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REDUCED–ORDER PERFECT NONLINEAR OBSERVERS OF FRACTIONAL DESCRIPTOR DISCRETE–TIME NONLINEAR SYSTEMS

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The purpose of this work is to propose and characterize fractional descriptor reduced-order perfect nonlinear observers for a class of fractional descriptor discrete-time nonlinear systems. Sufficient conditions for the existence of these observers are established. The design procedure of the observers is given and demonstrated on a numerical example.

Keywords: fractional, descriptor, nonlinear, discrete-time, design, reduced-order, perfect observer.

1. Introduction

Fractional linear systems have been considered in many papers and books (Kaczorek, 2013; 2008; 2012b; 2011a; 2011b; Oldham and Spanier, 1974; Ostalczyk, 2008; Podlubny, 1999; Vinagre *et al.*, 2002). Positive linear systems consisting of n subsystems with different fractional orders were proposed by Kaczorek (2011a; 2011b). Descriptor (singular) linear systems were investigated by Cuihong (2012), Dodig and Stosic (2009), Dai (1989), Duan (2010), Fahmy and O'Reill (1989), Gantmacher (1959), Kaczorek (2012b; 2013; 2004; 1992; 2012a), Kucera and Zagalak (1988), Lewis (1983), Luenberger (1977; 1978), Sajewski (2016), Van Dooren (1979) or Virnik (2008), and the positivity and stability of fractional descriptor time-varying discrete-time linear by Kaczorek (2016c), who also addressed the eigenvalues and invariants assignment by state and input feedbacks (Kaczorek, 2004; 1992; 2011b). The computation of Kronecker's canonical form of a singular pencil was analyzed by Van Dooren (1979).

A new concept of perfect observers for linear continuous-time systems was proposed Kaczorek (2001) and N'Doye *et al.* (2013). Observers for fractional linear systems were addressed by Kaczorek (2014b), Kociszewski (2013), and N'Doye *et al.* (2013) and for descriptor linear systems by Kaczorek (2015), who also discussed perfect nonlinear observers of descriptor nonlinear systems (Kaczorek, 2016a; 2016b). Fractional descriptor full-order observers for fractional descriptor continuous-time linear systems were proposed by Kaczorek (2014a), along with reduced-order observers (Kaczorek, 2016d; 2014). Stability of positive descriptor systems was investigated by Virnik (2008).

In this paper reduced-order perfect nonlinear observers for fractional descriptor nonlinear discrete-time systems will be proposed, conditions for their existence will be established and a design procedure will be given.

The paper is organized as follows. In Section 2 conditions for the existence of perfect full-order nonlinear observers for fractional descriptor nonlinear systems will be given. Conditions for the existence of reduced-order perfect observers of fractional discrete-time nonlinear systems will be established in Section 3. A design procedure and an illustrating numerical example for reduced-order perfect nonlinear observers will be presented in Section 4. Concluding remarks will be given in Section 5.

The following notation will be used: $\mathbb{R}$, the set of real numbers; $\mathbb{R}^{n\times m}$, the set of $n \times m$ real matrices; I_n, the $n \times n$ identity matrix; $\mathbb{Z}_+$, the set of nonnegative integers.

2. Perfect fractional discrete-time nonlinear observers

Consider the fractional descriptor discrete-time nonlinear system

$$E\Delta^\alpha x_{i+1} = Ax_i + f(x_i, u_i), \quad i \in \mathbb{Z}_+, \qquad (1a)$$

$$y_i = Cx_i, \tag{1b}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, $y_i \in \mathbb{R}^p$ are respectively the state, input and output vectors and $E, A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{p\times n}$, $f(x_i, u_i) \in \mathbb{R}^n$ is a continuous nonlinear vector function of x_i and u_i,

$$\Delta^\alpha x_i = \sum_{j=0}^{i} (-1)^j \binom{\alpha}{j} x_{i-j}, \tag{2a}$$

$$\binom{\alpha}{j} = \begin{cases} 1 & \text{for} \quad j = 0, \\ \frac{\alpha(\alpha-1)\ldots(\alpha-j+1)}{j!} & \text{for} \quad j = 1, 2, \ldots \end{cases} \tag{2b}$$

$\alpha \in \mathbb{R}$ is the fractional order difference of x_i.

Substituting (2) into (1) we obtain

$$Ex_{i+1} = A_\alpha x_i + \sum_{j=2}^{i+1} c_j E x_{i-j+1} + f(x_i, u_i), \tag{3a}$$

where

$$A_\alpha = A + E\alpha, c_j = (-1)^{j+1}\binom{\alpha}{j}. \tag{3b}$$

It is assumed that

$$\det E = 0, \quad \det[Ez - A] \neq 0. \tag{4}$$

for some $z \in \mathbb{C}$.

Definition 1. The fractional descriptor discrete-time nonlinear system

$$\begin{aligned} E\hat{x}_{i+1} = F\hat{x}_i &+ \sum_{j=2}^{i+1} c_j E x_{i-j+1} \\ &+ f(x_i, u_i) + Hy_i, \end{aligned} \tag{5}$$

where $\hat{x}_i$ is the estimate of x_i, u_i and $f(x_i, u_i)$, y_i are the same vectors as in (1), $E, F \in \mathbb{R}^{n\times n}$, $\det E = 0$, $H \in \mathbb{R}^{n\times p}$ is called a (full-order) *perfect observer* for the system (1) if

$$\hat{x}_i = x_i \quad \text{for } i = 1, 2, \ldots. \tag{6}$$

The following elementary row (column) operations will be used (Kaczorek, 1992):

1. Multiplication of the i-th row (column) by a real number c. Here and subsequently this operation will be denoted by $L[i \times c](R[i \times c])$.
2. Addition of the j-th row (column) multiplied by a real number c to the i-th row (column). This operation will be denoted by $L[i+j\times c](R[i+j\times c])$.
3. Intercharge of the i-th and j-th rows (columns). This operations will be denoted by $L[i, j](R[i, j])$.

Lemma 1. *If*

$$\text{rank}E = r < n, \tag{7}$$

then through elementary row and column operations the matrix E can be reduced to the following upper triangular form:

$$N = PEQ = \begin{bmatrix} 0 & E_{12} \\ 0 & 0 \end{bmatrix},$$
$$E_{12} = \begin{bmatrix} e_{11} & e_{12} & \cdots & e_{1r} \\ 0 & e_{22} & \cdots & e_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e_{rr} \end{bmatrix}, \tag{8}$$

where P and Q are matrices of the elementary row and column operations.

Proof. If (7) is satisfied, then by elementary row and column operations the matrix E can be reduced to the form

$$\begin{bmatrix} 0 & E'_{12} \\ 0 & 0 \end{bmatrix}, \quad E'_{12} \in \mathbb{R}^{r\times r}. \tag{9}$$

Next, applying elementary column operations, we can reduce the matrix E'_{12} to the upper triangular form E_{12}. ■

Definition 2. The smallest nonnegative integer q is called the *nilpotent index* of a nilpotent matrix N if $N^q = 0$ and $N^{q-1} \neq 0$.

Lemma 2. (Kaczorek, 2016b) *If*

$$\text{rank}\, E = r < \frac{n}{2}, \tag{10}$$

then the nilpotent index q of the matrix E is

$$q = 2 \quad \textit{for } r = 1, 2, \ldots, \frac{n}{2} - 1. \tag{11}$$

Lemma 3. (Kaczorek, 2016a) *If* (7) *is satisfied and N is the nilpotent matrix* (8)*, then the equation*

$$\begin{aligned} &Nx_{i+1} = Dx_i, \\ &x_i = [\, x_{1,i} \quad x_{2,i} \quad \cdots \quad x_{n,i}]^T, \quad i \in \mathbb{Z}_+ \end{aligned} \tag{12}$$

for a nonsingular diagonal matrix

$$D = \text{diag}[\, d_1 \quad \cdots \quad d_n], \tag{13}$$

with $d_k \neq 0$, $k = 1, \ldots, n$ has zero solution $x_i = 0$ for $i = 1, 2, \ldots$.

Theorem 1. (Kaczorek, 2016a) *The perfect observer (5) of the fractional descriptor nonlinear system (1) exists if and only if*

$$\text{rank} \begin{bmatrix} \bar{A} - D \\ \bar{C} \end{bmatrix} = \text{rank}\, \bar{C}, \tag{14}$$

where $\bar{A} = PA_\alpha Q$, $\bar{C} = CQ$ and the matrices P, Q are defined by (8).

To design the perfect observer (5) for the fractional descriptor nonlinear system (1) with given matrices A, C we have to choose the matrices F, H of the observer so that the conditions (14) and $\bar{F} = D$ are satisfied. Note that the conditions are met if and only if

$$\bar{A} - \bar{H}\bar{C} = D, \tag{15}$$

where $\bar{H} = PH$.

By the Kronecker–Capelli theorem, Eqn. (15) has a solution $\bar{H}$ for given $\bar{A}$, $\bar{C}$ and D if and only if the condition (14) is satisfied. Therefore, we have the following procedure for designing of the perfect observer (5) for the nonlinear system (1).

Procedure 1.

1. Find matrices P and Q of elementary row and column operations reducing the matrix E to its nilpotent form $N = PEQ$.

2. Using $\bar{A} = PA_\alpha Q$ and $\bar{C} = CQ$ compute the matrices $\bar{A}$ and $\bar{C}$.

3. Choose a diagonal matrix D so that the condition (14) is satisfied.

4. Find the solution $\bar{H}$ of Eqn. (15) for given $\bar{A}$, $\bar{C}$ and D.

5. Compute the matrices

$$F = A_\alpha - HC, \quad H = P^{-1}\bar{H} \tag{16}$$

of the perfect observer (5).

3. Reduced-order perfect observers of fractional discrete-time nonlinear systems

Consider the fractional descriptor discrete-time nonlinear system described by (3) and (1b). If

$$\operatorname{rank} C = p, \tag{17}$$

then there exists an elementary column operation matrix Q_1 such that (Kaczorek, 1992)

$$\bar{C} = CQ_1 = \begin{bmatrix} I_p & 0 \end{bmatrix}. \tag{18}$$

Substituting

$$x = Q_1\bar{x} \tag{19}$$

into (1b) and using (18), we obtain

$$\begin{aligned} y_i = Cx_i = CQ_1\bar{x}_i &= \begin{bmatrix} I_p & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_{1,i} \\ \bar{x}_{2,i} \end{bmatrix} \\ &= \bar{x}_{1,i}, \quad \bar{x}_{1,i} \in \mathbb{R}^p, \quad \bar{x}_{2,i} \in \mathbb{R}^{n-p}. \end{aligned} \tag{20}$$

From (20) it follows that for given y the subvector $\bar{x}_{1,i} \in \mathbb{R}^p$ is known. Therefore, the reduced-order observer of the fractional descriptor nonlinear system (1) should reconstruct only the subvector $\bar{x}_{2,i} \in \mathbb{R}^{n-p}$.

It is assumed that there exists a matrix of elementary row operations P_1 such that

$$P_1EQ_1 = \begin{bmatrix} E_{11} & 0 \\ E_{21} & E_{22} \end{bmatrix}, \quad E_{11} \in \mathbb{R}^{p\times p}, \quad E_{22} \in \mathbb{R}^{(n-p)\times(n-p)}, \tag{21a}$$

$$P_1A_\alpha Q_1 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A_{11} \in \mathbb{R}^{p\times p}, \quad A_{22} \in \mathbb{R}^{(n-p)\times(n-p)}, \tag{21b}$$

$$P_1f(x_i, u_i) = \begin{bmatrix} f_1(\bar{x}_{1,i}, u_i) \\ f_2(\bar{x}_i, u_i) \end{bmatrix}, \quad f_1(\bar{x}_{1,i}, u_i) \in \mathbb{R}^p, \quad f_2(\bar{x}_i, u_i) \in \mathbb{R}^{n-p}. \tag{21c}$$

Premultiplying (3a) by the matrix P_1 and using (20) and (21), we obtain

$$\begin{aligned} E_{11}\bar{x}_{1,i+1} &= A_{11}\bar{x}_{1,i} + A_{12}\bar{x}_{2,i} \\ &\quad + \sum_{j=2}^{i+1} c_jE_{11}\bar{x}_{1,i-j+1} + f_1(\bar{x}_{1,i}, u_i), \end{aligned} \tag{22a}$$

$$\begin{aligned} E_{21}\bar{x}_{1,i+1} + E_{22}\bar{x}_{2,i+1} &= A_{21}\bar{x}_{1,i} + A_{22}\bar{x}_{2,i} \\ &\quad + \sum_{j=2}^{i+1} c_j(E_{21}\bar{x}_{1,i-j+1} + E_{22}\bar{x}_{2,i-j+1}) \\ &\quad + f_2(\bar{x}_i, u_i). \end{aligned} \tag{22b}$$

Defining

$$\begin{aligned} \bar{y}_i = E_{11}\bar{x}_{1,i+1} - A_{11}\bar{x}_{1,i} \\ - \sum_{j=2}^{i+1} c_jE_{11}\bar{x}_{1,i-j+1} \\ - f_1(\bar{x}_{1,i}, u_i), \end{aligned} \tag{23a}$$

$$\begin{aligned} \bar{f}_2(\bar{x}_i, u_i) = f_2(\bar{x}_i, u_i) + A_{21}\bar{x}_{1,i} \\ + \sum_{j=2}^{i+1} c_jE_{21}\bar{x}_{1,i-j+1} \\ - E_{21}\bar{x}_{1,i+1} \end{aligned} \tag{23b}$$

as the output and input of the subsystem, respectively, from (22) we obtain

$$E_{22}\bar{x}_{2,i+1} = A_{22}\bar{x}_{2,i} + \sum_{j=2}^{i+1} c_j E_{22}\bar{x}_{2,i-j+1} + \bar{f}_2(\bar{x}_i, u_i), \tag{24a}$$

$$\bar{y}_i = A_{12}\bar{x}_{2,i}. \tag{24b}$$

If $\det E_{22} \neq 0$, then premultiplying (23a) by $\det E_{22}^{-1}$ we obtain the standard fractional discrete-time nonlinear system which can be analyzed by the well-known method (Kaczorek, 2016a).

Let

$$\operatorname{rank} E_{22} = r < n - p. \tag{25}$$

In this case the method presented in Section 2 can be used to design the perfect descriptor fractional nonlinear observer to the nonlinear system (1).

Therefore, the following theorem has been proved.

Theorem 2. *A reduced-order perfect nonlinear observer for the fractional descriptor nonlinear system (1) exists if the following conditions are satisfied:*

1. *The condition* (17) *is met.*
2. *There exists a matrix P_1 of elementary row operations such that (21) is satisfied.*
3. *The condition* (25) *is met.*
4. *The condition* (14) *is satisfied for the subsystem (24).*

4. Design procedure and an illustrating example

From Section 3 we have the following procedure for designing the perfect nonlinear observer for the fractional descriptor nonlinear system (24).

Procedure 2.

1. Using elementary column operations, find a matrix Q_1 satisfying the condition (18) and a subvectors $\bar{x}_{1,i} \in \mathbb{R}^p$ and $\bar{x}_{2,i} \in \mathbb{R}^{n-p}$.
2. Find the output $\bar{y}_i$ and the input $\bar{f}_2(\bar{x}_i, u_i)$ defined by (23) and the equations of the subsystem (24).
3. Using Procedure 1, find the desired perfect observer of the subsystem (24).

Example 1. Consider the fractional descriptor nonlinear system (1) with $\alpha = 0.5$ and

$$E = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 2 \end{bmatrix},$$

$$f(x_i, u_i) = \begin{bmatrix} x_{4,i}^2 + u_i \\ x_{1,i}x_{2,i} + x_{3,i}^2 u_i \\ 3u_i^2 \\ (x_{2,i} - 2x_{1,i} + 2x_{4,i})x_{4,i} - 2u_i^2 \end{bmatrix}. \tag{26}$$

The system satisfies the assumption (4) since

$$\begin{aligned} &\det[Ez - A_\alpha] \\ &= \det[E(z-\alpha) - A] \\ &= \begin{vmatrix} -1 & 0 & 0 & z-1.5 \\ 0 & z-0.5 & -1 & 0 \\ z+0.5 & 2z-2 & 0 & z-0.5 \\ 0 & -z-1.5 & -1 & -1 \end{vmatrix} \\ &= -2z^3 - z^2 + 4.5z - 0.75 \neq 0. \end{aligned} \tag{27}$$

Using Procedure 2 we obtain the following:

Step 1. Interchanging the first and fourth columns of the matrix C, we obtain

$$\begin{aligned} \hat{C} &= CQ_0 \\ &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \\ &= [\ C_1 \quad C_2], \end{aligned} \tag{28}$$

$$C_1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

and

$$\begin{aligned} \bar{C} = \hat{C}Q_2 &= [\ C_1 \quad C_2] \begin{bmatrix} C_1^{-1} & -C_1^{-1}C_2 \\ 0 & I_{n-p} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \end{aligned} \tag{29}$$

$$Q_1 = Q_0 Q_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{30}$$

Step 2. The new state vector has the form

$$\begin{aligned} \bar{x}_i = Q_1^{-1} x_i &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} x_{1,i} \\ x_{2,i} \\ x_{3,i} \\ x_{4,i} \end{bmatrix} \\ &= \begin{bmatrix} x_{4,i} \\ 2x_{1,i} + x_{2,i} + 2x_{4,i} \\ x_{3,i} \\ x_{1,i} \end{bmatrix} = \begin{bmatrix} \bar{x}_{1,i} \\ \bar{x}_{2,i} \end{bmatrix} \end{aligned} \tag{31}$$

and the subvector $\bar{x}_{1,i}$ is known since $y_i = \bar{x}_{1,i}$, $i \in \mathbb{Z}_+$. Therefore, the reduced-order perfect observer should reconstruct only the subvector $\bar{x}_{2,i}$. In this case we have

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -2 & 0 \end{bmatrix} \tag{32}$$

and

$$\begin{aligned} P_1 E Q_1 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix} \\ &\times \begin{bmatrix} 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} E_{11} & 0 \\ E_{21} & E_{22} \end{bmatrix}, \\ E_{11} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad E_{21} = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}, \\ E_{22} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \end{aligned} \tag{33}$$

$$\begin{aligned} P_1 A_\alpha Q_1 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1.5 \\ 0 & 0.5 & 1 & 0 \\ -0.5 & 2 & 0 & 0.5 \\ 0 & 1.5 & 1 & 1 \end{bmatrix} \\ &\times \begin{bmatrix} 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1.5 & 0 & 0 & 1 \\ -13 & 8.5 & 3 & -18 \\ 2.5 & -1.5 & 1 & 3.5 \\ 4 & -2.5 & 3 & 6 \end{bmatrix} \\ &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \\ A_{11} &= \begin{bmatrix} 1.5 & 0 \\ -13 & 8.5 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 & 1 \\ 3 & -18 \end{bmatrix} \\ A_{21} &= \begin{bmatrix} 2.5 & -1.5 \\ 4 & -2.5 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 1 & 3.5 \\ 3 & 6 \end{bmatrix}, \end{aligned} \tag{34}$$

$$\begin{aligned} P_1 f(x_i, u_i) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -2 & 0 \end{bmatrix} \\ &\times \begin{bmatrix} x_{4,i}^2 + u_i \\ x_{1,i}x_{2,i} + x_{3,i}^2 u_i \\ 3u_i^2 \\ (x_{2,i} - 2x_{1,i} + 2x_{4,i})x_{4,i} - 2u_i^2 \end{bmatrix} \\ &= \begin{bmatrix} x_{4,i}^2 + u_i \\ 3(x_{2,i} - 2x_{1,i} + 2x_{4,i})x_{4,i} \\ x_{1,i}x_{2,i} + x_{3,i}^2 u_i - 3u_i^2 \\ 3x_{1,i}x_{2,i} + 3x_{3,i}^2 u - 6u_i^2 \end{bmatrix} \\ &= \begin{bmatrix} f_1(\bar{x}_{1,i}, u_i) \\ f_2(\bar{x}_i, u_i) \end{bmatrix}, \\ f_1(\bar{x}_{1,i}, u_i) &= \begin{bmatrix} x_{4,i}^2 + u_i \\ 3(x_{2,i} - 2x_{1,i} + 2x_{4,i})x_{4,i} \end{bmatrix}, \\ f_2(\bar{x}_i, u_i) &= \begin{bmatrix} x_{1,i}x_{2,i} + x_{3,i}^2 u_i - 3u_i^2 \\ 3x_{1,i}x_{2,i} + 3x_{3,i}^2 u - 6u_i^2 \end{bmatrix}. \end{aligned} \tag{35}$$

The descriptor subsystem (24) is given by the equations

$$\begin{aligned} &\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \bar{x}_{2,i+1} \\ &\quad = \begin{bmatrix} 1 & 3.5 \\ 3 & 6 \end{bmatrix} \bar{x}_{2,i} \\ &\quad + \sum_{j=2}^{i+1} (-1)^{j+1} \binom{0.5}{j} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \bar{x}_{2,i-j+1} \\ &\quad + \begin{bmatrix} x_{1,i}x_{2,i} + x_{3,i}^2 u_i - 3u_i^2 \\ 3x_{1,i}x_{2,i} + 3x_{3,i}^2 u - 6u_i^2 \end{bmatrix}, \end{aligned} \tag{36a}$$

$$\bar{y}_i = \begin{bmatrix} 0 & 1 \\ 3 & -18 \end{bmatrix} \bar{x}_{2,i}, \tag{36b}$$

where

$$\begin{aligned} \bar{y}_i = &\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bar{x}_{1,i+1} \\ &- \sum_{j=2}^{i+1} (-1)^{j+1} \begin{pmatrix} 0.5 \\ j \end{pmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bar{x}_{1,i-j+1} \\ &- \begin{bmatrix} x_{4,i}^2 + u_i \\ 3(x_{2,i} - 2x_{1,i} + 2x_{4,i})x_{4,i} \end{bmatrix}. \end{aligned} \tag{36c}$$

Step 3. Using Procedure 1, we obtain the following. We have

$$N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \bar{A} = A_{22} + \alpha N, \quad \bar{C} = A_{12}$$

and we choose

$$D = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}.$$

Note that the condition (14) is satisfied and the equation (15) has the form

$$\begin{aligned} HA_{12} &= H \begin{bmatrix} 0 & 1 \\ 3 & -18 \end{bmatrix} \\ &= A_{22} + \alpha N - D = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}. \end{aligned} \tag{37}$$

Its solution is

$$H = \begin{bmatrix} 10 & \frac{1}{3} \\ 20 & 1 \end{bmatrix}. \tag{38}$$

Using (16), we obtain in our case

$$\begin{aligned} F &= A_\alpha - HA_{12} \\ &= \begin{bmatrix} 4 & 4 \\ 3 & 6 \end{bmatrix} - \begin{bmatrix} 10 & \frac{1}{3} \\ 20 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 3 & -18 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}. \end{aligned} \tag{39}$$

The desired reduced-order perfect observer is described by

$$\begin{aligned} &\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \hat{x}_{i+1} \\ &= \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \hat{x}_i \\ &\quad + \sum_{j=2}^{i+1} (-1)^{j+1} \begin{pmatrix} 0.5 \\ j \end{pmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_{i-j+1} \\ &\quad + f_2(\bar{x}_i, u_i) + \begin{bmatrix} 10 & \frac{1}{3} \\ 20 & 1 \end{bmatrix} \bar{y}_i. \end{aligned} \tag{40}$$

♦

5. Concluding remarks

Reduced-order perfect fractional descriptor nonlinear observers for fractional descriptor discrete-time nonlinear systems have been proposed. Conditions for the existence of the reduced-order perfect observers have been established (Theorem 2). A procedure for designing the reduced-order perfect observers has been proposed and illustrated with a numerical example.

An open problem is the extension of those considerations to fractional continuous-discrete nonlinear systems and to positive continuous-time and discrete-time nonlinear systems.

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