CIVIL AND ENVIRONMENTAL ENGINEERING REPORTS

No. 1 2005

A METHOD FOR SOLVING RECTANGULAR BEAMS AND DISCS

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An analytic method for solving two-dimensional elasticity problems of orthotropic body has been worked out. A solution for rectangular elements of the beam or disc type has been developed. This solution satisfies the fundamental equations of the elasticity theory of two-dimensional body exactly and the boundary conditions with large accuracy.

Keywords: thick plate, beam, disc, orthotropy, analytic method

1. FORMULATION OF THE PROBLEM

Beams and discs are widely used as structural members. Depending on the kind of reinforcement they may divided into two groups: isotropic or orthotropic. The former comprise all homogenous members like, for instance, steel elements. The reinforced members may be treated as orthotropic with varying degree of orthotropy. Typical examples are beams and discs made of wood, composite fibres and concrete.

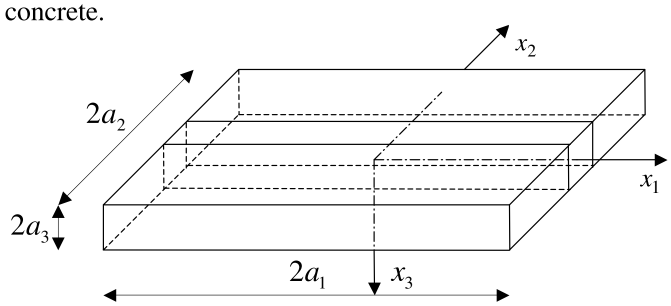

Fig.1. A beam as a plate strip

ISBN 83-89712-71-7

This work focuses on development of a method for solving of orthotropic structural members: beams and discs. Beams are treated as strips cut out of a thick orthotropic plate subject to cylindrical bending (fig. 1). Under axial stretch we face the membrane state.

Equilibrium of such a strip, treated as a beam or a disc depending on the way it is loaded, is governed by a system of two homogeneous differential equations [2,5]:

$$b_{55}u_{3,11} + (b_{31} + b_{55})u_{1,31} + b_{33}u_{3,33} = 0\,, \qquad (1)$$
$$b_{11}u_{1,11} + (b_{13} + b_{55})u_{3,31} + b_{55}u_{1,33} = 0\,,$$

where: u_j $(j = 1,3)$ are the components of the displacement vector and b_{kl} $(k,l = 1,3,5)$ correspond to the components of the stiffness tensor for an orthotropic material. Specifically, in (1) the following convention has been adopted: $b_{11} = b_{1111}$, $b_{13} = b_{1133} = b_{31}$, $b_{55} = b_{1313}$, $b_{33} = b_{3333}$.

2. METHOD OF SOLUTION

Following the approach taken in [1,4], we assume the solution to a non-symmetrical problems in the form:

$$u_1 = \sum_{m=1}^{\infty}\Big\{f^{[1]}_{2(m)}(x_1)\cos(\delta^{[3]}_{(m)}x_3) + f^{[1]}_{1(m)}(x_1)\sin(\delta^{[3]}_{(m)}x_3) +$$
$$+ f^{[3]}_{2(m)}(x_3)\cos(\delta^{[1]}_{(m)}x_1) + f^{[3]}_{1(m)}(x_3)\sin(\delta^{[1]}_{(m)}x_1)\Big\},$$
$$u_3 = \sum_{m=1}^{\infty}\Big\{f^{[1]}_{4(m)}(x_1)\cos(\delta^{[3]}_{(m)}x_3) + f^{[1]}_{3(m)}(x_1)\sin(\delta^{[3]}_{(m)}x_3) + \qquad (2)$$
$$+ f^{[3]}_{4(m)}(x_3)\cos(\delta^{[1]}_{(m)}x_1) + f^{[3]}_{3(m)}(x_3)\sin(\delta^{[1]}_{(m)}x_1)\Big\},$$

where the functions: $f^{[j]}_{p(m)}(x_j)$, $p = \overline{1,4}$ are the unknowns and: $\delta^{[j]}_{m} = \dfrac{(2m-1)\pi}{2a_j}$ the parameters.

Substituting (2) to (1), and separating the variables, we obtain a system of ordinary differential equations for unknown functions $f^{[j]}_{p(m)}(x_j)$:

$$\sum_{r=1}^{3}\Big\{A^{[j]}_{kr(m)} f^{[j]\,(3-r)}_{1(m)}(x_j) + B^{[j]}_{kr(m)} f^{[j]\,(3-r)}_{2(m)}(x_j) + C^{[j]}_{kr(m)} f^{[j]\,(3-r)}_{3(m)}(x_j) +$$
$$+ D^{[j]}_{kr(m)} f^{[j]\,(3-r)}_{4(m)}(x_j)\Big\} = 0\,, \quad k = \overline{1,4} \quad, r = \overline{1,3}\,. \qquad (3)$$

The coefficients: $A^{[j]}_{kr(m)}, B^{[j]}_{kr(m)}, C^{[j]}_{kr(m)}, D^{[j]}_{kr(m)}$ in (3) can be expressed via the components of the stiffness tensor for the orthotropic material and appropriate parameters:

$$\begin{gathered}
A^{[1]}_{11(m)} = b_{11}, \quad A^{[1]}_{13(m)} = -b_{55}\delta^{[3]\,2}_{(m)}, \\
B^{[1]}_{21(m)} = b_{11}, \;\; C^{[1]}_{22(m)} = (b_{13}+b_{55})\delta^{[3]}_{(m)}, \;\; B^{[1]}_{23(m)} = -b_{55}\delta^{[3]\,2}_{(m)}, \\
C^{[1]}_{31(m)} = b_{55}, \;\; B^{[1]}_{32(m)} = -(b_{13}+b_{55})\delta^{[3]}_{(m)}, \;\; C^{[1]}_{33(m)} = -b_{33}\delta^{[3]\,2}_{(m)}, \\
D^{[1]}_{41(m)} = b_{55}, \;\; A^{[1]}_{42(m)} = (b_{13}+b_{55})\delta^{[3]}_{(m)}, \;\; D^{[1]}_{43(m)} = -b_{33}\delta^{[3]\,2}_{(m)}, \\
A^{[3]}_{11(m)} = b_{55}, \;\; D^{[3]}_{12(m)} = -(b_{13}+b_{55})\delta^{[1]}_{(m)}, \;\; A^{[3]}_{13(m)} = -b_{11}\delta^{[1]\,2}_{(m)}, \\
B^{[3]}_{21(m)} = b_{55}, \;\; C^{[3]}_{22(m)} = (b_{13}+b_{55})\delta^{[1]}_{(m)}, \;\; B^{[3]}_{23(m)} = -b_{11}\delta^{[1]\,2}_{(m)}, \\
C^{[1]}_{31(m)} = b_{33}, \;\; B^{[1]}_{32(m)} = -(b_{13}+b_{55})\delta^{[1]}_{(m)}, \;\; C^{[1]}_{33(m)} = -b_{55}\delta^{[1]\,2}_{(m)}, \\
D^{[3]}_{41(m)} = b_{33}, \;\; A^{[3]}_{42(m)} = (b_{13}+b_{55})\delta^{[1]}_{(m)}, \;\; D^{[3]}_{43(m)} = -b_{55}\delta^{[1]\,2}_{(m)}.
\end{gathered} \tag{4}$$

The remaining parameters have zero values.
We seek the following particular solution to (3):

$$f^{[j]}_{p(m)}\left(x_j\right) = R^{[j]}_{p(m)} \exp\left[\lambda^{[j]}_{(m)} x_j\right], \tag{5}$$

where $R^{[j]}_{p(m)}$ are unknown parameters. The quantities $\lambda^{[j]}_{(m)}$ are the roots of the characteristic equation [4]:

$$\begin{gathered}
(b_{11}\lambda^{[1]2}_{2(m)} - b_{55}\delta^{[3]2}_{(m)})(b_{55}\lambda^{[1]2}_{2(m)} - b_{33}\delta^{[3]2}_{(m)}) + (b_{13}+b_{55})^2\,\delta^{[3]2}_{(m)}\lambda^{[1]2}_{2(m)} = 0, \\
(b_{55}\lambda^{[3]2}_{2(m)} - b_{11}\delta^{[1]2}_{(m)})(b_{33}\lambda^{[3]2}_{2(m)} - b_{55}\delta^{[1]2}_{(m)}) + (b_{13}+b_{55})^2\,\delta^{[1]2}_{(m)}\lambda^{[3]2}_{2(m)} = 0,
\end{gathered} \tag{6}$$

Each of the above equations is a fourth-order equation (biharmonic equation) and, therefore, each has four roots (two positive and two negative):

$$\lambda^{[j]}_{1(m)} = -\lambda^{[j]}_{3(m)}, \quad \lambda^{[j]}_{2(m)} = -\lambda^{[j]}_{4(m)}. \tag{7}$$

Numerical computations show that for orthotropic material the roots $\lambda^{[j]}_{(m)}$ are all real.

General solution of the system takes the form:

$$f^{[j]}_{p(m)} = \sum_{\nu=1}^{4} R^{[j]}_{p\,\nu(m)} \exp \lambda^{[j]}_{\nu(m)} x_j \,. \tag{8}$$

Observe that within the set $\left\{R_{p_{\nu(m)}}^{[j]}\right\}$ only two groups of coefficients $R_{1\nu(m)}^{[j]}$ and $R_{2\nu(m)}^{[j]}$ are linearly independent. All the remaining coefficients are linearly related by the equations:

$$R_{3\nu(m)}^{[j]} = R_{2\nu(m)}^{[j]} K_{2\nu(m)}^{[j]}, \quad R_{4\nu(m)}^{[j]} = R_{1\nu(m)}^{[j]} K_{1\nu(m)}^{[j]}. \tag{9}$$

In the above:

$$K_{1\nu(m)}^{[1]} = \frac{b_{11}\lambda_{\nu(m)}^{[1]^2} - b_{55}\delta_{(m)}^{[3]^2}}{(b_{13}+b_{55})\lambda_{\nu(m)}^{[1]}\delta_{(m)}^{[3]}} = \frac{b_{11}\gamma_\nu^{[1]^2} - b_{55}}{(b_{13}+b_{55})\gamma_\nu^{[1]}} = K_{1\nu}^{[1]},$$

$$K_{2\nu(m)}^{[1]} = -\frac{b_{11}\lambda_{\nu(m)}^{[1]^2} - b_{55}\delta_{(m)}^{[3]^2}}{(b_{13}+b_{55})\lambda_{\nu(m)}^{[1]}\delta_{(m)}^{[3]}} = -\frac{b_{11}\gamma_\nu^{[1]^2} - b_{55}}{(b_{13}+b_{55})\gamma_\nu^{[1]}} = K_{2\nu}^{[1]}, \tag{10}$$

$$K_{1\nu(m)}^{[3]} = \frac{b_{55}\lambda_{\nu(m)}^{[3]^2} - b_{11}\delta_{(m)}^{[1]^2}}{(b_{13}+b_{55})\lambda_{\nu(m)}^{[3]}\delta_{(m)}^{[1]}} = \frac{b_{55}\gamma_\nu^{[3]^2} - b_{11}}{(b_{13}+b_{55})\gamma_\nu^{[3]}} = K_{1\nu}^{[3]},$$

$$K_{2\nu(m)}^{[3]} = -\frac{b_{55}\lambda_{\nu(m)}^{[3]^2} - b_{11}\delta_{(m)}^{[1]^2}}{(b_{13}+b_{55})\lambda_{\nu(m)}^{[3]}\delta_{(m)}^{[1]}} = -\frac{b_{55}\gamma_\nu^{[3]^2} - b_{11}}{(b_{13}+b_{55})\gamma_\nu^{[3]}} = K_{2\nu}^{[3]},$$

where: $\gamma_\nu^{[j]} = \lambda_{\nu(m)}^{[j]} / \delta_m^{[j]}$.

Making use of (2) and (8) we can determine the components of the displacement vector as:

$$\begin{aligned} u_1 = \sum_{m=1}^{\infty}\sum_{\nu=1}^{4}\Big\{ & R_{1\nu(m)}^{[1]}\left[U_{1\nu(m)}^{[1]}(x_1)\sin(\delta_m^{[3]}x_3)\right] + R_{2\nu(m)}^{[1]}\left[U_{2\nu(m)}^{[1]}(x_1)\cos(\delta_m^{[3]}x_3)\right] + \\ & + R_{1\nu(m)}^{[3]}\left[U_{1\nu(m)}^{[3]}(x_3)\sin(\delta_m^{[1]}x_1)\right] + R_{2\nu(m)}^{[3]}\left[U_{2\nu(m)}^{[3]}(x_3)\cos(\delta_m^{[1]}x_1)\right]\Big\}, \end{aligned} \tag{11}$$

$$\begin{aligned} u_3 = \sum_{m=1}^{\infty}\sum_{\nu=1}^{4}\Big\{ & R_{1\nu(m)}^{[1]}\left[W_{1\nu(m)}^{[1]}(x_1)\cos(\delta_m^{[3]}x_3)\right] + R_{2\nu(m)}^{[1]}\left[W_{2\nu(m)}^{[1]}(x_1)\sin(\delta_m^{[3]}x_3)\right] + \\ & + R_{1\nu(m)}^{[3]}\left[W_{1\nu(m)}^{[3]}(x_3)\cos(\delta_m^{[1]}x_1)\right] + R_{2\nu(m)}^{[3]}\left[W_{2\nu(m)}^{[3]}(x_3)\sin(\delta_m^{[1]}x_1)\right]\Big\}. \end{aligned}$$

We introduced the following notation in (11):

$$U_{p\nu(m)}^{[j]}(x_j) = \begin{cases} \sinh\lambda_{\nu(m)}^{[j]}x_j, & \nu = 1,2, \\ \cosh\lambda_{(\nu-2)(m)}^{[j]}x_j, & \nu = 3,4 \end{cases}$$

$$W^{[j]}_{p\nu(m)}(x_j) = K^{[j]}_{p\nu(m)} \begin{cases} \cosh \lambda^{[j]}_{\nu(m)} x_j, \nu = 1,2, \\ \\ \sinh \lambda^{[j]}_{(\nu-3)(m)} x_j, \nu = 3,4, \end{cases} \qquad \mathrm{p} = 1,2.. \tag{12}$$

The components of the strain tensor follow from the kinematic relations:

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{13}$$

Substituting (11) to the above, we derive the formulas for the components of the strain tensor:

$$\varepsilon_{11} = \sum_{m=1}^{\infty}\sum_{\nu=1}^{4}\left\{R^{[1]}_{1\nu(m)}\left[U^{[1]\,\prime}_{1\nu(m)}(x_1)\sin(\delta^{[3]}_m x_3)\right] + R^{[3]}_{1\nu(m)}\left[\delta^{[1]}_m U^{[3]}_{1\nu(m)}(x_3)\cos(\delta^{[1]}_m x_1)\right] + \right.$$
$$\left. + R^{[1]}_{2\nu(m)}\left[U^{[1]\,\prime}_{2\nu(m)}(x_1)\cos(\delta^{[3]}_m x_3)\right] + R^{[3]}_{2\nu(m)}\left[-\delta^{[1]}_m U^{[3]}_{2\nu(m)}(x_3)\sin(\delta^{[1]}_m x_1)\right]\right\},$$

$$\varepsilon_{33} = \sum_{m=1}^{\infty}\sum_{\nu=1}^{4}\left\{R^{[1]}_{1\nu(m)}\left[-\delta^{[3]}_m W^{[1]}_{1\nu(m)}(x_1)\sin(\delta^{[3]}_m x_3)\right] + R^{[3]}_{1\nu(m)}\left[W^{[3]\,\prime}_{1\nu(m)}(x_3)\cos(\delta^{[1]}_m x_1)\right] + \right.$$
$$\left. + R^{[1]}_{2\nu(m)}\left[\delta^{[3]}_m W^{[1]}_{2\nu(m)}(x_1)\cos(\delta^{[1]}_m x_1)\right] + R^{[3]}_{2\nu(m)}\left[W^{[3]\,\prime}_{2\nu(m)}(x_3)\sin(\delta^{[1]}_m x_1)\right]\right\},$$

$$\begin{aligned} \gamma_{13} = \sum_{m=1}^{\infty}\sum_{\nu=1}^{4}\Bigg\{ & R^{[1]}_{1\nu(m)}\left[\left(\delta^{[3]}_m U^{[1]}_{1\nu(m)}(x_1) + W^{[1]\,\prime}_{1\nu(m)}(x_1)\right)\cos(\delta^{[3]}_m x_3)\right] + \\ & + R^{[3]}_{1\nu(m)}\left[\left(U^{[3]\,\prime}_{1\nu(m)}(x_3) - \delta^{[1]}_m W^{[3]}_{1\nu(m)}(x_3)\right)\sin(\delta^{[1]}_m x_1)\right] + \\ & + R^{[1]}_{2\nu(m)}\left[\left(\delta^{[3]}_m U^{[1]}_{2\nu(m)}(x_1) + W^{[1]\,\prime}_{2\nu(m)}(x_1)\right)\cos(\delta^{[3]}_m x_3)\right] + \\ & + R^{[3]}_{2\nu(m)}\left[\left(U^{[3]\,\prime}_{2\nu(m)}(x_3) - \delta^{[1]}_m W^{[3]}_{2\nu(m)}(x_3)\right)\sin(\delta^{[1]}_m x_1)\right]\Bigg\}. \end{aligned} \tag{14}$$

Let us introduce the following notation:

$$G^{[1]}_{1\nu(m)}(x_1) = U^{[1]\,\prime}_{1\nu(m)}(x_1), \quad G^{[3]}_{1\nu(m)}(x_3) = \delta^{[1]}_m U^{[3]}_{1\nu(m)}(x_3),$$
$$G^{[1]}_{2\nu(m)}(x_1) = U^{[1]\,\prime}_{2\nu(m)}(x_1), \quad G^{[3]}_{2\nu(m)}(x_3) = -\delta^{[1]}_m U^{[3]}_{2\nu(m)}(x_3),$$
$$H^{[3]}_{1\nu(m)}(x_3) = W^{[3]\,\prime}_{1\nu(m)}(x_3), \quad H^{[1]}_{1\nu(m)}(x_1) = -\delta^{[3]}_m W^{[1]}_{1\nu(m)}(x_1),$$

$$H^{[3]}_{2\nu(m)}(x_3)=W^{[3]\,\prime}_{2\nu(m)}(x_3)\,,\quad H^{[1]}_{2\nu(m)}(x_1)=\delta^{[3]}_m W^{[1]}_{2\nu(m)}(x_1)\,,$$
$$T^{[1]}_{1\nu(m)}(x_1)=\delta^{[3]}_m U^{[1]}_{1\nu(m)}(x_1)+W^{[1]\,\prime}_{1\nu(m)}(x_1)\,,$$
$$T^{[3]}_{1\nu(m)}(x_3)=U^{[3]\,\prime}_{1\nu(m)}(x_3)-\delta^{[1]}_m W^{[3]}_{1\nu(m)}(x_1)$$
$$T^{[1]}_{2\nu(m)}(x_1)=\delta^{[3]}_m U^{[1]}_{2\nu(m)}(x_1)+W^{[1]\,\prime}_{2\nu(m)}(x_1),$$
$$T^{[3]}_{2\nu(m)}(x_3)=U^{[3]\,\prime}_{2\nu(m)}(x_3)-\delta^{[1]}_m W^{[3]}_{2\nu(m)}(x_3)\,. \tag{15}$$

Then the functions determining the components of the strain tensor take the form:

$$\varepsilon_{11}=\sum_{m=1}^{\infty}\sum_{\nu=1}^{4}\Big\{R^{[1]}_{1\nu(m)}\Big[G^{[1]}_{1\nu(m)}(x_1)\sin(\delta^{[3]}_m x_3)\Big]+R^{[3]}_{1\nu(m)}\Big[G^{[3]}_{1\nu(m)}(x_3)\cos(\delta^{[1]}_m x_1)\Big]+$$
$$+R^{[1]}_{2\nu(m)}\Big[G^{[1]}_{2\nu(m)}(x_1)\cos(\delta^{[3]}_m x_3)\Big]+R^{[3]}_{2\nu(m)}\Big[G^{[3]}_{1\nu(m)}(x_3)\sin(\delta^{[1]}_m x_1)\Big]\Big\},$$
$$\varepsilon_{33}=\sum_{m=1}^{\infty}\sum_{\nu=1}^{4}\Big\{R^{[1]}_{1\nu(m)}\Big[H^{[1]}_{1\nu(m)}(x_1)\sin(\delta^{[3]}_m x_3)\Big]+R^{[3]}_{1\nu(m)}\Big[H^{[3]}_{1\nu(m)}(x_3)\cos(\delta^{[1]}_m x_1)\Big]+$$
$$+R^{[1]}_{2\nu(m)}\Big[H^{[1]}_{2\nu(m)}(x_1)\cos(\delta^{[1]}_m x_1)\Big]+R^{[3]}_{2\nu(m)}\Big[H^{[3]}_{2\nu(m)}(x_3)\sin(\delta^{[1]}_m x_1)\Big]\Big\}, \tag{16}$$

$$\gamma_{13}=\sum_{m=1}^{\infty}\sum_{\nu=1}^{4}\Big\{R^{[1]}_{1\nu(m)}\Big[T^{[1]}_{1\nu(m)}(x_1)\cos(\delta^{[3]}_m x_3)\Big]+R^{[3]}_{1\nu(m)}\Big[T^{[3]}_{1\nu(m)}(x_3)\sin(\delta^{[1]}_m x_1)\Big]+$$
$$+R^{[1]}_{2\nu(m)}\Big[\Big(T^{[1]}_{2\nu(m)}(x_1)\Big)\cos(\delta^{[3]}_m x_3)\Big]+R^{[3]}_{2\nu(m)}\Big[T^{[3]}_{2\nu(m)}(x_3)\sin(\delta^{[1]}_m x_1)\Big]\Big\}\,,$$

The components of the stress tensor follow from the constitutive relations:

$$\sigma_{ij}=b_{ijkl}\varepsilon_{kl} \tag{17}$$

Substituting the right-hand sides of (16) to (17), we obtain:

$$\sigma_{11}=\sum_{m=1}^{\infty}\sum_{\nu=1}^{4}\Big\{R^{[1]}_{1\nu(m)}\Big[X^{[1]}_{1\nu(m)}(x_1)\sin(\delta^{[3]}_m x_3)\Big]+R^{[3]}_{1\nu(m)}\Big[X^{[3]}_{1\nu(m)}(x_3)\cos(\delta^{[1]}_m x_1)\Big]+$$
$$+R^{[1]}_{2\nu(m)}\Big[X^{[1]}_{2\nu(m)}(x_1)\cos(\delta^{[3]}_m x_3)\Big]+R^{[3]}_{2\nu(m)}\Big[Y^{[3]}_{1\nu(m)}(x_3)\sin(\delta^{[1]}_m x_1)\Big]\Big\},$$

$$\sigma_{22}=\sum_{m=1}^{\infty}\sum_{\nu=1}^{4}\Big\{R_{1\nu(m)}^{[1]}\Big[Y_{1\nu(m)}^{[1]}(x_1)\sin(\delta_m^{[3]}x_3)\Big]+R_{1\nu(m)}^{[3]}\Big[Y_{1\nu(m)}^{[3]}(x_3)\cos(\delta_m^{[1]}x_1)\Big]+ \\ +R_{2\nu(m)}^{[1]}\Big[Y_{2\nu(m)}^{[1]}\big(x_1\big)\cos(\delta_m^{[3]}x_3)\Big]+R_{2\nu(m)}^{[3]}\Big[Y_{1\nu(m)}^{[3]}(x_3)\sin(\delta_m^{[1]}x_1)\Big]\Big\}, \tag{18}$$

$$\sigma_{13}=\sum_{m=1}^{\infty}\sum_{\nu=1}^{4}\Big\{R_{1\nu(m)}^{[1]}\Big[T_{1\nu(m)}^{[1]}(x_1)\cos(\delta_m^{[3]}x_3)\Big]+R_{1\nu(m)}^{[3]}\Big[T_{1\nu(m)}^{[3]}(x_3)\sin(\delta_m^{[1]}x_1)\Big]+ \\ +R_{2\nu(m)}^{[1]}\Big[\Big(T_{2\nu(m)}^{[1]}(x_1)\Big)\cos(\delta_m^{[3]}x_3)\Big]+R_{2\nu(m)}^{[3]}\Big[T_{2\nu(m)}^{[3]}(x_3)\sin(\delta_m^{[1]}x_1)\Big]\Big\}.$$

In the above the following notation has been introduced:

$$X_{p\nu(m)}^{[j]}=b_{1111}G_{p\nu(m)}^{[j]}+b_{1133}H_{p\nu(m)}^{[j]},\ Y_{p\nu(m)}^{[j]}=b_{3311}G_{p\nu(m)}^{[j]}+b_{3333}H_{p\nu(m)}^{[j]}, \\ Z_{p\nu(m)}^{[j]}=b_{1313}T_{p\nu(m)}^{[j]},\ \ p,j=1,3\,. \tag{19}$$

The unknown parameters $R_{\nu p(m)}^{[j]}$ should be determined from the boundary conditions imposed on the surface of the structural member under consideration.

3. EXEMPLARY COMPUTATIONS

Example 1.

A beam with the span $2a_1$ and height $2a_3$ is fixed at the ends and loaded at the upper face with a symmetrically distributed transverse force (rys.2). The load is given by the function:

$$q(x_1)=p\cos(\delta_1^{[1]}x_1) \tag{20}$$

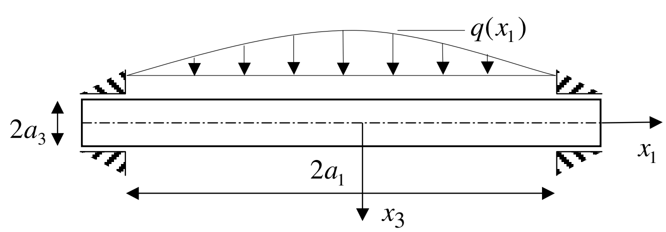

Fig. 2. Loads applied to the beam

With this load the axial displacement u_1 is antisymmetric and the deflection u_3 is symmetric with respekt to x_1. With respect to x_3 these displacements are non-symmetric. To satisfy these conditions, we have to assume $R^{[1]}_{2\nu(m)} = R^{[3]}_{2\nu(m)} = 0$ in (11). The following kinematic boundary conditions must be satisfied at the ends surfaces of the beam:

$$u_3\big|_{x_1=\pm a_1} = 0, \quad \left.\frac{\partial u_3}{\partial x_1}\right|_{x_1=\pm a_1} = 0. \tag{21}$$

Static boundary conditions at the upper and lower face of the beam take the following form:

$$\sigma_{33}\big|_{x_3=-a_3} = -q(x_1), \;\; \sigma_{31}\big|_{x_3=a_3} = 0, \;\; \sigma_{33}\big|_{x_3=a_3} = 0, \;\; \sigma_{31}\big|_{x_3=-a_3} = 0. \tag{22}$$

The following numerical values for the dimensions of the beam have been assumed: $a_1 = 3m$, $a_3 = 0{,}3m$. The load is determined by $p = 1\,Pa$ and the material constants are: $E_1 = 5{,}7 \cdot 10^{10}\,Pa$, $E_3 = 1{,}4 \cdot 10^{10}\,Pa$, $G_{13} = 0{,}57 \cdot 10^{10}\,Pa$, $\nu_{13} = 0{,}068$.
Fig. 3 shows distribution of the stress component σ_{33} at the surfaces $\xi_3 = x_3 / a_3 = \pm 1$. The curve 1 shows the stress distribution at the upper face of the beam, the curve 2 at its middle surface ($\xi_3 = 0$), and the curve 3 at the lower face. To obtain sufficient accuracy, 15 approximations has been applied in the process of the numerical computations.

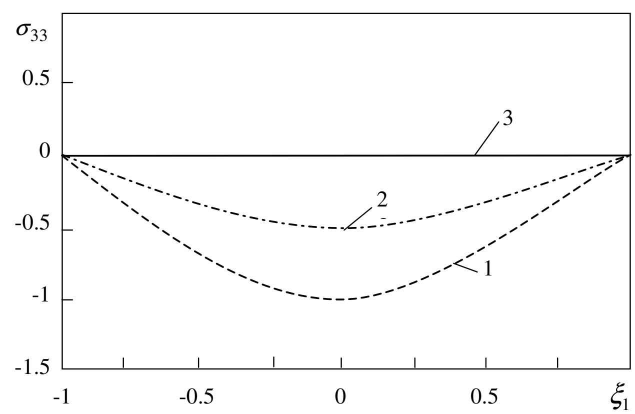

Fig.3. Stress distributions σ_{33} at $\xi_3 = \pm 1{,}0$

The graphs from fig.3 have been obtained by taking the mean value from the results of computations. The standard deviation of the stress σ_{33}, arising from the action of the load $q(x_1)$ on the upper face, does not exceed 10%. Larger peaks of σ_{33} are located at the corners; they are caused by clamping of the ends of the beam. The boundary conditions at the lower face are satisfied strictly. So are the conditions for the stress components σ_{31} at the surfaces $\xi_3 = \pm 1$ of the beam. Their maximal value does not exceed $5 \cdot 10^{-15}$ Pa.

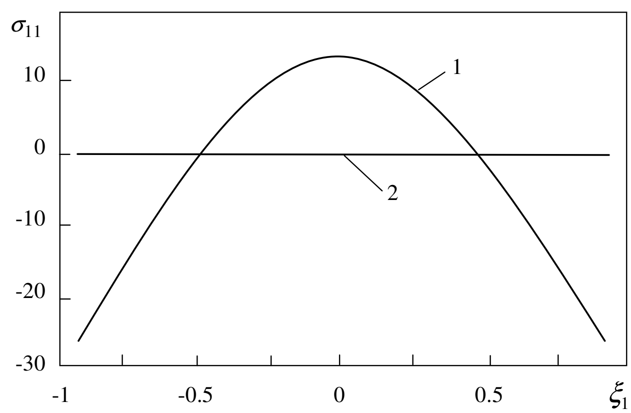

Fig.4. Stress distribution σ_{11} along the beam

Fig.4 shows stress distribution σ_{11} along the beam. The curve 1 shows the distribution at the lower face of the beam. The maximal value of the stress is located in the middle of the beam, whereas the maximal values with the opposite sign at its ends, where it is fixed. Graph 2 pertains to the axis of the beam.

Example 2.

Consider a disc subject to stretch along the Ox_1 axis (fig.5). The load is given by the function $p(x_3) = p \cos \delta_1^{[3]} x_3$.

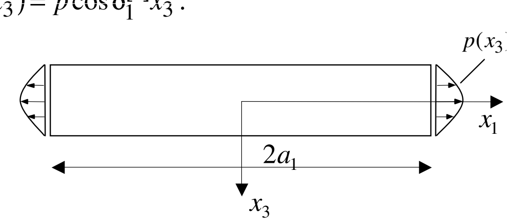

Fig.5. Loads applied to the disc.

The problem of the axial stretching of a rectangular orthotropic disc has been discussed in [3].
The origin of the coordinate system is located in the center of the disc and its axes coincide with the axes of symmetry of the disc. The boundary conditions take the form:

$$\sigma_{11}\big|_{x_1=\pm a_1} = p(x_3), \qquad \sigma_{13}\big|_{x_1=\pm a_1} = 0,$$

$$\sigma_{33}\big|_{x_3=\pm a_3} = 0, \quad \sigma_{31}\big|_{x_3=\pm a_3} = 0. \tag{23}$$

For the load given, the u_1 component of displacement is symmetric with respect to x_3 and antisymmetric with respect to x_1. The displacement u_3 is antisymmetric with respect to x_3 and symmetric with respect to x_1. These conditions will be satisfied if in (11) we put: $R^{[1]}_{1\nu(m)} = R^{[3]}_{2\nu(m)} = 0$ and $R^{[3]}_{1\nu(m)} = 0$ for $\nu = 1,2$ and $R^{[1]}_{2\nu(m)} = 0$ for $\nu = 3,4$.
To satisfy the boundary conditions at the surfaces of the disc Fourier series has been applied.
Fig.6 shows the graph of the σ_{11} stress at the longitudinal cross-section ($\xi_2 = 0$). The σ_{11} stresses diminish quickly with the increase of distance from the loaded surface and sufficiently far away assume constant values. The number of approximations had little impact on the values of the stresses in the central cross-section of the disc.

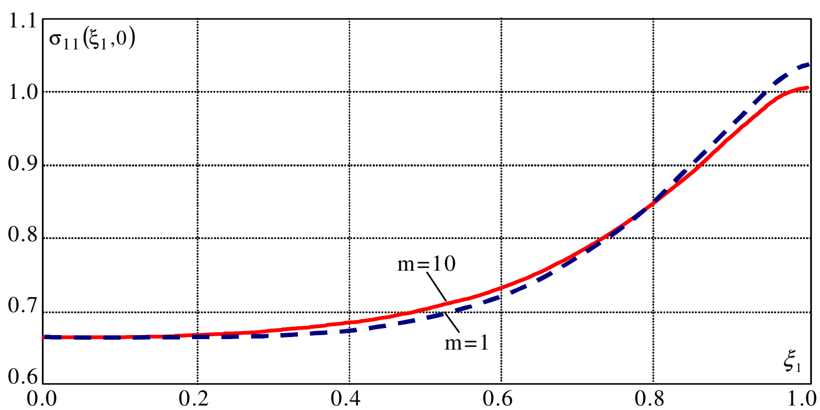

Fig.6. Stress distribution $\sigma_{11}(\xi_1, 0)$ in the central cross-section $(\xi_2 = 0)$ of the disc

Fig.7 shows the graphs of the stress distribution $\sigma_{11}(0, \xi_2)$ in the central cross-section $(\xi_1 = 0)$ for different values of the parameter m. The stresses σ_{11}

in the center of the disc are constant and for $m \geq 5$ their values do not depend on the number of approximations. The stresses σ_{22} have zero values at the edges of the disc (boundary conditions) and assume relatively small values in the interior. The tangent stresses on the surfaces of the disc diminish with the growing numer of iterations.

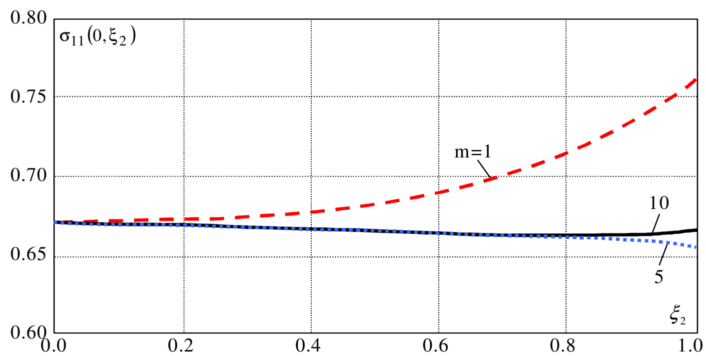

Fig.7. Stress distributions $\sigma_{11}(0,\xi_2)$ in the main cross-section of the disc

4. CONCLUSION

The analytic method of solving two-dimensional problems of the elasticity theory of an orthotropic body proposed in this paper is based on the three-dimensional theory. The paper covers the structural members of the beam and disc type. Beams are treated as strips cut out of a thick orthotropic plate subject to cylindrical bending. When the strip is stretched we face the membrane state. The solution obtained strictly satisfies the fundamental equations of the elasticity theory of a two-dimensional body and also fulfills the boundary conditions with high accuracy.

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METODA ROZWIĄZYWANIA PROSTOKĄTNYCH ORTOTROPOWYCH BELEK I TARCZ

S t r e s z c z e n i e

Opracowano metodę analityczną rozwiązywania dwuwymiarowych zagadnień teorii sprężystości ciała ortotropowego. Zaproponowano rozwiązanie dla elementów prostokątnych typu belka lub tarcza. Otrzymane rozwiązanie dokładnie spełnia wszystkie podstawowe równania teorii sprężystości ciała dwuwymiarowego i z dużą dokładnością spełnia warunki brzegowe.