A(s)\|^2:s\in[0,T]\},\\m_3&=\max\{\|B(s)\|^2:s\in[0,T]\},\\M_1&=\max\{\|S(t;T)\|^2:t\in[0,T]\},\\M_2&=\|\bar{S}(T)^{-1}\|^2.\end{aligned}$$

Theorem 1. *Assume that the functions involved in the stochastic impulsive system given by (10) satisfy the conditions (H_1)–(H_2) required to ensure the existence and uniqueness of a solution process $x(t)$ in $\mathcal{B}_2$ and that the hypotheses of Lemma 3 hold. Then, for every $x_0,x_T\in\mathbb{R}^n$ and prescribed values for the controls $u_0,u_T\in\mathbb{R}^m$, the nonlinear stochastic impulsive system (10) is completely controllable provided that*

$$6m_1(1+M_1M_2)\left(L_1+\rho\sum_{k=1}^{\rho}\alpha_k\right)(1+T)T<1. \quad (13)$$

Proof. To prove complete controllability, it is enough to show that $\mathcal{Q}$ has a fixed point in $\mathcal{B}_2$. To do this, we use the contraction mapping principle. To apply it, first we show that $\mathcal{Q}$ maps $\mathcal{B}_2$ into itself. For that we have

$$\begin{aligned}&\mathbf{E}\|(\mathcal{Q}z)(t)\|^2\\&=\mathbf{E}\Big\|\Phi(t,0)x_0+P(t;t)u_0+S(t;T)y(T)\\&\quad+\frac{1}{T}(u_T-u_0)\int_0^t P(s;t)\,\mathrm{d}s\\&\quad+\int_0^t\Phi(t,s)f(s,x(s))\,\mathrm{d}s\\&\quad+\int_0^t\Phi(t,s)\sigma(s,x(s))\,\mathrm{d}w(s)\\&\quad+\sum_{k=1}^{\rho}\Phi(t,t_k)I_k(t_k,x(t_k^-))\Big\|^2\\&\le 7\|\Phi(t,0)\|^2\|x_0\|^2+7\|P(t;t)\|^2\|u_0\|^2\\&\quad+\frac{7}{T^2}\|u_T-u_0\|^2\|\int_0^t P(s;t)\,\mathrm{d}s\|^2\\&\quad+7\|S(t;T)\|^2\mathbf{E}\|y(T)\|^2\\&\quad+7\mathbf{E}\|\int_0^t\Phi(t,s)f(s,x(s))\,\mathrm{d}s\|^2\\&\quad+7\mathbf{E}\|\int_0^t\Phi(t,s)\sigma(s,x(s))\,\mathrm{d}w(s)\|^2\\&\quad+7\mathbf{E}\Big\|\sum_{k=1}^{\rho}\Phi(t,t_k)I_k(t_k,x(t_k^-))\Big\|^2.\end{aligned}$$

Now we estimate $\mathbf{E}\|y(T)\|^2$,

$$\begin{aligned}\mathbf{E}\|y(T)\|^2\le 7\|\bar{S}(T)^{-1}\|^2\Big(&\|x_T\|^2\\&+\|\Phi(T,0)\|^2\|x_0\|^2+\|P(T;t)\|^2\|u_0\|^2\\&+\mathbf{E}\|\int_0^T\Phi(T,s)f(s,x(s))\,\mathrm{d}s\|^2\\&+\mathbf{E}\|\int_0^T\Phi(T,s)\sigma(s,x(s))\,\mathrm{d}w(s)\|^2\\&+\frac{1}{T}\|u_T-u_0\|^2\|\int_0^T P(s;t)\,\mathrm{d}s\|^2\end{aligned}$$

$$+\mathbf{E}\Big\|\sum_{k=1}^{\rho}\Phi(t,t_k)I_k(t_k,x(t_k^-))\Big\|^2\Big)$$
$$\le 7M_2\Big(\|x_T\|^2+m_1\|x_0\|^2+T^2m_1m_3\|u_0\|^2$$
$$+m_1m_3T^2\|u_T-u_0\|^2$$
$$+m_1\left(L_2+\rho\sum_{k=1}^{\rho}\beta_k\right)$$
$$\times\mathbf{E}\int_0^T(1+\|x(s)\|^2)\,\mathrm{d}s\Big).$$

Therefore,

$$\begin{aligned}
&\mathbf{E}\|(\mathcal{Q}x)(t)\|^2\\
&\le 49M_1M_2\|x_T\|^2+7(1+7M_1M_2)\Big(m_1\|x_0\|^2\\
&\quad+T^2m_1m_3\|u_0\|^2+m_1m_3T^2\|u_T-u_0\|^2\\
&\quad+m_1\left(L_2+\rho\sum_{k=1}^{\rho}\beta_k\right)(1+T)\\
&\quad\times\mathbf{E}\int_0^T(1+\|x(s)\|^2)\,\mathrm{d}s\Big).
\end{aligned}\tag{14}$$

From (14) and the condition (H_2) it follows, that there exists $C_1>0$ such that

$$\mathbf{E}\|(\mathcal{Q}x)(t)\|^2 \le C_1\Big(1+T\sup_{0\le s\le T}\mathbf{E}\|x(s)\|^2\Big),$$

for all $t\in[0,T]$. Therefore, $\mathcal{Q}$ maps $\mathcal{B}_2$ into itself.

Next we show that $\mathcal{Q}$ is a contraction mapping. Indeed,

$$\begin{aligned}
&\mathbb{E}\|(\mathcal{Q}x_1)(t)-(\mathcal{Q}x_2)(t))\|^2\\
&=\mathbb{E}\Big\|\int_{t_0}^{t}\Phi(t,s)[f(s,x_1(s))-f(s,x_2(s))]\,\mathrm{d}s\\
&\quad+\int_{t_0}^{t}\Phi(t,s)[\sigma(s,x_1(s))-\sigma(s,x_2(s))]\,\mathrm{d}w(s)\\
&\quad+\sum_{k=1}^{\rho}\Phi(t,t_k)[I_k(t_k,x_1(t_k^-))-I_k(t_k,x_2(t_k^-))]\\
&\quad+S(t;T)\bar{S}(T)^{-1}\\
&\quad\times\left(\int_{t_0}^{T}\Phi(T,s)[f(s,x_2(s))-f(s,x_1(s))]\,\mathrm{d}s\right.\\
&\quad+\int_{t_0}^{T}\Phi(T,s)[\sigma(s,x_2(s))-\sigma(s,x_1(s))]\,\mathrm{d}w(s)\\
&\quad\left.+\sum_{k=1}^{\rho}\Phi(T,t_k)[I_k(t_k,x_2(t_k^-))-I_k(t_k,x_1(t_k^-))]\right)\Big\|^2\\
&\le 6m_1L_1(1+T)\int_{t_0}^{T}\mathbb{E}\|x_1(s)-x_2(s)\|^2\,\mathrm{d}s\\
&\quad+6m_1\rho\sum_{k=1}^{\rho}\alpha_k\mathbb{E}\|x_1(t)-x_2(t)\|^2\\
&\quad+6M_1M_2\Big[m_1L_1(1+T)\int_{t_0}^{T}\mathbb{E}\|x_1(s)-x_2(s)\|^2\,\mathrm{d}s\\
&\quad+m_1\rho\sum_{k=1}^{\rho}\alpha_k\mathbb{E}\|x_1(t)-x_2(t)\|^2\Big]\\
&\le 6m_1(1+M_1M_2)(1+T)\left(L_1+\rho\sum_{k=1}^{\rho}\alpha_k\right)\\
&\quad\times\int_{t_0}^{T}\mathbb{E}\|x_1(s)-x_2(s)\|^2\,\mathrm{d}s.
\end{aligned}$$

Accordingly,

$$\begin{aligned}
&\sup_{t\in[t_0,T]}\mathbb{E}\|(\mathcal{Q}x_1)(t)-(\mathcal{Q}x_2)(t))\|^2\\
&\quad\le 6m_1(1+M_1M_2)\left(L_1+\rho\sum_{k=1}^{\rho}\alpha_k\right)(1+T)T\\
&\qquad\times\sup_{t\in[t_0,T]}\mathbb{E}\|x_1(t)-x_2(t)\|^2.
\end{aligned}$$

Therefore, from (13) we conclude that $\mathcal{Q}$ is a contraction mapping on $\mathcal{B}_2$. Then the mapping $\mathcal{Q}$ has a unique fixed point $x(\cdot)\in\mathcal{B}_2$, which is the solution of Eqn. (10). Thus the nonlinear stochastic impulsive system (10) is completely controllable. ■

4. Neutral stochastic impulsive system

Now, we consider a class of Itô type nonlinear neutral stochastic impulsive systems as follows:

$$\begin{aligned}
\mathrm{d}[x(t)-g(t,x(t))]&=[A(t)x(t)+B(t)u(t)\\
&\quad+f(t,x(t))]\,\mathrm{d}t\\
&\quad+\sigma(t,x(t))\,\mathrm{d}w(t),\quad t\ne t_k,\\
\Delta x(t_k)&=I_k(t_k,x(t_k^-)),\\
x(t_0)&=x_0,\quad x(T)=x_T,\\
u(t_0)&=u_0,\quad u(T)=u_T
\end{aligned}\tag{15}$$

for $t=t_k,\ k=1,2,\dots,\rho$, where $g:[t_0,T]\times\mathbb{R}^n\to\mathbb{R}^n$ is continuously differentiable. The controllability of this type for nonlinear systems with no constraints on the control function has been investigated by Karthikeyan and Balachandran (2009). The solution of the system (15) in the interval $[t_0,T]$ is given by the nonlinear integral equation

$$
\begin{aligned}
x(t) &= \Phi(t,t_0)[x_0 - g(t_0,x_0)] + g(t,x(t)) \\
&\quad + P(t;t)u_0 + \frac{1}{T}(u_T - u_0)\int_0^t P(s;t)\,\mathrm{d}s \\
&\quad + S(t;T)y(T) + \int_{t_0}^t A(s)\Phi(t,s)g(s,x(s))\,\mathrm{d}s \\
&\quad + \int_{t_0}^t \Phi(t,s)f(s,x(s))\,\mathrm{d}s \\
&\quad + \int_{t_0}^t \Phi(t,s)\sigma(s,x(s))\,\mathrm{d}w(s) \\
&\quad + \sum_{k=1}^{\rho}\Phi(t,t_k)I_k(t_k,x(t_k^-)).
\end{aligned} \tag{16}
$$

In order to apply the contraction principle, we set

$$
\begin{aligned}
(\mathcal{P}x)(t) &= \Phi(t,t_0)[x_0 - g(t_0,x_0)] + g(t,x(t)) \\
&\quad + P(t;t)u_0 + S(t;T)y(T) \\
&\quad + \frac{1}{T}(u_T - u_0)\int_0^t P(s;t)\,\mathrm{d}s \\
&\quad + \int_{t_0}^t A(s)\Phi(t,s)g(s,x(s))\,\mathrm{d}s \\
&\quad + \int_{t_0}^t \Phi(t,s)f(s,x(s))\,\mathrm{d}s \\
&\quad + \int_{t_0}^t \Phi(t,s)\sigma(s,x(s))\,\mathrm{d}w(s) \\
&\quad + \sum_{k=1}^{\rho}\Phi(t,t_k)I_k(t_k,x(t_k^-)), \\
u(t) &= \left(1 - \frac{t}{T}\right)u_0 + \frac{t}{T}u_T + \bar{C}(t;T)y(T),
\end{aligned}
$$

where

$$
\begin{aligned}
y(T) = \mathbf{E}\Big\{ &[\bar{S}(T)]^{-1}\Big(x_T - \Phi(T,0)[x_0 - g(t_0,x_0)] \\
&- g(T,x(T)) - P(T;T)u_0 \\
&- \frac{1}{T}(u_T - u_0)\int_0^T P(s;t)\,\mathrm{d}s \\
&- \int_0^T \Phi(T,s)f(s,x(s))\,\mathrm{d}s \\
&- \int_0^T \Phi(T,s)\sigma(s,x(s))\,\mathrm{d}w(s) \\
&- \int_{t_0}^T A(s)\Phi(T,s)g(s,x(s))\,\mathrm{d}s \\
&- \sum_{k=1}^{\rho}\Phi(T,t_k)I_k(t_k,x(t_k^-))\Big)\Big|\mathcal{F}_t\Big\}.
\end{aligned}
$$

Along with the hypotheses (H_1) and (H_2) we assume the following conditions on the problem data:

(H3) The function g satisfies the following Lipschitz condition: There exist a constant $L_2 > 0$ for $x, y \in \mathbb{R}^n$ and $t_0 \le t \le T$ such that

$$\|g(t,x) - g(t,y)\|^2 \le L_2\|x - y\|^2.$$

Theorem 2. *Under the conditions (H_1)–(H_3) and the hypotheses of Lemma 3, the nonlinear stochastic system (15) is completely controllable provided that*

$$
\begin{aligned}
&\Big[9L_2 + 9m_1(1 + M_1M_2)(1 + m_2) \\
&\qquad \times \Big(L_1 + L_2 + \rho\sum_{k=1}^{\rho}\alpha_k\Big)\Big](1 + T)T < 1.
\end{aligned} \tag{17}
$$

Proof. The proof is similar to that of Theorem 2 and therefore it is omitted. ■

5. Example

Consider the nonlinear stochastic impulsive system of the form

$$
\begin{aligned}
&\mathrm{d}[x_1 - x_2] \\
&\quad = \left[e^{-t}x_2 + 1.2u_1 - 0.2u_2 + \frac{x_1\cos x_2}{5}\right]\mathrm{d}t \\
&\qquad + \frac{x_1e^{-t}}{8(1 + x_2)}\,\mathrm{d}w_1(t), \quad t \ne t_k, \\
&\mathrm{d}[x_2 - 2\sin x_1] \\
&\quad = \left[e^{-t}x_2 + 0.6u_1 + 2.4u_2 + \frac{x_2\sin x_1}{6}\right]\mathrm{d}t \\
&\qquad + \frac{x_2e^{-t}}{7(1 + x_1)}\,\mathrm{d}w_2(t), \quad t \ne t_k, \\
&\begin{bmatrix} \Delta x_1(t_k) & \\ & \Delta x_2(t_k) \end{bmatrix} \\
&\quad = e^{-0.1k}\begin{bmatrix} 0.5 & -0.15 \\ 0.12 & 0.6 \end{bmatrix}\begin{bmatrix} x_1(t_k^-) \\ x_2(t_k^-) \end{bmatrix},
\end{aligned} \tag{18}
$$

with $t = t_k$, where $t_k = t_{k-1} + 0.15, \ \ k = 1, 2, \dots, \rho$. This above equation can be rewritten in the form (15) with $x(t) = (x_1(t), x_2(t)) \in \mathbb{R}^2, \ t_0 = 0$,

$$
\begin{aligned}
A(t) &= \begin{bmatrix} 0 & e^{-t} \\ 0 & e^{-t} \end{bmatrix}, \\
B(t) &= \begin{bmatrix} 1.2 & -0.2 \\ 0.6 & 2.4 \end{bmatrix}, \\
g(t,x(t)) &= \begin{bmatrix} x_2 \\ 2\sin x_1 \end{bmatrix}
\end{aligned}
$$

$$f(t,x(t)) = \begin{bmatrix} \dfrac{x_1 \cos x_2}{5} \\ \dfrac{x_2 \sin x_1}{6} \end{bmatrix},$$

$$\sigma(t,x(t)) = \begin{bmatrix} \dfrac{x_1 e^{-t}}{8(1+x_2)} & 0 \\ 0 & \dfrac{x_2 e^{-t}}{7(1+x_1)} \end{bmatrix}.$$

The fundamental matrix associated with the linear control system is

$$\Phi(t,0) = \begin{bmatrix} 1 & \exp(1-e^{-t}) - 1 \\ 0 & \exp(1-e^{-t}) \end{bmatrix}.$$

Take the final point as $x_T \in \mathbb{R}^2$. Moreover, it is easy to show that for all $x \in \mathbb{R}^2$, $\|f(t,x(t))\|^2 + \|\sigma(t,x(t))\|^2 \leq \|x\|^2/25$ and $\|g(t,x(t))\|^2 \leq 4\|x\|^2$. Also $L_1 = 1/25$, $L_2 = 4$, $\beta_k = 0.6469e^{-0.1k}$, $m_1 = 2(1+e^{2(1-e^{-T})})$, $m_2 = 2$, and $m_3 = 7.6$. Using the values of m_1 and m_2, we can easily obtain M_1 and M_2. Choose $T > 0$ in such a way that (17) is satisfied. One can see that all other conditions stated in Theorem 2 are satisfied. Hence, the stochastic impulsive system (18) is completely controllable on $[0,T]$ with arbitrarily prescribed initial and final values of control.

Remark 1. It is very important to note that the controllability results for stochastic integro-differential systems discussed by Shena *et al.* (2010) using Schaefer's fixed point theorem are invalid since the compactness of a bounded linear operator implies that its range space must be finite dimensional (Hernandez and O'Regan, 2009). It should be pointed out that for stochastic dynamical systems the state space $L_2(\Omega, \mathcal{F}_t, \mathbb{R}^n)$ is in fact an infinite-dimensional space, which is incompatible with the requirement that the mapping be compact. Even for a finite dimensional system, the finite set $\{y_i,\ 1 \leq i \leq m\}$ may well depend on the sample point $\omega \in \Omega$, and therefore proving the desired compactness is extremely difficult. Thus, Schauder's or Schaefer's fixed point theorem cannot be applied to study nonlinear stochastic control problems.

6. Concluding remarks

In the paper, sufficient conditions for complete controllability of linear and nonlinear stochastic systems with prescribed control were formulated and proved. It should be pointed out that these results constitute an extension of the controllability conditions for deterministic control systems given by Anichini (1980; 1983), Balachandran and Lalitha (1992) as well as Luke (1972) to stochastic impulsive systems with prescribed controls. As a possible application of the theoretical results, an example of a nonlinear stochastic system was presented. Some important comments regarding fixed point theorems involving compactness results for nonlinear stochastic control problems were explained.

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