

## FEEDBACK DESIGN OF DIFFERENTIAL EQUATIONS OF RECONSTRUCTION FOR SECOND-ORDER DISTRIBUTED PARAMETER SYSTEMS

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The paper aims at studying a class of second-order partial differential equations subject to uncertainty involving unknown inputs for which no probabilistic information is available. Developing an approach of feedback control with a model, we derive an efficient reconstruction procedure and thereby design differential equations of reconstruction. A characteristic feature of the obtained equations is that their inputs formed by the feedback control principle constructively approximate unknown inputs of the given second-order distributed parameter system.

**Keywords:** second-order partial differential equation, equations of reconstruction.

### 1. Introduction

We consider the following second-order differential equation:

$$\ddot{y}(t, \eta) - \Delta y(t, \eta) + m y(t, \eta) + \gamma \dot{y}(t, \eta) = g(y(t, \eta)) + (Bv(t))(\eta) + f(t, \eta) \quad (1)$$

for almost every (a.e.)  $(t, \eta) \in T \times \Omega$ ,

with the boundary condition  $y(t)|_G = 0$  for a.e.  $t \in T$ , and the initial conditions.

$$\begin{aligned} y(0) &= y_0 \in V = H_0^1(\Omega), \\ \dot{y}(0) &= y_{10} \in H = L_2(\Omega) \end{aligned} \quad (2)$$

Here,  $T = [0, \vartheta]$ ,  $0 < \vartheta < +\infty$ ,  $\Omega$  is an open bounded domain with the Lipschitz boundary  $G$ ,  $\Delta$  is the Laplace

operator,  $m = \text{const} > 0$ ,  $\gamma = \text{const} > 0$ ,  $g(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$  is a Lipschitz function with some constant  $L$ ,  $g(0) = 0$ ,  $f(\cdot) \in L_\infty(T; H)$  is a given function, and  $B$  is a linear continuous operator acting from a Hilbert space  $U$  with a norm  $|\cdot|_U$  and an inner product  $(\cdot, \cdot)_U$  (the space of disturbances) into the space  $H$  ( $B \in L(U; H)$ ).

Before formulating our main problem, we define a solution of Eqn. (1). Any function  $y(\cdot) \in C(T; V)$  such that

$$\begin{aligned} \dot{y}(\cdot) &\in W(T; V) = \{x(\cdot) \in C(T; H) : \\ &\dot{x}(\cdot) \in L_2(T; V^*)\} \end{aligned}$$

and  $y(\cdot)$  satisfies

$$\begin{aligned} \ddot{y}(t) - \Delta y(t) + m y(t) + \gamma \dot{y}(t) \\ = g(y(t)) + Bv(t) + f(t) \end{aligned} \quad \text{in } V^* \text{ for a.e. } t \in T$$

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is called a solution of Eqn. (1) on the interval  $T$  and is defined by  $y(\cdot) = y(\cdot; y_{10}, y_0, u(\cdot))$ . Due to Gajewski *et al.* (1974), for any  $v(\cdot) \in L_2(T; U)$ , there exists a unique solution of Eqn. (1) on the interval  $T$ .

A function  $v(\cdot)$  (an input) on the right-hand side of Eqn. (1) is unknown. It is only known that this function is an element of the space  $L_2(T; U)$ . Along with Eqn. (1), there is one more equation of a similar form, namely

$$\ddot{x}(t) - \Delta x(t) + mx(t) + \gamma \dot{x}(t) = g(x(t)) + Bu(t) + f(t) \quad \text{in } V^* \text{ for a.e. } t \in T \quad (3)$$

with the initial condition

$$x(0) = x_0^h \in V, \quad \dot{x}(0) = \dot{x}_0^h \in H. \quad (4)$$

At every time moment  $t$ , the derivatives of solutions of Eqns. (1) and (3) are measured; i.e., the values  $\dot{y}(t)$  and  $\dot{x}(t)$  are defined. These measurements can be performed with errors; i.e., instead of functions  $\dot{y}(\cdot)$  and  $\dot{x}(\cdot)$ , we know some functions  $\xi^h(\cdot) \in L_\infty(T; H)$  and  $\psi^h(\cdot) \in L_\infty(T; H)$  with the properties

$$|\dot{y}(t) - \xi^h(t)|_H \leq h, \quad (5)$$

$$|\dot{x}(t) - \psi^h(t)|_H \leq h \quad \text{for a.e. } t \in T.$$

In the latter case, we assume that the initial states of Eqns. (1) and (3) satisfy the relations

$$|x_{10}^h - y_{10}|_H \leq h, \quad |x_0^h - y_0|_V \leq h. \quad (6)$$

Here and below,  $h \in (0, 1)$  is a value of the measurement error, the symbol  $|\cdot|_H$  ( $|\cdot|_V$ ) stands for the norm in the space  $H(V)$ , and the symbol  $(\cdot, \cdot)_H$  denotes the scalar product in the space  $H$ .

In the case where the solutions of Eqns. (1) and (3) are measured with no error (then  $x_0^h = y_0$  and  $x_{10}^h = y_{10}$ ), it is necessary to specify a family of functions  $u^\alpha(\cdot)$  (depending on a parameter  $\alpha \in (0, 1)$ ) with the following properties. First, at every time  $t \in T$ , the functions  $u^\alpha(\cdot)$  depend on the derivatives of solutions  $\dot{y}(t)$  and  $\dot{x}^\alpha(t)$ , i.e.,

$$u^\alpha(t) = u^\alpha(\dot{y}(t), \dot{x}^\alpha(t)),$$

where  $x^\alpha(\cdot) = x(\cdot; y_{10}, y_0, u^\alpha(\cdot))$ . Second, the following convergences:

$$u^\alpha(\cdot) \rightarrow u_*(\cdot) \quad \text{in } L_2(T; U), \quad (7)$$

$$x^\alpha(\cdot) \rightarrow y(\cdot) \quad \text{in } W(V; H) \text{ as } \alpha \rightarrow 0 \quad (8)$$

take place. Here, the symbol  $x^\alpha(\cdot) = x(\cdot; y_{10}, y_0, u^\alpha(\cdot))$  denotes the solution of Eqn. (3) with the right-hand side  $u(t) = u^\alpha(t)$ ; i.e.,  $x^\alpha(\cdot)$  is the solution of the equation

$$\begin{aligned} \ddot{x}(t) - \Delta x(t) + mx(t) + \gamma \dot{x}(t) \\ = g(x(t)) + Bu^\alpha(t) + f(t) \\ \text{in } V^* \text{ for a.e. } t \in T \end{aligned} \quad (9)$$

with the initial condition (4) (with  $x_0^h = y_0, x_{10}^h = y_{10}$ ),

$$W(V; H) = \{w(\cdot) \in C(T; V) : \dot{w}(\cdot) \in C(T; H)\}.$$

Further,  $u_*(\cdot) = u_*(\cdot; y(\cdot))$  means an element of the set  $U(y(\cdot))$  of minimal  $L_2(T; U)$ -norm and  $U(y(\cdot))$  is the set of all functions  $u(\cdot) \in L_2(T; U)$  generating the solution  $y(\cdot)$  of Eqn. (1), i.e.,

$$\begin{aligned} U(y(\cdot)) \\ = \left\{ u(\cdot) \in L_2(T; U) : \right. \\ \left. (\ddot{y}(t) - \Delta y(t) + my(t) + \gamma \dot{y}(t), z)_H \right. \\ \left. = (Bu(t) + f(t), z)_H \text{ for a.e. } t \in T \quad \forall z \in V \right\}. \end{aligned} \quad (10)$$

Note that the set  $U(y(\cdot))$  (10) is convex and closed in the space  $L_2(T; U)$ . Therefore, the element  $u_*(\cdot)$  is unique. In addition, when defining the set  $U(y(\cdot))$ , we use the solution  $y(\cdot)$  of Eqn. (1) generated by some function  $v(\cdot) \in L_2(T; U)$ . Therefore, the set  $U(y(\cdot))$  is non-empty, since it contains at least this function  $v(\cdot)$ .

**Remark 1.** Note that one and the same solution of Eqn. (1) can be derived by multiple inputs. In compliance with the approach conventional in the theory of ill-posed problems we reconstruct  $u_*(\cdot)$ .

In the case where solutions of Eqns. (1) and (3) are inaccurately measured (then inequalities (5) and (6) are fulfilled), it is necessary to specify a two-parameter family of functions  $u^{\alpha, h}(\cdot)$  (depending on  $\alpha \in (0, 1)$  and  $h \in (0, 1)$ ) with the properties

$$\begin{aligned} u^{\alpha, h}(t) = u^{\alpha, h}(\xi^h(t), \psi^h(t)), \\ |\dot{x}^h(t) - \psi^h(t)|_H \leq h, \end{aligned}$$

and the convergences

$$u^h(\cdot) = u^{\alpha(h), h}(\cdot) \rightarrow u_*(\cdot) \quad \text{in } L_2(T; U), \quad (11)$$

$$x^h(\cdot) \rightarrow y(\cdot) \quad \text{in } W(V; H) \text{ as } h \rightarrow 0, \quad (12)$$

for an appropriate concordance of  $\alpha = \alpha(h)$  and  $h$ . Here the symbol  $x^h(\cdot) = x(\cdot; x_{10}^h, x_0^h, u^h(\cdot))$  stands for the solution of Eqn. (3) with the right-hand side  $u(t) = u^{\alpha, h}(t)$ ; i.e.,  $x^h(\cdot)$  is the solution of the equation

$$\begin{aligned} \ddot{x}(t) - \Delta x(t) + mx(t) + \gamma \dot{x}(t) \\ = g(x(t)) + Bu^h(t) + f(t) \\ \text{in } V^* \text{ for a.e. } t \in T \end{aligned} \quad (13)$$

with the initial condition (4). Equations of the form (9) and (4) with properties (7) and (8) (respectively, equations of the form (13) and (4) with the properties (11) and (12)) are called *differential equations of reconstruction*. Thus,

the problem under consideration in this paper consists in designing differential equations of reconstruction.

The problem described above belongs to the class of problems of dynamical inversion (Lavrentiev *et al.*, 1980; Schwaller *et al.*, 2013; Banks and Kunisch, 1989; Lasiecka *et al.*, 1999; Mordukhovich and Zhang, 1997; Mordukhovich, 2008; 2011). The methodology of solving this problem suggested below uses an approach described, e.g., by Kryazhinskii and Osipov (1995), Maksimov (2002; 1996; 1995), Maksimov and Pandolfi (2001), Maksimov and Tröltzsch (2006), Kryazhinskii and Maksimov (2010), or Kapustyan and Maksimov (2014). This approach is based on the combination of the principle of feedback control (known in the theory of guaranteed control) with a model (Krasovskii and Subbotin, 1988) and one of the basic methods of ill-posed problems—that of the smoothing functional (Lavrentiev *et al.*, 1980).

Note that problems of dynamical reconstruction were studied by Kryazhinskii and Osipov (1995), Maksimov (2002), Maksimov and Pandolfi (2001), Maksimov and Tröltzsch (2006) or Kryazhinskii and Maksimov (2010). In these papers, systems described by ordinary differential and parabolic equations were considered. For hyperbolic equations and variational inequalities, this approach was developed by Maksimov (1995; 1996). In the works of Mordukhovich (2011), Maksimov and Tröltzsch (2006), Kryazhinskii and Maksimov (2010) or Maksimov (1995; 1996), the case where an input is subject to instantaneous constraints  $u(t) \in P$ , with  $P$  being a convex, closed and bounded set from a uniformly convex Banach space, was considered. In the present paper, continuing a series of works (Maksimov, 2002; Maksimov and Pandolfi, 2001), we consider the case where such a constraint is absent. Let us emphasize that the developed approach to the study of dynamic systems in uncertainty conditions is significantly different from the well-known stochastic approach to deal with problems under uncertainties, which in fact has not been largely developed for distributed parameter systems. In our case we do not assume the availability of any probabilistic information on perturbations. The absence of such information is quite realistic in many practical problems, in particular, those governed by PDE systems (see, e.g., Mordukhovich, 2008; 2011).

## 2. Equations of reconstruction: The case of precise measurement of solutions

First, we consider the case where the derivatives of the solutions  $y(\cdot)$  and  $x(\cdot)$  are measured without any error. Namely, we assume that, at every time  $t \in T$ , elements  $\dot{x}(t) \in H$  and  $\dot{y}(t) \in H$  are known. Let the function  $u^\alpha(\cdot)$  on the right-hand side of Eqn. (9) be defined by the

formula

$$u^\alpha(t) = \alpha^{-1} B^*(\dot{y}(t) - \dot{x}^\alpha(t)), \quad (14)$$

where  $B^*$  is the conjugate operator. Then Eqn. (9) has the form

$$\begin{aligned} \ddot{x}^\alpha(t) - \Delta x^\alpha(t) + m x^\alpha(t) + \gamma \dot{x}^\alpha(t) \\ = g(x^\alpha(t)) + \alpha^{-1} B B^*(\dot{y}(t) - \dot{x}^\alpha(t)) + f(t) \\ \text{in } V^* \text{ for a.e. } t \in T. \end{aligned} \quad (15)$$

**Theorem 1.** *Let  $\gamma > L\vartheta$ . Then the convergences (7) and (8) take place.*

The assertion of Theorem 1 follows from Theorem 1.2.1 of Maksimov (2002, p. 23) and Lemma 2 given below. To prove this lemma, the following statement is necessary. This result can be treated as a variant of the classical Gronwall lemma, while being different from the usual formulations of the latter (see, e.g., Warga, 1972).

**Lemma 1.** (Barbashin, 1970) *Let  $\phi(\cdot)$  and  $F(\cdot)$  be non-negative and integrable functions on some interval  $t_0 \leq t \leq t_0 + a$ ,  $a > 0$ . Let  $L$  be a positive constant. If the inequality*

$$\phi(t) \leq F(t) + L \int_{t_0}^t \phi(s) ds, \quad t_0 \leq t \leq t_0 + a,$$

*is valid, then the estimate*

$$\phi(t) \leq F(t) + L \int_{t_0}^t \exp\{L(t-s)\} F(s) ds$$

*takes place.*

**Lemma 2.** *Let the functions  $u^\alpha(\cdot)$  on the right-hand side of Eqn. (9) have the following properties:*

$$\sup_{t \in T} \{ |y(t) - x^\alpha(t)|_V^2 + |\dot{y}(t) - \dot{x}^\alpha(t)|_H^2 \} \leq C_0 \alpha, \quad (16)$$

$$|u^\alpha(\cdot)|_{L_2(T;U)} \leq |u_*(\cdot)|_{L_2(T;U)}. \quad (17)$$

*Then the convergences (7) and (8) take place.*

Here  $C_0$  is some constant independent of  $t$  and  $\alpha$ . In turn, the inequalities (16) and (17) follow from the following result:

**Lemma 3.** *Let the conditions of Theorem 1 be fulfilled. Then the function  $u^\alpha(\cdot)$  of the form (14) satisfies the conditions of Lemma 2.*

*Proof.* Let  $\mu_\alpha(t) = x^\alpha(t) - y(t)$ . By (15), we conclude that  $\mu_\alpha(t)$  is a solution of the equation

$$\begin{aligned} & \ddot{\mu}_\alpha(t) - \Delta\mu_\alpha(t) + m\mu_\alpha(t) + \gamma\dot{\mu}_\alpha(t) \\ &= g(x^\alpha(t)) - g(y(t)) + (Bu^\alpha(t) - Bu_*(t)), \quad t \in T \end{aligned} \quad (18)$$

with the initial conditions

$$\mu_\alpha(0) = 0, \quad \dot{\mu}_\alpha(0) = 0.$$

Introduce the Lyapunov function

$$\begin{aligned} V(t) &= |\dot{\mu}_\alpha(t)|_H^2 + |\mu_\alpha(t)|_V^2 + m|\mu_\alpha(t)|_H^2 \\ &+ 2\gamma \int_0^t |\dot{\mu}_\alpha(\tau)|_H^2 d\tau \\ &+ \alpha \int_0^t |u^\alpha(\tau)|_U^2 d\tau, \quad t \in T. \end{aligned} \quad (19)$$

Taking (19) into account, we deduce that

$$\begin{aligned} & \dot{V}(t) - \alpha|u_*(t)|_U^2 \\ &= 2(g(x^\alpha(t)) - g(y(t)), \dot{\mu}_\alpha(t))_H \\ &+ 2(B(u^\alpha(t) - u_*(t)), \dot{\mu}_\alpha(t))_H \\ &+ \alpha|u^\alpha(t)|_U^2 - \alpha|u_*(t)|_U^2 \quad \text{for a.e. } t \in T. \end{aligned} \quad (20)$$

Note that (see (14)) for a.e.  $t \in T$

$$\begin{aligned} u^\alpha(t) &= \arg \min \left\{ \alpha|v|_U^2 \right. \\ &\quad \left. - 2(B^*(\dot{y}(t) - \dot{x}^\alpha(t)), v)_U : v \in U \right\}. \end{aligned} \quad (21)$$

Using (21), we see that

$$\begin{aligned} & \dot{V}(t) - \alpha|u_*(t)|_U^2 \\ &\leq 2(g(x^\alpha(t)) - g(y(t)), \dot{\mu}_\alpha(t))_H \\ &\leq 2L|\mu_\alpha(t)|_H |\dot{\mu}_\alpha(t)|_H \\ &\leq |\dot{\mu}_\alpha(t)|_H^2 + L^2|\mu_\alpha(t)|_H^2 \leq cV(t), \end{aligned} \quad (22)$$

where  $c = \max\{1, m\} \max\{1, L^2\}$ . In addition,

$$V(0) = 0. \quad (23)$$

In this case, using Lemma 1, the inequality

$$\int_0^t e^{c(t-\tau)} d\tau \leq c^{-1} e^{ct},$$

(22) and (23), we get

$$\begin{aligned} V(t) &\leq \alpha \int_0^t |u_*(\tau)|_U^2 d\tau \\ &+ c\alpha \int_0^t \left\{ e^{c(t-\tau)} \int_0^\tau |u_*(p)|_U^2 dp \right\} d\tau \\ &\leq \alpha(1 + e^{c\vartheta}) \int_0^\vartheta |u_*(\tau)|_U^2 d\tau \leq C_0\alpha. \end{aligned} \quad (24)$$

The inequality (16) is proved. Let us verify the inequality (17). It is easily seen that the inequality

$$\begin{aligned} V(t) - \alpha \int_0^t |u_*(\tau)|_U^2 d\tau \\ \leq V(0) + 2L\vartheta \int_0^t |\dot{\mu}_\alpha(\tau)|_H^2 d\tau \end{aligned} \quad (25)$$

is true. In this case, from (25) it follows that

$$\begin{aligned} & |\mu_\alpha(t)|_V^2 + |\dot{\mu}_\alpha(t)|_H^2 + m|\mu_\alpha(t)|_H^2 \\ &+ \alpha \int_0^t \{|u^\alpha(\tau)|_U^2 - |u_*(\tau)|_U^2\} d\tau \leq V(0). \end{aligned} \quad (26)$$

By virtue of (23) and (26), we obtain (17). The proof is complete. ■

From Theorem 1 it follows that Eqn. (15) is the differential equation of reconstruction in the case of precise measurements of the solution.

### 3. Equations of reconstruction: The case of inaccurate measurement of solutions

Consider the case where the derivatives of the solution  $y(\cdot)$  of Eqn. (1) and the solution  $x(\cdot)$  of Eqn. (3) are inaccurately measured. Namely, we assume that, at every time  $t \in T$ , some elements  $\xi^h(t)$  and  $\psi^h(t)$  satisfying (5) are known. Let the function  $u^h(\cdot)$  in (13) be defined by

$$u^h(t) = \alpha^{-1} B^*(\xi^h(t) - \psi^h(t)). \quad (27)$$

In this case, Eqn. (13) takes the form

$$\begin{aligned} & \ddot{x}(t) - \Delta x(t) + mx(t) + \gamma\dot{x}(t) \\ &= g(x(t)) + \alpha^{-1} BB^*(\xi^h(t) - \psi^h(t)) + f(t) \\ &\quad \text{in } V^* \text{ for a.e. } t \in T. \end{aligned} \quad (28)$$

Let, as above,  $u_*(\cdot) = u_*(\cdot; y(\cdot))$  be the element of the set  $U(y(\cdot))$  of minimal  $L_2(T; U)$ -norm.

**Theorem 2.** Let  $\alpha = \alpha(h) \in (0, 1)$  for  $h \in (0, 1)$ . Let also  $\gamma > L\vartheta$  and  $h\alpha^{-2}(h) \leq C = \text{const} > 0$  for  $h \in (0, 1)$ . Then the convergences (11) and (12) take place.

The assertion of Theorem 2 follows from the results below.

**Lemma 4.** Let the conditions of Theorem 2 be fulfilled. Then there exists  $h_* \in (0, 1)$  such that the inequalities

$$\sup_{t \in T} \{ |y(t) - x^h(t)|_V^2 + |\dot{y}(t) - \dot{x}^h(t)|_H^2 \} \leq C_1(\alpha(h) + h), \quad (29)$$

$$|u^h(\cdot)|_{L_2(T;U)}^2 \leq r_1(h)|u_*(\cdot)|_{L_2(T;U)}^2 + r_2(h) \quad (30)$$

are fulfilled for  $h \in (0, h_*)$ . Here

$$r_1(h) \rightarrow 1, \quad r_2(h) \rightarrow 0 \quad \text{as } h \rightarrow 0;$$

$C_1$  is some constant independent of  $t$ ,  $\alpha$ , and  $h$ .

*Proof.* By (27) and (5), the inequality

$$|u^h(t)|_U^2 \leq 2b^2\alpha^{-2}(h^2 + |\dot{\mu}_h(t)|_H^2), \quad t \in T,$$

is valid. Here  $\alpha = \alpha(h)$ ,  $\mu_h(t) = x^h(t) - y(t)$ , and  $b = |B^*|_{L(H;U)}$  is the norm of the linear continuous operator  $B^* \in L(H; U)$ . In this case, for  $t \in T$ , we have

$$\int_0^t |u^h(\tau)|_U^2 d\tau \leq 2b^2\alpha^{-2} \int_0^t |\dot{\mu}_h(\tau)|_H^2 d\tau + c_1 h^2 \alpha^{-2}. \quad (31)$$

It is easily seen that

$$\begin{aligned} & (B(u^h(t) - u_*(t)), \dot{\mu}_h(t))_H \\ & \leq (B(u^h(t) - u_*(t)), \psi^h(t) - \xi^h(t))_H \\ & + c_2 h \{ |u_*(t)|_U + |u^h(t)|_U \} \quad \text{for a.e. } t \in T. \end{aligned} \quad (32)$$

From (1) and (13), it follows that the function  $\mu_h(t)$  is a solution of the equation

$$\begin{aligned} & \ddot{\mu}_h(t) - \Delta\mu_h(t) + m\mu_h(t) + \gamma\dot{\mu}_h(t) \\ & = g(x^h(t)) - g(y(t)) + B(u^h(t) - u_*(t)) \\ & \quad \text{in } V^* \text{ for a.e. } t \in T \end{aligned} \quad (33)$$

with the initial conditions

$$\mu_h(0) = y_0 - x_0^h, \quad \dot{\mu}_h(0) = y_{10} - x_{10}^h.$$

Introduce the Lyapunov function (see (19))

$$\begin{aligned} V(t) &= |\dot{\mu}_h(t)|_H^2 + |\mu_h(t)|_V^2 \\ &+ m|\mu_h(t)|_H^2 + 2\gamma \int_0^t |\dot{\mu}_h(\tau)|_H^2 d\tau \\ &+ \alpha \int_0^t |u^h(\tau)|_U^2 d\tau, \quad t \in T. \end{aligned} \quad (34)$$

By virtue of (34), we conclude that for a.e.  $t \in T$

$$\begin{aligned} & \dot{V}(t) - \alpha|u_*(t)|_U^2 \\ &= 2(g(x^h(t)) - g(y(t)), \dot{\mu}_h(t))_H \\ &+ 2(B(u^h(t) - u_*(t)), \dot{\mu}_h(t))_H \\ &+ \alpha|u^h(t)|_U^2 - \alpha|u_*(t)|_U^2. \end{aligned} \quad (35)$$

Note that the control  $u^h(t)$  of the form (27) is defined by the rule

$$\begin{aligned} u^h(t) &= \arg \min \left\{ \alpha|v|_U^2 \right. \\ &\quad \left. - 2(B^*(\xi^h(t) - \psi^h(t)), v)_U : v \in U \right\}. \end{aligned} \quad (36)$$

From (35), (32) and (36), we deduce that

$$\begin{aligned} \dot{V}(t) - \alpha|u_*(t)|_U^2 &\leq cV(t) + 2c_2 h \{ |u_*(t)|_U \\ &\quad + |u^h(t)|_U \} \quad \text{for a.e. } t \in T. \end{aligned} \quad (37)$$

Using the inclusion  $u_*(\cdot) \in L_2(T; U)$ , we have

$$\int_0^{\vartheta} |u_*(\tau)|_U d\tau \leq c_3.$$

It follows from this inequality, (37), and the inequality

$$\begin{aligned} V(0) &= |\dot{\mu}_h(0)|_H^2 + |\mu_h(0)|_V^2 + m|\mu_h(0)|_H^2 \\ &\leq (2 + mb_0^2)h^2, \end{aligned}$$

by analogy with (24), that

$$V(t) \leq c_4(h + \alpha) + c_5 h \int_0^t |u^h(\tau)|_U^2 d\tau. \quad (38)$$

Here  $b_0 > 0$  is a constant such that

$$|z|_H \leq b_0 |z|_V \quad \text{for every } z \in V.$$

In turn, from (38), by virtue of (31), we conclude that

$$\begin{aligned} V(t) &\leq c_4(h + \alpha) \\ &+ c_6 h \alpha^{-2} (h^2 + \int_0^t |\dot{\mu}_h(\tau)|_H^2 d\tau). \end{aligned} \quad (39)$$

From (39), it follows that the estimate

$$\begin{aligned} |\dot{\mu}_h(t)|_H^2 &\leq c_4(h + \alpha) \\ &+ c_6 h \alpha^{-2} (h^2 + \int_0^t |\dot{\mu}_h(\tau)|_H^2 d\tau) \end{aligned} \quad (40)$$

is valid. Taking into account the Gronwall lemma, we have, for  $t \in T$ ,

$$|\dot{\mu}_h(t)|_H^2 \leq (c_4(h + \alpha) + c_6 h^3 \alpha^{-2}) \exp c_6 t h \alpha^{-2}. \quad (41)$$

Due to the condition of the theorem, for  $h \in (0, 1)$ , we get

$$h \alpha^{-2}(h) \leq C. \quad (42)$$

Then, using (41) and (42), we have

$$|\dot{\mu}_h(t)|_H^2 \leq c_7(h + \alpha(h)), \quad t \in T. \quad (43)$$

In this case, (39) and (43) imply the inequality

$$V(t) \leq c_8(h + \alpha), \quad t \in T.$$

The inequality (29) follows from this estimate. Then taking into account the Lipschitz property of the function  $g(\cdot)$ , from (26) we obtain the estimate

$$\begin{aligned} V(t) - \alpha \int_0^t |u_*(\tau)|_U^2 d\tau &\leq V(0) + L^2 h^2 \vartheta \mu^{-1} \\ &+ (2L\vartheta + \mu) \int_0^t |\dot{\mu}_h(\tau)|_H^2 d\tau + c_9 h \\ &+ c_{10} h \int_0^t |u^h(\tau)|_H^2 d\tau, \end{aligned} \quad (44)$$

where the number  $\mu > 0$  is such that

$$2\gamma > 2L\vartheta + \mu.$$

In this case, using (44), we obtain

$$\begin{aligned} (\alpha - c_{10}h) \int_0^t |u^h(\tau)|_U^2 d\tau &\leq \alpha \int_0^t |u_*(\tau)|_U^2 d\tau + c_{11}h, \quad t \in T. \end{aligned} \quad (45)$$

The validity of the inequality (30) follows from (45) and the convergence  $h\alpha^{-1}(h) \rightarrow 0$  (as  $h \rightarrow 0$ ). In this situation,

$$r_1(h) = \alpha(h) \{\alpha(h) - c_{10}h\}^{-1},$$

$$r_2(h) = c_{11}h \{\alpha(h) - c_{10}h\}^{-1}.$$

The lemma is proved. ■

From Theorem 2, it follows that Eqn. (28) is a differential equation of reconstruction in the case of inaccurate measurements of the solution.

Under some additional conditions, one can rewrite the estimate of the convergence rate (see Theorem 3 below). To derive this estimate, we need the following result.

**Lemma 5.** (Maksimov, 2002, p. 47) Let  $u(\cdot) \in L_\infty(T; V^*)$ ,  $v(\cdot) \in W_V(T)$ ,

$$\left| \int_0^t u(\tau) d\tau \right|_{V^*} \leq \varepsilon, \quad |v(t)|_V \leq K \quad \forall t \in T.$$

Then for all  $t \in T$ , the inequality

$$\left| \int_0^t \langle u(\tau), v(\tau) \rangle d\tau \right| \leq \varepsilon(K + \text{var}(T; v(\cdot)))$$

is valid.

Here the symbol  $\text{var}(T; v(\cdot))$  means the variation of the function  $v(\cdot)$  over the interval  $T$ , the symbol  $W_V(T)$  means the set of functions  $y(\cdot) : T \rightarrow V$  of bounded variation, and the symbol  $\langle \cdot, \cdot \rangle$  means the duality between the spaces  $V^*$  and  $V$ .

**Theorem 3.** Let the conditions of Theorem 2 hold. Let also  $U = V$ , and let  $B$  be the operator of canonical embedding of space  $V$  into space  $H$ , and  $u_*(\cdot) \in W_V(T)$ . Then the inequality

$$\begin{aligned} |u_*(\cdot) - u^h(\cdot)|_{L_2(T; H)}^2 &\leq K \{h^{1/4} + \alpha^{1/4} + |r_1(h) - 1| + r_2(h)\} \end{aligned}$$

is valid. Here  $K$  is some constant independent of  $t$ ,  $\alpha$ , and  $h$ ; the symbol  $|\cdot|$  means the absolute value of its argument.

*Proof.* Taking into account equality (33) and multiplying its right- and left-hand sides by  $\dot{\mu}_h(t)$ , then integrating, we see that

$$\begin{aligned} |\dot{\mu}_h(t)|_H^2 + |\mu_h(t)|_V^2 + m|\mu_h(t)|_H^2 + 2\gamma \int_0^t |\dot{\mu}_h(\tau)|_H^2 d\tau &\leq |\dot{\mu}_h(0)|_H^2 + |\mu_h(0)|_V^2 + m|\mu_h(0)|_H^2 \\ &+ 2L \int_0^t |\mu_h(\tau)|_H |\dot{\mu}_h(\tau)|_H d\tau \\ &+ K_1 \int_0^t |u_*(\tau) - u^h(\tau)|_V |\dot{\mu}_h(\tau)|_{V^*} d\tau. \end{aligned} \quad (46)$$



By the Cauchy–Bunyakovsky inequality and Lemma 4 (see (29)), the last term on the right-hand side of (46) is estimated from above by the value

$$K_2(h^{1/2} + \alpha^{1/2}). \quad (47)$$

Using (6), (46), and (47), we conclude that the estimate

$$|\dot{\mu}_h(t)|_H^2 + |\mu_h(t)|_V^2 \leq K_3(h^{1/2} + \alpha^{1/2}) \quad (48)$$

is valid. Note that, for any  $t \in T$ , the inequality

$$\begin{aligned} & \left| \int_0^t B(u_*(\tau) - u^h(\tau)) d\tau \right|_{V^*} \\ &= \sup_{|v|_{V^*} \leq 1} \left| \left\langle \int_0^t \left\{ \ddot{\mu}_h(\tau) - \Delta \mu_h(\tau) + m \mu_h(\tau) \right. \right. \right. \\ & \quad \left. \left. \left. + \gamma \dot{\mu}_h(\tau) - g(x^h(\tau)) + g(y(\tau)) \right\} d\tau, v \right\rangle \right| \\ &\leq K_4 \left\{ |\dot{\mu}_h(t) - \dot{\mu}_h(0)|_H + |\mu_h(t) - \mu_h(0)|_V \right. \\ & \quad \left. + \int_0^t |\mu_h(\tau)|_V d\tau \right\} \end{aligned} \quad (49)$$

is fulfilled. Then, by using (48) and (6), we conclude that

$$\left| \int_0^t B(u_*(\tau) - u^h(\tau)) d\tau \right|_{V^*} \leq K_5(h^{1/4} + \alpha^{1/4}). \quad (50)$$

By virtue of Lemma 5 and the relations (30) and (50), we obtain

$$\begin{aligned} & |u_*(\cdot) - u^h(\cdot)|_{L_2(T;H)}^2 \\ &\leq (1 + r_1(h)) |u_*(\cdot)|_{L_2(T;H)}^2 \\ &\quad - 2 \int_0^\vartheta (Bu_*(\tau), u^h(\tau))_H d\tau + r_2(h) \\ &= 2 \int_0^\vartheta (B(u_*(\tau) - u^h(\tau)), u_*(\tau))_H d\tau \\ &\quad + |r_1(h) - 1| \int_0^\vartheta |Bu_*(\tau)|_H^2 d\tau + r_2(h) \\ &\leq K \{ h^{1/4} + \alpha^{1/4} + |r_1(h) - 1| + r_2(h) \}. \end{aligned}$$

The proof of the theorem is complete. ■

#### 4. Numerical example

In this section, we present a numerical example. The problem described in Section 3 is solved. It is assumed that the parameters of Eqn. (1) are as follows:

$$\Omega = [0, 1], \quad \vartheta = 2, \quad m = 2, \quad \gamma = 1, \quad U = \mathbb{R},$$

$$f(t, \eta) = 0, \quad g(x) = \sin x,$$

$$(Bv)(\eta) = \omega(\eta)v, \quad \omega(\eta) = 1 \quad \text{for a.e. } \eta \in \Omega.$$

As the initial state of (1), we take the functions  $y_0(\eta) = \eta(1 - \eta)$ ,  $y_{10}(\eta) = \eta$  for a.e.  $\eta \in \Omega$ . The control on the right-hand side of Eqn. (1) is  $v(t) = t^2$  and the control on the right-hand side of Eqn. (3) is calculated by (27). Equations (1) and (3) are solved by the grid method with the step  $\Delta\omega$  in the domain  $\Omega$  and the step  $\delta$  in the time interval  $T$ .

The results of computer modelling are presented in Figs. 1–6 for the following case:  $\Delta\omega = 1/15$ ,  $\delta = 2/150$ ,  $\tau_i = \tau_{i-1} + \delta$ ,  $\tau_0 = 0$ . In the experiment, we assume  $\xi^h(\tau_i, \nu_j) = \dot{y}(\tau_i, \nu_j) + h$ ,  $\psi^h(\tau_i, \nu_j) = \dot{x}(\tau_i, \nu_j) + h$ , where  $\nu_j = j\Delta\omega$ ,  $j = 0, \dots, 1/\Delta\omega$ . Figures 1 and 2 correspond to the case of  $h = 0$ , Figs. 3 and 4 to the case of  $h = 0.01$ , and Figs. 5 and 6 to the case  $h = 0.1$ . In Figs. 1, 3, and 5, the solid line represents the section of the function  $\dot{y}(t, \eta)$  by the hyperplane  $\eta = 0.5$ ; the dashed line—a similar section of the function  $\dot{x}(t, \eta)$ . By analogy, in Figs. 2, 4 and 6, the solid line corresponds to the control  $v(t)$  and the dashed line to the control  $u^h(t)$ . As can be seen from Fig. 1, the corresponding curves actually coincide.

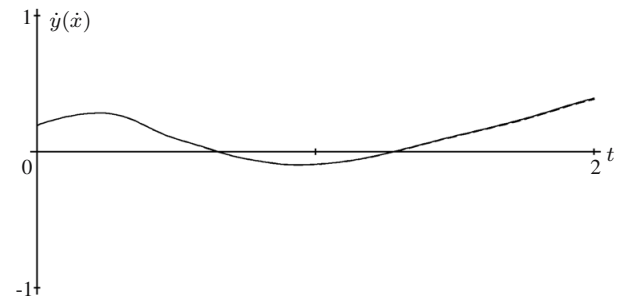


Fig. 1.  $h = 0$ .

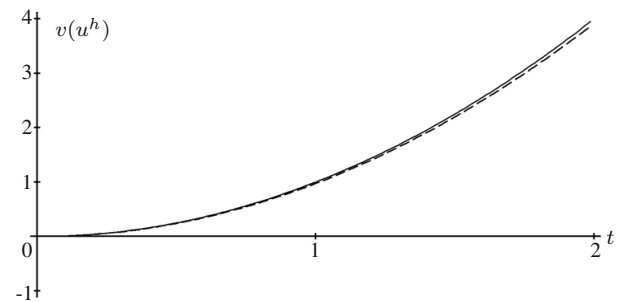


Fig. 2.  $h = 0$ .

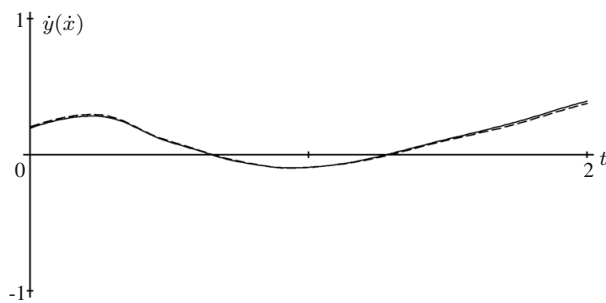


Fig. 3.  $h = 0.01$ .

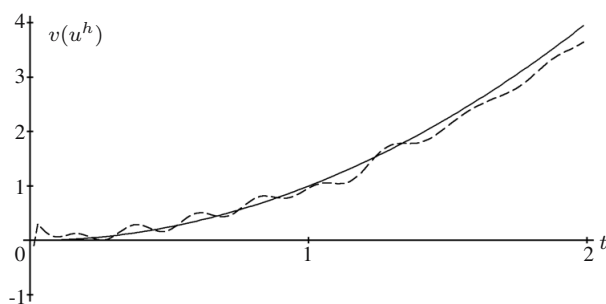


Fig. 4.  $h = 0.01$ .

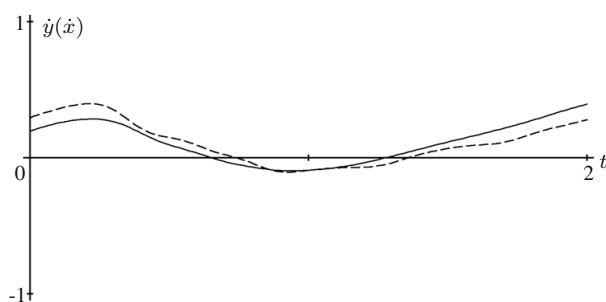


Fig. 5.  $h = 0.1$ .

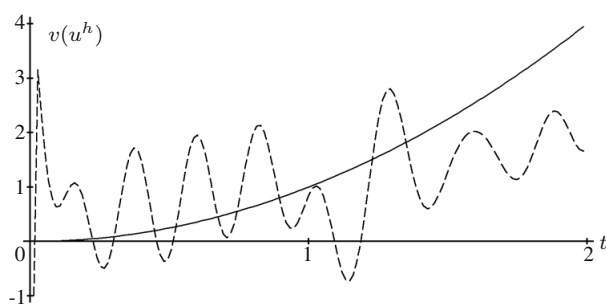


Fig. 6.  $h = 0.1$ .

## 5. Conclusions

For a second-order partial differential equation, an algorithm for constructing a differential equation of reconstruction has been suggested. The problem consists in the design of a reconstruction equation with a feedback control providing the closeness of solutions (and controls) of two equations: the given one (with an unknown control and a solution measured inaccurately) and the sought one (with a control formed by an appropriate way). The performance of the algorithm has been tested on a model example.

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