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# TIME–OPTIMAL CONTROL OF PARABOLIC TIME LAG SYSTEM†

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A time–optimal control problem for a distributed parabolic system in which constant time lags appear both in the state equation and in the boundary condition is presented. Some particular properties of the optimal control are discussed.

## 1. Introduction

Various optimization problems associated with the optimal control of distributed parameter systems with time lags appearing in the boundary conditions have been studied recently by Knowles (1978), Kowalewski (1987a; 1987b; 1988a; 1988b; 1988c; 1990a; 1990b; 1990c; 1990d; 1991), Kowalewski and Duda (1992), Wang (1975) and Wong (1987).

In this paper, we consider the time–optimal control problem for a linear parabolic system in which constant time lags appear in the state equation and in the Neumann boundary condition simultaneously. This equation constitutes a universal mathematical model in a linear approximation for many diffusion processes in which time–delayed feedback signals are introduced at the boundary of the system's spatial domain. Then, the signal at the boundary depends at any time on the signal which escaped earlier. This leads to boundary conditions involving time lags.

Existence and uniqueness of solutions of such parabolic equations are discussed. The optimal control is characterized by the adjoint equation. Using this characterization particular properties of the optimal control are proved.

## 2. Existence and Uniqueness of Solutions

Consider now a distributed–parameter system described by the following parabolic equation:

$$\frac{\partial y}{\partial t} + A(t)y + b(x,t)y(x,t-h) = u \qquad x \in \Omega,\ \ t \in (0,T) \tag{1}$$

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$$y(x,t') = \Phi_0(x,t') \qquad x \in \Omega,\ t' \in [-h,0) \tag{2}$$

$$y(x,0) = y_0(x) \qquad x \in \Omega \tag{3}$$

$$\frac{\partial y}{\partial \eta_A} = c(x,t)y(x,t-h) + v \qquad x \in \Gamma,\ t \in (0,T) \tag{4}$$

$$y(x,t') = \Psi_0(x,t') \qquad x \in \Gamma,\ t' \in [-h,0) \tag{5}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded, open set with boundary $\Gamma$, which is a $C^\infty$–manifold of dimension $(n-1)$. Locally, $\Omega$ is totally on one side of $\Gamma$. Moreover, we assume that

$$y \equiv y(x,t;u), \qquad u \equiv u(x,t), \qquad v \equiv v(x,t)$$
$$Q \equiv \Omega \times (0,T), \qquad \overline{Q} = \overline{\Omega} \times [0,T], \qquad Q_0 = \Omega \times [-h,0)$$
$$\Sigma = \Gamma \times (0,T), \qquad \Sigma_0 = \Gamma \times [-h,0)$$

$T$ is a specified positive number representing a time horizon, $b$ – a given real $C^\infty$–function defined on $\overline{Q}$, $c$ – a given real $C^\infty$–function defined on $\Sigma$, $h$ – a specified positive number representing a time lag, $\Phi_0$ – an initial function defined on $Q_0$, $\Psi_0$ – an initial function defined on $\Sigma_0$.

The parabolic operator $\dfrac{\partial}{\partial t} + A(t)$ in the state equation (1) satisfies the hypothesis of Section 1, Chapter 4 of (Lions and Magenes, 1972: v.2, p.2) and $A(t)$ is given by

$$A(t)y = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial y(x,t)}{\partial x_j}\right) \tag{6}$$

and the functions $a_{ij}(x,t)$ are real $C^\infty$–functions defined on $\overline{Q}$ (closure of $Q$) satisfying the ellipticity condition

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\varphi_i\varphi_j \geq \alpha \sum_{i=1}^{n} \varphi_i^2, \quad \alpha > 0,\ \forall (x,t) \in \overline{Q},\ \forall \varphi_i \in \mathbb{R} \tag{7}$$

Equations (1)–(5) constitute a Neumann problem. Then, the left–hand side of (4) is written in the following form

$$\frac{\partial y}{\partial \eta_A} = \sum_{i,j=1}^{n} a_{ij}(x,t)\cos(\widehat{n},x_i)\frac{\partial y(x,t)}{\partial x_j} = q(x,t) \tag{8}$$

where $\dfrac{\partial y}{\partial \eta_A}$ is the normal derivative of $y$ at $\Gamma$, directed towards the exterior of $\Omega$, $\cos(n,x_i)$ – the $i$–th direction cosine of $\widehat{n}$, with $\widehat{n}$ being the normal at $\Gamma$ exterior to $\Omega$, and

$$q(x,t) = c(x,t)y(x,t-h) + v(x,t) \tag{9}$$

For simplicity, we introduce the following notation:

$$E_j \stackrel{\text{def}}{=} ((j-1)h, jh), \qquad Q_j = \Omega \times E_j, \qquad Q_0 = \Omega \times [-h, 0)$$
$$\Sigma_j = \Gamma \times E_j, \qquad \Sigma_0 = \Gamma \times [-h, 0) \qquad \text{for} \quad j = 1, \dots, K$$

The existence of a unique solution of the mixed initial–boundary value problem (1)–(5) was verified by Kowalewski (1990d). It was shown that the following theorem holds:

**Theorem 1.** *Let* $y_0, \Phi_0, \Psi_0, v, u$ *be given with* $y_0 \in H^1(\Omega)$, $\Phi_0 \in H^{2,1}(Q_0)$, $\Psi_0 \in H^{1/2,1/4}(\Sigma_0)$, $v \in H^{1/2,1/4}(\Sigma)$ *and* $u \in L^2(Q)$. *Then, there exists a unique solution* $y \in H^{2,1}(Q)$ *for the mixed initial–boundary value problem (1)–(5). Moreover,* $y(\cdot, jh) \in H^1(\Omega)$ *for* $j = 1, \dots, K$.

Next, we shall establish $v \in H^{1/2,1/4}(\Sigma)$.

## 3. Problem Formulation and Optimization Theorems

Now, we shall formulate the minimum–time problem for (1)–(5) in the context of Theorem 1, that is

$$u \in U = \Big\{ u \in L^2(Q) : \big|u(x,t)\big| \leq 1 \ \text{a.e.} \Big\} \tag{10}$$

We shall define the reachable set $Y$ such that

$$Y = \Big\{ y \in L^2(Q) : \big\|y - z_d\big\|_{L^2(\Omega)} \leq \varepsilon \Big\} \tag{11}$$

where $z_d$ and $\varepsilon$ are given with $z_d \in L^2(\Omega)$ and $\varepsilon > 0$.

The solution of the stated minimum–time problem is equivalent to achieving the target set $Y$ in minimum time, that is, minimizing the time $t$, for which $y(t;u) \in Y$ and $u \in U$.

Moreover, we assume that

$$\text{there exist} \quad T > 0 \quad \text{and} \quad u \in U \quad \text{such that} \quad y(T;u) \in Y \tag{12}$$

In (Knowles, 1978) the following result was proven:

**Theorem 2.** *If assumption (12) holds, then the set* $Y$ *is reached in minimum time* $t^*$ *by an admissible control* $u^* \in U$. *Moreover*

$$\int_\Omega \big(z_d - y(t^*; u^*)\big)\big(y(t^*; u) - y(t^*; u^*)\big)\, \mathrm{d}x \leq 0, \qquad \forall\, u \in U \tag{13}$$

We shall apply Theorem 2 to the control of (1)–(5).

To simplify (13), we introduce the adjoint equation and for every $u \in U$, we define the adjoint variable $p = p(u) = p(x,t;u)$ as the solution of the equation

$$-\frac{\partial p(u)}{\partial t} + A^*(t)p(u) + b(x, t+h)p(x, t+h; u) = 0, \quad x \in \Omega, \ t \in (0, t^* - h) \tag{14}$$

$$-\frac{\partial p(u)}{\partial t} + A^*(t)p(u) = 0, \qquad x \in \Omega,\ t \in (t^* - h, t^*) \tag{15}$$

$$p(x,t^*;u) = z_d(x) - y(x,t^*;u), \qquad x \in \Omega \tag{16}$$

$$\frac{\partial p(u)}{\partial \eta_{A^*}}(x,t) = c(x,t+h)p(x,t+h;u), \quad x \in \Gamma,\ t \in (0,t^* - h) \tag{17}$$

$$\frac{\partial p(u)}{\partial \eta_{A^*}}(x,t) = 0, \qquad x \in \Gamma,\ t \in (t^* - h, t^*) \tag{18}$$

where

$$\begin{cases} \dfrac{\partial p(u)}{\partial \eta_{A^*}}(x,t) = \sum_{i,j=1}^{n} a_{ij}(x,t)\cos(\widehat{n}, x_i)\dfrac{\partial p(u)}{\partial x_j}(x,t) \\ A^*(t)p = -\sum_{i,j=1}^{n} \dfrac{\partial}{\partial x_j}\Big(a_{ij}(x,t)\dfrac{\partial p}{\partial x_i}\Big) \end{cases} \tag{19}$$

The existence of a unique solution for problem (14)–(18) on the cylinder $\Omega \times (0,t^*)$ can be proven using a constructive method. It is easy to notice that for a given $z_d$ and $u$, problem (14)–(18) can be solved backwards in time starting from $t = t^*$, i.e. solving (14)–(18) first on the subcylinder $Q_K$ and then on $Q_{K-1}$, etc. until the procedure covers the whole cylinder $\Omega \times (0,t^*)$. To this end, we may apply Theorem 1 (with an obvious change of variables). It is easy to verify that the following result holds.

**Theorem 3.** *Let the hypothesis of Theorem 1 be satisfied. Then for a given* $z_d \in L^2(\Omega)$ *and any* $u \in L^2(Q)$, *there exists a unique solution* $p(u) \in H^{2,1}(\Omega \times (0,t^*))$ *for the adjoint problem (14)–(18).*

We simplify (13) using the adjoint equation (14)–(18). After setting $u = u^*$ in (14)–(18), multiplying both sides of (14), (15) by $y(u) - y(u^*)$, then integrating over $\Omega \times (0, t^* - h)$ and $\Omega \times (t^* - h, t^*)$, respectively, and then adding both sides of (14) and (15) we get

$$\begin{aligned} &\int_0^{t^*}\int_\Omega \Big(-\frac{\partial p(u^*)}{\partial t} + A^*(t)p(u^*)\Big)\,(y(u) - y(u^*))\,\mathrm{d}x\mathrm{d}t \\ &\quad = -\int_\Omega p(x,t^*;u^*)y\big((x,t^*;u) - y(x,t^*;u^*)\big)\,\mathrm{d}x \\ &\quad + \int_0^{t^*}\int_\Omega p(u^*)\frac{\partial}{\partial t}(y(u) - y(u^*))\,\mathrm{d}x\mathrm{d}t + \int_0^{t^*}\int_\Omega A^*(t)p(u^*)(y(u) - y(u^*))\,\mathrm{d}x\mathrm{d}t \\ &\quad + \int_0^{t^*-h}\int_\Omega b(x,t+h)p(x,t+h;u^*)\big(y(x,t;u) - y(x,t;u^*)\big)\,\mathrm{d}x\mathrm{d}t = 0 \end{aligned} \tag{20}$$

Then, after applying (16), formula (20) can be expressed as

$$\int_{\Omega} \left(z_d - y(t^*;u^*)\right)\left(y(t^*;u) - y(t^*;u^*)\right) \mathrm{d}x$$

$$= \int_0^{t^*}\int_{\Omega} p(u^*)\frac{\partial}{\partial t}\left(y(u) - y(u^*)\right) \mathrm{d}x\mathrm{d}t + \int_0^{t^*}\int_{\Omega} A^*(t)p(u^*)\left(y(u) - y(u^*)\right) \mathrm{d}x\mathrm{d}t$$

$$+ \int_0^{t^*-h}\int_{\Omega} b(x,t+h)p(x,t+h;u^*)\left(y(x,t;u) - y(x,t;u^*)\right) \mathrm{d}x\mathrm{d}t \qquad (21)$$

Using equation (1), the first integral on the right–hand side of (21) can be rewritten as

$$\int_0^{t^*}\int_{\Omega} p(u^*)\frac{\partial}{\partial t}\left(y(u) - y(u^*)\right) \mathrm{d}x\mathrm{d}t$$

$$= \int_0^{t^*}\int_{\Omega} p(u^*)(u - u^*)\,\mathrm{d}x\mathrm{d}t - \int_0^{t^*}\int_{\Omega} p(u^*)A(t)\left(y(u) - y(u^*)\right) \mathrm{d}x\mathrm{d}t$$

$$- \int_0^{t^*}\int_{\Omega} p(x,t;u^*)b(x,t)\left(y(x,t-h;u) - y(x,t-h;u^*)\right) \mathrm{d}x\mathrm{d}t$$

$$= \int_0^{t^*}\int_{\Omega} p(u^*)(u - u^*)\,\mathrm{d}x\mathrm{d}t - \int_0^{t^*}\int_{\Omega} p(u^*)A(t)\left(y(u) - y(u^*)\right) \mathrm{d}x\mathrm{d}t$$

$$- \int_{-h}^{t^*-h}\int_{\Omega} p(x,t'+h;u^*)b(x,t'+h)\left(y(x,t';u) - y(x,t';u^*)\right) \mathrm{d}x\mathrm{d}t' \qquad (22)$$

The second integral on the right–hand side of (21), in view of Green's formula, can be expressed as

$$\int_0^{t^*}\int_{\Omega} A^*(t)p(u^*)\left(y(u) - y(u^*)\right) \mathrm{d}x\mathrm{d}t = \int_0^{t^*}\int_{\Omega} p(u^*)A(t)\left(y(u) - y(u^*)\right) \mathrm{d}x\mathrm{d}t$$

$$+ \int_0^{t^*}\int_{\Gamma} p(u^*)\left(\frac{\partial y(u)}{\partial \eta_A} - \frac{\partial y(u^*)}{\partial \eta_A}\right) \mathrm{d}\Gamma\mathrm{d}t - \int_0^{t^*}\int_{\Gamma} \frac{\partial y(u^*)}{\partial \eta_{A^*}}\left(y(u) - y(u^*)\right) \mathrm{d}\Gamma\mathrm{d}t \qquad (23)$$

Applying the boundary condition (4), the second integral on the right–hand side of (23) can be expressed as

$$\int_0^{t^*}\int_\Gamma p(u^*)\left(\frac{\partial y(u)}{\partial \eta_A}-\frac{\partial (u^*)}{\partial \eta_A}\right)\,\mathrm{d}\Gamma \mathrm{d}t$$

$$=\int_0^{t^*}\int_\Gamma p(x,t;u^*)c(x,t)\big(y(x,t-h;u)-y(x,t-h;u^*)\big)\,\mathrm{d}\Gamma \mathrm{d}t$$

$$=\int_{-h}^{t^*-h}\int_\Gamma p(x,t'+h;u^*)c(x,t'+h)\big(y(x,t';u)-y(x,t';u^*)\big)\,\mathrm{d}\Gamma \mathrm{d}t' \tag{24}$$

The last component in (23) can be rewritten as

$$\int_0^{t^*}\int_\Gamma \frac{\partial p(u^*)}{\partial \eta_{A^*}}\big(y(u)-y(u^*)\big)\,\mathrm{d}\Gamma \mathrm{d}t$$

$$=\int_0^{t^*-h}\int_\Gamma \frac{\partial p(u^*)}{\partial \eta_{A^*}}\big(y(u)-y(u^*)\big)\,\mathrm{d}\Gamma \mathrm{d}t+\int_{t^*-h}^{t^*}\int_\Gamma \frac{\partial p(u^*)}{\partial \eta_{A^*}}\big(y(u)-y(u^*)\big)\,\mathrm{d}\Gamma \mathrm{d}t \tag{25}$$

Substituting (24), (25) into (23) and then (22), (23) into (21) we obtain

$$\int_\Omega \big(z_d-y(t^*;u^*)\big)\big(y(t^*;u)-y(t^*;u^*)\big)\,\mathrm{d}x$$

$$=\int_0^{t^*}\int_\Omega p(u^*)(u-u^*)\,\mathrm{d}x\mathrm{d}t-\int_0^{t^*}\int_\Omega p(u^*)A(t)\big(y(u)-y(u^*)\big)\,\mathrm{d}x\mathrm{d}t$$

$$-\int_{-h}^{0}\int_\Omega b(x,t+h)p(x,t+h;u^*)\big(y(x,t;u)-y(x,t;u^*)\big)\,\mathrm{d}x\mathrm{d}t$$

$$-\int_0^{t^*-h}\int_\Omega b(x,t+h)p(x,t+h;u^*)\big(y(x,t;u)-y(x,t;u^*)\big)\,\mathrm{d}x\mathrm{d}t$$

$$+\int_0^{t^*}\int_\Omega p(u^*)A(t)\big(y(u)-y(u^*)\big)\,\mathrm{d}x\mathrm{d}t$$

$$+\int_{-h}^{0}\int_{\Gamma} p(x,t+h;u^*)c(x,t+h)\big(y(x,t;u)-y(x,t;u^*)\big)\,\mathrm{d}\Gamma\mathrm{d}t$$

$$+\int_{0}^{t^*-h}\int_{\Gamma} p(x,t+h;u^*)c(x,t+h)\big(y(x,t;u)-y(x,t;u^*)\big)\,\mathrm{d}\Gamma\mathrm{d}t$$

$$-\int_{0}^{t^*-h}\int_{\Gamma} \frac{\partial p(u^*)}{\partial\eta_{A^*}}\big(y(u)-y(u^*)\big)\,\mathrm{d}\Gamma\mathrm{d}t-\int_{t^*-h}^{t^*}\int_{\Gamma} \frac{\partial p(u^*)}{\partial\eta_{A^*}}\big(y(u)-y(u^*)\big)\,\mathrm{d}\Gamma\mathrm{d}t$$

$$+\int_{0}^{t^*-h}\int_{\Omega} b(x,t+h)p(x,t+h;u^*)\big(y(x,t';u)-y(x,t';u^*)\big)\,\mathrm{d}x\mathrm{d}t$$

$$=\int_{0}^{t^*}\int_{\Omega} p(u^*)(u-u^*)\,\mathrm{d}x\mathrm{d}t \tag{26}$$

Substituting (26) into (13) gives

$$\int_{0}^{t^*}\int_{\Omega} p(u^*)(u-u^*)\,\mathrm{d}x\mathrm{d}t\le 0,\qquad \forall\, u\in U \tag{27}$$

The above result can now be summarized.

**Theorem 4.** *The optimal control* $u^*$ *is characterized by condition (27). Moreover, in particular case*

$$u^*(x,t)=\mathrm{sign}\big(p(x,t;u^*)\big),\qquad x\in\Omega,\ \ t\in(0,t^*) \tag{28}$$

*where* $p(x,t)\neq 0$.

This property leads to the following result:

**Theorem 5.** *If the coefficients of the operator* $A(t)$ *and the functions* $b(x,t)$, $c(x,t)$ *are analytic in* $\overline{\Omega}\times[0,T]$, *and* $\Omega$ *has analytic boundary* $\Gamma$, *then there exists a unique optimal control for the mixed initial–boundary value problem (1)–(5). Moreover, the optimal control is bang–bang, that is* $|u^*(x,t)|\equiv 1$ *almost everywhere, and the unique solution of (1)–(5), (14)–(18), (27).*

*Outline of the proof.* We have to verify that $p(x,t)\neq 0$ for almost all $(x,t)\in\Omega\times(0,t^*)$. We shall show this fact by contradiction. Therefore, we suppose that

$$p(x,t)=0\quad\text{for}\quad (x,t)\in K\subset\Omega\times(0,t^*) \tag{29}$$

where $K\neq\emptyset$.

Let us denote by $k_0$ the largest nonnegative integer $k$ such that $t^*-kh>0$.

Apart from that, we suppose that $K \cap \Omega \times (t^* - h, t^*) \neq \emptyset$.

Then $p(u^*)$ satisfies the following adjoint equation in the cylinder $\Omega \times (t^* - h, t^*)$

$$-\frac{\partial p(u^*)}{\partial t} + A^*(t)p(u^*) = 0, \qquad x \in \Omega, \ t \in (t^* - h, t^*) \tag{30}$$

$$\frac{\partial p(u^*)}{\partial \eta_{A^*}} = 0, \qquad x \in \Gamma, \ t \in (t^* - h, t^*) \tag{31}$$

It is easy to verify (Tanabe, 1965) that $p(u^*)$ must be analytic in the cylinder $\Omega \times (t^* - h, t^*)$. Then, from (29) it follows that

$$p(x,t) \equiv 0 \qquad \text{for} \qquad (x,t) \in \overline{\Omega} \times (t^* - h, t^*) \tag{32}$$

Using Theorem 3.1. of (Lions and Magenes, 1972: v.1, p.19) we can verify that $p(u^*) \in H^{2,1}(Q)$ implies that $t \longrightarrow p(t; u^*)$ is a continuous mapping of $[0,T]$ into $H^1(\Omega) \subset L^2(\Omega)$. Thus, $p(t; u^*) \in L^2(\Omega)$, and so

$$p(t^*; u^*) = 0 = y(t^*; u^*) - z_d \tag{33}$$

Hence (33) leads to a contradiction that $z_d \neq y(t^*; u^*)$.

Now, we shall extend our result to any cylinder $\overline{\Omega} \times \big(t^* - kh, t^* - (k-1)h\big)$, $k = 2, 3, \ldots, k_0$.

It is easy to notice that $p(u^*)$ satisfies the adjoint equation

$$-\frac{\partial p(u^*)}{\partial t} + A^*(t)p(u^*) + b(x+h)p(x,t+h;u^*) = 0 \qquad x \in \Omega, \quad t \in (t^* - 2h, t^* - h) \tag{34}$$

$$\frac{\partial p(u^*)}{\partial \eta_{A^*}} = c(x,t+h)p(x,t+h;u^*) \qquad x \in \Gamma, \quad t \in (t^* - 2h, t^* - h) \tag{35}$$

in the cylinder $\overline{\Omega} \times (t^* - 2h, t^* - h)$.

Then, $p\big|_{\Omega}(x, t+h; u^*)$ and $p\big|_{\Gamma}(x, t+h; u^*)$ are analytic for $x \in \overline{\Omega}$, $t \in (t^* - 2h, t^* - h)$ and $x \in \Gamma$, $t \in (t^* - 2h, t^* - h)$, respectively, and consequently $p(u^*)$ must be analytic in $\overline{\Omega} \times (t^* - 2h, t^* - h)$, since (34), (35) have analytic coefficients (Tanabe, 1965). Thus, $p(u^*)$ must be analytic in any cylinder

$$\overline{\Omega} \times \big(t^* - kh, t^* - (k-1)h\big), \quad k = 2, 3, \ldots, k_0, \quad \text{and} \quad \overline{\Omega} \times (0, t^* - k_0 h)$$

Now we suppose that

$$p(u^*) = 0 \qquad \text{for} \quad (x,t) \in K \cap \Omega \times \big(t^* - kh, t^* - (k-1)h\big) \tag{36}$$

for some $k = 2, 3, \ldots, k_0$.

Then, by analyticity and continuity, it follows from (36) that

$$p(u^*) \equiv 0 \qquad \text{for} \quad (x,t) \in \overline{\Omega} \times \big(t^* - kh, t^* - (k-1)h\big) \tag{37}$$

Substituting (37) into (17) gives

$$\frac{\partial p}{\partial \eta_{A^*}}(x,t) = 0 \quad \text{for} \quad (x,t) \in \Gamma \times \big(t^* - (k-1)h, t^* - (k-2)h\big) \tag{38}$$

We can observe that $p(u^*)$ satisfies

$$-\frac{\partial p(u^*)}{\partial t} + A^*(t)p(u^*) = 0, \quad x \in \Omega,\ t \in \big(t^* - (k-1)h, t^* - (k-2)h\big) \tag{39}$$

$$\frac{\partial p(u^*)}{\partial \eta_{A^*}}(x,t) = 0, \quad x \in \Gamma,\ t \in \big(t^* - (k-1)h, t^* - (k-2)h\big) \tag{40}$$

$$p(\cdot, t^* - (k-1)h; u^*) = 0 \tag{41}$$

in the cylinder $\overline{\Omega} \times \big(t^* - (k-1)h, t^* - (k-2)h\big)$.

Then, using the property of backward uniqueness, we have

$$p(u^*) \equiv 0 \quad \text{in} \quad \overline{\Omega} \times \big(t^* - (k-1)h, t^* - (k-2)h\big) \tag{42}$$

We repeat this procedure again until $p(t^*; u^*) = 0$, which leads to a contradiction that $z_d \neq y(t^*; u^*)$. ■

## 4. Conclusions

The results presented in the paper can be treated as a generalization of the results obtained by Knowles (1978) onto the case of additional constant time lags appearing in the state equations. Moreover, the time–optimal control problems presented here can be extended to certain cases of nonlinear control without convexity and to certain fixed–time problems (Knowles, 1978).

Using condition (12), we can also prove that the parabolic system (1)–(5) is approximately controllable in $L^2(\Omega)$ in any finite time $T > 0$, that is the set $\big\{y(T;u) : u \in L^2(Q)\big\}$ is dense in $L^2(\Omega)$.

Finally, one may consider time–optimal control problems for hyperbolic systems in which constant time lags appear both in the state equations and in the boundary conditions.

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