GENERALIZED PRACTICAL STABILITY ANALYSIS OF DISCONTINUOUS DYNAMICAL SYSTEMS

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In practice, one is not only interested in the qualitative characterizations provided by the Lyapunov stability, but also in quantitative information concerning the system behavior, including estimates of trajectory bounds, possibly over finite time intervals. This type of information has been ascertained in the past in a systematic manner using the concept of practical stability. In the present paper, we give a new definition of generalized practical stability (abbreviated as GP-stability) and establish some sufficient conditions concerning GP-stability for a wide class of discontinuous dynamical systems. As in the classical Lyapunov theory, our results constitute a Direct Method, making use of auxiliary scalar-valued Lyapunov-like functions. These functions, however, have properties that differ significantly from the usual Lyapunov functions. We demonstrate the applicability of our results by means of several specific examples.

Keywords: discontinuous dynamical system, quantitative analysis, generalized practical stability (GP-stability), Lyapunov-like function

1. Introduction

It is well known that a discontinuous dynamical system can be regarded as a hybrid model that is composed of a family of continuous-time subsystems and a rule indicating which subsystem should be activated at a series of time instants. Recently, there has been increasing interest in a qualitative analysis of such systems. Most of this work (see, e.g., (DeCarlo *et al.*, 2000; Michel, 1999) and the references therein) is concerned with the stability of such systems in the Lyapunov sense.

In many problems of practical importance, one is not only interested in the qualitative information provided by Lyapunov stability results, but also in quantitative information concerning the system behavior, including estimates of trajectory bounds over a finite or an infinite time interval. For example, a system could be asymptotically stable in the Lyapunov sense, yet completely useless because of undesirable transient characteristics (e.g., it may exceed certain limits imposed on the trajectory bounds). On the other hand, a system which is unstable in the Lyapunov sense may exhibit dynamic behavior which is entirely acceptable over a specified finite time interval. Problems of this type have given rise to alternative notions of stability, called practical stability, and sometimes finite time stability. These stability concepts are phrased in terms of prespecified time intervals (finite or infinite) and in terms of prespecified subsets of the state space. As such, practical (or finite time) stability and the Lyapunov stability are distinct concepts, and, in general, neither implies the other. For some of the results concerning practical and finite time stability, refer, e.g., to (Lakshmikantham *et al.*, 1991; Michel, 1970; Michel and Porter, 1971; Weiss and Infante, 1967) and the references cited therein. Especially, the monograph (Lakshmikantham *et al.*, 1991) presented a systemetic study of the theory of practical stability. As in the case of the classical Lyapunov theory (see, e.g., (Michel *et al.*, 2000)), results of the type given in (Lakshmikantham *et al.*, 1991; Michel, 1970; Michel and Porter, 1971; Weiss and Infante, 1967) constitute a Direct Method, making use of auxiliary functions (or V-functions). We emphasize, however, that these V-functions (which we will call Lyapunov-like functions) have properties that differ significantly from the usual Lyapunov functions encountered in the classical Lyapunov theory.

Motivated by the practical considerations described above and addressed in (Lakshmikantham *et al.*, 1991; Michel, 1970; Michel and Porter, 1971; Weiss and Infante, 1967), in the recent paper (Zhai and Michel, 2002) we considered the practical stability for the discontinuous dynamical systems described by

$$\begin{cases} \dot{x}(t) = f_i\big(x(t), x(t_i), t\big) + u_i\big(x(t), t\big), \\ \qquad\qquad\qquad\qquad t_i \le t < t_{i+1}, \\ x(t) = g_i\big(x(t^-), x(t_i), t\big), \quad t = t_{i+1}, \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, t_0 is the initial time, $t_1, t_2, \dots, t_i, \dots \in I$ are discontinuity points and $I = [t_0, t_0 + T)$, where T is a specified (finite) constant or $T = \infty$. For all i, we assume that $f_i \in C^1[\mathbb{R}^n \times \mathbb{R}^n \times I, \mathbb{R}^n]$, $u_i \in C^1[\mathbb{R}^n \times I, \mathbb{R}^n]$ and $g_i \in C^1[\mathbb{R}^n \times \mathbb{R}^n \times I, \mathbb{R}^n]$, but, in general, we do not assume that $f_i(0,0,t) \equiv 0$, $u_i(0,t) \equiv 0$ and $g_i(0,0,t) \equiv 0$ so that stability with respect to a set, rather than a point, can be discussed. Clearly, the differential equation in (1) determines the dynamical behavior of the system over the indicated time intervals, where $u_i(x(t), t)$ denotes some persistent perturbing forces (resp., external inputs), while the second equation specifies the amount of the state jumps when discontinuities occur. In (Zhai and Michel, 2002), we used the concept of the practical stability proposed in (Lakshmikantham *et al.*, 1991; Michel, 1970; Michel and Porter, 1971; Weiss and Infante, 1967). More precisely, we established several sufficient conditions for the system (1) such that if the norm of the initial state is less than a positive scalar α, then the norm of the system state will never exceed a positive scalar β ($\beta \ge \alpha$) over a finite or an infinite time interval.

In this paper, we generalize the concept of the practical stability proposed in (Lakshmikantham *et al.*, 1991; Michel, 1970; Michel and Porter, 1971; Weiss and Infante, 1967). Rather than considering norm specification of the system state, we deal with two sets Ω_1 and Ω_2 satisfying $\Omega_1 \subset \Omega_2$, which are specified for the initial state and the entire system state, respectively. Ω_1 and Ω_2 do not have to include the origin, and they can be specified flexibly in real applications. Then, our generalized practical stability (GP-stability) requires that if the initial state is in Ω_1, then the system state should always stay in Ω_2. Obviously, GP-stability is a significant extension of practical stability. When there is no perturbation in (1), we will establish several sufficient conditions for some classes of the system (1) to be GP-stable, uniformly GP-stable and GP-unstable. When there is a perturbation in (1), we analyze the properties of the system (1) using the concept of totally GP-stable or totally uniformly GP-stable systems.

The remainder of this paper is organized as follows: In Section 2 we establish the notation used throughout the paper and provide the definitions of generalized practical stability. In Sections 3 and 4 we present results of various GP-stabilities for some classes of the system (1). We demonstrate the applicability of our results by considering some specific examples. The paper is concluded with some remarks in Section 5.

2. Preliminaries

Let $\mathbb{R}^n$ denote a real n-space and $\|\cdot\|$ a norm on $\mathbb{R}^n$. Set $I = [t_0, t_0 + T)$, where $t_0, T \in \mathbb{R}^+$ and T may be finite or infinite. A property is said to hold almost everywhere (abbreviated as a.e.) if the set of points where it fails is a set of measure zero.

Consider the continuous or discontinuous dynamical system represented by

$$\dot{x}(t) = f(x, t), \tag{2}$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \times I \to \mathbb{R}^n$. In general, it is assumed that $f(x,t)$ is measurable in a domain G of $\mathbb{R}^n \times I$, and for any closed bounded domain of $D \subset G$ it is assumed that there exists a summable function $M(t)$ such that almost everywhere in D we have $\|f(x,t)\| \le M(t)$. We do not require that $f(0,t) \equiv 0$.

An absolutely continuous function $x(\cdot, x_0, t_0) : [t_0, t_1) \to \mathbb{R}^n$ (where t_1 may be infinite), is a solution of (2) if it satisfies (2) a.e. on $[t_0, t_1)$ with $x(t_0, x_0, t_0) = x_0$. Throughout this paper, we assume that all conditions are satisfied such that for every $(x_0, t_0) \in \mathbb{R}^n \times I$ (2) possesses a unique solution $x(t, x_0, t_0)$ which exists a.e. on $[t_0, \infty)$.

Associated with (2), there is a system which is under the influence of persistent perturbing forces (resp., external inputs), i.e., the system which is represented by

$$\dot{x}(t) = f(x, t) + u(x, t), \tag{3}$$

where $u : \mathbb{R}^n \times I \to \mathbb{R}^n$. It is assumed that $u(x,t)$ is also measurable in a domain G of $\mathbb{R}^n \times I$, and for any closed bounded domain of $D \subset G$ it is assumed that there exists a summable function $N(t)$ such that almost everywhere in D we have $\|u(x,t)\| \le N(t)$. We do not require $u(0,t) \equiv 0$.

We define the solution of (3) similarly as for (2) and we assume that (3) possesses a unique solution $x(t, x_0, t_0)$ for every $(x_0, t_0) \in \mathbb{R}^n \times I$, which exists a.e. on $[t_0, \infty)$.

For a set $\Omega \in \mathbb{R}^n$, we use $\bar{\Omega}$ and Ω^c to denote the closure and the complement of Ω, respectively. For two sets $\Omega_1 \subset \Omega_2$, we use $\Omega_2 - \Omega_1$ to denote the set

$$[\Omega_2 - \Omega_1] = \{x \in \mathbb{R}^n : x \in \Omega_2,\ x \notin \Omega_1\}. \tag{4}$$

In the sequel, real-valued functions $V : \mathbb{R}^n \times I \to \mathbb{R}$ will be employed. If $V(x,t)$ possesses continuous first partial derivatives on $\mathbb{R}^n \times I$, for the case of the differential equations of the form (2) with continuous right-hand side, we can write the expression for the derivative $\dot{V}(x,t)$ along solutions $x(t)$ as

$$\dot{V}(x,t) = \big(\nabla V(x,t)\big)^T f(x,t) + \frac{\partial V}{\partial t}, \tag{5}$$

where ∇V denotes the gradient vector of V. Under the same conditions, for perturbed continuous systems of the form (3) we can write the expression

$$\dot{V}(x,t) = \dot{V}(x,t)\mid_{u\equiv 0} + \big(\nabla V(x,t)\big)^T u(x,t). \qquad (6)$$

For the case of piecewise continuous systems such as (1), the expressions (5) and (6) are valid almost everywhere.

We now give two definitions concerning the generalized practical stability of the systems (2) and (3). As in Section 1, we abbreviate "generalized practically stable" and "generalized practically unstable" as GP-stable and GP-unstable, respectively.

Definition 1. (GP-stability)

- System (2) is *GP-stable* with respect to $(\Omega_1, \Omega_2, t_0, T)$, $\Omega_1 \subset \Omega_2$, if $x(t_0) \in \Omega_1$ implies $x(t) \in \Omega_2$ for all $t \in I = [t_0, t_0 + T)$.
- System (2) is *uniformly GP-stable* with respect to (Ω_1, Ω_2, T), $\Omega_1 \subset \Omega_2$, if for each $t_i \in I$, $x(t_i) \in \Omega_1$ implies $x(t) \in \Omega_2$ for all $t \in [t_i, t_0 + T)$.
- System (2) is *GP-unstable* with respect to $(\Omega_1, \Omega_2, t_0, T)$, $\Omega_1 \subset \Omega_2$, if there exists an $x(t_0) \in \Omega_1$ and a $t_c \in (t_0, t_0 + T)$ such that $x(t_c) \notin \Omega_2 (\leftrightarrow x(t_c) \in \Omega_2^c)$.

Definition 2. (Total GP-stability)

- System (3) is *totally GP-stable* with respect to $(\Omega_1, \Omega_2, \epsilon, t_0, T, \|\cdot\|)$, $\Omega_1 \subset \Omega_2$, if the conditions (a) $x(t_0) \in \Omega_1$ and (b) $\|u(x,t)\| \le \epsilon$ a.e. $x \in \Omega_2$, $t \in I$, imply $x(t) \in \Omega_2$ for all $t \in I$.
- System (3) is *totally uniformly GP-stable* with respect to $(\Omega_1, \Omega_2, \epsilon, T, \|\cdot\|)$, $\Omega_1 \subset \Omega_2$, if for each $t_i \in I$, the conditions (a) $x(t_i) \in \Omega_1$ and (b) $\|u(x,t)\| \le \epsilon$ a.e. $x \in [\Omega_2 - \bar{\Omega}_1]$, $t \in I$, imply $x(t) \in \Omega_2$ for all $t \in [t_i, t_0 + T)$.

Remark 1. It is emphasized that the sets Ω_1 and Ω_2, the scalar ϵ and the norm $\|\cdot\|$ are all prespecified in a given problem. The set Ω_2 utilized in the above definitions yields a specific trajectory area for the system.

Remark 2. The system (2) is uniformly GP-stable with respect to (Ω_1, Ω_2, T), $\Omega_1 \subset \Omega_2$, if and only if it is GP-stable with respect to $(\Omega_1, \Omega_2, t_i, T)$ for each $t_i \in I$. System (3) is totally uniformly GP-stable with respect to $(\Omega_1, \Omega_2, \epsilon, T, \|\cdot\|)$, $\Omega_1 \subset \Omega_2$, if and only if it is totally GP-stable with respect to $(\Omega_1, \Omega_2, \epsilon, t_i, T, \|\cdot\|)$ for each $t_i \in I$.

Remark 3. A system which is Lyapunov-stable may be unstable in the sense of the above definitions, and vice versa.

3. Analysis for Unperturbed Systems

We first consider the discontinuous dynamical system described by

$$\begin{cases} \dot{x}(t) = f_i\big(x(t), t\big), & t_i \le t < t_{i+1}, \\ x(t) = g_i\big(x(t^-), t\big), & t = t_{i+1}, \end{cases} \qquad (7)$$

where $f_i \in C^1[\mathbb{R}^n \times I, \mathbb{R}^n]$ and $g_i \in C^1[\mathbb{R}^n \times I, \mathbb{R}^n]$. Obviously, this system is a special form of (1), where the discontinuities do not depend on the system state.

Theorem 1. *The system (7) is GP-stable with respect to $(\Omega_1, \Omega_2, t_0, T)$, $\Omega_1 \subset \Omega_2$, if there exist a real-valued Lyapunov-like function $V(x,t)$ satisfying local Lipschitz conditions in $\Omega_2 \times I$, a positive scalar μ and a function $\phi(t)$ which is Lebesgue-integrable on I, such that*

(i) $\dot{V}\big(x(t), t\big) \le \phi(t)$ *a.e.* $t \in I$, $x \in \Omega_2$,

(ii) $V\big(g_i(x,t), t\big) \le \mu V(x,t)$, $\forall i$, $\forall t \in I$, $\forall x \in \Omega_2$,

(iii) $\displaystyle\int_{t_0}^{t} \mu^{N(\tau,t)} \phi(\tau)\,\mathrm{d}\tau < \inf_{x\in\Omega_2^c} V(x,t) - \mu^{N(t_0,t)} \sup_{x\in\Omega_1} V(x,t_0)$, $\forall t \in I$,

where $N(a,b)$ denotes the number of discontinuities on the time interval $[a,b)$.

Proof. The proof is by contradiction. Let $x(t)$ be a solution of (7), with $x(t_0) \in \Omega_1$. Assume that there exists a $\bar{t} \in [t_0, t_0 + T)$, the first time such that $x(\bar{t}) \notin \Omega_2$. Let $t_1, \ldots, t_m$ denote the time instants where discontinuities occur before $\bar{t}$. Then, since $V(x,t)$ satisfies local Lipschitz conditions, we obtain

$$\begin{aligned} V\big(x(\bar{t}), \bar{t}\big) &= V\big(x(t_m), t_m\big) + \int_{t_m}^{\bar{t}} \dot{V}\big(x(\tau), \tau\big)\,\mathrm{d}\tau, \\ V\big(x(t_m^-), t_m^-\big) &= V\big(x(t_{m-1}), t_{m-1}\big) \\ &\quad + \int_{t_{m-1}}^{t_m} \dot{V}\big(x(\tau), \tau\big)\,\mathrm{d}\tau, \\ &\ \ \vdots \\ V\big(x(t_1^-), t_1^-\big) &= V\big(x(t_0), t_0\big) + \int_{t_0}^{t_1} \dot{V}\big(x(\tau), \tau\big)\,\mathrm{d}\tau. \end{aligned} \qquad (8)$$

According to the hypothesis (i), we have

$$\begin{aligned} V\big(x(\bar{t}), \bar{t}\big) &\le V\big(x(t_m), t_m\big) + \int_{t_m}^{\bar{t}} \phi(\tau)\,\mathrm{d}\tau \\ V\big(x(t_m^-), t_m^-\big) &\le V\big(x(t_{m-1}), t_{m-1}\big) \\ &\quad + \int_{t_{m-1}}^{t_m} \phi(\tau)\,\mathrm{d}\tau \\ &\ \ \vdots \\ V\big(x(t_1^-), t_1^-\big) &\le V\big(x(t_0), t_0\big) + \int_{t_0}^{t_1} \phi(\tau)\,\mathrm{d}\tau. \end{aligned} \qquad (9)$$

Hence, using the hypothesis (ii), we have $V(x(t_i),t_i) \le \mu V(x(t_i^-),t_i^-)$ for $i=1,\dots,m$, and obtain

$$\begin{aligned} V\big(x(\bar t),\bar t\big) &\le \mu^{N(t_0,\bar t)} V\big(x(t_0),t_0\big) + \int_{t_0}^{\bar t} \mu^{N(\tau,\bar t)}\phi(\tau)\,\mathrm{d}\tau \\ &\le \mu^{N(t_0,\bar t)} \sup_{x\in\Omega_1} V(x,t_0) \\ &\quad + \int_{t_0}^{\bar t} \mu^{N(\tau,\bar t)}\phi(\tau)\,\mathrm{d}\tau. \end{aligned} \tag{10}$$

In view of the hypothesis (iii), from (10) we obtain

$$V\big(x(\bar t),\bar t\big) < \inf_{x\in\Omega_2^c} V(x,\bar t), \tag{11}$$

which implies that $x(\bar t)\in\Omega_2^c$ is not true, which is a contradiction to the original assumption. Therefore, there does not exist a $\bar t\in[t_0,t_0+T)$ as asserted above, and thus $x(t)\in\Omega_2$ holds for every $t\in I$. This completes the proof. ■

The following two results address the uniform GP-stability and the GP-instability of the system (7):

Theorem 2. *The system (7) is uniformly GP-stable with respect to* (Ω_1,Ω_2,T), $\Omega_1\subset\Omega_2$, *if there exist a real-valued Lyapunov-like function* $V(x,t)$ *satisfying local Lipschitz conditions in* $[\Omega_2-\bar\Omega_1]\times I$, *a positive scalar* μ *and a function* $\phi(t)$ *which is Lebesgue integrable on* I, *such that*

(i) $\dot V\big(x(t),t\big)\le\phi(t)$ *a.e.* $t\in I$, $x\in[\Omega_2-\bar\Omega_1]$,

(ii) $V\big(g_i(x,t),t\big)\le\mu V(x,t)$ $\forall i$, $\forall t\in I$, $\forall x\in[\Omega_2-\bar\Omega_1]$,

(iii) $$\int_{t_1}^{t_2}\mu^{N(\tau,t_2)}\phi(\tau)\,\mathrm{d}\tau < \inf_{x\in\Omega_2^c}V(x,t_2)-\mu^{N(t_1,t_2)}\sup_{x\in\Omega_1}V(x,t_1), \quad \forall t_1,t_2\in I,\ t_2>t_1.$$

Theorem 3. *The system (7) is GP-unstable with respect to* $(\Omega_1,\Omega_2,t_0,T)$, $\Omega_1\subset\Omega_2$, *if there exist a real-valued Lyapunov-like function* $V(x,t)$ *satisfying local Lipschitz conditions in* $\Omega_2\times I$, *a* $\bar t\in(t_0,t_0+T)$, *an* $x_0\in\Omega_1$, *a solution* $x(t)$ *through the initial point* (t_0,x_0), *a positive scalar* μ *and a function* $\phi(t)$ *which is Lebesgue integrable on* I, *such that*

(i) $\dot V\big(x(t),t\big)\ge\phi(t)$ *a.e.* $t\in I$, $x\in\Omega_2$,

(ii) $V\big(g_i(x,t),t\big)\ge\mu V(x,t)$, $\forall i$, $\forall t\in[t_0,\bar t)$, $\forall x\in\Omega_2$,

(iii) $$\int_{t_0}^{\bar t}\mu^{N(\tau,\bar t)}\phi(\tau)\,\mathrm{d}\tau > \sup_{x\in\bar\Omega_2}V(x,\bar t)-\mu^{N(t_0,\bar t)}V(x_0,t_0).$$

The proofs of Theorems 2 and 3 are similar to that of Theorem 1, and are thus omitted.

Setting $\phi(t)=0$ in Theorems 1, 2 and 3 leads to the following results:

Corollary 1. *The system (7) is GP-stable with respect to* $(\Omega_1,\Omega_2,t_0,T)$, $\Omega_1\subset\Omega_2$, *if there exist a real-valued Lyapunov-like function* $V(x,t)$ *satisfying local Lipschitz conditions in* $\Omega_2\times I$ *and a positive scalar* μ *such that*

(i) $\dot V\big(x(t),t\big)\le 0$ *a.e.* $t\in I$, $x\in\Omega_2$,

(ii) $V\big(g_i(x,t),t\big)\le\mu V(x,t)$, $\forall i$, $\forall t\in I$, $\forall x\in\Omega_2$,

(iii) $\mu^{N(t_0,t)}\sup_{x\in\Omega_1}V(x,t_0)<\inf_{x\in\Omega_2^c}V(x,t)$, $\forall t\in I$.

Corollary 2. *The system (7) is uniformly GP-stable with respect to* (Ω_1,Ω_2,T), $\Omega_1\subset\Omega_2$, *if there exist a real-valued Lyapunov-like function* $V(x,t)$ *satisfying local Lipschitz conditions in* $[\Omega_2-\bar\Omega_1]\times I$ *and a positive scalar* μ *such that*

(i) $\dot V\big(x(t),t\big)\le 0$ *a.e.* $t\in I$, $x\in[\Omega_2-\bar\Omega_1]$,

(ii) $V\big(g_i(x,t),t\big)\le\mu V(x,t)$, $\forall i$, $\forall t\in I$, $\forall x\in[\Omega_2-\bar\Omega_1]$,

(iii) $\mu^{N(t_1,t_2)}\sup_{x\in\Omega_1}V(x,t_1) < \inf_{x\in\Omega_2^c}V(x,t_2)$, $\forall t_1,t_2\in I$, $t_2>t_1$.

Corollary 3. *The system (7) is GP-unstable with respect to* $(\Omega_1,\Omega_2,t_0,T)$, $\Omega_1\subset\Omega_2$, *if there exist a real-valued Lyapunov-like function* $V(x,t)$ *satisfying local Lipschitz conditions in* $\Omega_2\times I$, *a positive scalar* μ, *a* $\bar t\in(t_0,t_0+T)$, *an* $x_0\in\Omega_1$, *a solution* $x(t)$ *through the initial point* (t_0,x_0), *such that*

(i) $\dot V\big(x(t),t\big)\ge 0$ *a.e.* $t\in I$, $x\in\Omega_2$,

(ii) $V\big(g_i(x,t),t\big)\ge\mu V(x,t)$, $\forall i$, $\forall t\in[t_0,\bar t)$, $\forall x\in\Omega_2$,

(iii) $\mu^{N(t_0,\bar t)}V(x_0,t_0)>\sup_{x\in\bar\Omega_2}V(x,\bar t)$.

Remark 4. The real-valued Lyapunov-like V functions utilized in the above results are not Lyapunov functions in the usual sense since we do not require any particular definiteness conditions concerning these functions or their derivatives. We use the term “Lyapunov-like function” since, in much the same way as in the classical Lyapunov theory, these functions serve as auxiliary functions in a Direct Method.

We give two examples to demonstrate the above results.

Example 1. Consider the discontinuous dynamical system described by

$$\begin{cases} \dot{x}(t) = A_i(t)x(t), & t_i \le t < t_{i+1}, \\ x(t) = B_i(t)x(t^-), & t = t_{i+1}, \end{cases} \tag{12}$$

where $A_i(t), B_i(t) \in \mathbb{R}^{n\times n}$. Clearly, (12) is a special case of (7). Let $\|\cdot\|$ denote the Euclidean norm. Suppose that we deal with the sets

$$\begin{aligned} \Omega_1 &= \left\{x \in \mathbb{R}^n : x^T P x < \alpha^2\right\}, \\ \Omega_2 &= \left\{x \in \mathbb{R}^n : x^T P x < \beta^2\right\}, \end{aligned} \tag{13}$$

where α, β are two positive scalars satisfying $\alpha < \beta$, and P is a positive definite matrix.

1. First, we let $V(x,t) = \ln(x^T P x)$, $C_i(t) = \frac{1}{2}(P^{-\frac{1}{2}} A_i^T(t) P^{\frac{1}{2}} + P^{\frac{1}{2}} A_i(t) P^{-\frac{1}{2}})$ and we let $\Lambda_i(t)$ denote the maximum eigenvalue of $C_i(t)$. Then, on any time interval $[t_i, t_{i+1})$,

$$\begin{aligned} \dot{V}(x,t) &= \big(\nabla V(x)\big)^T \dot{x} \\ &= \frac{x^T(A_i^T P + P A_i)x}{x^T P x} \le 2\Lambda_i(t) \end{aligned} \tag{14}$$

holds for any $x \ne 0$. Hence, according to Theorem 1, the system (12) is GP-stable with respect to $(\Omega_1, \Omega_2, t_0, T)$ if $\|B_i(t)\| \le 1$ $(\mu = 1)$ for all i, and

$$\int_{t_0}^{t} \Lambda(\tau)\,\mathrm{d}\tau < \ln(\beta/\alpha), \quad \forall t \in [t_0, t_0 + T), \tag{15}$$

where $\Lambda(t) = \Lambda_i(t)$ when $t \in [t_i, t_{i+1})$. According to Theorem 2, the system (12) is uniformly GP-stable with respect to (Ω_1, Ω_2, T) if $\|B_i(t)\| \le 1$ for all i, and

$$\int_{t_1}^{t_2} \Lambda(\tau)\,\mathrm{d}\tau < \ln(\beta/\alpha), \quad \forall t_1, t_2 \in [t_0, t_0{+}T),\ t_2 > t_1. \tag{16}$$

2. Secondly, we let $V(x,t) = \sqrt{x^T P x}$. Then, on any time interval $[t_i, t_{i+1})$,

$$\dot{V}(x,t) = \big(\nabla V(x)\big)^T \dot{x} \le \Lambda_i(t) V(x,t). \tag{17}$$

If $\|B_i(t)\| \le 1$ holds for all i, we obtain

$$\begin{aligned} V(x,t) &\le V\big(x(t_m), t_m\big) \exp\left(\int_{t_m}^{t} \Lambda_m(\tau)\,\mathrm{d}\tau\right) \\ &\le V\big(x(t_{m-1}), t_{m-1}\big) \exp\left(\int_{t_{m-1}}^{t_m} \Lambda_{m-1}(\tau)\,\mathrm{d}\tau\right) \\ &\quad \times \exp\left(\int_{t_m}^{t} \Lambda_m(\tau)\,\mathrm{d}\tau\right) \\ &\le \cdots \le V\big(x(t_0), t_0\big) \exp\left(\int_{t_0}^{t} \Lambda(\tau)\,\mathrm{d}\tau\right). \end{aligned} \tag{18}$$

From this inequality, we also know that the system (12) is GP-stable with respect to $(\Omega_1, \Omega_2, t_0, T)$ if (15) is satisfied, and that system (12) is uniformly GP-stable with respect to (Ω_1, Ω_2, T) if (16) is fulfilled.

If $\|B_i(t)\| \le \mu$, $\mu > 1$ holds for all i, in a similar manner we obtain

$$\begin{aligned} &V\big(x(t), t\big) \\ &\le V\big(x(t_0), t_0\big) \mu^{2N(t_0,t)} \exp\left(\int_{t_0}^{t} \Lambda(\tau)\,\mathrm{d}\tau\right). \end{aligned} \tag{19}$$

Therefore, the system (12) is GP-stable with respect to $(\Omega_1, \Omega_2, t_0, T)$ if

$$N(t_0, t)\ln(\mu) + \int_{t_0}^{t} \Lambda(\tau)\,\mathrm{d}\tau < \ln(\beta/\alpha),\ \forall t \in [t_0, t_0{+}T), \tag{20}$$

and the system (12) is uniformly GP-stable with respect to (Ω_1, Ω_2, T) if

$$N(t_1, t_2)\ln(\mu) + \int_{t_1}^{t_2} \Lambda(\tau)\,\mathrm{d}\tau < \ln(\beta/\alpha) \tag{21}$$

for any $t_1, t_2 \in [t_0, t_0 + T)$, $t_2 > t_1$. As was also pointed out in (Zhai and Michel, 2002), we note here that the inequalities (20) and (21) are in fact the conditions on the average dwell time between discontinuities, and that the average dwell time approach was extensively discussed in the sense of the Lyapunov stability for switched systems in (Hespanha and Morse, 1999; Zhai *et al.*, 2000; 2001; 2002). ♦

Example 2. Consider the discontinuous dynamical system described by

$$\begin{cases} \dot{x}(t) = A_i x(t) + M_i x(t_i), & t_i \le t < t_{i+1}, \\ x(t) = B_i x(t^-), & t = t_{i+1}, \end{cases} \tag{22}$$

where A_i, M_i, $B_i \in \mathbb{R}^{n\times n}$. We deal with the same sets Ω_1 and Ω_2 as in Example 1, and assume that

$$\begin{gathered} \|P^{-\frac{1}{2}} A_i P^{-\frac{1}{2}}\| < \lambda, \quad \|P^{-\frac{1}{2}} M_i P^{-\frac{1}{2}}\| < \gamma, \\ \|B_i\| < \mu < 1. \end{gathered} \tag{23}$$

Let $V(x,t) = \sqrt{x^T P x}$. For any $t \in [t_0, t_0 + T)$, we assume that the discontinuous time instants on $[t_0, t)$ are $t_1, \ldots, t_m$. Hence

$$\begin{aligned} \dot{V}\big(x(t), t\big) &= \big(\nabla V(x,t)\big)^T \dot{x} \\ &\le \lambda V(x,t) + \gamma V\big(x(t_m), x(t_m)\big), \end{aligned} \tag{24}$$

and thus

$$V\big(x(t),t\big)\| \leq \Big[e^{\lambda(t-t_m)}+\gamma\int_{t_m}^{t} e^{\lambda(t-\tau)}\,\mathrm{d}\tau\Big] V\big(x(t_m),t_m\big) \leq (1+\gamma\lambda^{-1})e^{\lambda(t-t_m)} V\big(x(t_m),t_m\big). \quad (25)$$

Since $V(x(t_m),t_m) < \mu V(x(t_m^-),t_m^-)$, repeating the above computation, we obtain

$$V\big(x(t),t\big) \leq \big(\mu(1+\gamma\lambda^{-1})\big)^{N(t_0,t)}\, e^{\lambda(t-t_0)} V\big(x(t_0),t_0\big). \quad (26)$$

Therefore, the system (22) is GP-stable with respect to $(\Omega_1,\Omega_2,t_0,T)$ if

$$N(t_0,t)\ln\big(\mu(1+\gamma\lambda^{-1})\big)+\lambda(t-t_0) < \ln(\beta/\alpha) \quad (27)$$

for any $t \in [t_0, t_0+T)$, and the system (22) is uniformly GP-stable with respect to (Ω_1,Ω_2,T) if

$$N(t_1,t_2)\ln\big(\mu(1+\gamma\lambda^{-1})\big)+\lambda(t_2-t_1) < \ln(\beta/\alpha) \quad (28)$$

for any $t_1,t_2 \in [t_0,t_0+T)$, $t_2 > t_1$. Obviously, the conditions (27) and (28) yield also an average dwell time between discontinuities in (22). ♦

Remark 5. To show various kinds of the GP-stability of a discontinuous dynamical system, the key point is to find a Lyapunov-like function $V(x,t)$. However, even for a single nonlinear system with continuous right-hand side, the Lyapunov-like function candidate can take various forms, and it is not easy to establish a systemic way of calculating such $V(x,t)$. Examples 1 and 2 showed that the form of $\sqrt{x^TPx}$ or $\ln(x^TPx)$ with some positive-definite matrix P may be effective in many cases. For a more general form of the Lyapunov-like function candidate in the present case, we suggest the method proposed in (Lakshmikantham *et al.*, 1991), together with an average dwell time scheme which deals with the discontinuities. For example, in the case of Theorem 1, we may first use the methods in (Lakshmikantham *et al.*, 1991) to determine some Lyapunov function candidates satisfying the conditions (i) and (ii). Then we consider some average dwell time scheme, such as (21) or (28), to choose an appropriate one which satisfies furthermore the condition (iii). Examples 1 and 2 were analysed using this procedure.

4. Analysis for Perturbed Systems

In this section, we consider the discontinuous dynamical system under perturbing forces (resp., external inputs), described by

$$\begin{cases} \dot{x}(t) \;=\; f_i\big(x(t),t\big)+u_i\big(x(t),t\big), & t_i \leq t < t_{i+1}. \\ x(t) \;=\; g_i\big(x(t^-),t\big), & t = t_{i+1}, \end{cases} \quad (29)$$

where $u_i(x,t)$ is defined as in (1) and the notation is the same as in (7). It is assumed that $u_i(x,t)$ is measurable in a domain G of $\mathbb{R}^n \times I$, and for any closed bounded domain of $D \subset G$ it is assumed that there exists a summable function $N_i(t)$ such that almost everywhere in D we have $\|u_i(x,t)\| \leq N_i(t)$. $u_i(0,t)=0$.

Theorem 4. *System (29) is totally GP-stable with respect to* $(\Omega_1,\Omega_2,\epsilon,t_0,T,\|\cdot\|)$, $\Omega_1 \subset \Omega_2$, *if there exist a real-valued function* $V(x,t) \in C^1$, *a positive scalar* μ *and two integrable functions* $\phi(t),\eta(t)$ *on* I *such that*

(i) $\dot{V}\big(x(t),t\big)|_{u\equiv 0} \leq \phi(t)$ *a.e.* $t \in I$, $x \in \Omega_2$,

(ii) $\big\|\nabla V\big(x(t),t\big)\big\| \leq \eta(t)$ *a.e.* $t \in I$, $x \in \Omega_2$,

(iii) $V\big(g_i(x,t),t\big) \leq \mu V(x,t)$, $\forall i$, $\forall t \in I$, $\forall x \in \Omega_2$,

(iv) $\displaystyle\int_{t_0}^{t} \mu^{N(\tau,t)}\big(\phi(\tau)+\epsilon\eta(\tau)\big)\,\mathrm{d}\tau < \inf_{x\in\Omega_2^c} V(x,t) - \mu^{N(t_0,t)} \sup_{x\in\Omega_1} V(x,t_0)$, $\forall t \in I$.

Proof. The proof is by contradiction. Let $x(t)$ be a solution of (29), with $x(t_0) \in \Omega_1$. Assume that there exists a $\bar{t} \in [t_0,t_0+T)$, the first time such that $x(\bar{t}) \in \Omega_2^c$. Let $t_1,\ldots,t_m$ be the discontinuous time instants before $\bar{t}$. Then, similarly as in the proof of Theorem 1, we obtain

$$\begin{aligned} V\big(x(\bar{t}),\bar{t}\big) &\leq V\big(x(t_m),t_m\big)+\int_{t_m}^{\bar{t}} \Gamma(\tau)\,\mathrm{d}\tau, \\ V\big(x(t_m^-),t_m^-\big) &\leq V\big(x(t_{m-1}),t_{m-1}\big)+\int_{t_{m-1}}^{t_m} \Gamma(\tau)\,\mathrm{d}\tau, \\ &\vdots \\ V\big(x(t_1^-),t_1^-\big) &\leq V\big(x(t_0),t_0\big)+\int_{t_0}^{t_1} \Gamma(\tau)\,\mathrm{d}\tau, \end{aligned} \quad (30)$$

where $\Gamma(\tau)=\phi(\tau)+\epsilon\eta(\tau)$, along with the hypotheses (i) and (ii) used to estimate $\dot{V}(x,t)$. Hence, using the hypothesis (iii), we have $V(x(t_i),t_i) \leq \mu V(x(t_i^-),t_i^-)$ for $i=1,\ldots,m$, and we obtain

$$\begin{aligned} V\big(x(\bar{t}),\bar{t}\big) &\leq \mu^{N(t_0,\bar{t})} V\big(x(t_0),t_0\big)+\int_{t_0}^{\bar{t}} \mu^{N(\tau,\bar{t})}\Gamma(\tau)\,\mathrm{d}\tau \\ &\leq \mu^{N(t_0,\bar{t})} \sup_{x\in\Omega_1} V(x,t_0)+\int_{t_0}^{\bar{t}} \mu^{N(\tau,\bar{t})}\Gamma(\tau)\,\mathrm{d}\tau. \end{aligned} \quad (31)$$

Finally, in view of the hypothesis (iv), we can write

$$V\big(x(\bar{t}),\bar{t}\big) < \inf_{x\in\Omega_2^c} V(x,\bar{t}), \quad (32)$$

which implies that $x(\bar{t}) \in \Omega_2^c$ is not true, which is a contradiction to the original assumption. Therefore, there does not exist a $\bar{t} \in [t_0,t_0+T)$ as asserted above, and thus $x(t) \in \Omega_2$ holds for every $t \in I$. This completes the proof. ■

Theorem 5. *The system (29) is totally uniformly GP-stable with respect to* $(\Omega_1, \Omega_2, \epsilon, T, \|\cdot\|)$, $\Omega_1 \subset \Omega_2$, *if there exist a real-valued function* $V(x,t) \in C^1$, *a positive scalar* μ *and two integrable functions* $\phi(t), \eta(t)$ *on* I *such that*

(i) $\dot{V}\big(x(t),t\big)|_{u\equiv 0} \le \phi(t)$ *a.e.* $t \in I,\ x \in [\Omega_2 - \bar{\Omega}_1]$,

(ii) $\big\|\nabla V\big(x(t),t\big)\big\| \le \eta(t)$ *a.e.* $t \in I,\ x \in [\Omega_2 - \bar{\Omega}_1]$,

(iii) $V\big(g_i(x,t),t\big) \le \mu V(x,t),\ \forall i,\ \forall t \in I,\ \forall x \in [\Omega_2 - \bar{\Omega}_1]$.

(iv) $$\int_{t_1}^{t_2} \mu^{N(\tau,t_2)} \left(\phi(\tau) + \epsilon\eta(\tau)\right) \mathrm{d}\tau < \inf_{x\in\Omega_2^c} V(x,t_2) - \mu^{N(t_1,t_2)} \sup_{x\in\Omega_1} V(x,t_1), \quad \forall t_1, t_2 \in I,\ t_2 > t_1.$$

The proof of Theorem 5 is similar to that of Theorem 4, and is thus omitted.

Setting $\phi(t) = 0$ in Theorems 4 and 5 leads to the following results:

Corollary 4. *The system (29) is totally GP-stable with respect to* $(\Omega_1, \Omega_2, \epsilon, t_0, T, \|\cdot\|)$, $\Omega_1 \subset \Omega_2$, *if there exist a real-valued function* $V(x,t) \in C^1$, *a positive scalar* μ *and an integrable function* $\eta(t)$ *on* I *such that*

(i) $\dot{V}\big(x(t),t\big)|_{u\equiv 0} \le 0$ *a.e.* $t \in I,\ x \in \Omega_2$,

(ii) $\big\|\nabla V\big(x(t),t\big)\big\| \le \eta(t)$ *a.e.* $t \in I,\ x \in \Omega_2$,

(iii) $V\big(g_i(x,t),t\big) \le \mu V(x,t),\ \forall i,\ \forall t \in I,\ \forall x \in \Omega_2$,

(iv) $$\epsilon \int_{t_0}^{t} \mu^{N(\tau,t)}\eta(\tau)\,\mathrm{d}\tau + \mu^{N(t_0,t)} \sup_{x\in\Omega_1} V(x,t_0) < \inf_{x\in\Omega_2^c} V(x,t),\ \forall t \in I.$$

Corollary 5. *The system (29) is totally uniformly GP-stable with respect to* $(\Omega_1, \Omega_2, \epsilon, T, \|\cdot\|)$, $\Omega_1 \subset \Omega_2$, *if there exist a real-valued function* $V(x,t) \in C^1$, *a positive scalar* μ *and an integrable function* $\eta(t)$ *on* I *such that*

(i) $\dot{V}\big(x(t),t\big)|_{u\equiv 0} \le 0$ *a.e.* $t \in I,\ x \in [\Omega_2 - \bar{\Omega}_1]$,

(ii) $\big\|\nabla V\big(x(t),t\big)\big\| \le \eta(t)$ *a.e.* $t \in I,\ x \in [\Omega_2 - \bar{\Omega}_1]$,

(iii) $V\big(g_i(x,t),t\big) \le \mu V(x,t),\ \forall i,\ \forall t \in I,\ \forall x \in [\Omega_2 - \bar{\Omega}_1]$,

(iv) $$\epsilon \int_{t_1}^{t_2} \mu^{N(\tau,t_2)}\eta(\tau)\,\mathrm{d}\tau + \mu^{N(t_1,t_2)} \sup_{x\in\Omega_1} V(x,t_1) < \inf_{x\in\Omega_2^c} V(x,t_2),\ \forall t_1, t_2 \in I,\ t_2 > t_1.$$

Example 3. Consider the discontinuous dynamical system described by

$$\begin{cases} \dot{x}(t) \;=\; A_i(t)x(t) + u_i\big(x(t),t\big), & t_i \le t < t_{i+1}, \\ x(t) \;=\; B_i(t)x(t^-), & t = t_{i+1}, \end{cases} \tag{33}$$

where the notation is the same as in (12) except that $u_i(x(t),t)$ describes the perturbing forces. Let $\|\cdot\|$ denote the Euclidean norm.

As in Example 1, we deal with the sets Ω_1 and Ω_2 described in (13). Then, on any time interval $[t_i, t_{i+1})$ and $x \in [\Omega_2 - \bar{\Omega}_1]$,

$$\dot{V}(x,t) = \big(\nabla V(x)\big)^T \dot{x} = \frac{2x^T C_i x + 2x^T P u_i}{x^T P x} \le 2\Lambda_i(t) + 2\epsilon\zeta/\alpha, \qquad x \ne 0, \tag{34}$$

where ζ is the largest eigenvalue of $P^{\frac{1}{2}}$. Then, according to Theorem 5, the system (33) is uniformly totally GP-stable with respect to $(\Omega_1, \Omega_2, \epsilon, T, \|\cdot\|)$, if for any $t_1, t_2 \in I,\ t_2 > t_1$,

$$\int_{t_1}^{t_2} \mu^{N(\tau,t_2)} \left(\Lambda(\tau) + \epsilon\zeta\alpha^{-1}\right) \mathrm{d}\tau < \ln(\beta) - \mu^{N(t_1,t_2)} \ln(\alpha), \tag{35}$$

which degenerates to

$$\int_{t_1}^{t_2} \left(\Lambda(\tau) + \epsilon\zeta\alpha^{-1}\right) \mathrm{d}\tau < \ln(\beta/\alpha), \quad t_1, t_2 \in I,\ t_2 > t_1, \tag{36}$$

in the case of $\mu = 1$. ♦

Finally, we review Example 1 by dealing with two different sets for the system state trajectory. We do it here instead of immediately analysing it after Example 1 since the result turns out to be quite similar to the case where persistent perturbations exist.

Example 4. (Review of Example 1) Consider the system (12) with the following sets:

$$\begin{aligned} \tilde{\Omega}_1 &= \left\{x \in \mathbb{R}^n : (x-\Theta)^T P (x-\Theta) < \alpha^2\right\}, \\ \tilde{\Omega}_2 &= \left\{x \in \mathbb{R}^n : (x-\Theta)^T P (x-\Theta) < \beta^2\right\}, \end{aligned} \tag{37}$$

where $\Theta \in \mathbb{R}^n$ is a specified vector, and $0 < \alpha < \beta$.

We let $V(x,t) = \ln((x-\Theta)^T P (x-\Theta))$. Then, on any time interval $[t_i, t_{i+1})$,

$$\begin{aligned} \dot{V}(x,t) &= \big(\nabla V(x)\big)^T \dot{x} \\ &= \frac{(x-\Theta)^T P A_i x + x^T A_i^T(t) P (x-\Theta)}{(x-\Theta)^T P (x-\Theta)} \\ &\le 2\Lambda_i(t) + 2\psi_i(t)/\alpha, \quad x \ne \Theta, \end{aligned} \tag{38}$$

where $\psi_i(t) = \|P^{\frac{1}{2}} A_i(t)\Theta\|$. Then, according to Theorem 1, the system (12) is GP-stable with respect to $(\tilde{\Omega}_1, \tilde{\Omega}_2, t_0, T)$ if $\|B_i(t)\| \le 1$ ($\mu = 1$) for all i, and

$$\int_{t_0}^{t} \left(\Lambda(\tau) + \psi(\tau)\alpha^{-1}\right) \mathrm{d}\tau < \ln(\beta/\alpha), \quad \forall t \in [t_0, t_0 + T), \tag{39}$$

where $\psi(t) = \psi_i(t)$ when $t \in [t_i, t_{i+1})$. According to Theorem 2, the system (12) is uniformly GP-stable with respect to $(\tilde{\Omega}_1, \tilde{\Omega}_2, T)$ if $\|B_i(t)\| \le 1$ for all i, and

$$\int_{t_1}^{t_2} \left(\Lambda(\tau) + \psi(\tau)\alpha^{-1}\right) \mathrm{d}\tau < \ln(\beta/\alpha) \tag{40}$$

holds for any $t_1, t_2 \in [t_0, t_0 + T)$, $t_2 > t_1$. ♦

5. Conclusion

In the present paper we proposed a new concept of *generalized practical stability* and established sufficient conditions of various *GP-stabilities* for a wide class of discontinuous dynamical systems. We allowed for the case of systems subjected to persistent perturbing forces (resp., external inputs). Our results provide estimates of system trajectory areas. As in the classical Lyapunov theory, these results constitute a Direct Method, involving auxiliary scalar-valued Lyapunov-like functions. These functions, however, have properties that differ significantly from the usual Lyapunov functions. Some of our results turn out to be closely related to the existing results on switched systems which make use of the average dwell time approach. We demonstrated the applicability of the method advanced herein by means of several specific examples.

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