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SENSITIVITY ANALYSIS OF BUCKLING LOADS OF PHYSICALLY NON-LINEAR FRAMES USING DYNAMIC AND STATIC CRITERIA

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Small eigenvibrations around the prestressed equilibrium state are considered and the sensitivity of eigenfrequencies with respect to cross-sectional parameters of the frame is assessed. The sensitivity operators thus derived are used to determine the sensitivity of buckling load with dynamic criterion. For comparison, the static criterion is employed, too. The problem is solved with the use of variational formulation. The redistribution of prestress is evaluated by the adjoint-variable method. Numerical examples solved with the use of the Finite-Element Method illustrate the theory and demonstrate the quantitative contribution of prestress redistribution into sensitivity operators. The numerical efficiency of dynamic and static approaches is discussed.

1. Introduction

Sensitivity analysis (SA) of buckling loads and eigenfrequencies of frames finds wide application in the optimal structural design. An efficient method of computing the sensitivity of eigenfrequencies and buckling loads of linear elastic structures has been presented in (Cohen *et al.*, 1990; Mróz and Haftka, 1988). Design engineers are often faced with problems when the structure responds in the non-linear regime, therefore in the present paper physically non-linear structures are considered. The sensitivity analysis and optimal redesign of physically non-linear columns using the variational calculus and Pontriagin maximum principle were presented in (Garstecki and Glema, 1991; Glema, 1992). The implementation of a static criterion of stability with the use of the Finite-Element Method (FEM) was discussed in (Garstecki and Glema, 1992). The aim of this paper is to demonstrate the use of a dynamic criterion in sensitivity analysis of buckling loads of physically non-linear frames subjected to an initial prestress and to discuss its efficiency in comparison with the static criterion.

2. Problem Statement

Consider a frame structure made of a non-linear elastic material with an arbitrary, increasing $\sigma = \sigma(\varepsilon)$ relation. In the following, three steps of considerations and

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computations can be distinguished. In the first step, the non-linear analysis of the structure subjected to external load is carried out. The layered beam-type finite element and the Newton-Raphson procedure are implemented. The values referring to this state will be denoted by the subscript '0'. Let u_0 and w_0 denote longitudinal and transversal displacements, N_0 and M_0 denote the axial tensile force and bending moment, whereas x and z represent local longitudinal and transversal axes, respectively. In the next step, small free vibrations around the initial state of equilibrium are computed. The third step consists in sensitivity analysis using the results of previous steps.

In the present stage of study, we neglect the initial transverse displacements w_0 and the initial bending moment M_0 in the vibration analysis and in the prebuckling state of equilibrium. Damping is also neglected. Hence, in implementation of the FEM the tangent stiffness, initial stress and mass matrices will appear, whereas the geometric matrices and damping matrices will not be introduced.

For harmonic small vibrations the equilibrium condition can be written in the following variational (weak) form:

$$\int_L \Big[D^b w_1'' \delta w_1'' + D^a u_1' \delta u' + N_0 w_1' \delta w' - \omega^2 m (w_1 \delta w + u_1 \delta u)\Big] \, \mathrm{d}x = 0 \tag{1}$$

for all kinematically admissible variations of displacements δu and δw. Here w_1, u_1 denote displacement amplitudes and ω, m denote the eigenfrequency and distributed mass, respectively. Furthermore, D^b and D^a denote bending and longitudinal tangent stiffness cross-sectional coefficients computed in the non-linear analysis of the structure at the initial state of loading. In the following, $\int \ldots \mathrm{d}x$ will denote integration along all the members of the frame. Note that the initial axial force N_0 is a non-linear function of the cross-sectional design parameter s and strains ε_0

$$N_0 = N_0(s, \varepsilon_0) = \int_{A(s)} \sigma(\varepsilon_0) \, \mathrm{d}A, \qquad \varepsilon_0 = u_0' \tag{2}$$

and

$$D^a = \int_A E^t \, \mathrm{d}A, \qquad D^b = \int_A E^t z^2 \, \mathrm{d}A, \qquad E^t = \frac{\partial \sigma(\varepsilon)}{\partial \varepsilon} \tag{3}$$

where E^t and A denote the tangent modulus of elasticity and cross-sectional area, respectively.

Equation (1) describes the eigenvalue problem. In the FEM formulation the first two terms of (1) contribute to the tangent stiffness matrix, whereas the third and forth terms form the initial stress and mass matrices, respectively.

3. Sensitivity Analysis of Eigenfrequencies

Setting $\delta u = u_1$ and $\delta w = w_1$ in (1), we obtain

$$\int_L \Big[D^b w_1'' w_1'' + D^a u_1' u_1' + N_0 w_1' w_1' - \omega^2 m (w_1 w_1 + u_1 u_1)\Big] \, \mathrm{d}x = 0 \tag{4}$$

The variation of (4) with respect to the design parameter vector s leads to

$$\int_L \Big[w_1''w_1''\delta D^b + 2D^b w_1''\delta w_1'' + u_1'u_1'\delta D^a + 2D^a u_1'\delta u_1' + w_1'w_1'\delta N_0$$
$$+2N_0 w_1'\delta w_1' - \delta(\omega^2)\, m\,(w_1w_1 + u_1u_1) - \omega^2(w_1w_1 + u_1u_1)\delta m$$
$$-2\omega^2 m\,(w_1\delta w_1 + u_1\delta u_1)\Big]\,\mathrm{d}x = 0 \tag{5}$$

The terms standing with factor 2 vanish due to (1).

The variations δD^b and δD^a depend on s explicitly and also implicitly through redistribution of N_0 due to the variation δs

$$\delta D^b = \frac{\partial D^b}{\partial s}\delta s + \frac{\partial D^b}{\partial N_0}\,\delta_s N_0, \qquad \delta D^a = \frac{\partial D^a}{\partial s}\delta s + \frac{\partial D^a}{\partial N_0}\,\delta_s N_0 \tag{6}$$

Using (6) and normalization of the eigenvector $\{u_1, w_1\}$, i.e. based on the dependence $\int m(u_1\, u_1 + w_1\, w_1)\,\mathrm{d}x = 1$, we can rewrite (5) in the form

$$\delta(\omega^2) = \int_L \Bigg\{\Bigg[\frac{\partial D^b}{\partial s}\, w_1''w_1'' + \frac{\partial D^a}{\partial s}\, u_1'u_1' - \omega^2\frac{\partial m}{\partial s}(w_1w_1 + u_1u_1)\Bigg]\,\delta s$$
$$+ \Bigg[\frac{\partial D^b}{\partial N_0}\, w_1''w_1'' + \frac{\partial D^a}{\partial N_0}\, u_1'u_1' + w_1'w_1'\Bigg]\,\delta_s N_0\Bigg\}\,\mathrm{d}x \tag{7}$$

To compute the implicit variation $\delta_s N_0$ in (7), we introduce an adjoint structure which is linear and elastic with stiffness coefficients D^b and D^a. We assume that it is subjected to the initial axial strain

$$\varepsilon_2^i = \frac{\partial D^b}{\partial N_0} w_1''w_1'' + \frac{\partial D^a}{\partial N_0} u_1'u_1' + w_1'w_1' \tag{8}$$

In what follows, the strain, stress and displacement fields of the adjoint structure will be denoted by the subscript '2'.

The strain ε_2^i is kinematically inadmissible, therefore the elastic strain ε_2^e appears, so that the total strain $u_2' = \varepsilon_2 = \varepsilon_2^i + \varepsilon_2^e$ is kinematically admissible, whereas the induced stress N_2 is self-equilibrated. Hence

$$\int \Bigg(\frac{\partial D^b}{\partial N_0} w_1''w_1'' + \frac{\partial D^a}{\partial N_0} u_1'u_1' + w_1'w_1'\Bigg)\,\delta_s N_0\,\mathrm{d}x = \int \varepsilon_2^i \delta_s N_0\,\mathrm{d}x$$
$$= \int \big(\underline{\varepsilon_2\delta_s N_0} - \varepsilon_2^e\delta_s N_0\big)\,\mathrm{d}x = -\int \varepsilon_2^e\delta_s N_0\,\mathrm{d}x \tag{9}$$

The underlined term vanishes because $\delta_s N_0$ is statically admissible and self-equilibrated.

Variation of (2) with respect to s gives

$$\delta_s N_0 = \int_A \frac{\partial \sigma_0}{\partial \varepsilon} \delta\varepsilon_0 \, \mathrm{d}A + \int_{\delta A} \sigma_0 \, \mathrm{d}A = \int_A E^t \delta\varepsilon_0 \, \mathrm{d}A + \int_{\delta A} \sigma_0 \, \mathrm{d}A \tag{10}$$

Introducing (10) to (9) we obtain

$$\begin{aligned} -\int \varepsilon_2^e \delta_s N_0 \, \mathrm{d}x &= -\int \left(\int_A \varepsilon_2^e E^t \delta\varepsilon_0 \, \mathrm{d}A + \int_{\delta A} \varepsilon_2^e \sigma_0 \, \mathrm{d}A \right) \mathrm{d}x \\ &= -\int \underline{N_2 \delta\varepsilon_0} \, \mathrm{d}x - \int\int_{\delta A} \varepsilon_2^e \sigma_0 \, \mathrm{d}A \, \mathrm{d}x \end{aligned} \tag{11}$$

The underlined term vanishes because N_2 is self-equilibrated, whereas $\delta\varepsilon_0$ is kinematically admissible. Since δA is small, $\int_{\delta A}(\ldots)\,\mathrm{d}A = (\ldots)\delta A = (\ldots)\partial A/\partial s \delta(s)$.

Using (9), (11) and (7) we obtain the variation of the eigenvalue with respect to the design variable s:

$$\begin{aligned} \delta(\omega^2) = & \int_L \left(\frac{\partial D^b}{\partial s} w_1'' w_1'' + \frac{\partial D^a}{\partial s} u_1' u_1' \right. \\ & \left. - \omega^2 \frac{\partial m}{\partial s}(w_1 w_1 + u_1 u_1) - \varepsilon_2^e \sigma_0 \frac{\partial A}{\partial s} \right) \delta s \, \mathrm{d}x \end{aligned} \tag{12}$$

4. Sensitivity of Critical Load Using Dynamic Criterion

Assume one parameter load with load factor λ. Our aim is to find the sensitivity derivative of λ with respect to the distributed design parameter s. In the state of critical load $\lambda = \lambda^{cr}$, the eigenfrequency is equal to zero, $\omega = 0$, for the variable s, so that

$$\mathrm{d}(\omega^2) = \frac{\partial \omega^2}{\partial s} \delta s + \frac{\partial \omega^2}{\partial \lambda} \delta \lambda^{cr} = 0 \tag{13}$$

Hence

$$\delta \lambda^{cr} = -\frac{\partial \omega^2 / \partial s}{\partial \omega^2 / \partial \lambda} \delta s = -\frac{\delta(\omega^2)}{\delta_\lambda(\omega^2)} \tag{14}$$

where the numerator is equal to (12).

The denominator in (14) is a scalar parameter, and therefore the easiest way of its numerical evaluation is to apply the finite-difference scheme

$$\delta_\lambda(\omega^2) = \frac{\partial(\omega^2)}{\partial \lambda} \approx \left. \frac{\Delta(\omega^2)}{\Delta \lambda} \right|_{\lambda = \lambda^{cr}} = \frac{[\omega(s, \lambda^{cr})]^2 - [\omega(s, \lambda^{cr} - \Delta\lambda)]^2}{\Delta \lambda} \tag{15}$$

Hence the sensitivity derivative of critical load factor $\delta\lambda^{cr}$ is a function proportional to (12) evaluated at $\lambda = \lambda^{cr}$, with proportionality factor $-1/\delta_\lambda(\omega^2)$.

5. Sensitivity of Critical Load Using Static Criterion

Let us note that the normal force in the cross-sections of a frame in the pre-buckled state is a non-linear function of the design variable s and load factor λ, $N_0 = N_0(s, \lambda)$ and thus (6) takes the form

$$\begin{aligned} \delta D^b &= \frac{\partial D^b}{\partial s}\delta s + \frac{\partial D^b}{\partial N_0}\left(\delta_s N_0 + \frac{\partial N_0}{\partial \lambda}\delta\lambda\right) \\ \delta D^a &= \frac{\partial D^a}{\partial s}\delta s + \frac{\partial D^a}{\partial N_0}\left(\delta_s N_0 + \frac{\partial N_0}{\partial \lambda}\delta\lambda\right) \end{aligned} \tag{16}$$

Substituting (16) into (5) and neglecting the inertial terms with ω and $\delta\omega$ we obtain

$$\begin{aligned} \int_L \Bigg\{ &\left[\frac{\partial D^b}{\partial s}\delta s + \frac{\partial D^b}{\partial N_0}\left(\delta_s N_0 + \frac{\partial N_0}{\partial \lambda}\delta\lambda\right)\right] w_1'' w_1'' \\ &+ \left[\frac{\partial D^a}{\partial s}\delta s + \frac{\partial D^a}{\partial N_0}\left(\delta_s N_0 + \frac{\partial N_0}{\partial \lambda}\delta\lambda\right)\right] u_1' u_1' \\ &+ 2D^b w_1'' \delta w_1'' + 2D^a u_1' \delta u_1' \\ &+ \left(\delta_s N_0 + \frac{\partial N_0}{\partial \lambda}\delta\lambda\right) w_1' w_1' + 2N_0 w_1' \delta w_1' \Bigg\} \, \mathrm{d}x = 0 \end{aligned} \tag{17}$$

The terms with factor 2 vanish due to (1) with inertial terms neglected. Thus (17) can be transformed to

$$\begin{aligned} \delta\lambda^{cr} = &\frac{1}{-\int_L \frac{\partial N_0}{\partial \lambda}\left(\frac{\partial D^b}{\partial N_0} w_1'' w_1'' + \frac{\partial D^a}{\partial N_0} u_1' u_1' + w_1' w_1'\right) \mathrm{d}x} \\ &\times \int_L \Bigg[\left(\frac{\partial D^b}{\partial s} w_1'' w_1'' + \frac{\partial D^a}{\partial s} u_1' u_1'\right)\delta s \\ &+ \left(\frac{\partial D^b}{\partial N_0} w_1'' w_1'' + \frac{\partial D^a}{\partial N_0} u_1' u_1' + w_1' w_1'\right)\delta_s N_0\Bigg] \, \mathrm{d}x \end{aligned} \tag{18}$$

To transform the implicit variation $\delta_s N_0$ to an explicit form with respect to δs, we use again the adjoint-variable method (8) to (11) and we finally get

$$\begin{aligned} \delta\lambda^{cr} = &\frac{1}{-\int_L \frac{\partial N_0}{\partial \lambda}\left(\frac{\partial D^b}{\partial N_0} w_1'' w_1'' + \frac{\partial D^a}{\partial N_0} u_1' u_1' + w_1' w_1'\right) \mathrm{d}x} \\ &\times \int_L \left(\frac{\partial D^b}{\partial s} w_1'' w_1'' + \frac{\partial D^a}{\partial s} u_1' u_1' - \varepsilon_2^e \, \sigma_0 \frac{\partial A}{\partial s}\right)\delta s \, \mathrm{d}x \end{aligned} \tag{19}$$

Let us note that if we set $\omega = 0$ in (12), then the numerators of (14) and (19) are equal since the eigenmodes $\{w, u\}$ for vibrations with $\omega = 0$ are identical to the eigenmodes $\{w, u\}$ for stability. It can be proved that for conservative systems the formulae (14) and (15) can be transformed to the form (19). However, for numerical applications, the formulae (14) and (15) seem to be more advantageous than (19). This will be studied by means of numerical examples.

6. Numerical Examples

For brevity of description, the same cross-section of bars is assumed in all the examples, namely a box section made of 2 ⊏ 140E, reinforced by two cover plates attached to flanges. Each cover plate is 80 mm wide and 6 mm thick. The total cross-sectional area and moment of inertia around the y-axis, parallel to webs, are equal to $A = 40.8\,\mathrm{cm}^2$, $I = 980.48\,\mathrm{cm}^4$, respectively.

A FEM program has been developed for the non-linear analysis of structures subjected to initial loads and for the solution of the linear eigenvalue problem as well as for computing the sensitivity operators. A multi-layered beam finite element has been used. In the following examples, the cross-section is divided into 10 layers parallel to the y-axis. Four of them are placed in flanges of ⊏ 140E sections and four other ones in webs. The last two layers are reserved for cover plates. This discretization allows for a non-linear stress distribution within the depth of the cross-section.

The examples will illustrate the sensitivity of the eigenfrequency with respect to the variations of the width of cover plates δs and the load factor $\delta\lambda$, where $P = \lambda P^{cr}$. Sensitivity operators are calculated for the load factor $\lambda = 0.0$ (the structure without loading), $\lambda = 0.5$ and $\lambda = 1.0$ (the buckling load). The results are compared with those obtained by total finite-difference and semi-analytical methods.

The material satisfies the non-linear elastic law described by a bilinear relation with transition curve represented by a third-order polynomial which ensures continuity of the function $\sigma(\varepsilon)$ and its first derivative $\mathrm{d}\sigma(\varepsilon)/\mathrm{d}\varepsilon$. The Young modulus of elasticity is equal to $E = 210\,\mathrm{GPa}$. For the second linear part, the tangent modulus is equal to $E_1 = 10\,\mathrm{GPa}$. Both lines of the constant modulus would intersect at the stress $\sigma_p = 250\,\mathrm{MPa}$, while the transition portion is between $\sigma = 215\,\mathrm{MPa}$ and $\sigma = 251.7\,\mathrm{MPa}$. For comparison, calculations are also performed for the linear elastic material. The material mass density is equal to $\rho = 7850\,\mathrm{kg/m^3}$. The cross-sectional and structural dimensions are assumed such that the prebuckling stress remains in the non-linear regime of the above physical rule.

6.1. Simply-Supported Column with Concentrated Tip Force

This simple structure has an analytical solution for free vibrations, therefore it illustrates well the accuracy of eigenfrequencies and sensitivity operators obtained by computer methods. Let the length of the column be equal to 4.5 m. The first eigenvalue is examined, using 20 finite elements to discretize half of the column. The critical buckling load for the linear elastic material is $P_L^{cr} = 1003.535\,\mathrm{kN}$ and for the non-linear law $P_{NL}^{cr} = 905.140\,\mathrm{kN}$. The sensitivity derivatives of ω^2 obtained from formula (12) with the use of the FEM program are presented in Fig. 1.

The exact formula for the n-th eigenfrequency squared ω_n^2 of an elastic column is

$$\omega_n^2 = n^2\,\pi^4 \frac{E\,I}{m\,L}\left(1 - \frac{P\,L^2}{n^2\,\pi^2\,E\,I}\right) = \omega_{n\,0}^2\left(1 - \frac{P}{P^{cr}}\right) \tag{20}$$

where ω_0 and P^{cr} denote the eigenfrequency of the unloaded column and Euler criti-

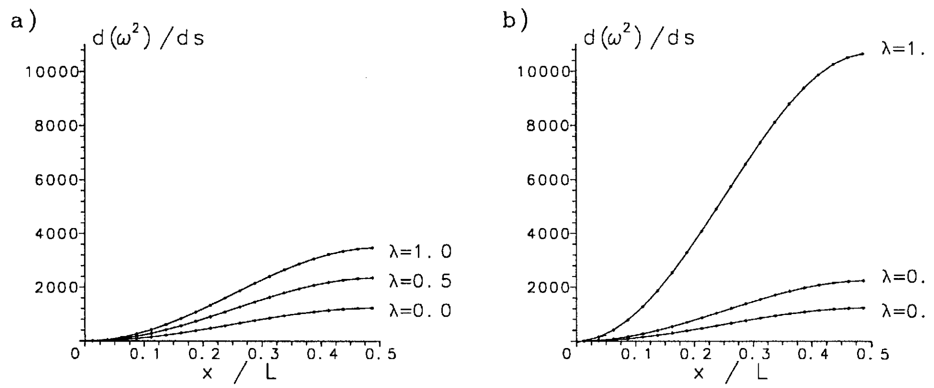

Fig. 1. Sensitivity operators with respect to the cross-sectional parameter s for the variable load factor $\lambda = P/P^{cr}$.
(a) Linear elastic column; (b) non-linear elastic column.

cal load, respectively. Variation of (20) with respect to the cross-sectional parameter s and load P, for n=1 leads to

$$\delta(\omega^2)\,(s,P;\delta s,\delta P) = \left(1 - \frac{P}{P^{cr}}\right)\delta(\omega_0^2)(s,0;\delta s) - \omega_0^2 \frac{P}{(P^{cr})^2}\delta P^{cr}(s;\delta s) - \frac{\omega_0^2}{P^{cr}}\delta P \quad (21)$$

A formal notation for the Gâteau variation is used, namely $\delta(\omega^2)\,(s,P;\delta s,\delta P)$ refers to the design s, P and the direction δs, δP . The first two right-hand side terms in (21) represent the variation of ω^2 due to δs, while the last term results from the independent variation of the prestress δP.

From (20) it follows that $\omega^2 = 15271.357\,(\mathrm{rad/s})^2$ and the same value was obtained by the FEM. Exact integration of formula (12) for a constant variation δs gives $\delta(\omega^2)/\delta s = 24687.5\,\mathrm{rad^2/s^2m}$ for $\lambda = 0$, $\delta(\omega^2)/\delta s = 47145.4\,\mathrm{rad^2/s^2m}$ for $\lambda = 0.5$ and $\delta(\omega^2)/\delta s = 69603.3\,\mathrm{rad^2/s^2m}$ for $\lambda = 1.0$.

The doubled surface under the respective curves in Fig. 1a, when compared with the above exact values $\delta(\omega^2)/\delta s$, shows the excellent accuracy keeping the error less than $4 \times 10^{-4}\%$. The total Finite-Difference Method (FDM), with forward step equal to 1.25% of s, has also been implemented for comparison with the values shown in Fig. 1a,b. The results of the total FDM give underestimation of $\delta(\omega^2)/\delta s$ with the error less than 0.2%. The error increases with the increase of the step.

The values shown in Fig. 1a,b for $\lambda = 1.0$, when multiplied by the factor $-1/\delta_\lambda(\omega^2)$ according to (14), provide the sensitivity of the buckling load $\delta(\lambda)/\delta s$ from the dynamic criterion. The sensitivity derivatives $\delta(\omega^2)/\delta s$ and $\delta(\lambda^{cr})/\delta s$ for the linear and non-linear structural material are shown in Table 1.

Tab. 1. Sensitivity derivatives $\delta(\lambda)/\delta s$ for the simply-supported column.

El	$\delta(\omega^2)/\delta s$		$\delta(\lambda^{cr})/\delta s$	
No	linear	non-linear	linear	non-linear
1	6946	21264	0.4548	0.3278
2	6861	21003	0.4493	0.3237
3	6692	20487	0.4382	0.3158
4	6444	19728	0.4220	0.3041
5	6124	18747	0.4010	0.2890
6	5738	17566	0.3757	0.2708
7	5297	16215	0.3468	0.2499
8	4811	14727	0.3150	0.2270
9	4292	13138	0.2810	0.2025
10	3753	11489	0.2458	0.1771
11	3207	9819	0.2100	0.1513
12	2668	8169	0.1747	0.1259
13	2150	6581	0.1408	0.1014
14	1664	5093	0.1089	0.0785
15	1222	3742	0.0800	0.0577
16	837	2561	0.0548	0.0395
17	516	1579	0.0338	0.0243
18	268	821	0.0176	0.0127
19	100	305	0.0065	0.0047
20	14	44	0.0009	0.0007

6.2. Portal Frame

Consider eigenvibrations of an orthogonal portal frame with the columns 3.6 m high and with the span of the beam equal to 6.0 m. The columns are rigidly fixed in foundations. Let the frame be subjected to two concentrated vertical loads, applied at the top of the columns. The FEM computations are carried out for the whole structure, although the results are presented only for its half. Symmetric discretization with 6 finite elements for a column and 6 for half the beam span was used. The critical buckling load for the linear case was $P_L^{cr} = 1015.986\,\mathrm{kN}$ and for the non-linear material $P_{NL}^{cr} = 917.920\,\mathrm{kN}$. The square of the first eigenfrequency was $\omega^2 = 2442.441\,(\mathrm{rad/s})^2$ and it was associated with the antisymmetric mode. The sensitivity derivatives $\delta(\omega^2)/\delta s$ with respect to the cross-sectional parameters for different λ are shown in Fig. 2. The sensitivity derivatives $\delta(\lambda)/\delta s$ were computed using both the dynamic method (12), (15), (14) and the static approach (19). The sensitivity derivatives are listed in Table 2.

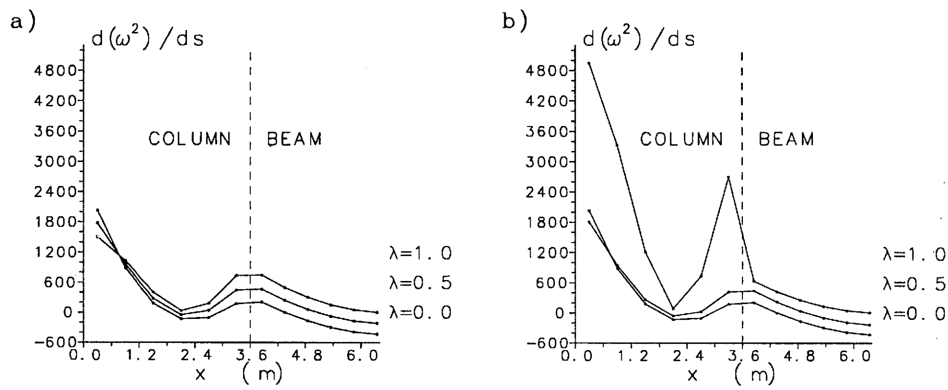

Fig. 2. Sensitivity operators for the portal frame.
(a) Linear elastic material; (b) non-linear elastic material.

Tab. 2. Sensitivity derivatives $\delta(\lambda)/\delta s$ for the portal frame.

El No	$\delta(\omega^2)/\delta s$ linear	$\delta(\omega^2)/\delta s$ non-linear	$\delta(\lambda^{cr})/\delta s$ linear	$\delta(\lambda^{cr})/\delta s$ non-linear
1	1503	4948	0.6038	0.5636
2	1030	3327	0.4140	0.3789
3	402	1216	0.1613	0.1385
4	38	89	0.0151	0.0102
5	182	732	0.0731	0.0834
6	738	2697	0.2965	0.3071
7	749	634	0.3010	0.0722
8	502	425	0.2017	0.0484
9	305	258	0.1224	0.0294
10	156	132	0.0628	0.0151
11	58	49	0.0232	0.0056
12	8	7	0.0033	0.0008

6.3. Three-Bar Structure

The dimensions of the structure and the discretization with 24 finite elements are shown in Fig. 3. The critical buckling load of the structure is $P_L^{cr} = 1037.441\,\mathrm{kN}$ for the linear case and $P_{NL}^{cr} = 912.885\,\mathrm{kN}$ for the non-linear law. The square of the eigenfrequency of the unprestressed structure is equal to $\omega^2 = 6431.932\,(\mathrm{rad/s})^2$.

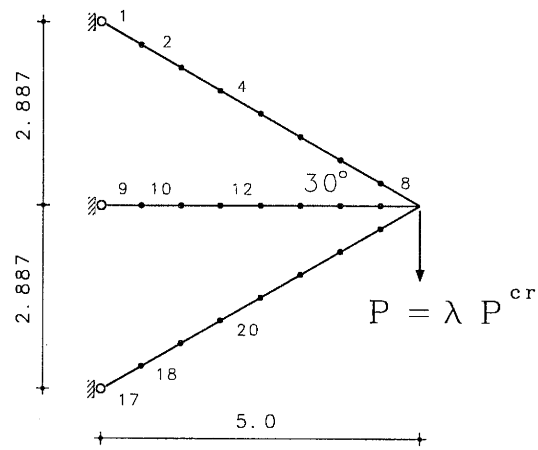

Fig. 3. Three-bar structure.

The sensitivity operators with respect to the variation of the width of cover plates are shown in Table 3 for the linear and non-linear elastic material. The sensitivity derivatives presented in Table 3 were computed using formula (12) with an analytic method of evaluating the derivatives of the corresponding stiffness and mass matrices. For comparison, a semi-analytical method, employing the FDM to calculate derivatives of the matrices, was used, too. The values presented in Table 3 were checked with the use of the total FDM. All sensitivity operators obtained using the three approaches described above were in satisfactory agreement (errors in the range of 0.5%).

Table 4 illustrates interesting phenomena regarding the quantitative contribution of various terms of (12) to the sensitivity operator. Column 3 pertains to the first two integrands in (12), whereas columns 4 and 5 refer to the third and forth integrand, respectively. A comparison of column 5 with columns 3, 4 and 6 demonstrates that the implicit redistribution of $\delta_s N_0$ contributes significantly to the total sensitivity.

As regards the comparison of numerical efficiency of the two approaches based on the static and dynamic criteria, the crucial point is the numerical effort of evaluation of denominators in (19) and (14). Therefore in all the examples presented above, the accuracy of formula (15) was studied. It appeared that ω^2 was nearly linear with respect to the load parameter λ in a vicinity of $\omega = 0$. Therefore formula (15) gave very good results for different steps $\Delta\lambda$, and was numerically very efficient. Hence the sensitivity derivatives computed using the static approach demonstrate very good agreement with differences remaining within computer round-off errors.

7. Concluding Remarks

The variational formulation and solution to the sensitivity analysis of linear and non-linear elastic frames subjected to on initial load were presented. The sensitivity

Tab. 3. Sensitivity derivatives $\delta(\lambda)/\delta s$ for a three-bar structure.

El	$\delta(\omega^2)/\delta s$		$\delta(\lambda^{cr})/\delta s$	
No	linear	non-linear	linear	non-linear
1	739	3109	0.0782	0.0778
2	741	3113	0.0783	0.0779
3	746	3128	0.0789	0.0782
4	763	3177	0.0807	0.0795
5	812	3318	0.0859	0.0830
6	949	3717	0.1004	0.0930
7	1333	4837	0.1409	0.1210
8	2402	7978	0.2540	0.1996
9	4	1	0.0004	0.0000
10	32	22	0.0033	0.0006
11	89	71	0.0094	0.0018
12	175	147	0.0186	0.0037
13	292	250	0.0308	0.0063
14	437	381	0.0462	0.0095
15	613	540	0.0648	0.0135
16	817	726	0.0864	0.0182
17	151	-254	0.0160	-0.0063
18	4451	13512	0.4706	0.3380
19	8919	27713	0.9431	0.6932
20	9261	28602	0.9792	0.7154
21	5148	15317	0.5443	0.3831
22	531	714	0.0562	0.0179
23	-153	-1076	-0.0162	-0.0269
24	3752	11675	0.3967	0.2920

Tab. 4. Contribution to $\delta(\omega^2)/\delta s$ for a three-bar structure (the non-linear material).

λ	Element	Stiffness	Mass	Redistr	Total
1	2	3	4	5	6
	4	2798	-1940	0	858
0.0	12	1268	-638	0	630
	20	2798	-1940	0	858
	4	374	-313	242	303
0.5	12	696	-281	-2	413
	20	7128	-2860	-243	4025
	4	76	0	3101	3177
1.0	12	169	0	-22	147
	20	31719	0	-3117	28602

derivatives of the square of the eigenfrequency thus derived were used in the evaluation of the sensitivity of the buckling load with the use of the dynamic criterion. The static approach to the sensitivity analysis of critical loads was used, too. It was shown that the sensitivity derivatives of the buckling load are proportional to the respective derivatives of the square of the eigenfrequency.

The method of SA based on the dynamic criterion was numerically more efficient than the method based on the static criterion. Writing the code of the program was easier. The results were identical within the limits of computer round-off errors. The effective method of computation of the sensitivity operators allowing for the redistribution of the pre-buckling load in the non-linear structure was demonstrated using the adjoint-variable method. The redistribution is often neglected in the optimal design. The numerical example of a three-bar structure demonstrated that the redistribution of prestress contributes significantly to the sensitivity operators and cannot be neglected.

The illustrative examples also demonstrated numerical efficiency and stability of computations using the derived formulae for sensitivity. An excellent accuracy of analytical sensitivity analysis and a good accuracy of semi-analytical sensitivity analysis were observed for linear and non-linear structures and for various load levels in the case of derivatives of the eigenfrequency.

References

Cohen G.A., Haftka R.T. and Mróz Z. (1990): *Derivatives of buckling loads and vibration frequencies with respect to stiffness and initial strain parameters.* — Trans. ASME, v.57, No.1, pp.18–24.

Garstecki A. and Glema A. (1991): *Sensitivity analysis and optimal redesign of columns in the state of initial distortions and prestress.* — Structural Optimization, v.5, No.3, pp.141–148.

Garstecki A. and Glema A. (1992): *Methods of optimal redesign of structures.* — Advanced Tempus Course on *Numerical Methods in Computer Optimal Aided Design*, Zakopane, May 1992, Lecture Notes, v.2, pp.117–141.

Glema A. (1992): *Optimal redesign of nonlinear bar structures in the state of initial stress.* — Ph.D. Thesis, Technical University of Poznań, (in Polish).

Mróz Z. and Haftka T. (1988): *Sensitivity of buckling loads and vibration frequencies of plates*, In: Josef Singer Anniversary Volume. — New York: Springer Verlag.