

## ANALYSIS OF THE DESCRIPTOR ROESSER MODEL WITH THE USE OF THE DRAZIN INVERSE

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A method of analysis for a class of descriptor 2D discrete-time linear systems described by the Roesser model with a regular pencil is proposed. The method is based on the transformation of the model to a special form with the use of elementary row and column operations and on the application of a Drazin inverse of matrices to handle the model. The method is illustrated with a numerical example.

**Keywords:** Drazin inverse, descriptor, Roesser model, descriptor-time, 2D linear system.

### 1. Introduction

Descriptor (singular) linear systems were considered in many papers and books (Bru *et al.*, 2003; 2000; 2002; Campbell *et al.*, 1976; Dai, 1989; Dodig and Stosic, 2009; Fahmy and O'Reill, 1989; Gantmecher, 1960; Duan, 2010; Kaczorek, 2014a; 2011a; 2004; 2013; 2011b; 2011c; 2011d; 1992; Kucera and Zagalak, 1988; Van Dooren, 1979). The eigenvalues and invariants assignment by state and output feedbacks was investigated by Fahmy and O'Reill (1989) as well as Kaczorek (2004; 1992), who also discussed the minimum energy control of descriptor linear systems (Kaczorek, 2014b; 2014c). Computation of Kronecker's canonical form of the singular pencil was analyzed by Van Dooren (1979), while positive linear systems with different fractional orders were addressed by Kaczorek (2010), along with selected problems in the theory of fractional linear systems (Kaczorek, 2011d).

Descriptor standard positive linear systems were addressed with the use of the Drazin inverse by Bru *et al.* (2003; 2000; 2002), Campbell *et al.* (1976), and Kaczorek (2014a; 2011d; 1992), who also applied the shuffle algorithm to check the positivity of descriptor linear systems (Kaczorek, 2011a). The stability of positive descriptor systems was investigated by Virnik (2008), while reduction and decomposition of descriptor fractional discrete-time linear systems were considered by Kaczorek (2011b), who also introduced a new class of descriptor fractional linear discrete-time systems

(Kaczorek, 2011c).

The Drazin inverse for finding the solution to the state equation of fractional continuous-time linear systems was applied by Kaczorek (2014a), and the controllability, reachability and minimum energy control of fractional discrete-time linear systems with delays in state were investigated by Busłowicz (2014).

In this paper, a Drazin inverse of matrices will be used in the analysis of descriptor discrete-time 2D linear systems regular pencils described by the Roesser model.

The paper is organized as follows. In Section 2, basic definitions and theorems concerning descriptor discrete-time linear systems with regular pencils are presented. The problem of the analysis of descriptor systems described by the Roesser model is formulated and solved in Section 3. The proposed method is illustrated with a numerical example in Section 4. Concluding remarks are given in Section 5.

The following notation will be used:  $\mathbb{R}$ , the set of real numbers;  $\mathbb{R}^{n \times m}$ , the set of  $n \times m$  real matrices and  $\mathbb{R}^n = \mathbb{R}^{n \times 1}$ ;  $\mathbb{Z}_+$ , the set of nonnegative integers;  $I_n$ , the  $n \times n$  identity matrix;  $\ker A$  ( $\text{im } A$ ), the kernel (image) of the matrix.

### 2. Descriptor discrete-time linear systems

Consider the descriptor discrete-time linear system

$$Ex_{i+1} = Ax_i + Bu_i, \quad i \in \mathbb{Z}_+, \quad (1)$$

where  $x_i \in \mathbb{R}^n$  is the state vector,  $u_i \in \mathbb{R}^m$  is the input vector,  $E, A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ .

It is assumed that  $\det E = 0$ , but

$$\det[Es - A] \neq 0 \quad \text{for some } c \in \mathbb{C}. \quad (2)$$

Assuming that, for some chosen  $c \in \mathbb{C}$ ,  $\det[Ec - A] \neq 0$  and premultiplying (1) by  $[Ec - A]^{-1}$ , we obtain

$$\bar{E}x_{i+1} = \bar{A}x_i + \bar{B}u_i, \quad (3a)$$

where

$$\begin{aligned} \bar{E} &= [Ec - A]^{-1}E, \\ \bar{A} &= [Ec - A]^{-1}A, \\ \bar{B} &= [Ec - A]^{-1}B. \end{aligned} \quad (3b)$$

Note that Eqns. (1) and (3a) have the same solution  $x_i$ ,  $i \in \mathbb{Z}_+$ .

**Definition 1.** (Campbell et al., 1976; Kaczorek, 1992) The smallest nonnegative integer  $q$  is called the *index* of the matrix  $\bar{E} \in \mathbb{R}^{n \times n}$  if

$$\text{rank } \bar{E}^q = \text{rank } \bar{E}^{q+1}. \quad (4)$$

**Definition 2.** (Campbell et al., 1976; Kaczorek, 1992) A matrix  $\bar{E}^D$  is called the *Drazin inverse* of  $\bar{E} \in \mathbb{R}^{n \times n}$  if it satisfies the conditions

$$\bar{E}\bar{E}^D = \bar{E}^D\bar{E}, \quad (5a)$$

$$\bar{E}^D\bar{E}\bar{E}^D = \bar{E}^D, \quad (5b)$$

$$\bar{E}^D\bar{E}^{q+1} = \bar{E}^q, \quad (5c)$$

where  $q$  is the index of  $\bar{E}$  defined by (4).

The Drazin inverse  $\bar{E}^D$  of a square matrix  $\bar{E}$  always exists and is unique (Campbell et al., 1976; Kaczorek, 1992). If  $\det \bar{E} \neq 0$ , then  $\bar{E}^D = \bar{E}^{-1}$ . Some methods for computation of the Drazin inverse are given by Kaczorek (1992) and Van Dooren (1979), and are summarized in Appendix.

**Theorem 1.** (Kaczorek, 1992) The matrices  $\bar{E}$  and  $\bar{A}$  defined by (3b) satisfy the following equalities:

$$\begin{aligned} \bar{A}\bar{E} &= \bar{E}\bar{A}, & \bar{A}^D\bar{E} &= \bar{E}\bar{A}^D, \\ \bar{E}^D\bar{A} &= \bar{A}\bar{E}^D, & \bar{A}^D\bar{E}^D &= \bar{E}^D\bar{A}^D, \end{aligned} \quad (6a)$$

$$\ker \bar{A} \cap \ker \bar{E} = \{0\}, \quad (6b)$$

$$\begin{aligned} \bar{E} &= T \begin{bmatrix} J & 0 \\ 0 & N \end{bmatrix} T^{-1}, \\ \bar{E}^D &= T \begin{bmatrix} J^{-1} & 0 \\ 0 & 0 \end{bmatrix} T^{-1}, \end{aligned} \quad (6c)$$

$$(I_n - \bar{E}\bar{E}^D)\bar{A}\bar{A}^D = I_n - \bar{E}\bar{E}^D, \quad (6d)$$

$$(I_n - \bar{E}\bar{E}^D)(\bar{E}\bar{A}^D)^q = 0, \quad (6d)$$

$\det T \neq 0$ ,  $J \in \mathbb{R}^{n_1 \times n_1}$ , is nonsingular,  $N \in \mathbb{R}^{n_2 \times n_2}$  is nilpotent,  $n_1 + n_2 = n$ .

**Theorem 2.** (Campbell et al., 1976; Kaczorek, 1992) The solution of Eqn. (3) is given by

$$\begin{aligned} x_i &= (\bar{E}^D\bar{A})^i\bar{E}^D\bar{E}v + \sum_{k=0}^{i-1} \bar{E}^D(\bar{E}^D\bar{A})^{i-k-1}\bar{B}u_k \\ &+ (\bar{E}\bar{E}^D - I_n) \sum_{k=0}^{q-1} (\bar{E}\bar{A}^D)^k\bar{A}^D\bar{B}u_{i+k}, \end{aligned} \quad (7)$$

where  $v \in \mathbb{R}^n$  is arbitrary and  $q$  is the index of  $E$ .

From (7), for  $i = 0$  we have

$$x_0 = \bar{E}^D\bar{E}v + (\bar{E}\bar{E}^D - I_n) \sum_{k=0}^{q-1} (\bar{E}\bar{A}^D)^k\bar{A}^D\bar{B}u_k. \quad (8)$$

Therefore, for given admissible  $u_i$ , the consistent initial conditions should satisfy the equality (8). In a particular case for  $u_i = 0$  we have  $x_0 = \bar{E}^D\bar{E}v$  and  $x_0 \in \text{Im}(\bar{E}^D\bar{E})$ , where  $\text{Im}$  denotes the image of  $\bar{E}^D\bar{E}$ .

**Theorem 3.** Let

$$P = \bar{E}\bar{E}^D, \quad (9)$$

$$Q = \bar{E}^D\bar{A}. \quad (10)$$

Then

(i)  $P^k = P \quad \text{for } k = 2, 3, \dots;$  (11)

(ii)  $PQ = QP = Q;$  (12)

(iii)  $P\bar{E}^D = \bar{E}^D P = \bar{E}^D;$  (13)

(iv) if there exists a vector  $v \in \mathbb{R}^n$  such that

$$v^T \bar{E} = 0, \quad (14)$$

then

$$v^T \bar{E}^D = 0. \quad (15)$$

*Proof.* Using (9), we obtain

$$P^2 = \bar{E}\bar{E}^D\bar{E}\bar{E}^D = \bar{E}\bar{E}^D = P \quad (16)$$

since, by (5b),  $\bar{E}^D\bar{E}\bar{E}^D = \bar{E}^D$  and, by induction,

$$P^k = P^{k-1}P = \bar{E}\bar{E}^D\bar{E}\bar{E}^D = P^2 = P \quad (17)$$

for  $k = 2, 3, \dots$

Using (9) and (10), we obtain

$$PQ = \bar{E}\bar{E}^D\bar{E}^D\bar{A} = \bar{E}^D\bar{E}\bar{E}^D\bar{A} = \bar{E}^D\bar{A} = Q \quad (18)$$

and

$$\begin{aligned} QP &= \bar{E}^D\bar{A}\bar{E}\bar{E}^D = \bar{E}^D\bar{E}\bar{A}\bar{E}^D \\ &= \bar{E}^D\bar{E}\bar{E}^D\bar{A} = \bar{E}^D\bar{A} = Q. \end{aligned} \quad (19)$$

Using (9), (5a) and (5b), we obtain

$$P\bar{E}^D = \bar{E}\bar{E}^D\bar{E}^D = \bar{E}^D\bar{E}\bar{E}^D = \bar{E}^D \quad (20)$$

and

$$\bar{E}^D P = \bar{E}^D \bar{E} \bar{E}^D = \bar{E}^D. \quad (21)$$

Taking into account that  $\bar{E} = VW$  (see Appendix)

and

$$\bar{E}^D = V[W\bar{E}V]^{-1}W, \quad (22)$$

we obtain

$$v^T \bar{E}^D = v^T V[W\bar{E}V]^{-1}W = 0 \quad (23)$$

since  $v^T V = 0$ . ■

The following elementary row (resp. column) operations will be used:

1. Multiplication of the  $i$ -th row (resp. column) by a real number  $c$ . This operation will be denoted by  $L[i \times c]$  (resp.  $R[i \times c]$ ).
2. Addition to the  $i$ -th row (resp. column) of the  $j$ -th row (resp. column) multiplied by a real number  $c$ . This operation will be denoted by  $L[i + j \times c]$  (resp.  $R[i + j \times c]$ ).
3. Interchange of the  $i$ -th and  $j$ -th rows (columns). This operation will be denoted by  $L[i, j]$  (resp.  $R[i, j]$ ).

### 3. Problem formulation and solution

Consider the descriptor Roesser model

$$E \begin{bmatrix} x_{i+1,j}^h \\ x_{i,j+1}^v \end{bmatrix} = A \begin{bmatrix} x_{i,j}^h \\ x_{i,j}^v \end{bmatrix} + B u_{i,j}, \quad (24)$$

where  $x_{i,j}^h \in \mathbb{R}^{n_1}$ ,  $x_{i,j}^v \in \mathbb{R}^{n_2}$  are the horizontal and vertical state vectors  $u_{i,j} \in \mathbb{R}^m$  is the input vector and  $E, A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ ,  $n = n_1 + n_2$ .

It is assumed that  $\det E = 0$ , but

$$\det \left[ E \begin{bmatrix} I_{n_1} z_1 & 0 \\ 0 & I_{n_2} z_2 \end{bmatrix} - A \right] \neq 0 \quad \text{for some } z_1, z_2 \in \mathbb{C}. \quad (25)$$

It is also assumed that, premultiplying (24) by a matrix  $P \in \mathbb{R}^{n \times n}$  of the elementary row operations and introducing the new state vector

$$\begin{bmatrix} \bar{x}_{i+1,j}^h \\ \bar{x}_{i,j+1}^v \end{bmatrix} = Q \begin{bmatrix} x_{i,j}^h \\ x_{i,j}^v \end{bmatrix}, \quad Q \in \mathbb{R}^{n \times n}, \quad \det Q \neq 0, \quad (26)$$

Eqn. (24) can be written in the following form:

Case 1:

$$\begin{bmatrix} E_h & 0 \\ 0 & E_v \end{bmatrix} \begin{bmatrix} \bar{x}_{i+1,j}^h \\ \bar{x}_{i,j+1}^v \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \bar{x}_{i,j}^h \\ \bar{x}_{i,j}^v \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{i,j}. \quad (27a)$$

Case 2:

$$\begin{bmatrix} E_h & 0 \\ 0 & E_v \end{bmatrix} \begin{bmatrix} \bar{x}_{i+1,j}^h \\ \bar{x}_{i,j+1}^v \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} \bar{x}_{i,j}^h \\ \bar{x}_{i,j}^v \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{i,j}, \quad (27b)$$

where

$$\begin{bmatrix} E_h & 0 \\ 0 & E_v \end{bmatrix} = PEQ^{-1}, \quad E_h \in \mathbb{R}^{n_1 \times n_1}, \quad E_v \in \mathbb{R}^{n_2 \times n_2}, \quad \det E_h = 0, \quad \det E_v = 0 \quad (27c)$$

$$\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} = PAQ^{-1}, \quad A_{11} \in \mathbb{R}^{n_1 \times n_1}, \quad A_{21} \in \mathbb{R}^{n_2 \times n_1}, \quad A_{22} \in \mathbb{R}^{n_2 \times n_2}, \quad (27d)$$

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} = PAQ^{-1}, \quad A_{11} \in \mathbb{R}^{n_1 \times n_1}, \quad A_{12} \in \mathbb{R}^{n_1 \times n_2}, \quad A_{22} \in \mathbb{R}^{n_2 \times n_2}, \quad (27e)$$

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = PB, \quad B_1 \in \mathbb{R}^{n_1 \times m}, \quad B_2 \in \mathbb{R}^{n_2 \times m}. \quad (27f)$$

In Case 1, from (27a) we have

$$E_h \bar{x}_{i+1,j}^h = A_{11} \bar{x}_{i,j}^h + B_1 u_{i,j}, \quad (28a)$$

$$E_v \bar{x}_{i,j+1}^v = A_{21} \bar{x}_{i,j}^h + A_{22} \bar{x}_{i,j}^v + B_2 u_{i,j}, \quad (28b)$$

and, in Case 2, from (27b) we have

$$E_h \bar{x}_{i+1,j}^h = A_{11} \bar{x}_{i,j}^h + A_{12} \bar{x}_{i,j}^v + B_1 u_{i,j}, \quad (29a)$$

$$E_v \bar{x}_{i,j+1}^v = A_{22} \bar{x}_{i,j}^v + B_2 u_{i,j}. \quad (29b)$$

From the assumption (25) for Case 1 it follows that

$$\det[E_h z_1 - A_{11}] \neq 0 \quad \text{for some } z_1 \in \mathbb{C}. \quad (30)$$

Therefore, there exists a number  $c_1 \in \mathbb{C}$  such that  $\det[E_h c_1 - A_{11}] \neq 0$  and, premultiplying (27a) by  $[E_h c_1 - A_{11}]^{-1}$ , we obtain

$$\bar{E}_h \bar{x}_{i+1,j}^h = \bar{A}_{11} \bar{x}_{i,j}^h + \bar{B}_1 u_{i,j}, \quad (31a)$$

where

$$\begin{aligned} \bar{E}_h &= [E_h c_1 - A_{11}]^{-1} E_h, \\ \bar{A}_{11} &= [E_h c_1 - A_{11}]^{-1} A_{11}, \\ \bar{B}_1 &= [E_h c_1 - A_{11}]^{-1} B_1. \end{aligned} \quad (31b)$$

Let  $\bar{E}_h^D$  ( $\bar{A}_{11}^D$ ) be the Drazin inverse of the matrix  $\bar{E}_h$

$(\bar{A}_{11})$ . Then, from Theorem 2, we have

$$\begin{aligned} \bar{x}_{i,j}^h &= (\bar{E}_h^D \bar{A}_{11})^i \bar{E}_h^D \bar{E}_h v_1 \\ &+ \sum_{k=0}^{i-1} \bar{E}_h^D (\bar{E}_h^D \bar{A}_{11})^{i-k-1} \bar{B}_1 u_{k,j} \\ &+ (\bar{E}_h \bar{E}_h^D - I_{n_1}) \sum_{k=0}^{q_1-1} (\bar{E}_h \bar{A}_{11}^D)^k \bar{A}_{11}^D \bar{B}_1 u_{i+k,j}, \end{aligned} \tag{32}$$

where  $q_1$  is the index of  $\bar{E}_h$  and  $v_1 \in \mathbb{R}^{n_1}$  is arbitrary depending on  $j$ .

Substituting (32) into (27b) yields

$$E_v \bar{x}_{i,j+1}^v = A_{22} \bar{x}_{i,j}^v + \hat{u}_{i,j} + \hat{w}_i, \tag{33a}$$

where

$$\begin{aligned} \hat{u}_{i,j} &= \sum_{k=0}^{i-1} A_{21} \bar{E}_h^D (\bar{E}_h^D \bar{A}_{11})^{i-k-1} \bar{B}_1 u_{k,j} \\ &+ A_{21} (\bar{E}_h \bar{E}_h^D - I_{n_1}) \\ &\times \sum_{k=0}^{q_1-1} (\bar{E}_h \bar{A}_{11}^D)^k \bar{A}_{11}^D \bar{B}_1 u_{i+k,j} + B_2 u_{i,j}, \end{aligned} \tag{33b}$$

$$\hat{w}_i = A_{21} (\bar{E}_h^D \bar{A}_{11})^i \bar{E}_h^D \bar{E}_h v_1. \tag{33c}$$

From the assumption (25) it follows that

$$\det[E_v z_2 - A_{22}] \neq 0 \quad \text{for some } z_2 \in \mathbb{C}. \tag{34}$$

Therefore, there exists a number  $c_2 \in \mathbb{C}$  such that  $\det[E_v c_2 - A_{22}] \neq 0$  and, premultiplying (33) by  $[E_v c_2 - A_{22}]^{-1}$ , we obtain

$$\bar{E}_v \bar{x}_{i,j+1}^v = \bar{A}_{22} \bar{x}_{i,j}^v + \bar{B}_2 (\hat{u}_{i,j} + \hat{w}_i), \tag{35a}$$

where

$$\begin{aligned} \bar{E}_v &= [E_v c_2 - A_{22}]^{-1} E_v, \\ \bar{A}_{22} &= [E_v c_2 - A_{22}]^{-1} A_{22}, \\ \bar{B}_2 &= [E_v c_2 - A_{22}]^{-1}. \end{aligned} \tag{35b}$$

Let  $\bar{E}_v^D (\bar{A}_{22}^D)$  be the Drazin inverse of the matrix  $\bar{E}_v (\bar{A}_{22})$ . Then, from Theorem 2, we have

$$\begin{aligned} \bar{x}_{i,j}^v &= (\bar{E}_v^D \bar{A}_{22})^j \bar{E}_v^D \bar{E}_v v_2 \\ &+ \sum_{l=0}^{j-1} \bar{E}_v^D (\bar{E}_v^D \bar{A}_{22})^{j-l-1} \bar{B}_2 (\hat{u}_{i,l} + \hat{w}_i) \\ &+ (\bar{E}_v \bar{E}_v^D - I_{n_2}) \sum_{l=0}^{q_2-1} (\bar{E}_v \bar{A}_{22}^D)^l \\ &\times \bar{A}_{22}^D \bar{B}_2 (\hat{u}_{i,j+l} + \hat{w}_i), \end{aligned} \tag{36}$$

where  $q_2$  is the index of  $\bar{E}_v$  and  $v_2 \in \mathbb{R}^{n_2}$  is arbitrary depending on  $i$ .

Knowing  $\bar{x}_{i,j}^h$  and  $\bar{x}_{i,j}^v$  we can find the solution of Eqn. (27) from (26) and obtain

$$\begin{bmatrix} \bar{x}_{i,j}^h \\ \bar{x}_{i,j}^v \end{bmatrix} = Q^{-1} \begin{bmatrix} \bar{x}_{i,j}^h \\ \bar{x}_{i,j}^v \end{bmatrix}. \tag{37}$$

Therefore, the following result has been proved.

**Theorem 4.** The solution of Eqn. (27a) is given by (37) and the vectors  $\bar{x}_{i,j}^h$  and  $\bar{x}_{i,j}^v$  are defined by (32) and (36), respectively.

From the assumption (25), for Case 2 it follows that

$$\det[E_v z_2 - A_{22}] \neq 0 \quad \text{for some } z_2 \in \mathbb{C}. \tag{38}$$

Therefore, there exists a number  $c_2 \in \mathbb{C}$  such that  $\det[E_v c_2 - A_{22}] \neq 0$  and, premultiplying (27b) by  $[E_v c_2 - A_{22}]^{-1}$ , we obtain

$$\bar{E}_v \bar{x}_{i,j+1}^v = \bar{A}_{22} \bar{x}_{i,j}^v + \bar{B}_2 u_{i,j}, \tag{39a}$$

where

$$\begin{aligned} \bar{E}_v &= [E_v c_2 - A_{22}]^{-1} E_v, \\ \bar{A}_{22} &= [E_v c_2 - A_{22}]^{-1} A_{22}, \\ \bar{B}_2 &= [E_v c_2 - A_{22}]^{-1} B_2. \end{aligned} \tag{39b}$$

Let  $\bar{E}_v^D (\bar{A}_{22}^D)$  be the Drazin inverse of the matrix  $\bar{E}_v (\bar{A}_{22})$ . Then, from Theorem 2, we have

$$\begin{aligned} \bar{x}_{i,j}^v &= (\bar{E}_v^D \bar{A}_{22})^j \bar{E}_v^D \bar{E}_v v_3 \\ &+ \sum_{l=0}^{j-1} \bar{E}_v^D (\bar{E}_v^D \bar{A}_{22})^{j-l-1} \bar{B}_2 u_{i,l} \\ &+ (\bar{E}_v \bar{E}_v^D - I_{n_2}) \sum_{l=0}^{q_2-1} (\bar{E}_v \bar{A}_{22}^D)^l \bar{A}_{22}^D \bar{B}_2 u_{i,j+l}, \end{aligned} \tag{40}$$

where  $q_2$  is the index of  $\bar{E}_v$  and  $v_3 \in \mathbb{R}^{n_2}$  is arbitrary.

Substituting (40) into (27a) yields

$$E_h \bar{x}_{i+1,j}^h = A_{11} \bar{x}_{i,j}^h + \tilde{u}_{i,j} + \tilde{w}_j, \tag{41a}$$

where

$$\begin{aligned} \tilde{u}_{i,j} &= \sum_{l=0}^{j-1} A_{21} \bar{E}_v^D (\bar{E}_v^D \bar{A}_{22})^{j-l-1} \bar{B}_2 u_{i,l} \\ &+ A_{12} (\bar{E}_v \bar{E}_v^D - I_{n_2}) \sum_{l=0}^{q_2-1} (\bar{E}_v \bar{A}_{22}^D)^l \\ &\times \bar{A}_{22}^D \bar{B}_2 u_{i,j+l} + B_1 u_{i,j}, \end{aligned} \tag{41b}$$

$$\tilde{w}_j = A_{12} (\bar{E}_v^D \bar{A}_{22})^j \bar{E}_v^D \bar{E}_v v_3. \tag{41c}$$

From the assumption (25) it follows that

$$\det[E_h z_1 - A_{11}] \neq 0 \quad \text{for some } z_1 \in \mathbb{C}. \tag{42}$$

Therefore, there exists a number  $c_1 \in \mathbb{C}$  such that  $\det[E_h c_1 - A_{11}] \neq 0$  and, premultiplying (41a) by  $[E_h c_1 - A_{11}]^{-1}$ , we obtain

$$\tilde{E}_h \tilde{x}_{i+1,j}^h = \tilde{A}_{11} \tilde{x}_{i,j}^h + \tilde{B}_1 (\tilde{u}_{i,j} + \tilde{w}_j), \quad (43a)$$

where

$$\begin{aligned} \tilde{E}_h &= [E_h c_1 - A_{11}]^{-1} E_h, \\ \tilde{A}_{11} &= [E_h c_1 - A_{11}]^{-1} A_{11}, \\ \tilde{B}_1 &= [E_h c_1 - A_{11}]^{-1}. \end{aligned} \quad (43b)$$

Let  $\tilde{E}_h^D (\tilde{A}_{11}^D)$  be the Drazin inverse of the matrix  $\tilde{E}_h (\tilde{A}_{11})$ . Then, from Theorem 2, we have

$$\begin{aligned} \tilde{x}_{i,j}^h &= (\tilde{E}_h^D \tilde{A}_{11})^i \tilde{E}_h^D \tilde{E}_h v_4 \\ &+ \sum_{k=0}^{i-1} \tilde{E}_h^D (\tilde{E}_h^D \tilde{A}_{11})^{i-k-1} \tilde{B}_1 (\tilde{u}_{k,j} + \tilde{w}_j) \\ &+ (\tilde{E}_h \tilde{E}_h^D - I_{n_1}) \sum_{k=0}^{q_1-1} (\tilde{E}_h \tilde{A}_{11}^D)^k \\ &\times \tilde{A}_{11}^D \tilde{B}_1 (\tilde{u}_{i+k,j} + \tilde{w}_j), \end{aligned} \quad (44)$$

where  $q_1$  is the index of  $\tilde{E}_h$  and  $v_4 \in \mathbb{R}^{n_1}$  is arbitrary.

Knowing  $\tilde{x}_{i,j}^h$  and  $\tilde{x}_{i,j}^v$ , we can find the solution of Eqn. (27b) from (37).

Therefore, the following result has been proved.

**Theorem 5.** *The solution of Eqn. (27b) is given by (37) and the vectors  $\tilde{x}_{i,j}^h$  and  $\tilde{x}_{i,j}^v$  are defined by (44) and (40), respectively.*

#### 4. Example

Consider the descriptor discrete-time linear system (24) with the matrices

$$\begin{aligned} E &= \begin{bmatrix} 0 & 0.5 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -2 \end{bmatrix}, \\ A &= \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ -1 & -1 & 0 & 2 \\ -2 & 0 & -1 & 5 \end{bmatrix}, \\ B &= \begin{bmatrix} 0 \\ 1 \\ -2 \\ 4 \end{bmatrix}. \end{aligned} \quad (45)$$

Premultiplying Eqn. (1) with (45) by the matrix

$$P = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} \quad (46)$$

of the elementary row operations  $L[1 \times 2], L[2+1 \times (-1)], L[3 \times (-1)], L[4+3 \times 2]$  and introducing the new state vector (26) with

$$Q = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad (47)$$

we obtain Eqn. (27a) with

$$\begin{aligned} \begin{bmatrix} E_h & 0 \\ 0 & E_v \end{bmatrix} &= PEQ^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\ \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} &= PAQ^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 1 & 1 & -2 & 0 \\ 2 & 0 & 1 & -1 \end{bmatrix}, \\ \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} &= PB = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 8 \end{bmatrix}. \end{aligned} \quad (48)$$

Using the procedure presented for Case 1, we obtain what follows.

For  $c_1 = 0$ , from (31b) we have

$$\begin{aligned} \bar{E}_h &= [-A_{11}]^{-1} E_h = \frac{1}{3} \begin{bmatrix} -1 & 0 \\ -2 & 0 \end{bmatrix}, \\ \bar{A}_{11} &= [-A_{11}]^{-1} A_{11} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \\ \bar{B}_1 &= [-A_{11}]^{-1} B_1 = \frac{1}{3} \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \end{aligned} \quad (49)$$

Taking into account that

$$\bar{E}_h = VW, \quad V = \frac{1}{3} \begin{bmatrix} -1 \\ -2 \end{bmatrix}, \quad W = [1 \ 0] \quad (50)$$

and using Procedure A1 from Appendix, we obtain

$$\begin{aligned} \bar{E}_h^D &= V[W \bar{E}_h V]^{-1} W \\ &= \frac{1}{3} \begin{bmatrix} -1 \\ -2 \end{bmatrix} \left[ \frac{1}{9} [1 \ 0] \begin{bmatrix} -1 & 0 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \end{bmatrix} \right]^{-1} \\ &\times [1 \ 0] \\ &= \begin{bmatrix} -3 & 0 \\ -6 & 0 \end{bmatrix} \end{aligned} \quad (51)$$

and

$$\bar{A}_{11}^D = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (52)$$

The index  $q_1$  of the matrix (49) is equal to one. Using (32) and taking into account that

$$\begin{aligned} \bar{E}_h^D \bar{A}_{11} &= \begin{bmatrix} -3 & 0 \\ -6 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 6 & 0 \end{bmatrix}, \\ \bar{E}_h^D \bar{E}_h &= \begin{bmatrix} -3 & 0 \\ -6 & 0 \end{bmatrix} \frac{1}{3} \begin{bmatrix} -1 & 0 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \end{aligned} \tag{53}$$

we obtain

$$\begin{aligned} \bar{x}_{i,j}^h &= (\bar{E}_h^D \bar{A}_{11})^i \bar{E}_h^D \bar{E}_h v_1(j) \\ &+ \sum_{k=0}^{i-1} \bar{E}_h^D (\bar{E}_h^D \bar{A}_{11})^{i-k-1} \bar{B}_1 u_{k,j} \\ &+ (\bar{E}_h \bar{E}_h^D - I_{n_1}) \bar{A}_{11}^D \bar{B}_1 u_{i,j} \\ &= \begin{bmatrix} 3 & 0 \\ 6 & 0 \end{bmatrix}^i \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} v_1(j) \\ &+ \sum_{k=0}^{i-1} \begin{bmatrix} -3 & 0 \\ -6 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 6 & 0 \end{bmatrix}^{i-k-1} \frac{1}{3} \begin{bmatrix} -1 \\ 1 \end{bmatrix} u_{k,j} \\ &+ \begin{bmatrix} 0 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} -1 \\ 1 \end{bmatrix} u_{i,j} \\ &= \begin{bmatrix} 3^i & 0 \\ 2(3)^i & 0 \end{bmatrix} v_1(j) \\ &+ \sum_{k=0}^{i-1} \begin{bmatrix} 3^{i-k-1} \\ 2(3)^{i-k-1} + 1 \end{bmatrix} u_{k,j} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_{i,j}, \end{aligned} \tag{54}$$

where  $v_1(j)$  is an arbitrary function of  $j$ .

Substituting (54) into Eqn. (28b), we obtain (31a) with

$$E_v = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix}, \tag{55a}$$

$$\begin{aligned} \hat{u}_{i,j} &= \sum_{k=0}^{i-1} \begin{bmatrix} 3^{i-k-1} \\ 2(3)^{i-k-1} + 1 \end{bmatrix} u_{k,j} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_{i,j}, \\ \hat{w}_i &= \begin{bmatrix} 3^i & 0 \\ 2(3)^i & 0 \end{bmatrix} v_1(j). \end{aligned} \tag{55b}$$

In this case, we choose  $c_2 = 0$  and, using (31), we obtain

$$\bar{E}_v \bar{x}_{i,j+1}^v = \bar{A}_{22} \bar{x}_{i,j}^v + \bar{B}_2 (\hat{u}_{i,j} + \hat{w}_i), \tag{56a}$$

where

$$\begin{aligned} \bar{E}_v &= [-A_{22}]^{-1} E_v = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \\ \bar{A}_{22} &= [-A_{22}]^{-1} A_{22} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \\ \bar{B}_2 &= [-A_{22}]^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}. \end{aligned} \tag{56b}$$

Taking into account that

$$\bar{E}_v = VW, \quad V = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad W = [1 \ 0] \tag{57}$$

and using Procedure A1 from Appendix, we obtain

$$\begin{aligned} \bar{E}_v^D &= V[W \bar{E}_v V]^{-1} W \\ &= \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left[ \frac{1}{4} [1 \ 0] \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right]^{-1} [1 \ 0] \\ &= \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} \end{aligned} \tag{58}$$

and

$$\bar{A}_{22}^D = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{59}$$

Using (58), (59) and (36), we obtain

$$\begin{aligned} \bar{x}_{i,j}^v &= \begin{bmatrix} -2 & 0 \\ -2 & 0 \end{bmatrix}^j v_2(i) \\ &+ \sum_{l=0}^{j-1} \begin{bmatrix} -2 & 0 \\ -2 & 0 \end{bmatrix}^{j-l-1} (\hat{u}_{i,l} + \hat{w}_i) \\ &+ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (\hat{u}_{i,l} + \hat{w}_i), \end{aligned} \tag{60}$$

where  $v_2(i)$  is an arbitrary function of  $i$ ,  $\bar{x}_{i,j}^{h1}$  and  $\bar{x}_{i,j}^{h2}$  are the components of  $\bar{x}_{i,j}^h$  given by (54).

Knowing the vectors (54), (60) and the matrix (47), we can find the solution of Eqn. (1) with (45) from (37).

### 5. Concluding remarks

A method of analysis for a class of descriptor 2D discrete-time linear systems described by the Roesser model with a regular pencil has been proposed. The method is based on transformation of the descriptor Roesser model (1) to the form (27) with the use of elementary row and column operations. To find a solution to Eqn. (27), a method based on application of the Drazin inverse has been proposed (Theorems 4 and 5). The method has been illustrated with a numerical example. It can be extended to fractional and fractional positive descriptor linear systems described by Roesser models with regular pencils. An extension of the method to 2D descriptor linear systems described by the Fornasini–Marchesini and Kurek models constitutes an open problem.

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## Appendix

### Procedure for computation of Drazin inverse matrices

To compute the Drazin inverse  $\bar{E}^D$  of the matrix  $\bar{E} \in \mathbb{R}^{n \times n}$  defined by (3b), the following procedure is recommended.

**Procedure A1.**

*Step 1.* Find a pair of matrices  $V \in \mathbb{R}^{n \times r}$ ,  $W \in \mathbb{R}^{r \times n}$  such that

$$\bar{E} = VW, \quad \text{rank } V = \text{rank } W = \text{rank } \bar{E} = r. \quad (\text{A1})$$

As the  $r$  columns (rows) of the matrix  $V$  ( $W$ ), the  $r$  linearly independent columns (rows) of the matrix  $\bar{E}$  can be chosen.

*Step 2.* Compute the nonsingular matrix

$$W\bar{E}V \in \mathbb{R}^{r \times r}. \quad (\text{A2})$$

*Step 3.* The desired Drazin inverse matrix is given by

$$\bar{E}^D = V[W\bar{E}V]^{-1}W. \quad (\text{A3})$$

*Proof.* It will be shown that the matrix (A3) satisfies the three conditions (5) of Definition 2. Taking into account that  $\det WV \neq 0$  and (A1), we obtain

$$[W\bar{E}V]^{-1} = [WVWV]^{-1} = [WV]^{-1}[WV]^{-1}. \quad (\text{A4})$$

Using (5), (A1) and (A4), we obtain

$$\begin{aligned} \bar{E}\bar{E}^D &= VWV[W\bar{E}V]^{-1}W \\ &= VWV[WV]^{-1}[WV]^{-1}W \\ &= V[WV]^{-1}W \end{aligned} \quad (\text{A5})$$

and

$$\begin{aligned} \bar{E}^D\bar{E} &= V[W\bar{E}V]^{-1}WVW \\ &= V[WV]^{-1}[WV]^{-1}WVW \\ &= V[WV]^{-1}W. \end{aligned} \quad (\text{A6})$$

Therefore, the condition (5) is satisfied.

To check the condition (5), we compute

$$\begin{aligned} \bar{E}^D\bar{E}\bar{E}^D &= V[W\bar{E}V]^{-1}WVWV[W\bar{E}V]^{-1}W \\ &= V[WVWV]^{-1}WVWV[W\bar{E}V]^{-1}W \\ &= V[W\bar{E}V]^{-1}W = \bar{E}^D. \end{aligned} \quad (\text{A7})$$

Therefore, the condition (5) is also satisfied.

Using (5), (A1), (A3) and (A4), we obtain

$$\begin{aligned} \bar{E}^D\bar{E}^{q+1} &= V[W\bar{E}V]^{-1}W(VW)^{q+1} \\ &= V[WV]^{-1}[WV]^{-1}WVW(VW)^q \\ &= V[WV]^{-1}W(VW)^q \\ &= VW(VW)^{q-1} \\ &= (VW)^q = \bar{E}^q, \end{aligned} \quad (\text{A8})$$

where  $q$  is the index of  $\bar{E}$ . Therefore, the condition (5) is also satisfied.

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