ON FINITE ELEMENT UNIQUENESS STUDIES FOR COULOMB'S FRICTIONAL CONTACT MODEL

PATRICK HILD*

* Laboratoire de Mathématiques, Université de Savoie, CNRS UMR 5127, 73376 Le Bourget-du-Lac, France,
e-mail: hild@univ-savoie.fr

We are interested in the finite element approximation of Coulomb's frictional unilateral contact problem in linear elasticity. Using a mixed finite element method and an appropriate regularization, it becomes possible to prove existence and uniqueness when the friction coefficient is less than $C\varepsilon^2|\log(h)|^{-1}$, where h and ε denote the discretization and regularization parameters, respectively. This bound converging very slowly towards 0 when h decreases (in comparison with the already known results of the non-regularized case) suggests a minor dependence of the mesh size on the uniqueness conditions, at least for practical engineering computations. Then we study the solutions of a simple finite element example in the non-regularized case. It can be shown that one, multiple or an infinity of solutions may occur and that, for a given loading, the number of solutions may eventually decrease when the friction coefficient increases.

Keywords: Coulomb's friction law, finite elements, mesh-size dependent uniqueness conditions, non-uniqueness example

1. Introduction and Problem Set-Up

Coulomb's friction model is currently chosen in the numerical approximation of contact problems arising in structural mechanics. From a mathematical point of view, the study of the continuous model in elastostatics using the associated variational formulation obtained in (Duvaut and Lions, 1972) leads to existence results when the friction coefficient is sufficiently small (Eck and Jarušek, 1998; Jarušek, 1983; Kato, 1987; Nečas *et al.*, 1980). As regards the associated finite element model, it was proved in (Haslinger, 1983; 1984) that it always admits a solution and that the solution is unique provided that the friction coefficient is lower than a positive value vanishing as the discretization parameter decreases. Also in (Haslinger, 1983), a convergence result of the finite element model towards the continuous model was established. Besides, in the finite dimensional context, numerous studies and examples of non-uniqueness using truss elements were exhibited, proving that the problem is in general not well posed (Alart, 1993; Janovský, 1981; Klarbring, 1990).

Our first aim in this paper is to study the influence of a specific regularization (i.e. the smoothing of the absolute value involved in the friction model) on the uniqueness conditions for the discrete problem. We consider a mixed finite element method in Section 2 and, denoting by h and ε the discretization and the regularization parameters, respectively, we show in Section 3 that the problem admits a unique solution if the friction coefficient is less than $C\varepsilon^2|\log(h)|^{-1}$, and we notice that a bound of only $Ch^{\frac{1}{2}}$ can be obtained in the case of the exact model (i.e. when $\varepsilon = 0$). As a consequence, we note that if ε is chosen as a parameter slowly decreasing towards zero (as h decreases), then the bound of the non-regularized case becomes more satisfactory than the one arising from the exact model.

Our second aim, in Section 4, is to choose a particular case of a finite dimensional problem in the non-regularized case: a simple example using finite elements. We study this problem and show that it may admit one, multiple or an infinity of solutions. Such an example completes and illustrates the already known results using truss elements, especially (Klarbring, 1990).

Let us now consider an elastic body occupying in the initial configuration a bounded subset $\overline{\Omega}$ of $\mathbb{R}^2$. The boundary $\partial\Omega$ of the domain Ω is supposed to be Lipschitz and consists of three non-overlapping parts Γ_D, Γ_N and Γ_C. The unit outward normal on $\partial\Omega$ is denoted by $\boldsymbol{n} = (n_1, n_2)$ and we set $\boldsymbol{t} = (n_2, -n_1)$. The body is submitted to volume forces $\boldsymbol{f} = (f_1, f_2) \in (L^2(\Omega))^2$ on Ω and to surface forces $\boldsymbol{F} = (F_1, F_2) \in (L^2(\Gamma_N))^2$ on Γ_N. The part Γ_D is embedded and we suppose that the surface measure of Γ_D does not vanish. Initially, the body is in contact with a rigid foundation on the straight line segment Γ_C.

The unilateral contact problem with Coulomb's friction consists in finding the displacement field $\boldsymbol{u} = (u_i)$, $1 \leq i \leq 2$ and the stress tensor field $\boldsymbol{\sigma} = (\sigma_{ij})$, $1 \leq i, j \leq 2$, satisfying the following conditions (1)–(4):

$$\operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}) + \boldsymbol{f} = \boldsymbol{0} \quad \text{in } \Omega, \quad \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \boldsymbol{F} \quad \text{on } \Gamma_N, \qquad \boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_D, \tag{1}$$

where $(\operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}))_i = \sigma_{ij,j}$, $1 \le i \le 2$, the notation $_{,j}$ denotes the j-th partial derivative and the summation convention of repeated indices is adopted. The stress tensor field is linked to the displacement field by the constitutive law of linear elasticity

$$\sigma_{ij}(\boldsymbol{u}) = \lambda \varepsilon_{kk}(\boldsymbol{u})\delta_{ij} + 2\mu\varepsilon_{ij}(\boldsymbol{u}), \tag{2}$$

where λ and μ are positive Lamé coefficients and $\varepsilon_{ij}(\boldsymbol{u}) = (1/2)(u_{i,j} + u_{j,i})$ denotes the linearized strain tensor field.

On the boundary $\partial\Omega$, we write $\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \sigma_n(\boldsymbol{u})\boldsymbol{n} + \sigma_t(\boldsymbol{u})\boldsymbol{t}$ and $\boldsymbol{u} = u_n\boldsymbol{n} + u_t\boldsymbol{t}$. Let $\mathcal{F} > 0$ stand for the friction coefficient on Γ_C. The conditions on the contact zone Γ_C are as follows:

$$u_n \le 0, \quad \sigma_n(\boldsymbol{u}) \le 0, \quad \sigma_n(\boldsymbol{u})\, u_n = 0, \tag{3}$$

$$|\sigma_t(\boldsymbol{u})| \le \mathcal{F}|\sigma_n(\boldsymbol{u})|, \quad \big(|\sigma_t(\boldsymbol{u})| - \mathcal{F}|\sigma_n(\boldsymbol{u})|\big)u_t = 0, \qquad \sigma_t(\boldsymbol{u})\, u_t \le 0. \tag{4}$$

Conditions (3) express unilateral contact and conditions (4) represent Coulomb's friction. The closed convex cone $\boldsymbol{K}$ of admissible displacements is a subset in the Sobolev space $(H^1(\Omega))^2$ of the displacement fields satisfying the embedding and the non-penetration conditions

$$\boldsymbol{K} = \Big\{\boldsymbol{v} = (v_1, v_2) \in \boldsymbol{V}, \quad v_n \le 0 \text{ on } \Gamma_C\Big\}, \tag{5}$$

where

$$\boldsymbol{V} = \Big\{\boldsymbol{v} = (v_1, v_2) \in \big(H^1(\Omega)\big)^2, \quad \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_D\Big\}.$$

As is done in (Nečas *et al.*, 1980), we consider the mapping $\Phi : M \to M$ with

$$M = \Big\{\alpha \in H^{-\frac{1}{2}}(\Gamma_C), \quad \alpha \ge 0\Big\},$$

defined for all $g \in M$ as $\Phi(g) = -\sigma_n(\boldsymbol{u}(g))$, where $\boldsymbol{u}(g) \in \boldsymbol{K}$ is the unique solution of the variational inequality

$$\begin{aligned} \boldsymbol{u}(g) \in \boldsymbol{K}, \quad & \int_\Omega \sigma_{ij}\big(\boldsymbol{u}(g)\big)\, \varepsilon_{ij}\big(\boldsymbol{v} - \boldsymbol{u}(g)\big)\, d\Omega \\ & + \langle \mathcal{F}g, |v_t| - |u_t(g)|\rangle_{\Gamma_C} \\ \ge & \int_\Omega f_i\big(v_i - u_i(g)\big)\, d\Omega \\ & + \int_{\Gamma_N} F_i\big(v_i - u_i(g)\big)\, d\Gamma, \ \forall \boldsymbol{v} \in \boldsymbol{K}, \end{aligned} \tag{6}$$

where $\langle \cdot, \cdot \rangle_{\Gamma_C}$ denotes the duality pairing between the fractional Sobolev space $H^{\frac{1}{2}}(\Gamma_C)$ (Adams, 1975) and its dual space $H^{-\frac{1}{2}}(\Gamma_C)$. Following (Nečas *et al.*, 1980; Haslinger *et al.*, 1996), a weak solution of the unilateral contact problem with Coulomb's friction is a pair $(\boldsymbol{u}, \gamma)$, where γ is a fixed point of Φ and $\boldsymbol{u}$ is the unique solution of the problem (6) with $g = \gamma$.

The first existence result for the unilateral contact problem with Coulomb's friction in the case of a sufficiently small friction coefficient $\mathcal{F}$ was proved in (Nečas *et al.*, 1980). Generalizations and/or improvements were established in (Eck and Jarušek, 1998; Jarušek, 1983; Kato, 1987). The uniqueness seems to remain an open problem.

2. The Discrete Problem

We discretize the domain Ω with a family of triangulations $(\mathscr{T}_h)_h$, where the notation $h > 0$ stands for the discretization parameter representing the greatest diameter of a triangle in $\mathscr{T}_h$. The chosen space of finite elements of degree one is

$$\boldsymbol{V}_h = \Big\{\boldsymbol{v}_h;\ \boldsymbol{v}_h \in \big(\mathscr{C}(\overline{\Omega})\big)^2,\ \boldsymbol{v}_h|_T \in (P_1(T))^2 \quad \forall T \in \mathscr{T}_h,\ \ \boldsymbol{v}_h = \boldsymbol{0} \text{ on } \Gamma_D\Big\},$$

where $\mathscr{C}(\overline{\Omega})$ and $P_1(T)$ denote the space of continuous functions on $\overline{\Omega}$ and the space of polynomial functions of degree one on T, respectively. We assume that the families of monodimensional traces of triangulations on Γ_C are quasi-uniform in order to use inverse inequalities (Ciarlet, 1991). Let W_h be the range of $\boldsymbol{V}_h$ by the normal trace operator on Γ_C:

$$W_h = \Big\{\mu_h;\ \mu_h = \boldsymbol{v}_h|_{\Gamma_C} \cdot \boldsymbol{n}, \quad \boldsymbol{v}_h \in \boldsymbol{V}_h\Big\}.$$

Clearly, the space W_h involves functions which are continuous and piecewise of degree one. We define M_h as the closed convex cone of Lagrange multipliers expressing non-negativity:

$$M_h = \Big\{\mu_h \in W_h, \quad \mu_h \ge 0\Big\}.$$

For any $\boldsymbol{u}$ and $\boldsymbol{v}$ in $(H^1(\Omega))^2$, define

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, d\Omega,$$

$$L(\boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, d\Omega + \int_{\Gamma_N} \boldsymbol{F} \cdot \boldsymbol{v}\, d\Gamma.$$

Finally, let us mention that we still keep the notation $\boldsymbol{v}_h = v_{hn}\boldsymbol{n} + v_{ht}\boldsymbol{t}$ on the boundary $\partial\Omega$, for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$.

To approximate Coulomb's frictional contact problem, we choose a mixed finite element method with a nonnegative parameter ε regularizing the absolute value (the case $\varepsilon = 0$ corresponds to the non-regularized problem). As in the continuous framework (6), the approximated problem requires the introduction of an intermediate setting with a given slip limit $g_h \in M_h$. It consists in *finding* $\boldsymbol{u}_h \in \boldsymbol{V}_h$ *and* $\lambda_h \in M_h$ *such that*

$$\left\{\begin{aligned} & a(\boldsymbol{u}_h, \boldsymbol{v}_h - \boldsymbol{u}_h) + \int_{\Gamma_C} \lambda_h (v_{hn} - u_{hn})\,\mathrm{d}\Gamma \\ & \quad + \int_{\Gamma_C} \mathcal{F} g_h \Big(\sqrt{v_{ht}^2 + \varepsilon^2} - \sqrt{u_{ht}^2 + \varepsilon^2}\Big)\,\mathrm{d}\Gamma \\ & \quad \geq L(\boldsymbol{v}_h - \boldsymbol{u}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ & \int_{\Gamma_C} (\mu_h - \lambda_h) u_{hn}\,\mathrm{d}\Gamma \leq 0, \quad \forall \mu_h \in M_h. \end{aligned}\right. \tag{7}$$

In what follows, the problem (7) will be denoted by $P_\varepsilon(g_h)$.

Remark 1. It can be checked that if $(\boldsymbol{u}_h, \lambda_h)$ solves (7), then $\boldsymbol{u}_h$ is also a solution of the variational inequality which consists in finding $\boldsymbol{u}_h \in \boldsymbol{K}_h$ satisfying

$$\begin{aligned} & a(\boldsymbol{u}_h, \boldsymbol{v}_h - \boldsymbol{u}_h) \\ & + \int_{\Gamma_C} \mathcal{F} g_h \Big(\sqrt{v_{ht}^2 + \varepsilon^2} - \sqrt{u_{ht}^2 + \varepsilon^2}\Big)\,\mathrm{d}\Gamma \geq L(\boldsymbol{v}_h - \boldsymbol{u}_h) \end{aligned}$$

for all $\boldsymbol{v}_h \in \boldsymbol{K}_h$. Here $\boldsymbol{K}_h$ stands for a finite dimensional approximation of $\boldsymbol{K}$ defined in (5):

$$\boldsymbol{K}_h = \Big\{\boldsymbol{v}_h \in \boldsymbol{V}_h, \quad \int_{\Gamma_C} \mu_h v_{hn}\,\mathrm{d}\Gamma \leq 0, \quad \forall \mu_h \in M_h\Big\}.$$

Problem $P_\varepsilon(g_h)$ is also equivalent to finding a saddle-point $(\boldsymbol{u}_h, \lambda_h) \in \boldsymbol{V}_h \times M_h$ satisfying

$$\mathscr{L}(\boldsymbol{u}_h, \mu_h) \leq \mathscr{L}(\boldsymbol{u}_h, \lambda_h) \leq \mathscr{L}(\boldsymbol{v}_h, \lambda_h), \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h,\ \forall \mu_h \in M_h,$$

where

$$\begin{aligned} \mathscr{L}(\boldsymbol{v}_h, \mu_h) = {} & \frac{1}{2} a(\boldsymbol{v}_h, \boldsymbol{v}_h) + \int_{\Gamma_C} \mu_h v_{hn}\,\mathrm{d}\Gamma \\ & + \int_{\Gamma_C} \mathcal{F} g_h \sqrt{v_{ht}^2 + \varepsilon^2}\,\mathrm{d}\Gamma - L(\boldsymbol{v}_h). \end{aligned}$$

From the results concerning saddle-point problems obtained in (Haslinger *et al.*, 1996), the existence of such a saddle-point follows. Moreover, the $\boldsymbol{V}$-ellipticity of $a(\cdot,\cdot)$ implies that the first argument $\boldsymbol{u}_h$ is unique. Besides, if for any $\mu_h \in W_h$ one has

$$\int_{\Gamma_C} \mu_h v_{hn}\,\mathrm{d}\Gamma = 0, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h \quad \Longrightarrow \mu_h = 0, \tag{8}$$

then the second argument λ_h is unique and $P_\varepsilon(g_h)$ admits a unique solution. Note that condition (8) is fulfilled because the space W_h coincides with the space obtained from $\boldsymbol{V}_h$ by the normal trace operator on Γ_C.

It becomes then possible to define two maps: the first one denoted by $\Psi_{\varepsilon h}$ yielding the first component (i.e. $\Psi_{\varepsilon h}(g_h) = \boldsymbol{u}_h$), and the other denoted by $\Phi_{\varepsilon h}$ such that

$$\Phi_{\varepsilon h} : \begin{array}{ccc} M_h & \longrightarrow & M_h, \\ g_h & \longmapsto & \lambda_h, \end{array}$$

where $(\boldsymbol{u}_h, \lambda_h)$ is the solution to $P_\varepsilon(g_h)$. The introduction of this map allows us to define a solution to Coulomb's discrete frictional contact problem.

Definition 1. A solution to Coulomb's discrete regularized (resp. non-regularized) frictional contact problem is a solution to $P_\varepsilon(\lambda_h)$ with $\varepsilon > 0$ (resp. $\varepsilon = 0$), where $\lambda_h \in M_h$ is a fixed point of $\Phi_{\varepsilon h}$.

Set

$$\tilde{\boldsymbol{V}}_h = \Big\{\boldsymbol{v}_h \in \boldsymbol{V}_h,\ v_{ht} = 0 \ \text{ on } \ \Gamma_C\Big\}.$$

It is easy to check that the definition of $\|\cdot\|_{-\frac{1}{2},h}$ given by

$$\|\nu\|_{-\frac{1}{2},h} = \sup_{\boldsymbol{v}_h \in \tilde{\boldsymbol{V}}_h} \frac{\displaystyle\int_{\Gamma_C} \nu v_{hn}\,\mathrm{d}\Gamma}{\|\boldsymbol{v}_h\|_1} \tag{9}$$

is a norm on W_h (since the condition (8) holds). The notation $\|\cdot\|_1$ represents the $(H^1(\Omega))^2$-norm.

3. Existence and Uniqueness Studies

We are now interested in the existence and uniqueness study for the discrete problem. In order to establish the existence, it suffices to show that the mapping $\Phi_{\varepsilon h}$ admits a fixed point in M_h by using Brouwer's theorem. The uniqueness is ensured if the mapping is contractive. Such a technique was already used in the non-regularized case with discontinuous and piecewise constant Lagrange multipliers (Haslinger, 1983; 1984). Our aim is to study the regularized case (and also the non-regularized one) when using Lagrange multipliers which are piecewise continuous of degree one.

Theorem 1. *Let* $\varepsilon > 0$. *The following results hold:*

(Existence) For any positive $\mathcal{F}$*, there exists a solution to Coulomb's discrete regularized frictional contact problem.*

(Uniqueness) Assume that $\overline{\Gamma}_D \cap \overline{\Gamma}_C = \emptyset$. *If* $\mathcal{F} \leq C\varepsilon^2 |\log(h)|^{-1}$*, then the problem admits a unique solution. The positive constant* C *depends on neither* h *nor* ε.

Proof. Let $(\boldsymbol{u}_h, \lambda_h)$ be the solution to $P_\varepsilon(g_h)$. Taking $\boldsymbol{v}_h = \boldsymbol{0}$ in (7) gives

$$a(\boldsymbol{u}_h, \boldsymbol{u}_h) + \int_{\Gamma_C} \lambda_h u_{hn} \, d\Gamma - \int_{\Gamma_C} \mathcal{F} g_h \Big(\varepsilon - \sqrt{u_{ht}^2 + \varepsilon^2}\Big) \, d\Gamma \le L(\boldsymbol{u}_h). \quad (10)$$

Since $g_h \ge 0$, $\varepsilon - \sqrt{u_{ht}^2 + \varepsilon^2} \le 0$, and according to

$$\int_{\Gamma_C} \lambda_h u_{hn} \, d\Gamma = 0,$$

it follows from (10), the $\boldsymbol{V}$-ellipticity of $a(\cdot,\cdot)$ and the continuity of $L(\cdot)$ that

$$\alpha \|\boldsymbol{u}_h\|_1^2 \le a(\boldsymbol{u}_h, \boldsymbol{u}_h) \le L(\boldsymbol{u}_h) \le C \|\boldsymbol{u}_h\|_1,$$

where α stands for the ellipticity constant of $a(\cdot,\cdot)$. Here, the constant C depends on the external loads $\boldsymbol{f}$ and $\boldsymbol{F}$. Therefore using the trace theorem yields

$$\|u_{ht}\|_{H^{\frac{1}{2}}(\Gamma_C)} \le C' \|\boldsymbol{u}_h\|_1 \le \frac{CC'}{\alpha}. \quad (11)$$

Besides, the equality in (7) implies

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) + \int_{\Gamma_C} \lambda_h v_{hn} \, d\Gamma = L(\boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \tilde{\boldsymbol{V}}_h.$$

Denoting by M' the continuity constant of $a(\cdot,\cdot)$ yields

$$\int_{\Gamma_C} \lambda_h v_{hn} \, d\Gamma \le M' \|\boldsymbol{u}_h\|_1 \, \|\boldsymbol{v}_h\|_1 + C \|\boldsymbol{v}_h\|_1, \quad \forall \boldsymbol{v}_h \in \tilde{\boldsymbol{V}}_h.$$

As a result,

$$\|\lambda_h\|_{-\frac{1}{2},h} \le M' \|\boldsymbol{u}_h\|_1 + C \le \Big(\frac{M'}{\alpha} + 1\Big) C.$$

So, we conclude that

$$\|\Phi_{\varepsilon h}(g_h)\|_{-\frac{1}{2},h} \le C', \quad \forall g_h \in M_h, \quad (12)$$

where C' only depends on the applied loads $\boldsymbol{f}, \boldsymbol{F}$, and on the continuity and ellipticity constants of $a(\cdot,\cdot)$.

The existence result of Theorem 1 consists now in showing that the mapping $\Phi_{\varepsilon h}$ is continuous.

Let $(\boldsymbol{u}_h, \lambda_h)$ and $(\overline{\boldsymbol{u}}_h, \overline{\lambda}_h)$ be the solutions to $P_\varepsilon(g_h)$ and $P_\varepsilon(\overline{g}_h)$, respectively (where $g_h \in M_h$ and $\overline{g}_h \in M_h$). From (7), we get

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) + \int_{\Gamma_C} \lambda_h v_{hn} \, d\Gamma = L(\boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \tilde{\boldsymbol{V}}_h,$$

and

$$a(\overline{\boldsymbol{u}}_h, \boldsymbol{v}_h) + \int_{\Gamma_C} \overline{\lambda}_h v_{hn} \, d\Gamma = L(\boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \tilde{\boldsymbol{V}}_h,$$

which implies by subtraction that

$$\int_{\Gamma_C} (\lambda_h - \overline{\lambda}_h) v_{hn} \, d\Gamma = a(\overline{\boldsymbol{u}}_h - \boldsymbol{u}_h, \boldsymbol{v}_h) \le M' \|\boldsymbol{u}_h - \overline{\boldsymbol{u}}_h\|_1 \, \|\boldsymbol{v}_h\|_1, \quad \forall \boldsymbol{v}_h \in \tilde{\boldsymbol{V}}_h,$$

where the continuity of the bilinear form $a(\cdot,\cdot)$ was used. So we get the following estimate:

$$\|\lambda_h - \overline{\lambda}_h\|_{-\frac{1}{2},h} \le M' \|\boldsymbol{u}_h - \overline{\boldsymbol{u}}_h\|_1. \quad (13)$$

Next, we show that $\Psi_{\varepsilon h}$ is continuous from M_h into $\boldsymbol{V}_h$. We consider again $(\boldsymbol{u}_h, \lambda_h)$ and $(\overline{\boldsymbol{u}}_h, \overline{\lambda}_h)$, the solutions to $P_\varepsilon(g_h)$ and $P_\varepsilon(\overline{g}_h)$, respectively. We have

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h - \boldsymbol{u}_h) + \int_{\Gamma_C} \lambda_h (v_{hn} - u_{hn}) \, d\Gamma + \int_{\Gamma_C} \mathcal{F} g_h \Big(\sqrt{v_{ht}^2 + \varepsilon^2} - \sqrt{u_{ht}^2 + \varepsilon^2}\Big) \, d\Gamma \ge L(\boldsymbol{v}_h - \boldsymbol{u}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h,$$

and

$$a(\overline{\boldsymbol{u}}_h, \boldsymbol{v}_h - \overline{\boldsymbol{u}}_h) + \int_{\Gamma_C} \overline{\lambda}_h (v_{hn} - \overline{u}_{hn}) \, d\Gamma + \int_{\Gamma_C} \mathcal{F} \, \overline{g}_h \Big(\sqrt{v_{ht}^2 + \varepsilon^2} - \sqrt{\overline{u}_{ht}^2 + \varepsilon^2}\Big) \, d\Gamma \ge L(\boldsymbol{v}_h - \overline{\boldsymbol{u}}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h.$$

Choosing $\boldsymbol{v}_h = \overline{\boldsymbol{u}}_h$ in the first inequality and $\boldsymbol{v}_h = \boldsymbol{u}_h$ in the second one, from (7) we obtain

$$a(\boldsymbol{u}_h, \overline{\boldsymbol{u}}_h - \boldsymbol{u}_h) + \int_{\Gamma_C} \mathcal{F} g_h \Big(\sqrt{\overline{u}_{ht}^2 + \varepsilon^2} - \sqrt{u_{ht}^2 + \varepsilon^2}\Big) \, d\Gamma \ge L(\overline{\boldsymbol{u}}_h - \boldsymbol{u}_h)$$

and

$$a(\overline{\boldsymbol{u}}_h, \boldsymbol{u}_h - \overline{\boldsymbol{u}}_h) + \int_{\Gamma_C} \mathcal{F} \overline{g}_h \Big(\sqrt{u_{ht}^2 + \varepsilon^2} - \sqrt{\overline{u}_{ht}^2 + \varepsilon^2}\Big) \, d\Gamma \ge L(\boldsymbol{u}_h - \overline{\boldsymbol{u}}_h).$$

Thus

$$a(\boldsymbol{u}_h - \overline{\boldsymbol{u}}_h, \boldsymbol{u}_h - \overline{\boldsymbol{u}}_h) \le \int_{\Gamma_C} \mathcal{F} (g_h - \overline{g}_h) \Big(\sqrt{\overline{u}_{ht}^2 + \varepsilon^2} - \sqrt{u_{ht}^2 + \varepsilon^2}\Big) \, d\Gamma. \quad (14)$$

Consequently,

$$\alpha \|\boldsymbol{u}_h - \overline{\boldsymbol{u}}_h\|_1^2 \le \mathcal{F} \|g_h - \overline{g}_h\|_{H^{-\frac{1}{2}}(\Gamma_C)} \times \Big\| \sqrt{\overline{u}_{ht}^2 + \varepsilon^2} - \sqrt{u_{ht}^2 + \varepsilon^2} \Big\|_{H^{\frac{1}{2}}(\Gamma_C)}. \quad (15)$$

The next step consists in estimating the $H^{\frac{1}{2}}$-norm term in (15). To attain our ends, we need to use two lemmas:

Lemma 1. *There exists a positive constant C satisfying*

$$\|fg\|_{H^{\frac{1}{2}}(\Gamma_C)} \leq C\Big(\|f\|_{H^{\frac{1}{2}}(\Gamma_C)}\|g\|_{L^\infty(\Gamma_C)} + \|f\|_{L^\infty(\Gamma_C)}\|g\|_{H^{\frac{1}{2}}(\Gamma_C)}\Big). \quad (16)$$

for all f and g in $H^{\frac{1}{2}}(\Gamma_C) \cap L^\infty(\Gamma_C)$.

Proof. From the definition of the $H^{\frac{1}{2}}(\Gamma_C)$-norm (Adams, 1975), we have

$$\|fg\|^2_{H^{\frac{1}{2}}(\Gamma_C)} = \|fg\|^2_{L^2(\Gamma_C)} + \int_{\Gamma_C}\int_{\Gamma_C} \frac{\big(f(x)g(x) - f(y)g(y)\big)^2}{(x-y)^2}\,\mathrm{d}\Gamma\,\mathrm{d}\Gamma.$$

Let us begin with bounding (roughly) the first term:

$$\|fg\|^2_{L^2(\Gamma_C)} = \int_{\Gamma_C} f^2(x)g^2(x)\,\mathrm{d}\Gamma \leq \|f\|^2_{L^2(\Gamma_C)}\|g\|^2_{L^\infty(\Gamma_C)}. \quad (17)$$

The second term is handled as follows:

$$\int_{\Gamma_C}\int_{\Gamma_C} \frac{(f(x)(g(x)-g(y))+g(y)(f(x)-f(y)))^2}{(x-y)^2}\,\mathrm{d}\Gamma\,\mathrm{d}\Gamma \leq 2\int_{\Gamma_C}\int_{\Gamma_C} \frac{f^2(x)(g(x)-g(y))^2}{(x-y)^2} + \frac{g^2(y)\big(f(x)-f(y)\big)^2}{(x-y)^2}\,\mathrm{d}\Gamma\,\mathrm{d}\Gamma \quad (18)$$
$$\leq 2\Big(\|f\|^2_{L^\infty(\Gamma_C)}\|g\|^2_{H^{\frac{1}{2}}(\Gamma_C)} + \|f\|^2_{H^{\frac{1}{2}}(\Gamma_C)}\|g\|^2_{L^\infty(\Gamma_C)}\Big).$$

Putting together (17) and (18) establishes (16). ■

Lemma 2. *For any real number $p \in [1,\infty[$, the following inequality holds:*

$$\|f\|_{L^p(\Gamma_C)} \leq C\sqrt{p}\|f\|_{H^{\frac{1}{2}}(\Gamma_C)}, \quad \forall f \in H^{\frac{1}{2}}(\Gamma_C), \quad (19)$$

where C is independent of p.

Proof. see (Ben Belgacem, 2000). ■

Proof of Theorem 1 (continued). We consider the $H^{\frac{1}{2}}$-norm term in (15). Employing the estimate (16) gives

$$\Big\|\sqrt{\overline{u}^2_{ht}+\varepsilon^2} - \sqrt{u^2_{ht}+\varepsilon^2}\Big\|_{H^{\frac{1}{2}}(\Gamma_C)} = \Big\|(u_{ht}-\overline{u}_{ht})\frac{u_{ht}+\overline{u}_{ht}}{\sqrt{\overline{u}^2_{ht}+\varepsilon^2}+\sqrt{u^2_{ht}+\varepsilon^2}}\Big\|_{H^{\frac{1}{2}}(\Gamma_C)}$$
$$\leq C\|u_{ht}-\overline{u}_{ht}\|_{L^\infty(\Gamma_C)} \times \Big\|\frac{u_{ht}+\overline{u}_{ht}}{\sqrt{\overline{u}^2_{ht}+\varepsilon^2}+\sqrt{u^2_{ht}+\varepsilon^2}}\Big\|_{H^{\frac{1}{2}}(\Gamma_C)} + C\|u_{ht}-\overline{u}_{ht}\|_{H^{\frac{1}{2}}(\Gamma_C)} \times \Big\|\frac{u_{ht}+\overline{u}_{ht}}{\sqrt{\overline{u}^2_{ht}+\varepsilon^2}+\sqrt{u^2_{ht}+\varepsilon^2}}\Big\|_{L^\infty(\Gamma_C)}. \quad (20)$$

In the previous estimate, we leave the third term unchanged whereas the last one is bounded by 1. It remains then to bound the first two terms, which is performed hereafter. We begin with the first one:

$$\|u_{ht}-\overline{u}_{ht}\|_{L^\infty(\Gamma_C)} \leq Ch^{-\frac{1}{p}}\|u_{ht}-\overline{u}_{ht}\|_{L^p(\Gamma_C)} \leq C\sqrt{p}h^{-\frac{1}{p}}\|u_{ht}-\overline{u}_{ht}\|_{H^{\frac{1}{2}}(\Gamma_C)}, \quad (21)$$

for any $p \in [1,\infty[$. In (21), we used an easily recoverable inverse inequality (Ciarlet, 1991), as well as (19). The second term of (20) is bounded due to (16):

$$\Big\|\frac{u_{ht}+\overline{u}_{ht}}{\sqrt{\overline{u}^2_{ht}+\varepsilon^2}+\sqrt{u^2_{ht}+\varepsilon^2}}\Big\|_{H^{\frac{1}{2}}(\Gamma_C)} \leq C\|u_{ht}+\overline{u}_{ht}\|_{L^\infty(\Gamma_C)} \times \Big\|\frac{1}{\sqrt{\overline{u}^2_{ht}+\varepsilon^2}+\sqrt{u^2_{ht}+\varepsilon^2}}\Big\|_{H^{\frac{1}{2}}(\Gamma_C)} + C\|u_{ht}+\overline{u}_{ht}\|_{H^{\frac{1}{2}}(\Gamma_C)} \times \Big\|\frac{1}{\sqrt{\overline{u}^2_{ht}+\varepsilon^2}+\sqrt{u^2_{ht}+\varepsilon^2}}\Big\|_{L^\infty(\Gamma_C)}$$
$$\leq C\sqrt{p}h^{-\frac{1}{p}}\|u_{ht}+\overline{u}_{ht}\|_{H^{\frac{1}{2}}(\Gamma_C)} \times \Big\|\frac{1}{\sqrt{\overline{u}^2_{ht}+\varepsilon^2}+\sqrt{u^2_{ht}+\varepsilon^2}}\Big\|_{H^{\frac{1}{2}}(\Gamma_C)} + \frac{1}{2\varepsilon}\|u_{ht}+\overline{u}_{ht}\|_{H^{\frac{1}{2}}(\Gamma_C)}, \quad (22)$$

where the first L^∞-norm term is bounded as in (21), whereas the other is roughly bounded by $1/2\varepsilon$. Next, we develop the first $H^{\frac{1}{2}}$-norm term in (22):

$$\begin{aligned}&\left\|\frac{1}{\sqrt{\overline{u}_{ht}^2+\varepsilon^2}+\sqrt{u_{ht}^2+\varepsilon^2}}\right\|^2_{H^{\frac{1}{2}}(\Gamma_C)}\\ &\quad=\left\|\frac{1}{\sqrt{\overline{u}_{ht}^2+\varepsilon^2}+\sqrt{u_{ht}^2+\varepsilon^2}}\right\|^2_{L^2(\Gamma_C)}\\ &\qquad+\int_{\Gamma_C}\int_{\Gamma_C}\frac{1}{(y-x)^2}\\ &\qquad\times\Bigg(\frac{1}{\sqrt{\overline{u}_{ht}^2(x)+\varepsilon^2}+\sqrt{u_{ht}^2(x)+\varepsilon^2}}\\ &\qquad-\frac{1}{\sqrt{\overline{u}_{ht}^2(y)+\varepsilon^2}+\sqrt{u_{ht}^2(y)+\varepsilon^2}}\Bigg)^2\,\mathrm{d}\Gamma\,\mathrm{d}\Gamma.\end{aligned}$$

It is easy to check that the L^2-norm term is less than $\mathrm{meas}(\Gamma_C)/4\varepsilon^2$. Developing the previous integral, bounding then the denominator and using the estimate $(a+b)^2\le 2a^2+2b^2$ furnishes the following upper bound:

$$\begin{aligned}&\frac{1}{8\varepsilon^4}\int_{\Gamma_C}\int_{\Gamma_C}\frac{\left(\sqrt{\overline{u}_{ht}^2(x)+\varepsilon^2}-\sqrt{\overline{u}_{ht}^2(y)+\varepsilon^2}\right)^2}{(y-x)^2}\\ &\qquad+\frac{\left(\sqrt{u_{ht}^2(x)+\varepsilon^2}-\sqrt{u_{ht}^2(y)+\varepsilon^2}\right)^2}{(y-x)^2}\,\mathrm{d}\Gamma\,\mathrm{d}\Gamma.\end{aligned}$$

We use the estimate $|\sqrt{a^2+\varepsilon^2}-\sqrt{b^2+\varepsilon^2}|\le|a-b|$ in the previous expression so that

$$\begin{aligned}&\left\|\frac{1}{\sqrt{\overline{u}_{ht}^2+\varepsilon^2}+\sqrt{u_{ht}^2+\varepsilon^2}}\right\|^2_{H^{\frac{1}{2}}(\Gamma_C)}\\ &\quad\le\frac{\mathrm{meas}(\Gamma_C)}{4\varepsilon^2}+\frac{1}{8\varepsilon^4}\Big(\|\overline{u}_{ht}\|^2_{H^{\frac{1}{2}}(\Gamma_C)}+\|u_{ht}\|^2_{H^{\frac{1}{2}}(\Gamma_C)}\Big).\end{aligned}$$

Therefore we deduce from (11) that there exists a positive constant C satisfying

$$\left\|\frac{1}{\sqrt{\overline{u}_{ht}^2+\varepsilon^2}+\sqrt{u_{ht}^2+\varepsilon^2}}\right\|_{H^{\frac{1}{2}}(\Gamma_C)}\le C\Big(\frac{1}{\varepsilon}+\frac{1}{\varepsilon^2}\Big).\quad(23)$$

Applying (23) to (22) and using (11) and (20), we get

$$\begin{aligned}&\Big\|\sqrt{\overline{u}_{ht}^2+\varepsilon^2}-\sqrt{u_{ht}^2+\varepsilon^2}\Big\|_{H^{\frac{1}{2}}(\Gamma_C)}\\ &\quad\le C\|u_{ht}-\overline{u}_{ht}\|_{H^{\frac{1}{2}}(\Gamma_C)}\\ &\qquad\times\Bigg(1+\sqrt{p}h^{-\frac{1}{p}}\Big(\frac{1}{\varepsilon}+\sqrt{p}h^{-\frac{1}{p}}\Big(\frac{1}{\varepsilon}+\frac{1}{\varepsilon^2}\Big)\Big)\Bigg).\end{aligned}$$

Choosing $p=-\log(h)$ (h is assumed to be sufficiently small) in the previous estimate, we obtain

$$\begin{aligned}&\Big\|\sqrt{\overline{u}_{ht}^2+\varepsilon^2}-\sqrt{u_{ht}^2+\varepsilon^2}\Big\|_{H^{\frac{1}{2}}(\Gamma_C)}\\ &\quad\le C\|u_{ht}-\overline{u}_{ht}\|_{H^{\frac{1}{2}}(\Gamma_C)}\\ &\qquad\times\Bigg(1+\frac{\sqrt{-\log h}}{\varepsilon}+\frac{-\log h}{\varepsilon}+\frac{-\log h}{\varepsilon^2}\Bigg).\quad(24)\end{aligned}$$

Inequality (15) together with (24) and the trace theorem becomes

$$\begin{aligned}&\|\boldsymbol{u}_h-\overline{\boldsymbol{u}}_h\|_1\\ &\quad\le C\mathcal{F}\|g_h-\overline{g}_h\|_{H^{-\frac{1}{2}}(\Gamma_C)}\\ &\qquad\times\Bigg(1+\frac{\sqrt{-\log h}}{\varepsilon}+\frac{-\log h}{\varepsilon}+\frac{-\log h}{\varepsilon^2}\Bigg),\quad(25)\end{aligned}$$

which proves that the mapping $\Psi_{\varepsilon h}$ is continuous. This, together with (13), implies that $\Phi_{\varepsilon h}$ is continuous. Then, from (12) and the Brouwer fixed point theorem, we conclude the existence of at least one solution to Coulomb's discrete regularized frictional contact problem.

We now consider the uniqueness. Under the assumption that $\overline{\Gamma}_D\cap\overline{\Gamma}_C=\emptyset$, it was proved in (Coorevits *et al.*, 2002) that there exists a positive constant β (independent of h) satisfying

$$\beta\|\mu_h\|_{H^{-\frac{1}{2}}(\Gamma_C)}\le\|\mu_h\|_{-\frac{1}{2},h},\quad\forall\mu_h\in W_h.\quad(26)$$

Assembling this result with (25) and (13) yields

$$\begin{aligned}&\|\lambda_h-\overline{\lambda}_h\|_{H^{-\frac{1}{2}}(\Gamma_C)}\\ &\quad\le C\mathcal{F}\|g_h-\overline{g}_h\|_{H^{-\frac{1}{2}}(\Gamma_C)}\\ &\qquad\times\Bigg(1+\frac{\sqrt{-\log h}}{\varepsilon}+\frac{-\log h}{\varepsilon}+\frac{-\log h}{\varepsilon^2}\Bigg).\end{aligned}$$

Supposing that h and ε are small enough, we deduce that the mapping $\Phi_{\varepsilon h}$ is contractive if the friction coefficient

$\mathcal{F}$ is less than $C\varepsilon^2|\log(h)|^{-1}$. This completes the proof of the theorem. ■

The non-regularized case (i.e. $\varepsilon = 0$) is handled in the proposition that follows.

Proposition 1. *Let $\varepsilon = 0$. The following results hold:*

(Existence) For any positive $\mathcal{F}$, there exists a solution to Coulomb's discrete frictional contact problem.

(Uniqueness) Assume that $\overline{\Gamma}_D \cap \overline{\Gamma}_C = \emptyset$. If $\mathcal{F} \leq Ch^{\frac{1}{2}}$, then the problem admits a unique solution. The positive constant C is independent of h.

Proof. Estimates (12) and (13) remain still valid when $\varepsilon = 0$. The starting point of the analysis is (14):

$$
\begin{aligned}
& a(\boldsymbol{u}_h - \overline{\boldsymbol{u}}_h, \boldsymbol{u}_h - \overline{\boldsymbol{u}}_h) \\
& \leq \int_{\Gamma_C} \mathcal{F}(g_h - \overline{g}_h)(|\overline{u}_{ht}| - |u_{ht}|)\,\mathrm{d}\Gamma \\
& \leq \mathcal{F}\,\|\overline{g}_h - g_h\|_{L^2(\Gamma_C)}\|\,|\overline{u}_{ht}| - |u_{ht}|\,\|_{L^2(\Gamma_C)} \\
& \leq C\mathcal{F}\,h^{-\frac{1}{2}}\|\overline{g}_h - g_h\|_{H^{-\frac{1}{2}}(\Gamma_C)}\|\overline{u}_{ht} - u_{ht}\|_{L^2(\Gamma_C)} \\
& \leq C'\mathcal{F}\,h^{-\frac{1}{2}}\|\overline{g}_h - g_h\|_{H^{-\frac{1}{2}}(\Gamma_C)}\|\overline{\boldsymbol{u}}_h - \boldsymbol{u}_h\|_1,
\end{aligned}
$$

where an inverse inequality between $L^2(\Gamma_C)$ and $H^{-\frac{1}{2}}(\Gamma_C)$ was used. From the last bound, combined with (13) and (26), we deduce that

$$\|\lambda_h - \overline{\lambda}_h\|_{H^{-\frac{1}{2}}(\Gamma_C)} \leq C\mathcal{F}h^{-\frac{1}{2}}\|g_h - \overline{g}_h\|_{H^{-\frac{1}{2}}(\Gamma_C)}.$$

This proves the proposition. ■

Remark 2. *1. In the proof of Proposition 1, we are not able to remove the mesh dependent uniqueness condition, also when avoiding the $L^2(\Gamma_C)$-norms and using only $H^{\frac{1}{2}}(\Gamma_C)$-norms and $H^{-\frac{1}{2}}(\Gamma_C)$-norms. More precisely, there does not exist a positive constant C independent of h such that*

$$\|\,|\overline{g}_h - g_h|\,\|_{H^{-\frac{1}{2}}(\Gamma_C)} \leq C\|\overline{g}_h - g_h\|_{H^{-\frac{1}{2}}(\Gamma_C)}$$

or

$$\|\,|\overline{u}_{ht}| - |u_{ht}|\,\|_{H^{\frac{1}{2}}(\Gamma_C)} \leq C\|\overline{u}_{ht} - u_{ht}\|_{H^{\frac{1}{2}}(\Gamma_C)}.$$

2. The use of inverse inequalities in the proofs of Theorem 1 and Proposition 1 implies that it is not possible to generalize the calculus to the continuous problem.

4. The Study of a Simple Finite Element Example

We consider the triangle Ω of vertices $A = (0,0)$, $B = (\ell, 0)$ and $C = (0,\ell)$ with $\ell > 0$. We define $\Gamma_D = [B,C]$, $\Gamma_N = [A,C]$, $\Gamma_C = [A,B]$, and $\{X_1, X_2\}$ denotes the canonical orthonormal basis (see Fig. 1). We suppose that the volume forces $\boldsymbol{f}$ are absent and that the surface forces denoted by $\boldsymbol{F} = F_1X_1 + F_2X_2$ are such that F_1 and F_2 are constant on Γ_N.

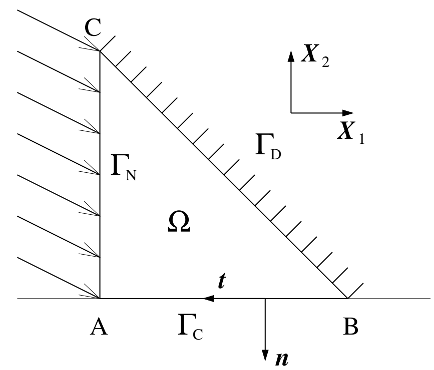

Fig. 1. Problem setting.

We suppose that Ω is discretized with a single finite element of degree one. Consequently, the finite element space becomes

$$\boldsymbol{V}_h = \Big\{\boldsymbol{v}_h = (v_{h1}, v_{h2}) \in \big(P_1(\Omega)\big)^2, \quad \boldsymbol{v}_h|_{\Gamma_D} = \boldsymbol{0}\Big\}.$$

In this case, we have

$$M_h = \Big\{g_h \in P_1(\Gamma_C), \quad g_h \geq 0, \quad g_h(B) = 0\Big\}.$$

Clearly, $\boldsymbol{V}_h$ is of dimension two and M_h belongs to the space W_h of linear functions on Γ_C vanishing at B, which is of dimension one. Moreover, since (8), or equivalently (9), is satisfied, it follows that the existence is ensured for all $\varepsilon \geq 0$ according to Theorem 1 and Proposition 1.

Let $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $\mu_h \in M_h$. Then we denote by (V_T, V_N) the value of $\boldsymbol{v}_h(A)$, corresponding to the tangential and the normal displacements at point A, respectively (in our example, we have $V_T = -v_{h1}(A)$ and $V_N = -v_{h2}(A)$). We also denote by Θ the value of μ_h at point A. Then, for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $\mu_h \in M_h$, we obtain

$$\boldsymbol{\varepsilon}(\boldsymbol{v}_h) = \frac{1}{2\ell}\begin{pmatrix} 2V_T & V_T + V_N \\ V_T + V_N & 2V_N \end{pmatrix}$$

and

$$\boldsymbol{\sigma}(\boldsymbol{v}_h) = \frac{1}{\ell}\begin{pmatrix} (\lambda+2\mu)V_T + \lambda V_N & \mu(V_T + V_N) \\ \mu(V_T + V_N) & (\lambda+2\mu)V_N + \lambda V_T \end{pmatrix}.$$

Therefore

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) = \frac{1}{2}\Big((\lambda+3\mu)(U_T V_T + U_N V_N) + (\lambda+\mu)(U_T V_N + U_N V_T)\Big)$$

and

$$L(\boldsymbol{v}_h) = -\frac{1}{2}\ell(F_1 V_T + F_2 V_N).$$

Besides,

$$\int_{\Gamma_C} \mu_h v_{hn}\, \mathrm{d}\Gamma = \frac{\Theta V_N \ell}{3}$$

and

$$\int_{\Gamma_C} \mathcal{F}\mu_h |v_{ht}|\, \mathrm{d}\Gamma = \frac{\mathcal{F}\Theta |V_T| \ell}{3}.$$

Let $(\boldsymbol{u}_h, \lambda_h)$ be a solution to the discrete unilateral contact problem with Coulomb's friction and without regularization (i.e. with $\varepsilon = 0$ in (7)). As was mentioned above, the notation (U_T, U_N) stands for the value of $\boldsymbol{u}_h(A)$ ($U_T = -u_{h1}(A)$ and $U_N = -u_{h2}(A)$). We also denote by Λ' the value of λ_h at point A. To simplify the notation and the forthcoming calculations, we set $\Lambda = 2\Lambda'/3$.

The discrete unilateral contact problem with Coulomb's friction and without regularization issued from (7) and Definition 1 consists then in finding $(U_T, U_N, \Lambda) \in \mathbb{R}^3$ such that

$$\begin{cases} (\lambda+3\mu)(U_T V_T + U_N V_N) + (\lambda+\mu)(U_T V_N + U_N V_T) \\ \qquad + \Lambda\ell V_N + \mathcal{F}\Lambda\ell|V_T| \\ \qquad \geq -\ell(F_1 V_T + F_2 V_N), \quad \forall V_T \in \mathbb{R}, \quad \forall V_N \in \mathbb{R}, \\ (\lambda+3\mu)(U_T^2 + U_N^2) + 2(\lambda+\mu)(U_T U_N) + \mathcal{F}\Lambda\ell|U_T| \\ \qquad = -\ell(F_1 U_T + F_2 U_N), \\ \Lambda \geq 0, \qquad U_N \leq 0, \qquad \Lambda U_N = 0, \end{cases}$$

or equivalently,

$$\begin{cases} (\lambda+3\mu)U_N + (\lambda+\mu)U_T + \Lambda\ell = -\ell F_2, \\ (\lambda+\mu)U_N + (\lambda+3\mu)U_T + \mathcal{F}\Lambda\ell \geq -\ell F_1, \\ (\lambda+\mu)U_N + (\lambda+3\mu)U_T - \mathcal{F}\Lambda\ell \leq -\ell F_1, \\ (\lambda+3\mu)(U_T^2 + U_N^2) + 2(\lambda+\mu)(U_T U_N) + \mathcal{F}\Lambda\ell|U_T| \\ \qquad = -\ell(F_1 U_T + F_2 U_N), \\ \Lambda \geq 0, \qquad U_N \leq 0, \quad \Lambda U_N = 0. \end{cases} \tag{27}$$

Let us now look for solutions to (27). Clearly, a solution to (27) satisfies either $U_N = 0$ or $\Lambda = 0$.

(i) Case 1: $U_N = 0$. Equations (27) become

$$\begin{cases} (\lambda+\mu)U_T + \Lambda\ell = -\ell F_2, \\ (\lambda+3\mu)U_T + \mathcal{F}\Lambda\ell \geq -\ell F_1, \\ (\lambda+3\mu)U_T - \mathcal{F}\Lambda\ell \leq -\ell F_1, \\ (\lambda+3\mu)U_T^2 + \mathcal{F}\Lambda\ell|U_T| = -\ell F_1 U_T, \\ \Lambda \geq 0. \end{cases}$$

- Suppose that $U_T = 0$. Then

$$\Lambda = -F_2, \quad F_2 \leq 0, \quad |F_1| \leq \mathcal{F}|F_2|.$$

- Suppose that $U_T > 0$. Then

$$\begin{cases} (\lambda+\mu)U_T + \Lambda\ell = -\ell F_2, \\ (\lambda+3\mu)U_T + \mathcal{F}\Lambda\ell = -\ell F_1, \\ \Lambda \geq 0. \end{cases}$$

  – Assume that $\mathcal{F} \neq (\lambda+3\mu)/(\lambda+\mu)$. Then

$$U_T = \frac{\ell(\mathcal{F}F_2 - F_1)}{(\lambda+3\mu) - \mathcal{F}(\lambda+\mu)} > 0,$$

$$\Lambda = \frac{(\lambda+\mu)F_1 - (\lambda+3\mu)F_2}{(\lambda+3\mu) - \mathcal{F}(\lambda+\mu)} \geq 0.$$

  – Assume that $\mathcal{F} = (\lambda+3\mu)/(\lambda+\mu)$. Then
    * If $F_1 = \mathcal{F}F_2$, the solutions are

$$(\lambda+\mu)U_T + \Lambda\ell = -\ell F_2, \quad U_T > 0, \quad \Lambda \geq 0.$$

    * If $F_1 \neq \mathcal{F}F_2$, then there are no solutions.

- Suppose that $U_T < 0$. Then

$$\begin{cases} (\lambda+\mu)U_T + \Lambda\ell = -\ell F_2, \\ (\lambda+3\mu)U_T - \mathcal{F}\Lambda\ell = -\ell F_1, \\ \Lambda \geq 0, \end{cases}$$

  which gives

$$U_T = \frac{\ell(\mathcal{F}F_2 + F_1)}{-(\lambda+3\mu) - \mathcal{F}(\lambda+\mu)} < 0,$$

$$\Lambda = \frac{-(\lambda+\mu)F_1 + (\lambda+3\mu)F_2}{-(\lambda+3\mu) - \mathcal{F}(\lambda+\mu)} \geq 0.$$

(ii) Case 2: $\Lambda = 0$.

$$\begin{cases} (\lambda+3\mu)U_N + (\lambda+\mu)U_T = -\ell F_2, \\ (\lambda+\mu)U_N + (\lambda+3\mu)U_T = -\ell F_1, \\ U_N \le 0, \end{cases}$$

so that

$$U_T = \frac{\ell\big((\lambda+\mu)F_2 - (\lambda+3\mu)F_1\big)}{4\mu(\lambda+2\mu)},$$

$$U_N = \frac{\ell\big((\lambda+\mu)F_1 - (\lambda+3\mu)F_2\big)}{4\mu(\lambda+2\mu)} \le 0.$$

All the results are reported in the proposition that follows. There are three cases which consist in comparing the friction coefficient $\mathcal{F}$ with the critical value $(\lambda+3\mu)/(\lambda+\mu) = 3-4\nu$ (ν denotes Poisson's ratio with $0 < \nu < 1/2$). The results are also depicted in Figs. 2–4.

Proposition 2. *1. If $\mathcal{F} < (\lambda+3\mu|/(\lambda+\mu)$, then the problem (27) admits a unique solution:*

(Separation) If $F_2 > ((\lambda+\mu)/(\lambda+3\mu))F_1,$ then

$$\begin{aligned} U_T &= \frac{\ell\big((\lambda+\mu)F_2 - (\lambda+3\mu)F_1\big)}{4\mu(\lambda+2\mu)}, \\ U_N &= \frac{\ell\big((\lambda+\mu)F_1 - (\lambda+3\mu)F_2\big)}{4\mu(\lambda+2\mu)}, \ \Lambda = 0. \end{aligned} \tag{28}$$

(Stick) If $|F_1| \le \mathcal{F}|F_2|$ and $F_2 \le 0$, then

$$U_T = 0, \quad U_N = 0, \quad \Lambda = -F_2. \tag{29}$$

(Right slip) If $F_2 \le ((\lambda+\mu)/(\lambda+3\mu))F_1,\ \mathcal{F}F_2 + F_1 > 0$, then

$$\begin{aligned} U_T &= \frac{\ell(\mathcal{F}F_2 + F_1)}{-(\lambda+3\mu) - \mathcal{F}(\lambda+\mu)}, \\ U_N &= 0, \quad \Lambda = \frac{-(\lambda+\mu)F_1 + (\lambda+3\mu)F_2}{-(\lambda+3\mu) - \mathcal{F}(\lambda+\mu)}. \end{aligned} \tag{30}$$

(Left slip) If $F_2 \le ((\lambda+\mu)/(\lambda+3\mu))F_1,\ \mathcal{F}F_2 - F_1 > 0$, then

$$\begin{aligned} U_T &= \frac{\ell(\mathcal{F}F_2 - F_1)}{(\lambda+3\mu) - \mathcal{F}(\lambda+\mu)}, \\ U_N &= 0, \quad \Lambda = \frac{(\lambda+\mu)F_1 - (\lambda+3\mu)F_2}{(\lambda+3\mu) - \mathcal{F}(\lambda+\mu)}. \end{aligned} \tag{31}$$

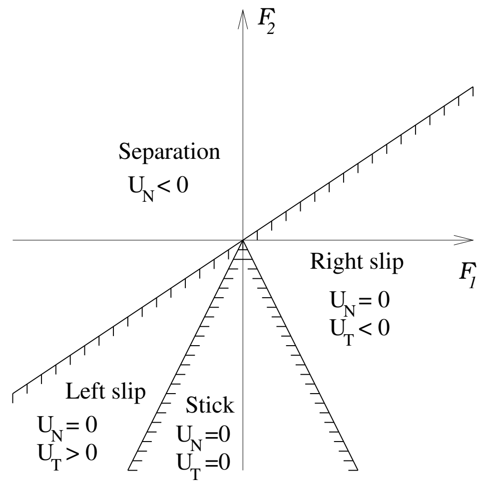

Fig. 2. Case $\mathcal{F} < \frac{\lambda+3\mu}{\lambda+\mu} = 3-4\nu$. Problem (27) admits a unique solution.

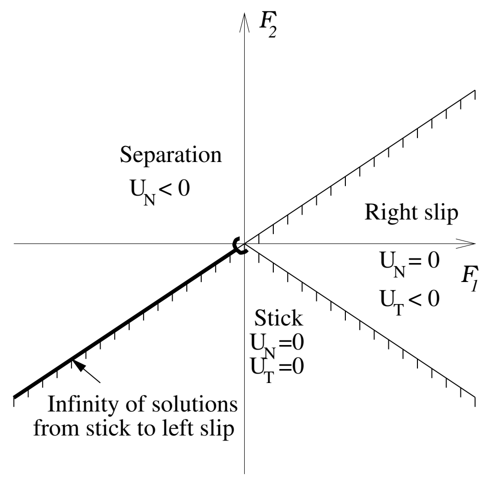

Fig. 3. Case $\mathcal{F} = \frac{\lambda+3\mu}{\lambda+\mu} = 3-4\nu$. Problem (27) admits either a unique or an infinity of solutions.

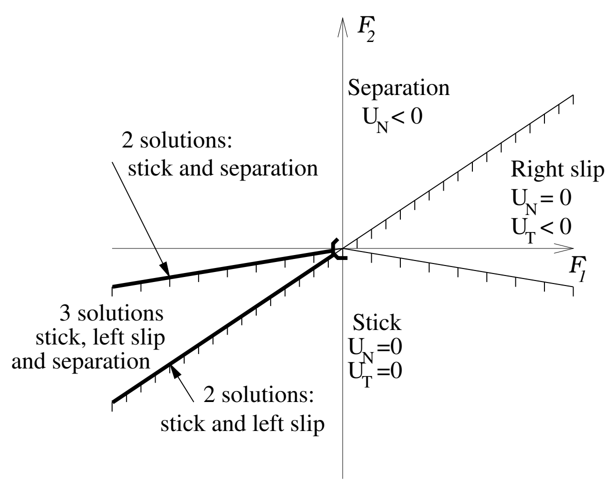

Fig. 4. Case $\mathcal{F} > \frac{\lambda+3\mu}{\lambda+\mu} = 3-4\nu$. Problem (27) admits a unique, two or three solutions.

2. If $\mathcal{F} = (\lambda + 3\mu)/(\lambda + \mu)$, then, depending on the loadings, the problem (27) admits either a unique solution or an infinity of solutions:

(Separation) If $F_2 > ((\lambda + \mu)/(\lambda + 3\mu))F_1$, then the solution is given by (28).

(Stick) If ($-\mathcal{F}|F_2| < F_1 \leq \mathcal{F}|F_2|$ and $F_2 \leq 0$) or $F_1 = F_2 = 0$, then the solution is given by (29).

(Right slip) If $F_2 \leq ((\lambda + \mu)/(\lambda + 3\mu))F_1,\ \mathcal{F}F_2 + F_1 > 0$, then the solution is given by (30).

(From stick to left slip) If $F_1 = \mathcal{F}F_2$ and $F_2 < 0$, then there exists an infinity of solutions:

$$\begin{aligned} U_T &= \frac{-\ell(F_2+\beta)}{\lambda+\mu}, \\ U_N &= 0, \quad \Lambda = \beta, \end{aligned} \qquad \text{for all } 0 \leq \beta \leq -F_2.$$

3. If $\mathcal{F} > (\lambda + 3\mu)/(\lambda + \mu)$, then, depending on the loadings, the problem (27) admits one, two or three solutions:

(Separation) If $F_2 > ((\lambda + \mu)/(\lambda + 3\mu))F_1$ and $\mathcal{F}F_2 - F_1 > 0$, then the solution is given by (28).

(Stick) If ($-((\lambda + 3\mu)/(\lambda + \mu))|F_2| < F_1 \leq \mathcal{F}|F_2|$ and $F_2 \leq 0$) or $F_1 = F_2 = 0$, then the solution is given by (29).

(Right slip) If $F_2 \leq ((\lambda + \mu)/(\lambda + 3\mu))F_1,\ \mathcal{F}F_2 + F_1 > 0$, then the solution is given by (30).

(Separation and stick) If $F_1 = \mathcal{F}F_2$ and $F_2 < 0$, then there are two solutions given by (28) and (29).

(Stick and left slip) If $F_1 = ((\lambda + 3\mu)/(\lambda + \mu))F_2$ and $F_2 < 0$, then there are two solutions given by (29) and (31).

(Separation, stick and left slip) If $-\mathcal{F}|F_2| < F_1 < -((\lambda + 3\mu)/(\lambda + \mu))|F_2|$ and $F_2 \leq 0$ then there are three solutions given by (28), (29) and (31).

The study of sufficient conditions of non-uniqueness for Coulomb's frictional contact problem in the continuous framework is actually under consideration in (Hassani *et al.*, 2001).

References

Adams R.A. (1975): *Sobolev Spaces.* — New-York: Academic Press.

Alart P. (1993): *Critères d'injectivité et de surjectivité pour certaines applications de $\mathbb{R}^n$ dans lui même: application à la mécanique du contact.* — Math. Model. Numer. Anal., Vol. 27, No. 2, pp. 203–222.

Ben Belgacem F. (2000): *Numerical simulation of some variational inequalities arisen from unilateral contact problems by the finite element method.* — SIAM J. Numer. Anal., Vol. 37, No. 4, pp. 1198–1216.

Ciarlet P.G. (1991): *The finite element method for elliptic problems*, In: Handbook of Numerical Analysis, Volume II (P.G. Ciarlet and J.L. Lions, Eds.). — Amsterdam: North Holland, pp.17–352.

Coorevits P., Hild P., Lhalouani K. and Sassi T. (2002): *Mixed finite element methods for unilateral problems: convergence analysis and numerical studies.* — Internal report of Laboratoire de Mathématiques de l'Université de Savoie n^o 00-01c, to appear in Mathematics of Computation (published online May 21, 2001).

Duvaut G. and Lions J.-L. (1972): *Les Inéquations en Mécanique et en Physique.* — Paris: Dunod.

Eck C. and Jarušek J. (1998): *Existence results for the static contact problem with Coulomb friction.* — Math. Mod. Meth. Appl. Sci., Vol. 8, No. 3, pp. 445–468.

Haslinger J. (1983): *Approximation of the Signorini problem with friction, obeying the Coulomb law.* — Math. Meth. Appl. Sci., Vol. 5, No. 3, pp. 422–437.

Haslinger J. (1984): *Least square method for solving contact problems with friction obeying Coulomb's law.* — Apl. Mat., Vol. 29, No. 3, pp. 212–224.

Haslinger J., Hlaváček I. and Nečas J. (1996): *Numerical methods for unilateral problems in solid mechanics*, In: Handbook of Numerical Analysis, Vol. IV (P.G. Ciarlet and J.L. Lions, Eds.). — Amsterdam: North Holland, pp. 313–485.

Hassani R., Hild P. and Ionescu I. (2001): *On non-uniqueness of the elastic equilibrium with Coulomb friction: A spectral approach.* — Internal report of Laboratoire de Mathématiques de l'Université de Savoie n^o 01-04c. Submitted.

Janovský V. (1981): *Catastrophic features of Coulomb friction model*, In: The Mathematics of Finite Elements and Aplications (J.R. Whiteman, Ed.). — London: Academic Press, pp.259–264.

Jarušek J. (1983): *Contact problems with bounded friction. Coercive case.* — Czechoslovak. Math. J., Vol. 33, No. 2, pp. 237–261.

Kato Y. (1987): *Signorini's problem with friction in linear elasticity.* – Japan J. Appl. Math., Vol. 4, No. 2, pp. 237–268.

Klarbring A. (1990): *Examples of non-uniqueness and non-existence of solutions to quasistatic contact problems with friction.* — Ing. Archiv, Vol. 60, pp. 529–541.

Nečas J., Jarušek J. and Haslinger J. (1980): *On the solution of the variational inequality to the Signorini problem with small friction.* — Boll. Unione Mat. Ital., Vol. 17-B(5), No. 2, pp. 796–811.