Int. J. Appl. Math. Comput. Sci., 2018, Vol. 28, No. 3, 457–472
DOI: 10.2478/amcs-2018-0035

BIBO STABILISATION OF CONTINUOUS–TIME TAKAGI–SUGENO SYSTEMS UNDER PERSISTENT PERTURBATIONS AND INPUT SATURATION

JOSÉ V. SALCEDO [a,*], MIGUÉL MARTÍNEZ [a], SERGIO GARCÍA-NIETO [a], ADOLFO HILARIO [a]

[a] Institute of Control Systems and Industrial Computing (ai2)
Polytechnic University of Valencia, Camino de Vera S/N, 46022 Valencia, Spain
e-mail: jsalcedo@upv.es

This paper presents a novel approach to the design of fuzzy state feedback controllers for continuous-time non-linear systems with input saturation under persistent perturbations. It is assumed that all the states of the Takagi–Sugeno (TS) fuzzy model representing a non-linear system are measurable. Such controllers achieve bounded input bounded output (BIBO) stabilisation in closed loop based on the computation of inescapable ellipsoids. These ellipsoids are computed with linear matrix inequalities (LMIs) that guarantee stabilisation with input saturation and persistent perturbations. In particular, two kinds of inescapable ellipsoids are computed when solving a multiobjective optimization problem: the maximum volume inescapable ellipsoids contained inside the validity domain of the TS fuzzy model and the smallest inescapable ellipsoids which guarantee a minimum $\star$-norm (upper bound of the 1-norm) of the perturbed system. For every initial point contained in the maximum volume ellipsoid, the closed loop will enter the minimum $\star$-norm ellipsoid after a finite time, and it will remain inside afterwards. Consequently, the designed controllers have a large domain of validity and ensure a small value for the 1-norm of closed loop.

Keywords: LMIs, fuzzy systems, non-linear systems, input saturation, disturbances.

1. Introduction

The design of fuzzy controllers for non-linear systems using LMIs has been an important and relevant topic for researchers since the mid-1990s (Tanaka *et al.*, 1998; Tanaka and Wang, 2001). TS fuzzy models can exactly represent non-linear systems in a certain domain of validity. TS fuzzy models allow the design of several kinds of controllers using LMIs. One of the most common is parallel distributed compensation (PDC) (Tanaka and Wang, 2001) which has the same premises as the TS model and its consequents are linear state feedback laws.

Several sorts of conditions can be taken into account during the design stage (Tanaka and Wang, 2001): stability, decay rate, state and input constraints, $\mathcal{H}_\infty$-norm, etc. Recently, some papers (Saifia *et al.*, 2012; Chang and Shih, 2015; Nguyen *et al.*, 2015; 2016; Klug *et al.*, 2015; Duan *et al.*, 2016; Vafamand *et al.*, 2016) have reported research on designing fuzzy controllers for TS fuzzy models when perturbations are present and there exists actuator saturation and state constraints. These publications improve previous ways of handling input constraints (Tanaka and Wang, 2001; Du and Zhang, 2009; Zhao and Gao, 2012; da Silva *et al.*, 2013; Bezzaoucha *et al.*, 2013; Nguyen *et al.*, 2014; Benzaouia *et al.*, 2015; Yang and Tong, 2015) and use invariant set theory to guarantee that in closed loop non-linear systems will evolve inside a robust positively invariant ellipsoid.

In this paper we want to improve results from these publications by extending previous results of Salcedo and Martinez (2008) as well as Salcedo *et al.* (2008) for continuous-time TS fuzzy models under persistent perturbations using the concept in $\star$-norm in the case of actuator saturation and state constraints.

A first step in this direction is to remove the need for bilinear matrix inequalities (BMIs) (Salcedo and Martinez, 2008) when there is a direct coupling between performance output and perturbation. In Section 3 Theorem 2 will show that it is possible to take into account LMI conditions only. This step is really important since BMIs are complex non-convex conditions (Goh *et al.*, 1996).

*Corresponding author

Nguyen *et al.* (2015; 2016) and Vafamand *et al.* (2016) claim that they outperform previous results (Saifia *et al.*, 2012; Chang and Shih, 2015; Klug *et al.*, 2015). Therefore this paper will only focus on those articles.

Nguyen *et al.* (2015) and Vafamand *et al.* (2016) deal with continuous-time TS fuzzy systems, whereas the paper by Nguyen *et al.* (2016) is related to discrete time models. Nguyen *et al.* (2015; 2016) use a generalised sector bound condition employ to manage input saturation, and Vafamand *et al.* (2016) employ an inequality involving a parameter to guarantee stability and $\mathcal{L}_1$-performance. The generalized sector condition is more powerful since parameters are computed when solving LMIs conditions, whereas the parameter used by Vafamand *et al.* (2016) must be known in advance. In this paper we will adapt the generalised sector condition to inescapable ellipsoids, avoiding the use of additional parameters.

The validity domain of controllers in the works of Nguyen *et al.* (2015; 2016) and Vafamand *et al.* (2016) is given in terms of robust positively invariant ellipsoids. This is claimed by Vafamand *et al.* (2016), but the authors did not prove it. In fact, in Theorem 1 therein an additional constraint must be added in order to ensure such invariance. On the other hand, Vafamand *et al.* (2016) require to know in advance several parameters, but any guidelines on how to choose them are missing. Thus, the design procedure has not been clearly stated. In the works of Nguyen *et al.* (2015; 2016) the computed ellipsoids are robust positively invariant, but they only estimate a single ellipsoid to achieve a trade-off between the size of the validity domain and the output performance, for persistent perturbations and finite energy perturbations (those which have finite 2-norm) (Nguyen *et al.*, 2016) and only finite energy perturbations (Nguyen *et al.*, 2015).

Nguyen *et al.* (2015; 2016) take into account state constraints using polyhedral sets where robust positively invariant sets must be contained, whereas Vafamand *et al.* (2016) do not take into account any condition related to state constraints. Conditions of Salcedo and Martinez (2008), Salcedo *et al.* (2008) and Nguyen *et al.* (2015) use an upper bound to the persistent perturbation. However, Theorem 1 of Vafamand *et al.* (2016) seems to be independent of such bound. This is surprising when there are constraints on inputs and/or states.

In this paper a novel approach is proposed to improve previous results related to persistent perturbations, which is based on avoiding the requirement that design parameters be known in advance, and computing two kinds of inescapable ellipsoids: the maximum volume inescapable ellipsoids contained inside the domain of validity of the TS fuzzy model, and the smallest inescapable ellipsoids which guarantees the minimum $\star$-norm (upper bound of 1-norm). On the whole, the larger is the inescapable ellisoid, the higher the $\star$-norm of the closed loop. As a consequence, there is a trade-off between obtaining maximum volume and minimum $\star$-norm ellipsoids. In this paper we propose a multi-objective optimization to provide valid solutions to this trade-off.

This novel approach can be characterized by the following features:

- Extension of the concept of inescapable ellipsoids and $\star$-norm (Salcedo and Martinez, 2008) to continuous-time TS fuzzy systems with input saturation.
- Use of LMIs conditions only for the computation of $\star$-norm instead of some BMIs as was stated in (Salcedo and Martinez, 2008).
- Computation of fuzzy PDC state feedback controllers related to maximum volume and minimum $\star$-norm inescapable ellipsoids for continuous-time TS fuzzy systems. To the best of our knowledge, these procedures are new.
- Development of algorithms to obtain fuzzy PDC state feedback controllers for continuous-time TS fuzzy systems which solve the multi-objective optimization trade-off between maximum volume and minimum $\star$-norm inescapable ellipsoids. For Vafamand *et al.* (2016), minimum 1-norm was the only objective.
- With these algorithms controllers have a large domain of validity and ensure a small value for the 1-norm of the closed loop for continuous-time TS fuzzy models compared with those given by Vafamand *et al.* (2016). It is important to emphasize that Nguyen *et al.* (2016) deal with discrete TS fuzzy systems, but also consider finite energy perturbations instead of persistent ones (Nguyen *et al.*, 2015). Consequently, it is not possible to establish a theoretical comparison of these works (Nguyen *et al.*, 2015; 2016).

The rest of the paper is organized as follows: Section 2 presents theoretical background. Section 3 discusses the $\star$-norm and its relation with the 1-norm. Main results of this paper are developed in Section 4. Algorithms for designing fuzzy PDC state feedback controllers which yield a solution to the multi-objective optimization trade-off between maximum volume and minimum $\star$-norm inescapable ellipsoids are described in Section 5. Sections 6 and 7 are devoted to application examples. Finally, in Section 8 conclusions are discussed.

2. Theoretical background

A linear matrix inequality (LMI) is an expression of the form (Boyd *et al.*, 1994)

$$\boldsymbol{H}(\boldsymbol{x}) \triangleq \boldsymbol{H}_0 + \sum_{i=1}^{m} x_i \boldsymbol{H}_i > 0, \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^m$ is an unknown vector and the symmetric matrices $\boldsymbol{H}_i = \boldsymbol{H}_i^T \in \mathbb{R}^{n\times n}$, $i = 0, \dots, m$ are given. The inequality symbol $>$ means that $\boldsymbol{H}(\boldsymbol{x})$ is a positive-definite matrix. By definition, the previous LMI is strict, although it is possible to consider non-strict LMIs using $\geq$ instead of $>$.

If the set $\{\boldsymbol{x} : \boldsymbol{H}(\boldsymbol{x}) > 0\}$ is not empty, i.e., if it admits solutions, it is convex. In general, LMIs do not have analytical solutions but they can be solved using highly efficient numerical algorithms in polynomial time (Boyd *et al.*, 1994; El Ghaoui and Niculescu, 2000). Some of these algorithms have been incorporated into various computer tools (Gahinet *et al.*, 1995; Sturm, 1999; Löfberg, 2004) for solution of LMI problems.

The use of rule-based fuzzy models to represent non-linear systems is an idea that has been gaining in popularity in past years (Tanaka and Wang, 2001; Guerra *et al.*, 2006). It is a method that has simplified a controller design by eliminating the need to design a controller specifically for the non-linear system. Instead, the controller is designed for the fuzzy system it represents. This paper uses the TS fuzzy model (Takagi and Sugeno, 1985), where each rule in this fuzzy model represents a linear state space model:

$$\begin{aligned} &\text{Rule } i\text{:} \quad \text{IF } z_1(t) \text{ is } M_{i,1} \text{ and} \dots \text{ and } z_p(t) \text{ is } M_{ip} \\ &\qquad \text{THEN } \dot{\boldsymbol{x}}(t) = \boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_{1i}\boldsymbol{u}(t) + \boldsymbol{B}_{2i}\boldsymbol{\phi}(t), \\ &\qquad\qquad \boldsymbol{y}(t) = \boldsymbol{C}_i\boldsymbol{x}(t) + \boldsymbol{D}_i\boldsymbol{\phi}(t), \end{aligned} \tag{2}$$

where $i = 1, 2, \dots, r$ and r is the number of rules, $\boldsymbol{x}(t) \in \mathbb{R}^{n_x}$ is the state vector, $z_1(t)$, $z_2(t)$, ..., $z_p(t)$ are the premise variables, M_{ij} signifies the degree of membership of the variable $z_j(t)$ to rule i ($j = 1, 2, \dots, p$), $\boldsymbol{u}(t) \in \mathbb{R}^{n_u}$ is the control input vector, $\boldsymbol{\phi} \in \mathbb{R}^{n_\phi}$ is the disturbance vector, $\boldsymbol{y}(t) \in \mathbb{R}^{n_y}$ is the controlled output. It is assumed that all the states and premise variables are measurable.

By using the inference method with a singleton fuzzifier, a product inference engine and a defuzzifier based on the centre average (Tanaka and Wang, 2001), the dynamic fuzzy model (2) is

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \frac{\sum_{i=1}^{r} w_i(t)\left(\boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_{1i}\boldsymbol{u}(t) + \boldsymbol{B}_{2i}\boldsymbol{\phi}(t)\right)}{\sum_{i=1}^{r} w_i(t)} \\ &= \sum_{i=1}^{r} h_i(t)\left(\boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_{1i}\boldsymbol{u}(t) + \boldsymbol{B}_{2i}\boldsymbol{\phi}(t)\right), \\ \boldsymbol{y}(t) &= \sum_{i=1}^{r} h_i(t)\left(\boldsymbol{C}_i\boldsymbol{x}(t) + \boldsymbol{D}_i\boldsymbol{\phi}(t)\right), \end{aligned} \tag{3}$$

with

$$\begin{aligned} w_i(t) &= \prod_{j=1}^{p} M_{ij}(z_j(t)), \\ h_i(t) &= \frac{w_i(t)}{\sum_{i=1}^{r} w_i(t)}. \end{aligned} \tag{4}$$

$M_{ij}(z_j(t))$ is the degree of membership of $z_j(t)$ to M_{ij}. It is assumed that

$$\begin{aligned} w_i(t) \geq 0, \quad i = 1, \dots, r \quad \forall\, t, \\ \sum_{i=1}^{r} w_i(t) > 0, \quad \forall t. \end{aligned}$$

Therefore,

$$h_i(t) \geq 0, \quad \sum_{i=1}^{r} h_i(t) = 1, \quad \forall t. \tag{5}$$

The domain of validity $\mathcal{P}_x$ (polyhedral) of this dynamic fuzzy model is defined as

$$\mathcal{P}_x \triangleq \left\{ \boldsymbol{x} \in \mathbb{R}^{n_x} \ : \ \boldsymbol{h}_m^T\boldsymbol{x} \leq 1,\ m = 1, \dots, s \right\}, \tag{6}$$

where the vectors $\boldsymbol{h}_m$ are given and can be computed from the state constraints of (3). Consequently, $\mathcal{P}_x$ also represents the state constraints of the fuzzy model.

To simplify the presentation of fuzzy systems, the following notation will be used:

$$\begin{aligned} \boldsymbol{Y}_z &= \sum_{i=1}^{r} h_i(z(t))\boldsymbol{Y}_i, \\ \boldsymbol{Y}_{zz} &= \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))h_j(z(t))\boldsymbol{Y}_{ij} \end{aligned} \tag{7}$$

where $\boldsymbol{Y}_i$ and $\boldsymbol{Y}_{ij}$ are constant matrices. Then the fuzzy system (3) takes the form

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}_z\boldsymbol{x}(t) + \boldsymbol{B}_{1z}\boldsymbol{u}(t) + \boldsymbol{B}_{2z}\boldsymbol{\phi}(t), \\ \boldsymbol{y}(t) &= \boldsymbol{C}_z\boldsymbol{x}(t) + \boldsymbol{D}_z\boldsymbol{\phi}(t). \end{aligned} \tag{8}$$

When TS fuzzy systems have input saturation, their dynamic model transforms into

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}_z\boldsymbol{x}(t) + \boldsymbol{B}_{1z}\mathbf{sat}(\boldsymbol{u}(t)) + \boldsymbol{B}_{2z}\boldsymbol{\phi}(t), \\ \boldsymbol{y}(t) &= \boldsymbol{C}_z\boldsymbol{x}(t) + \boldsymbol{D}_z\boldsymbol{\phi}(t), \end{aligned} \tag{9}$$

where

$$\mathbf{sat}(\boldsymbol{u}) \triangleq (\mathrm{sat}(u_1)\dots\mathrm{sat}(u_{n_u}))^T,$$
$$\mathrm{sat}(u_l) \triangleq \mathrm{sign}(u_l)\min(|u_l|, u_{\max,l}), \quad l = 1,\dots,n_u. \tag{10}$$

Therefore each component of the control action applied to the TS fuzzy system will satisfy

$$-u_{\max,l} \le u_l \le u_{\max,l}, \quad l = 1,\dots,n_u. \tag{11}$$

3. 1-norm and ⋆-norm for continuous-time TS fuzzy systems without input saturation

The main objective in this work is to design fuzzy state-feedback controllers for TS fuzzy systems with input saturation, which are capable of stabilizing the system when the disturbance vector $\boldsymbol{\phi}$ is bounded for the entire time interval, i.e.,

$$\boldsymbol{\phi}(t)^T\boldsymbol{\phi}(t) \le \delta^2, \quad \forall t,\ \delta > 0, \tag{12}$$

where the signal did not necessarily tend asymptotically to 0 as $t \to \infty$. This type of disturbance is called persistent. The required stabilization condition is BIBO stability, which means that the output vector will always be bounded when the system is affected by such types of disturbances:

$$\exists \mu > 0\ :\ \boldsymbol{y}(t)^T\boldsymbol{y}(t) \le \mu^2, \quad \forall t. \tag{13}$$

The 1-norm rather than the $\mathcal{H}_\infty$-norm is used when working with persistent disturbances.[1] The 1-norm is defined by (Boyd *et al.*, 1994; Abedor *et al.*, 1996; Sanchez Peña and Sznaier, 1998)

$$||\boldsymbol{G}_{\phi\to y}||_1 \triangleq \sup_{||\boldsymbol{\phi}(t)||_\infty \ne 0} \frac{||\boldsymbol{y}(t)||_\infty}{||\boldsymbol{\phi}(t)||_\infty}, \tag{14}$$

where the ∞-norm of a vector signal is defined as

$$||\boldsymbol{\phi}(t)||_\infty^2 \triangleq \sup_{t\ge 0} \boldsymbol{\phi}(t)^T\boldsymbol{\phi}(t) = \delta^2. \tag{15}$$

This paper proposes an extension of the method presented by Salcedo and Martinez (2008) when a fuzzy state-feedback controller is designed for minimizing in closed loop the 1-norm between $\boldsymbol{\phi}(t)$ and $\boldsymbol{y}(t)$ with input saturation. In the work of Salcedo and Martinez (2008) TS fuzzy systems did not have input saturation.

It is more complicated to determine the 1-norm than the 2-norm or the $\mathcal{H}_\infty$-norm (Sanchez Peña and Sznaier, 1998), although it is possible to get an upper bound for the same, called star (⋆) norm, by means of LMIs (Abedor *et al.*, 1996; Sanchez Peña and Sznaier, 1998; Salcedo *et al.*, 2007). This alternative makes it possible to use the existing techniques for fuzzy controllers design via LMIs (Tanaka and Wang, 2001; Liu and Zhang, 2003; Teixeira *et al.*, 2003; Guerra *et al.*, 2006).

In this paper PDC state-feedback fuzzy controllers (Tanaka and Wang, 2001) with the same premise variables as the TS fuzzy model (2) and linear state feedback control laws will be designed:

Controller Rule i:

IF $z_1(t)$ is $M_{i,1}$ and ... and $z_p(t)$ is M_{ip}

THEN $\boldsymbol{u}(t) = \boldsymbol{F}_i\boldsymbol{x}(t), \quad i = 1, 2, \dots, r,$ (16)

where $\boldsymbol{F}_i$ is the local feedback matrix associated with the i-th rule. The final model for this PDC fuzzy controller is expressed by

$$\begin{aligned}\boldsymbol{u}(t) &= \frac{\sum_{i=1}^r w_i(t)\boldsymbol{F}_i\boldsymbol{x}(t)}{\sum_{i=1}^r w_i(t)}\\ &= \sum_{i=1}^r h_i(t)\boldsymbol{F}_i\boldsymbol{x}(t) = \boldsymbol{F}_z\boldsymbol{x}(t).\end{aligned} \tag{17}$$

When the PDC fuzzy state-feedback controller (17) is applied to the open-loop fuzzy system (8) without input saturation, the following closed-loop generic fuzzy system is obtained:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}_z^{CL}\boldsymbol{x} + \boldsymbol{B}_z^{CL}\boldsymbol{\phi}, \quad \boldsymbol{y} = \boldsymbol{C}_z^{CL}\boldsymbol{x} + \boldsymbol{D}_z^{CL}\boldsymbol{\phi}, \tag{18}$$

where

$$\boldsymbol{A}_z^{CL} = \boldsymbol{A}_z + \boldsymbol{B}_{1z}\boldsymbol{F}_z, \qquad \boldsymbol{B}_z^{CL} = \boldsymbol{B}_{2z},$$
$$\boldsymbol{C}_z^{CL} = \boldsymbol{C}_z, \qquad \boldsymbol{D}_z^{CL} = \boldsymbol{D}_z.$$

Theorem 1 of Salcedo and Martinez (2008) shows a method to compute the ⋆-norm of (18).

Theorem 1. (Computation of ⋆-norm) *The ⋆-norm between the $\boldsymbol{y}$ output and the $\boldsymbol{\phi}$ input for the system* (18) *is obtained by solving the problem*

$$||\boldsymbol{G}_{\phi\to y}^{CL}||_\star = \inf_{\alpha>0} N(\alpha), \tag{19}$$

where $N(\alpha)$ is calculated of each fixed $\alpha > 0$, as follows:

$$N(\alpha) \triangleq \frac{1}{\delta}\min\Big\{\mu \ge 0 :\ \bar{\boldsymbol{P}} = \bar{\boldsymbol{P}}^T > 0,\ \sigma > 0,$$
$$\text{subject to (20) and (21)}\Big\},$$

$$\begin{pmatrix} \boldsymbol{A_z^{CL}}^T\bar{\boldsymbol{P}} + \bar{\boldsymbol{P}}\boldsymbol{A}_z^{CL} + \alpha\bar{\boldsymbol{P}} & \delta\bar{\boldsymbol{P}}\boldsymbol{B}_z^{CL} \\ \delta{\boldsymbol{B}_z^{CL}}^T\bar{\boldsymbol{P}} & -\alpha\boldsymbol{I}\end{pmatrix} \le 0, \tag{20}$$

$$\begin{pmatrix} \sigma\bar{\boldsymbol{P}} & \boldsymbol{0} & {\boldsymbol{C}_z^{CL}}^T \\ \boldsymbol{0} & (\mu^2-\sigma)\boldsymbol{I} & \delta{\boldsymbol{D}_z^{CL}}^T \\ \boldsymbol{C}_z^{CL} & \delta\boldsymbol{D}_z^{CL} & \boldsymbol{I}\end{pmatrix} \ge 0. \tag{21}$$

Condition (20) is an LMI in the unknown $\bar{\boldsymbol{P}}$, while (21) is not an LMI owing to the product of unknowns $\bar{\boldsymbol{P}}$

[1] Given that the 2-norm of persistent disturbances is not finite.

and σ. In the work of Salcedo and Martinez (2008) an iterative LMI-based method was presented to overcome this problem. Optimization with respect to α (19) is carried out by obtaining the values of $N(\alpha)$ for a sufficiently representative (Salcedo *et al.*, 2007) finite set of values for α (a grid), and the value producing a minimum of $N(\alpha)$ is taken.

The positive-definite matrix $\bar{\boldsymbol{P}}$ defines an inescapable ellipsoid (20) (Abedor *et al.*, 1996; Salcedo and Martinez, 2008):

$$\mathcal{E}(\bar{\boldsymbol{P}}) \triangleq \{\boldsymbol{x} : \boldsymbol{x}^T \bar{\boldsymbol{P}} \boldsymbol{x} \leq 1\} \tag{22}$$

Thus, it is a robust control positively invariant set and if $\boldsymbol{x}(0) \notin \mathcal{E}(\bar{\boldsymbol{P}})$ after a finite time t_0, $\boldsymbol{x}(t) \in \mathcal{E}(\bar{\boldsymbol{P}})$, $\forall t \geq t_0$. Moreover, procedure (19) estimates the invariant ellipsoid which assures the smallest upper bound for the 1-norm.

Theorem 2 below shows a new alternative method to compute $\star$-norm only with LMI conditions.

Theorem 2. ($\star$-norm with LMIs) *The $\star$-norm* (19) *can be computed substituting* (20) *and* (21) *by the following LMIs conditions for* $0 \leq \beta \leq \alpha$*:*

$$\begin{pmatrix} \boldsymbol{A}_{\boldsymbol{z}}^{\boldsymbol{CL}^T} \bar{\boldsymbol{P}} + \bar{\boldsymbol{P}} \boldsymbol{A}_z^{CL} + \alpha \bar{\boldsymbol{P}} & \delta \bar{\boldsymbol{P}} \boldsymbol{B}_z^{CL} \\ \delta \boldsymbol{B}_z^{CL^T} \bar{\boldsymbol{P}} & -\beta \boldsymbol{I} \end{pmatrix} \leq 0, \tag{23}$$

$$\begin{pmatrix} \alpha \bar{\boldsymbol{P}} & \boldsymbol{0} & \boldsymbol{C}_z^{CL^T} \\ \boldsymbol{0} & (\mu - \beta)\boldsymbol{I} & \delta \boldsymbol{D}_z^{CL^T} \\ \boldsymbol{C}_z^{CL} & \delta \boldsymbol{D}_z^{CL} & \mu \boldsymbol{I} \end{pmatrix} \geq 0. \tag{24}$$

Proof. See Appendix. ■

Remark 1. Condition (24) is equivalent to condition (14) of Theorem 1 by Vafamand *et al.* (2016) when $\delta = 1$. Note that Theorem 1 by Vafamand *et al.* (2016) does not take into account any bound on the persistent perturbation.

Remark 2. $\mathcal{E}(\bar{\boldsymbol{P}})$ in Theorem 2 is an inescapable ellipsoid and hence it is robust positively invariant. However, in Theorem 1 of Vafamand *et al.* (2016) the computed ellipsoid $\{\boldsymbol{x} : \boldsymbol{x}^T P^{-1} \boldsymbol{x} < \rho\}$ is not robust positively invariant although this is claimed by the authors. In order to guarantee such a statement, the following condition must be added:

$$\frac{\beta}{\alpha} ||\phi(t)||_\infty^2 \leq \rho. \tag{25}$$

4. State feedback controller synthesis for continuous-time TS systems with input saturation under persistent perturbations

The closed loop of saturated TS fuzzy system (9) with controller (17) is

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= (\boldsymbol{A}_z + \boldsymbol{B}_{1z}\boldsymbol{F}_z)\, \boldsymbol{x}(t) - \boldsymbol{B}_{1z}\boldsymbol{\psi}(t) + \boldsymbol{B}_{2z}\boldsymbol{\phi}(t), \\ \boldsymbol{y}(t) &= \boldsymbol{C}_z \boldsymbol{x}(t) + \boldsymbol{D}_z \boldsymbol{\phi}(t), \end{aligned} \tag{26}$$

where

$$\boldsymbol{\psi} \triangleq \boldsymbol{u} - \mathbf{sat}(\boldsymbol{u}). \tag{27}$$

Hereafter, some useful preliminary results for theoretical developments are presented. First, Lemma 1 of (Nguyen *et al.*, 2016) is recalled.

Lemma 1. ($\mathcal{P}_u$ set) *Given matrices $\boldsymbol{F}_i$, $\boldsymbol{W}_i \in \mathbb{R}^{n_u \times n_x}$ for $i = 1, \ldots, r$, define the following (polyhedral) set:*

$$\mathcal{P}_u \triangleq \{\boldsymbol{x} : \; |(\boldsymbol{F}_z - \boldsymbol{W}_z)_l\, \boldsymbol{x}| \leq u_{\max,l}, \; l = 1, \ldots, n_u\}. \tag{28}$$

If $\boldsymbol{x} \in \mathcal{P}_u$, then the inequality on the dead-zone nonlinearity $\boldsymbol{\psi}(\boldsymbol{u})$ defined in (27)*,*

$$\boldsymbol{\psi}^T(\boldsymbol{u}) \boldsymbol{S}_z^{-1} [\boldsymbol{\psi}(\boldsymbol{u}) - \boldsymbol{W}_z \boldsymbol{x}] \leq 0, \tag{29}$$

holds for any positive diagonal matrices $\boldsymbol{S}_i \in \mathbb{R}^{n_u \times n_u}$ and for any scalar functions $h_i(t)$ $i = 1, \ldots, r$ satisfying the convex sum property (5)*.*

Lemma 2. (Tuan *et al.*, 2001) *Given symmetric matrices $\boldsymbol{\Upsilon}_{ij}$ of appropriate dimensions, the inequality*

$$\boldsymbol{\Upsilon}_{zz} = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(z(t)) h_j(z(t)) \boldsymbol{\Upsilon}_{ij} < 0 \tag{30}$$

is satisfied if

$$\begin{aligned} &\boldsymbol{\Upsilon}_{ii} < 0, \quad i = 1, \ldots, r, \\ &\frac{2}{r-1} \boldsymbol{\Upsilon}_{ii} + \boldsymbol{\Upsilon}_{ij} + \boldsymbol{\Upsilon}_{ji} < 0, \quad i, j = 1, \ldots, r, \quad j \neq i. \end{aligned} \tag{31}$$

Theorem 3. (Minimum $\star$-norm state feedback controllers with input saturation) *The minimum $\star$-norm state feedback controller between the $\boldsymbol{y}$ output and the $\boldsymbol{\phi}$ input for the TS fuzzy system* (26) *subject to the state constraints* (6) *and input saturation* (11) *is obtained by solving the following optimization problem:*

$$||\boldsymbol{G}_{\phi \to y}^{CL}||_\star^* = \inf_{\alpha > 0} N(\alpha), \tag{32}$$

where

$$\begin{aligned} N(\alpha) \triangleq \Big\{ &\frac{1}{\delta} \min \mu \geq 0 : \exists\, \boldsymbol{X} = \boldsymbol{X}^T > 0, \\ &0 < \beta \leq \alpha, \boldsymbol{Y}_i, \boldsymbol{Z}_i \in \mathbb{R}^{n_u \times n_x}, \\ &\textit{positive diagonal matrices } \boldsymbol{S}_i \in \mathbb{R}^{n_u \times n_u} \\ &\textit{subject to LMIs } (33)\text{–}(37) \Big\}, \end{aligned}$$

$$\begin{bmatrix} \boldsymbol{X} & \boldsymbol{Y}_{i,l}^T - \boldsymbol{Z}_{i,l}^T \\ * & u_{\max,l}^2 \end{bmatrix} \geq 0, \quad i = 1, \ldots, r, \quad l = 1, \ldots, n_u, \tag{33}$$

$$\begin{bmatrix} \boldsymbol{X} & \boldsymbol{X}\boldsymbol{h}_m \\ * & 1 \end{bmatrix} \geq 0, \quad m = 1, \ldots, s, \tag{34}$$

$$\begin{bmatrix} \alpha\boldsymbol{X} & * & * \\ \boldsymbol{0} & (\mu - \beta)\,\mathbf{I} & * \\ \boldsymbol{C}_i\boldsymbol{X} & \delta\boldsymbol{D}_i & \mu\mathbf{I} \end{bmatrix} \geq 0, \quad i = 1, \ldots, r, \tag{35}$$

$$\boldsymbol{\Upsilon}_{ii} < 0, \quad i = 1, \ldots, r, \tag{36}$$

$$\frac{2}{r-1}\boldsymbol{\Upsilon}_{ii} + \boldsymbol{\Upsilon}_{ij} + \boldsymbol{\Upsilon}_{ji} < 0, \quad i, j = 1, \ldots, r, \quad i \neq j, \tag{37}$$

with $\boldsymbol{Y}_{i,l}$ and $\boldsymbol{Z}_{i,l}$ signifying the l-th rows of $\boldsymbol{Y}_i$ and $\boldsymbol{Z}_i$, respectively,

$$\boldsymbol{\Upsilon}_{ij} = \begin{bmatrix} \boldsymbol{X}\boldsymbol{A}_i^T + \boldsymbol{A}_i\boldsymbol{X} + \boldsymbol{B}_{1i}\boldsymbol{Y}_j + \boldsymbol{Y}_j^T\boldsymbol{B}_{1i}^T + \alpha\boldsymbol{X} & * & * \\ -\boldsymbol{S}_i\boldsymbol{B}_{1i}^T + \boldsymbol{Z}_i & -2\boldsymbol{S}_i & * \\ \delta\boldsymbol{B}_{2i}^T & \boldsymbol{0} & -\beta\mathbf{I} \end{bmatrix}$$

The controller gains are defined as

$$\boldsymbol{F}_i = \boldsymbol{Y}_i\boldsymbol{X}^{-1}, \quad i = 1, \ldots, r$$

and the inescapable ellipsoid is $\mathcal{E}(\boldsymbol{X}^{-1})$.

Proof. See Appendix. ■

Remark 3. LMI conditions (33) are related to actuator saturation (11) and set (28), and LMI conditions (34) are related to state constraints (6).

Remark 4. Theorem 3 provides a controller with the minimum upper bound for 1-norm in closed loop with input saturation using $\star$-norm. Also, the inescapable ellipsoid $\mathcal{E}(\boldsymbol{X}^{-1}) \subset \mathcal{P}_x \cap \mathcal{P}_u$ and this implies it is a validity domain for the obtained fuzzy state feedback controller. However, this inescapable ellipsoid could not be large enough for real applications.

Remark 5. One possible solution to overcome the size of inescapable ellipsoids is to obtain a state feedback controller which maximizes the size of this ellipsoid keeping the $\star$-norm below some prescribed level. This idea is presented in Theorem 4.

Remark 6. It is possible to compare Theorem 3 with Theorem 1 of Vafamand *et al.* (2016). Both provide a state feedback TS fuzzy controllers which minimize an upper bound of the 1-norm. However, Theorem 1 of Vafamand *et al.* (2016) requires that three parameters (ϵ, τ and ρ) be chosen in advance, whereas it is not the case for Theorem 3. On the other hand, in the work of Vafamand *et al.* (2016) guidelines on choosing such parameters are missing. Secondly, their Theorem 1 does not take into account any type of state constraints. Theorem 3 uses polytopic constraints for states (34).

Theorem 4. (Maximum volume inescapable ellipsoid state feedback controllers with input saturation) *The state feedback controller which achieves the maximum volume inescapable ellipsoid ($\mathcal{E}$(max. vol.)) guaranteeing a prescribed value $||\boldsymbol{G}_{\phi\to y}^{CL}||_\star^\Delta$ for the $\star$-norm between the $\boldsymbol{y}$ output and the $\boldsymbol{\phi}$ input for the TS fuzzy system* (26) *subject to the state constraints* (6) *and input saturation* (11) *is obtained by solving the following optimization problem:*

$$\mathcal{E}(\textit{max. vol.}) = \max_{\alpha>0} \textit{Vol}(\mathcal{E}(\boldsymbol{X}^{-1})) \tag{38}$$

where Vol($\mathcal{E}(\boldsymbol{X}^{-1})$) is calculated of each fixed $\alpha > 0$, as follows:

$$\begin{aligned} &\textit{Vol}(\mathcal{E}(\boldsymbol{X}^{-1})) \\ &\quad \triangleq -\min\Big\{-\log\det(\boldsymbol{X}) : \exists\, \boldsymbol{X} = \boldsymbol{X}^T > 0, \\ &\quad\ 0 < \beta \leq \alpha, \boldsymbol{Y}_i, \boldsymbol{Z}_i \in \mathbb{R}^{n_u \times n_x}, \\ &\quad\ \textit{positive diagonal matrices } \boldsymbol{S}_i \in \mathbb{R}^{n_u \times n_u} \\ &\quad\ \textit{subject to LMIs } (33), (34), (36), (37) \textit{ and } (39)\Big\}, \end{aligned}$$

$$\begin{bmatrix} \alpha\boldsymbol{X} & * & * \\ \boldsymbol{0} & \left(\mu^\Delta - \beta\right)\mathbf{I} & * \\ \boldsymbol{C}_i\boldsymbol{X} & \delta\boldsymbol{D}_i & \mu^\Delta\mathbf{I} \end{bmatrix} \geq 0, \quad i = 1, \ldots, r, \tag{39}$$

where $\boldsymbol{Y}_{i,l}$, $\boldsymbol{Z}_{i,l}$ and $\boldsymbol{\Upsilon}_{ij}$ are the same as the ones defined in Theorem 3 and

$$\mu^\Delta = ||\boldsymbol{G}_{\phi\to y}^{CL}||_\star^\Delta \cdot \delta.$$

The controller gains are recovered with

$$\boldsymbol{F}_i = \boldsymbol{Y}_i\boldsymbol{X}^{-1}, \quad i = 1, \ldots, r,$$

and the inescapable ellipsoid is $\mathcal{E}(\boldsymbol{X}^{-1})$.

Proof. See Appendix. ■

Remark 7. In Theorem 4 $||\boldsymbol{G}_{\phi\to y}^{CL}||_\star^\Delta$ cannot have a lower value than $||\boldsymbol{G}_{\phi\to y}^{CL}||_\star^*$ in Theorem 3.

Remark 8. The maximum volume inescapable ellipsoids of Theorem 4 are useful to extend the validity domain of the computed fuzzy state feedback controllers. However, with the computed fuzzy state feedback controller the correct value for the $\star$-norm could be lower than $||\boldsymbol{G}^{CL}_{\phi\to y}||^{\Delta}_{\star}$, because it can be related to a different inescapable ellipsoid $\mathcal{E}(\hat{\boldsymbol{X}}^{-1})$. In the next section this question will be analyzed in depth.

Remark 9. If an LMI solver based on interior point methods (Boyd *et al.*, 1994) is used, the computational cost of the LMI optimization problem can be estimated as being proportional to $N^3_{\text{var}} \times N_{\text{row}}$, where N_{var} is the total number of scalar decision variables and N_{row} the total row size of the LMIs (Gahinet *et al.*, 1995). In the proposed theorems, we have the following:

- Theorem 3:

$$N_{\text{var}} = 2 + \frac{1}{2}n_x(n_x+1) + rn_u(2n_x+1),$$

$$N_{\text{row}} = 1 + rn_u(n_x+1) + s(n_x+1) + \dots + r(n_x+n_y+n_\phi) + r^2(n_x+n_u+n_\phi).$$

- Theorem 4:

$$N_{\text{var}} = N^{\text{Theorem 3}}_{\text{var}} - 1,$$

$$N_{\text{row}} = N^{\text{Theorem 3}}_{\text{row}}.$$

Theorem 1 of Vafamand *et al.* (2016) with invariance condition (25) and state constraints (6) is characterized by the following figures:

$$N_{\text{var}} = 2 + \frac{1}{2}n_x(n_x+1) + rn_un_x,$$

$$N_{\text{row}} = 1 + rn_u(n_x+1) + s(n_x+1) + \dots + r(n_x+n_y+n_\phi) + r^2(n_x+2n_u+n_\phi).$$

Comparing these numbers, we conclude that all of them have the same order of complexity. Theorem 1 of Vafamand *et al.* (2016) rn_un_x fewer variables than Theorem 3, but r^2n_u more rows. However, Theorem 1 of Vafamand *et al.* (2016) requires that three parameters (ϵ, τ and ρ) be chosen in advance. This implies that a gridding technique should be additionally applied to find their best values. This kind of techniques is highly demanding from a computational point of view.

5. Algorithms for estimation of inescapable ellipsoids

Remark 8 shows that the same fuzzy state feedback controller may have an infinite number inescapable ellipsoids. All these ellipsoids can be obtained using LMIs (33), (34), (36) and (37) substituting $\boldsymbol{Y}_i$ by $\boldsymbol{F}_i\boldsymbol{X}$, because now $\boldsymbol{F}_i$ are known matrices.

On the other hand, there is a trade-off between maximum volume inescapable ellipsoids which provide a large domain of validity for the controller, and minimum $\star$-norm inescapable ellipsoids which provide the lowest upper bound for the 1-norm of $\boldsymbol{y}$.

If both the previous paragraphs are put together, it can be concluded that a possible solution to this trade-off is to compute two ellipsoids for the same controller:

1. Maximum volume ellipsoid: $\mathcal{E}(\boldsymbol{X}_v^{-1})$. This one can be calculated using Theorem 4 without LMI (39) and substituting $\boldsymbol{Y}_i$ with $\boldsymbol{F}_i\boldsymbol{X}$.

2. Minimum $\star$-norm ellipsoid $\mathcal{E}(\boldsymbol{X}_\star^{-1})$. This ellipsoid can be obtained using Theorem 3 and substituting $\boldsymbol{Y}_i$ with $\boldsymbol{F}_i\boldsymbol{X}$.

Remark 10. Both ellipsoids satisfy $\mathcal{E}(\boldsymbol{X}_\star^{-1}) \cap \mathcal{E}(\boldsymbol{X}_v^{-1}) \neq \emptyset$, because by definition both contain the origin (22). In particular, inside its intersection there is a ball centred at the origin with radius equal to minimum of the lowest eigenvalues of $\boldsymbol{X}_v^{-1}$ and $\boldsymbol{X}_\star^{-1}$.

The fuzzy state feedback controller will be valid inside $\mathcal{E}(\boldsymbol{X}_v^{-1})$ and for every initial state $\boldsymbol{x}(0) \in \mathcal{E}(\boldsymbol{X}_v^{-1})$ there exists a finite time t_0 such that $\boldsymbol{x}(t) \in \mathcal{E}(\boldsymbol{X}_\star^{-1})\ \forall t \geq t_0$, since both ellipsoids are inescapable and $\mathcal{E}(\boldsymbol{X}_\star^{-1}) \cap \mathcal{E}(\boldsymbol{X}_v^{-1}) \neq \emptyset$.

Remark 11. Generally speaking, both inescapable ellipsoids, $\mathcal{E}(\boldsymbol{X}_\star^{-1})$ and $\mathcal{E}(\boldsymbol{X}_v^{-1})$, are related to different values of parameter α. However, a new problem appears: Which method is to be applied for obtaining the fuzzy state feedback controller? A general solution to this problem is the following multi-objective optimization: Find

$$\boldsymbol{P} = \boldsymbol{P}^T > 0, \quad \boldsymbol{R} = \boldsymbol{R}^T > 0, \quad 0 < \beta \leq \alpha,$$

$$0 < \beta_1 \leq \alpha_1, \quad \boldsymbol{F}_i, \boldsymbol{W}_i \in \mathbb{R}^{n_u \times n_x}, \quad \boldsymbol{S}_i \in \mathbb{R}^{n_u \times n_u},$$

such that

$$\boldsymbol{P} = \arg\max_{\alpha>0} \log\det(\boldsymbol{P}^{-1})$$

$$\boldsymbol{R} = \arg\min_{\alpha_1>0} \mu,$$

subject to

$$\begin{bmatrix} (\boldsymbol{A}_z + \boldsymbol{B}_{1z}\boldsymbol{F}_z)^T\boldsymbol{P} + \boldsymbol{P}(\boldsymbol{A}_z + \boldsymbol{B}_{1z}\boldsymbol{F}_z) + \alpha\boldsymbol{P} & * & * \\ -\boldsymbol{B}_{1z}^T\boldsymbol{P} + \boldsymbol{S}_z^{-1}\boldsymbol{W}_z & -2\boldsymbol{S}_z^{-1} & * \\ \delta\boldsymbol{B}_{2z}^T\boldsymbol{P} & \boldsymbol{0} & -\beta\mathbf{I} \end{bmatrix} \leq 0, \quad (40)$$

$$\begin{bmatrix} \boldsymbol{P} & * \\ \boldsymbol{F}_{z,l} - \boldsymbol{W}_{z,l} & u^2_{\max,l} \end{bmatrix} \geq 0, \quad l = 1, \dots, n_u, \quad (41)$$

$$\begin{bmatrix} \boldsymbol{P} & * \\ \boldsymbol{h}_m^T & 1 \end{bmatrix} \geq 0, \quad m = 1, \dots, s, \tag{42}$$

$$\begin{bmatrix} (\boldsymbol{A}_z + \boldsymbol{B}_{1z}\boldsymbol{F}_z)^T \boldsymbol{R} + \boldsymbol{R}(\boldsymbol{A}_z + \boldsymbol{B}_{1z}\boldsymbol{F}_z) + \alpha_1 \boldsymbol{R} & * & * \\ -\boldsymbol{B}_{1z}^T \boldsymbol{R} + \boldsymbol{S}_z^{-1}\boldsymbol{W}_z & -2\boldsymbol{S}_z^{-1} & * \\ \delta \boldsymbol{B}_{2z}^T \boldsymbol{R} & \boldsymbol{0} & -\beta_1 \mathbf{I} \end{bmatrix} \leq 0, \tag{43}$$

$$\begin{bmatrix} \alpha_1 \boldsymbol{R} & * & * \\ \boldsymbol{0} & (\mu - \beta_1)\mathbf{I} & * \\ \boldsymbol{C}_z & \delta \boldsymbol{D}_z & \mu \mathbf{I} \end{bmatrix} \geq 0. \tag{44}$$

Remark 12. The proof of Theorem 3 guarantees that (40) and (43) imply that $\mathcal{E}(\boldsymbol{P})$ and $\mathcal{E}(\boldsymbol{R})$ are inescapable ellipsoids. $\mathcal{E}(\boldsymbol{P})$ will be the maximum volume inescapable ellipsoid and $\mathcal{E}(\boldsymbol{R})$ the minimum $\star$-norm inescapable ellipsoid.

Remark 13. This multi-objective optimization is a trade-off between the maximization of the volume of $\mathcal{E}(\boldsymbol{P})$ and the minimization of the $\star$-norm inside $\mathcal{E}(\boldsymbol{R})$.

Remark 14. To the best of our knowledge, conditions (40) and (43) cannot be recast as LMIs unless $\boldsymbol{P} = \boldsymbol{R}$. However, this solution is not appropriate to solve the multiobjective optimization problem. As an alternative, Algorithms (1) and (2) are proposed to provide possible optimal solutions to this multi-objective optimization.

Algorithm 1. Multiobjective optimal solution A.

Step 1. Using Theorem 3 compute a fuzzy state feedback controller which minimizes the 1-norm of $\boldsymbol{y}$. $\boldsymbol{F}_i$ and $||\boldsymbol{G}_{\phi\to y}^{CL}||_\star^{*,A} = \inf_{\alpha>0} N(\alpha)$ are obtained.

Step 2. Let $\boldsymbol{R} = \boldsymbol{X}^{-1}$ and $\mathcal{E}(\boldsymbol{X}_\star^{-1}) = \mathcal{E}(\boldsymbol{R})$.

Step 3. Compute the maximum volume ellipsoid $\mathcal{E}(\boldsymbol{X}_v^{-1})$ related to $\boldsymbol{F}_i$. It be calculated using Theorem 4 without LMI (39) and substituting $\boldsymbol{Y}_i$ by $\boldsymbol{F}_i\boldsymbol{X}$.

Step 4. Let $\boldsymbol{P} = \boldsymbol{X}^{-1}$ and $\mathcal{E}(\boldsymbol{X}_v^{-1}) = \mathcal{E}(\boldsymbol{P})$. We have $\max(\text{Vol})_A = \text{Vol}(\mathcal{E}(\boldsymbol{P}))$.

Remark 15. Obviously, $||\boldsymbol{G}_{\phi\to y}^{CL}||_\star^{*,A} \leq ||\boldsymbol{G}_{\phi\to y}^{CL}||_\star^{*,B}$ and $\max(\text{Vol})_B \geq \max(\text{Vol})_A$. Consequently, optimal solutions of both algorithms are non-dominant from a multi-objective point of view. Depending on the application, one of them must be chosen.

Remark 16. There are no fuzzy state feedback controllers with $||\boldsymbol{G}_{\phi\to y}^{CL}||_\star^{*} < ||\boldsymbol{G}_{\phi\to y}^{CL}||_\star^{*,A}$, nor with $\max(\text{Vol}) > \max(\text{Vol})_B$.

Algorithm 2. Multiobjective optimal solution B.

Step 1. Using Theorem 4 compute a fuzzy state feedback controller which maximizes the volume of the inescapable ellipsoid. Thus $\boldsymbol{F}_i$ are obtained.

Step 2. Let $\boldsymbol{P} = \boldsymbol{X}^{-1}$ and $\mathcal{E}(\boldsymbol{X}_v^{-1}) = \mathcal{E}(\boldsymbol{P})$. We have $\max(\text{Vol})_B = \text{Vol}(\mathcal{E}(\boldsymbol{P}))$.

Step 3. Compute the minimum $\star$-norm ellipsoid $\mathcal{E}(\boldsymbol{X}_\star^{-1})$ related to $\boldsymbol{F}_i$. This ellipsoid can be obtained using Theorem 3 and substituting $\boldsymbol{Y}_i$ by $\boldsymbol{F}_i\boldsymbol{X}$. $||\boldsymbol{G}_{\phi\to y}^{CL}||_\star^{*,B} = \inf_{\alpha>0} N(\alpha)$

Step 4. Let $\boldsymbol{R} = \boldsymbol{X}^{-1}$ and $\mathcal{E}(\boldsymbol{X}_\star^{-1}) = \mathcal{E}(\boldsymbol{R})$.

Remark 17. Consequently, both non-dominant and optimal solutions belong to the Pareto front of this multi-objective optimization.

Remark 18. Algorithms 1 and 2 in Steps 1 and 3 solve LMI conditions of Theorems 3 and 4 separately. Therefore they solve two sets of LMI conditions before getting final results. Nevertheless, Theorem 1 of Vafamand *et al.* (2016) and Theorem 1 of Nguyen *et al.* (2015) only perform one step with solely a set of LMI conditions. Their methodologies are, consequently, single-objective instead of multi-objective.

6. First application example

Consider the following non-linear unstable open-loop system (Example 3 of Vafamand *et al.* (2016)):

$$\begin{aligned} \dot{x}_1 &= -x_1 + \left(0.1 + 0.12x_2^2\right) x_2 \\ &\quad + \left(1.48 + 0.16x_2^3\right) u + 0.1\phi, \end{aligned}$$

$$\begin{aligned} \dot{x}_2 &= x_1 + 0.1\phi, \\ y &= x_2 + 0.2\phi. \end{aligned} \tag{45}$$

Non-linearities $0.1 + 0.12x_2^2$ and $1.48 + 0.16x_2^3$ are unbounded functions of x_2. Consequently, it is impossible to obtain a TS fuzzy model which globally represents the non-linear system. To overcome this problem, x_2 is constrained to belong to interval $[-1.5, 1.5]$. It is also considered the same constraint in the first state, leading to validity domain $\mathcal{P}_x \triangleq \{\boldsymbol{x} : |x_1| \leq 1.5, |x_1| \leq 1.5\}$. Inside this validity domain the following four-rule TS fuzzy model exactly represents the non-linear system:

Rule 1:

$$\boldsymbol{A}_1 = \begin{pmatrix} -1 & 0.1 \\ 1 & 0 \end{pmatrix},$$

$$\boldsymbol{B}_{11} = \begin{pmatrix} 0.94 \\ 0 \end{pmatrix}, \quad \boldsymbol{B}_{21} = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix},$$

$$\boldsymbol{C}_1 = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad \boldsymbol{D}_1 = 0.2. \tag{46}$$

Rule 2:

$$\boldsymbol{A}_2 = \begin{pmatrix} -1 & 0.1 \\ 1 & 0 \end{pmatrix},$$
$$\boldsymbol{B}_{12} = \begin{pmatrix} 2.02 \\ 0 \end{pmatrix}, \quad \boldsymbol{B}_{22} = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix},$$
$$\boldsymbol{C}_2 = \begin{pmatrix} 0 & 1 \end{pmatrix} \quad \boldsymbol{D}_2 = 0.2. \tag{47}$$

Rule 3:

$$\boldsymbol{A}_3 = \begin{pmatrix} -1 & 0.37 \\ 1 & 0 \end{pmatrix},$$
$$\boldsymbol{B}_{13} = \begin{pmatrix} 0.94 \\ 0 \end{pmatrix}, \quad \boldsymbol{B}_{23} = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix},$$
$$\boldsymbol{C}_3 = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad \boldsymbol{D}_3 = 0.2. \tag{48}$$

Rule 4:

$$\boldsymbol{A}_4 = \begin{pmatrix} -1 & 0.37 \\ 1 & 0 \end{pmatrix},$$
$$\boldsymbol{B}_{14} = \begin{pmatrix} 2.02 \\ 0 \end{pmatrix}, \quad \boldsymbol{B}_{24} = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix},$$
$$\boldsymbol{C}_4 = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad \boldsymbol{D}_4 = 0.2. \tag{49}$$

The control saturation limit will be taken as $u_{\max,1} = 1$, and $\phi(t)^2 \le 1$ ($\delta = 1$). The optimal solution produced by Algorithm 1 is

- $||\boldsymbol{G}^{CL}_{\phi\to y}||^{*,A}_{\star} = 0.3231$, $\alpha = 1.05$,
- $\boldsymbol{F}_1 = \begin{bmatrix} -35.244 & -33.098 \end{bmatrix}$,
- $\boldsymbol{F}_2 = \begin{bmatrix} -25.702 & -24.668 \end{bmatrix}$,
- $\boldsymbol{F}_3 = \begin{bmatrix} -22.313 & -21.615 \end{bmatrix}$,
- $\boldsymbol{F}_4 = \begin{bmatrix} -30.544 & -29.093 \end{bmatrix}$,
- $\boldsymbol{X}_\star = \begin{bmatrix} 0.1278 & -0.0827 \\ -0.0827 & 0.0925 \end{bmatrix}$,
- $\max(\mathrm{Vol})_A = 1.5722$, $\alpha = 0.21$,
- $\boldsymbol{X}_v = \begin{bmatrix} 2.2499 & -1.6604 \\ -1.6604 & 1.9242 \end{bmatrix}$.

Both the ellipsoids are shown in Fig. 1. Step 1 of Algorithm 1 required for each value of α an average time[2] of 0.3861 s. Step 3 required 0.2234 s.

The optimal solution produced by Algorithm 2 is

- $\max(\mathrm{Vol})_B = 2.0559$, $\alpha = 0.11$,
- $\boldsymbol{F}_1 = \begin{bmatrix} -10.781 & -4.5365 \end{bmatrix}$,
- $\boldsymbol{F}_2 = \begin{bmatrix} -15.632 & -6.4885 \end{bmatrix}$,
- $\boldsymbol{F}_3 = \begin{bmatrix} -3.7749 & -1.6307 \end{bmatrix}$,
- $\boldsymbol{F}_4 = \begin{bmatrix} -12.144 & -5.1083 \end{bmatrix}$,

[2]Under Matlab 2017b and Intel Core i7 860 at 2.8 GHz using the LMILAB solver (Gahinet *et al.*, 1995).

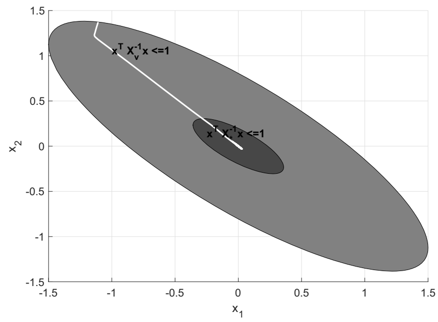

Fig. 1. Inescapable ellipsoids for Algorithm 1.

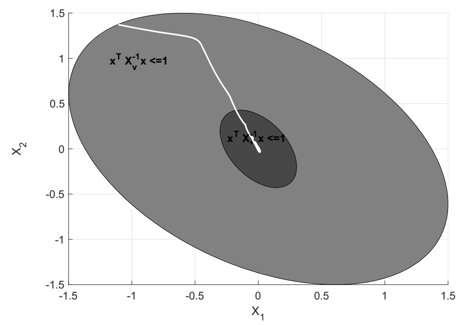

Fig. 2. Inescapable ellipsoids for Algorithm 2.

- $\boldsymbol{X}_v = \begin{bmatrix} 2.2496 & -0.91348 \\ -0.91348 & 2.2497 \end{bmatrix}$,
- $||\boldsymbol{G}^{CL}_{\phi\to y}||^{*,B}_{\star} = 0.8174$, $\alpha = 0.35$,
- $\boldsymbol{X}_\star = \begin{bmatrix} 0.12756 & -0.082216 \\ -0.082216 & 0.25613 \end{bmatrix}$.

Both the ellipsoids are shown in Fig. 2. Step 1 of Algorithm 2 required for each value of α an average time of 0.3002 s. Step 3 required 0.2137 s.

These results show that both the solutions are non-dominant. In Fig. 3 the maximum-volume inescapable ellipsoids of both the algorithms are represented. As expected, Algorithm 2 provides a larger domain of validity. However, there are points which are valid for Algorithm 1 but not for Algorithm 2. Moreover, the minimum $\star$-norm for Algorithm 1 is also smaller than the minimum $\star$-norm for Algorithm 2. Nevertheless, the controller gains of Algorithm 1 are higher and this will imply a more aggressive controller.

Next, closed loop simulations of non-linear system (45) have been performed with both the controllers taking as initial point $\boldsymbol{x}_0 = \begin{bmatrix} -1.108 & 1.372 \end{bmatrix}^T$ and using $\phi(t) = \sin(\pi t + \pi/2)$. Both the trajectories are drawn in white in Figs. 1 and 2, respectively. Note that $\boldsymbol{x}_0$ belongs to the boundary of both the maximum volume inescapable ellipsoids.

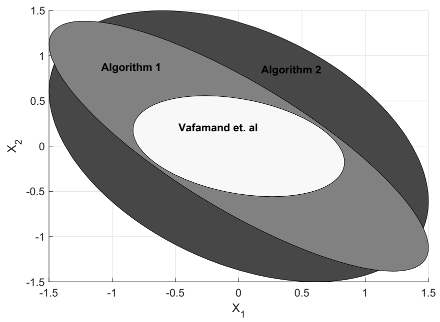

Fig. 3. Maximum volume inescapable ellipsoids for both the algorithms and the result of Vafamand *et al.* (2016).

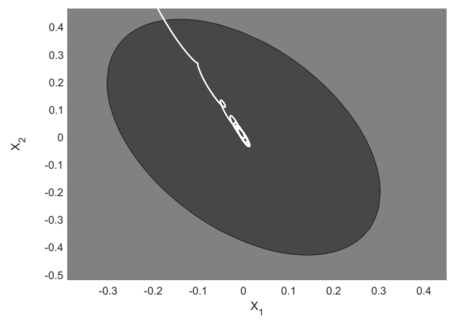

Fig. 5. Zoom of the trajectory for Algorithm 2.

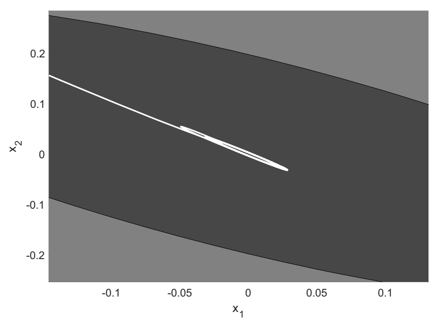

Fig. 4. Zoom of the trajectory for Algorithm 1.

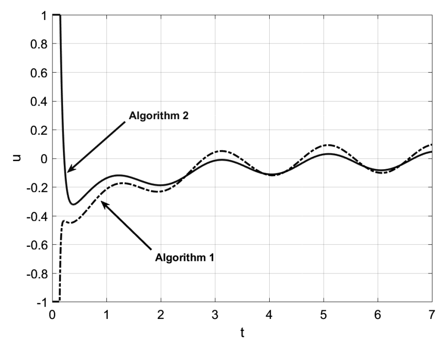

Fig. 6. Control actions.

In Figs. 4 and 5 final parts of both the trajectories have been zoomed. As expected, in steady state they are inside $\mathcal{E}(\boldsymbol{X}_\star^{-1})$ ellipsoids. From both the trajectories it is possible to compute the exact 1-norm using data from steady state. The 1-norm for Algorithm 1 is 0.211 and for Algorithm 2 it is 0.207. These values are almost equal and below $||\boldsymbol{G}^{CL}_{\phi\to y}||^{*,A}_\star$ and $||\boldsymbol{G}^{CL}_{\phi\to y}||^{*,B}_\star$, respectively. Finally, in Fig. 6 both control actions are represented. The controller from Algorithm 1 saturates at -1 at the beginning, whereas the one from Algorithm 2 saturates at 1 at the beginning, for a small period of time in both cases. In steady state, the controller from Algorithm 1 produces slightly higher control actions (between ± 0.1 instead of ± 0.062). Therefore, it can be concluded that both the controllers have very similar performances. However, the controller from Algorithm 2 has a larger inescapable ellipsoid. Consequently, in this example the controller from Algorithm 2 will be chosen.

It is possible to compare the results of these algorithms with Theorem 1 of Vafamand *et al.* (2016). In Introduction it has been commented that Vafamand *et al.* (2016) only compute one ellipsoid such that an upper bound of the 1-norm is minimized. On the other hand, this theorem requires to specify in advance the value of several design parameters (apart from α and β): ϵ, τ and ρ. For this example, if $\tau = 1$, $\epsilon = 0.95$ and $\rho \geq 1.58$, it is impossible to find any solution to the LMIs of Theorem 1 of Vafamand *et al.* (2016) for any $\alpha > 0$. The robust positively invariant ellipsoid which corresponds to $\tau = 1$, $\epsilon = 0.95$ and $\rho = 0.55$ is shown in Fig. 3. As can be seen, this ellipsoid is contained inside the inescapable ellipsoids of Algorithms 1 and 2. The obtained upper bound for the 1-norm is 0.7431 which is higher than the values provided by the algorithms presented here. Consequently, Theorem 1 of Vafamand *et al.* (2016) is less efficient than the algorithms presented here because it only computes one ellipsoid and there are three parameters which have to be specified in advance, and there are not clear rules to choose them. Therefore, the design procedure has not been clearly stated.

Nguyen *et al.* (2015) also use this non-linear system in Example 2. However, it is only possible to perform a partial comparison because this reference manages finite energy perturbations instead of persistent ones. In Fig. 7 we compare the inescapable ellipsoids of Algorithms 1 and 2 with the largest ellipsoid of attraction obtained by Nguyen *et al.* (2015). It can be concluded that Algorithm 2 provides a larger ellipsoid, and Algorithm 1 includes points which do not belong to the largest ellipsoid of Nguyen *et al.* (2015). Also, for this largest ellipsoid a poor $\mathcal{L}_2$-gain performance is obtained (4.7607). Otherwise, if the $\mathcal{L}_2$-gain is minimized instead, a small ellipsoid of attraction is obtained (see Fig. 7)

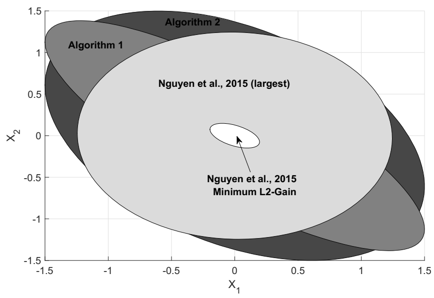

Fig. 7. Maximum volume inescapable ellipsoids for both algorithms and ellipsoids of attraction by Nguyen *et al.* (2015).

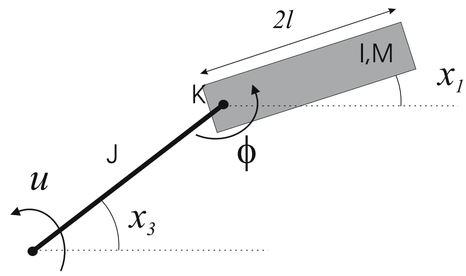

Fig. 8. Mechanical system composed of two rotating bars.

but a good $\mathcal{L}_2$-gain performance is achieved: (0.2432). Consequently, the proposed single ellipsoid of Nguyen *et al.* (2015) to solve the trade-off between the size of the validity domain and the output performance is outperformed by Algorithms 1 and 2, since they yield two ellipsoids, and Algorithm 2 can provide a larger ellipsoid of attraction.

Results of Nguyen *et al.* (2016) cannot be applied here because they are related to discrete TS fuzzy systems.

7. Second application example

Consider the following non-linear marginally stable open-loop system (Chen, 2006; Salcedo *et al.*, 2008):

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -\frac{Mgl}{I}\sin(x_1) - \frac{K}{I}(x_1 - x_3) + \frac{1}{I}\phi,\\ \dot{x}_3 &= x_4,\\ \dot{x}_4 &= \frac{K}{J}(x_1 - x_3) + \frac{1}{J}u.\end{aligned} \tag{50}$$

It is a mechanical system composed of two rotating bars (see Fig. 8), where x_1 and x_2 are, respectively, the angular position and the angular velocity of the first bar, and x_3 and x_4 are, respectively, the angular position and the angular velocity of the second bar, and where u is the torque applied to the second bar, g is the gravity constant, $I = 1$ kg·m^2 is the moment of inertia of the first bar, $J = 10$ kg·m^2 is the moment of inertia of the second bar, $l = 1$ m is half of the length of the first bar, $M = 1$ kg is the mass of the first bar, and $K = 5$ N·m/rad the elastic rigidity at the intersection of the two bars.

The non-linearity $\sin(x_1)$ can be exactly represented in the interval $x_1 \in [-\pi, \pi]$ by

$$\begin{aligned}\sin(x_1) &= h_1(x_1)\cdot x_1 + h_2(x_1)\cdot 0,\\ h_1(x_1) + h_2(x_1) &= 1,\\ h_1(x_1) &= \begin{cases} 1, & x_1 = 0,\\ \dfrac{\sin(x_1)}{x_1}, & x_1 \neq 0.\end{cases}\end{aligned} \tag{51}$$

Consequently, the non-linear model (50) can be exactly represented in $x_1, x_3 \in [-\pi, \pi]$ by the following two-rule TS fuzzy model:

Rule 1:

$$\boldsymbol{A}_1 = \begin{pmatrix} 0 & 1 & 0 & 0\\ -\dfrac{Mgl+K}{I} & 0 & \dfrac{K}{I} & 0\\ 0 & 0 & 0 & 1\\ \dfrac{K}{J} & 0 & -\dfrac{K}{J} & 0\end{pmatrix}, \quad \boldsymbol{B}_{11} = \begin{pmatrix} 0\\ 0\\ 0\\ \dfrac{1}{J}\end{pmatrix},$$

$$\boldsymbol{B}_{21} = \begin{pmatrix} 0\\ \dfrac{1}{I}\\ 0\\ 0\end{pmatrix}, \quad \boldsymbol{C}_1 = \begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}, \quad \boldsymbol{D}_1 = 0, \tag{52}$$

Rule 2:

$$\boldsymbol{A}_2 = \begin{pmatrix} 0 & 1 & 0 & 0\\ -\dfrac{K}{I} & 0 & \dfrac{K}{I} & 0\\ 0 & 0 & 0 & 1\\ \dfrac{K}{J} & 0 & -\dfrac{K}{J} & 0\end{pmatrix}, \quad \boldsymbol{B}_{12} = \boldsymbol{B}_{11},$$

$$\boldsymbol{B}_{22} = \boldsymbol{B}_{21}, \quad \boldsymbol{C}_2 = \boldsymbol{C}_1, \quad \boldsymbol{D}_2 = \boldsymbol{D}_1. \tag{53}$$

The control saturation limit is taken as $u_{\max,1} = 50$, and $\phi(t)^2 \leq 5^2$ implies $\delta = 5$ (persistent perturbation).

A comparison with Theorem 1 of Vafamand *et al.* (2016) is going to be performed. Recall that Nguyen *et al.* (2015) deal with finite energy perturbations and their other work (Nguyen *et al.*, 2016) is related to discrete TS fuzzy systems. Consequently, it is not possible to establish a comparison with the work of Nguyen *et al.* (2015; 2016).

The optimal solution of Algorithm 1 is

- $||\boldsymbol{G}^{CL}_{\phi\rightarrow y}||^{*,A}_{\star} = 0.0668$, $\alpha = 0.475$,
- $\boldsymbol{F}_1 = \begin{bmatrix}-179.8363 & -234.7233 & -49.9103 & -10.5381\end{bmatrix}$,
- $\boldsymbol{F}_2 = \begin{bmatrix}-199.8036 & -245.7261 & -52.0368 & -10.8828\end{bmatrix}$,
- $\max(\mathrm{Vol})_A = 7.2373$, $\alpha = 0.605$.

Step 1 of Algorithm 1 required for each value of α an average time[3] of 0.1055 s. Step 3 required 0.3905 s.

The optimal solution of Algorithm 2 is

- $\max(\text{Vol})_B = 28.0083, \alpha = 0.265$,
- $\boldsymbol{F}_1 = \begin{bmatrix} -29.5225 & -78.7107 & -37.5464 & -3.4463 \end{bmatrix}$,
- $\boldsymbol{F}_2 = \begin{bmatrix} -36.3120 & -79.7710 & -37.6327 & -3.4417 \end{bmatrix}$,
- $||\boldsymbol{G}^{CL}_{\phi \to y}||^{*,B}_{\star} = 0.9884, \alpha = 0.355$.

Step 1 of Algorithm 2 required for each value of α an average time of 0.1247 s. Step 3 required 0.0085 s.

These results show, again, that both the solutions are non-dominant. For this example, if $\tau = 11$, $\epsilon = 0.95$ and $\rho = 1$, it is possible to find solutions to the LMIs of Theorem 1 of Vafamand *et al.* (2016) with some $\alpha > 0$ adding invariant condition (25) and state constraints (34). The minimum value for the upper bound of the 1-norm is 0.0812 for $\alpha = 1.285$ and the volume of the corresponding invariant ellipsoid is 0.3481. These results do not improve the solutions provided by Algorithms 1 and 2.

8. Conclusions

We have presented a novel approach to the design of fuzzy PDC state feedback controllers for continuous-time non-linear systems with input saturation under persistent perturbations. Such controllers achieve BIBO stabilization in closed loop based on the computation of inescapable ellipsoids. These ellipsoids are computed with LMIs. Two ellipsoids are computed for each controller: the maximum-volume inescapable ellipsoid contained inside the domain of validity, and the smallest inescapable ellipsoid which guarantees a minimum $\star$-norm of the perturbed system. For every initial point contained in the first ellipsoid, the closed loop will enter the second one after a finite time, and will remain inside afterwards. Consequently, the designed controllers have a large domain of validity and ensure a small value for the 1-norm of the closed loop. Two algorithms have been proposed to compute controllers which solve a multi-objective optimization problem based on the trade-off between obtaining the maximum volume inescapable ellipsoid and the minimum $\star$-norm inescapable ellipsoid. Both the algorithms have been successfully applied to illustrative examples.

As possible topics for future research, the results of this paper can be relaxed, improved and extended using other existing techniques in the literature:

- replacing the quadratic Lypunov function by fuzzy or non-quadratic Lyapunov functions (Abdelmalek *et al.*, 2007; Guerra *et al.*, 2012; Pan *et al.*, 2012; Jaadari *et al.*, 2012; Bai *et al.*, 2015; Liu *et al.*, 2017; Nguyen *et al.*, 2017; Vafamand *et al.*, 2017b);
- replacing the PDC control law by non-PDC laws (Guerra *et al.*, 2012; Pan *et al.*, 2012; Jaadari *et al.*, 2012; Liu *et al.*, 2017; Vafamand *et al.*, 2017b);
- using piecewise-affine continuous-time TS fuzzy models and piecewise-affine Lyapunov functions (Tognetti and Oliveira, 2010; Qiu *et al.*, 2013; 2017);
- considering uncertain continuous-time TS fuzzy systems in order to design robust controllers (Vafamand *et al.*, 2018; 2017b).

Another direction of future research can be to analyze how to implement the designed controllers for sampled-data real processes instead of their continuous-time TS fuzzy models used in this paper.

[3]Under Matlab 2017b and Intel Core i7 860 at 2.8 GHz using the Mosek solver (`www.mosek.com`).

Acknowledgment

The authors wish to thank the Editor-in-Chief and the anonymous reviewers for their valuable comments and suggestions. This work has been funded by Ministerio de Economía y Competitividad (Spain) through the research project DPI2015-71443-R and by Generalitat Valenciana (Valencia, Spain) through the research project GV/2017/029.

References

Abdelmalek, I., Goléa, N. and Hadjili, M.L. (2007). A new fuzzy Lyapunov approach to non-quadratic stabilization of Takagi-Sugeno fuzzy models, *International Journal of Applied Mathematics and Computer Science* **17**(1): 39–51, DOI: 10.2478/v10006-007-0005-4.

Abedor, J., Nagpal, K. and Poolla, K. (1996). Linear matrix inequality approach to peak-to-peak gain minimization, *International Journal of Robust and Nonlinear Control* **6**(9–10): 899–927.

Bai, J., Lu, R., Liu, X., Xue, A. and Shi, Z. (2015). Fuzzy regional pole placement based on fuzzy Lyapunov functions, *Neurocomputing* **167**: 467–473.

Benzaouia, A., El Hajjaji, A., Hmamed, A. and Oubah, R. (2015). Fault tolerant saturated control for T–S fuzzy discrete-time systems with delays, *Nonlinear Analysis: Hybrid Systems* **18**: 60–71.

Bezzaoucha, S., Marx, B., Maquin, D. and Ragot, J. (2013). Stabilization of nonlinear systems subject to actuator saturation, *Proceedings of the IEEE International Conference Fuzzy Systems (FUZZ-IEEE), Hyderabad, India*, pp. 1–6.

Boyd, S., El Ghaoui, L., Feron, E. and Balakrishnan, V. (1994). *Linear Matrix Inequalities in System and Control Theory*, SIAM, Philadelphia, PA.

Chang, W.-J. and Shih, Y.-J. (2015). Fuzzy control of multiplicative noised nonlinear systems subject to actuator saturation and h_∞ performance constraints, *Neurocomputing* **148**: 512–520.

Chen, C.W. (2006). Stability conditions of fuzzy systems and its application to structural and mechanical systems, *Advances in Engineering Software* **37**(9): 624–629.

da Silva, J.G., Castelan, E., Corso, J. and Eckhard, D. (2013). Dynamic output feedback stabilization for systems with sector-bounded nonlinearities and saturating actuators, *Journal of The Franklin Institute* **350**(3): 464–484.

Du, H. and Zhang, N. (2009). Fuzzy control for nonlinear uncertain electrohydraulic active suspensions with input constraint, *IEEE Transactions on Fuzzy Systems* **17**(2): 343–356.

Duan, R., Li, J., Zhang, Y., Yang, Y. and Chen, G. (2016). Stability analysis and H_∞ control of discrete T–S fuzzy hyperbolic systems, *International Journal of Applied Mathematics and Computer Science* **26**(1): 133–145, DOI: 10.1515/amcs-2016-0009.

El Ghaoui, L. and Niculescu, S. (Eds.) (2000). *Advances in Linear Matrix Inequality Methods in Control*, SIAM, Philadelphia, PA.

Gahinet, P., Nemirovski, A., Laub, A.J. and Chilali, M. (1995). LMI control toolbox, *Technical report*, The Mathworks Inc., Natick, MA.

Goh, K.G., Safonov, M.G. and Ly, J.H. (1996). Robust synthesis via bilinear matrix inequalities, *International Journal of Robust and Nonlinear Control* **6**: 1079–1095.

Guerra, T.M., Bernal, M., Guelton, K. and Labiod, S. (2012). Non-quadratic local stabilization for continuous-time Takagi–Sugeno models, *Fuzzy Sets and Systems* **201**: 40–54.

Guerra, T.M., Kruszewski, A., Vermeiren, L. and Tirmant, H. (2006). Conditions of output stabilization for nonlinear models in the Takagi–Sugeno's form, *Fuzzy Sets and Systems* **157**: 1248–1259.

Jaadari, A., Guerra, T.M., Sala, A., Bernal, M. and Guelton, K. (2012). New controllers and new designs for continuous-time Takagi–Sugeno models, *Proceedings of the IEEE Mathematical Conference on Fuzzy Systems (FUZZ-IEEE), Brisbane, Australia*, pp. 1–7.

Klug, M., Castelan, E.B., Leite, V.J. and Silva, L.F. (2015). Fuzzy dynamic output feedback control through nonlinear Takagi–Sugeno models, *Fuzzy Sets and Systems* **263**: 92–111.

Liu, X. and Zhang, Q. (2003). Approaches to quadratic stability conditions and H_∞ control designs for T–S fuzzy systems, *IEEE Transactions on Fuzzy Systems* **11**(6): 830–838.

Liu, Y., Wu, F. and Ban, X. (2017). Dynamic output feedback control for continuous-time T–S fuzzy systems using fuzzy Lyapunov functions, *IEEE Transactions on Fuzzy Systems* **25**(5): 1155–1167.

Löfberg, J. (2004). YALMIP: A toolbox for modeling and optimization in MATLAB, *Proceedings of the CACSD Conference*, Taipei, Taiwan, pp. 284–289.

Nguyen, A., Dambrine, M. and Lauber, J. (2014). Lyapunov-based robust control design for a class of switching non-linear systems subject to input saturation: Application to engine control, *IET Control Theory and Applications* **8**(17): 1789–1802.

Nguyen, A., Dequidt, A. and Dambrine, M. (2015). Anti-windup based dynamic output feedback controller design with performance consideration for constrained Takagi–Sugeno systems, *Engineering Applications of Artificial Intelligence* **40**: 76–83.

Nguyen, A.-T., Laurain, T., Palhares, R., Lauber, J., Sentouh, C. and Popieul, J.-C. (2016). LMI-based control synthesis of constrained Takagi–Sugeno fuzzy systems subject to L_2 or L_∞ disturbances, *Neurocomputing* **207**: 793–804.

Nguyen, A.-T., Márquez, R. and Dequidt, A. (2017). An augmented system approach for LMI-based control design of constrained Takagi–Sugeno fuzzy systems, *Engineering Applications of Artificial Intelligence* **61**: 96–102.

Pan, J., Fei, S., Guerra, T. and Jaadari, A. (2012). Non-quadratic local stabilisation for continuous-time Takagi–Sugeno fuzzy models: A descriptor system method, *IET Control Theory and Applications* **6**(12): 1909–1917.

Qiu, J., Tian, H., Lu, Q. and Gao, H. (2013). Nonsynchronized robust filtering design for continuous-time T–S fuzzy affine dynamic systems based on piecewise Lyapunov functions, *IEEE Transactions on Cybernetics* **43**(6): 1755–1766.

Qiu, J., Wei, Y. and Wu, L. (2017). A novel approach to reliable control of piecewise affine systems with actuator faults, *IEEE Transactions on Circuits and Systems II: Express Briefs* **64**(8): 957–961.

Saifia, D., Chadli, M., Labiod, S. and Guerra, T.M. (2012). Robust h_∞ static output feedback stabilization of TS fuzzy systems subject to actuator saturation, *International Journal of Control, Automation and Systems* **10**(3): 613–622.

Salcedo, J. and Martinez, M. (2008). BIBO stabilisation of Takagi–Sugeno fuzzy systems under persistent perturbations using fuzzy output-feedback controllers, *IET Control Theory and Applications* **2**(6): 513–523.

Salcedo, J., Martínez, M., Blasco, X. and Sanchis, J. (2007). BIBO fuzzy stabilization of nonlinear systems under persistent perturbations, *Proceedings of the European Control Conference, Kos, Greece*, pp. 763–769.

Salcedo, J. V., Martínez, M. and García-Nieto, S. (2008). Stabilization conditions of fuzzy systems under persistent perturbations and their application in nonlinear systems, *Engineering Applications of Artificial Intelligence* **21**(8): 1264–1276.

Sanchez Peña, R.S. and Sznaier, M. (1998). *Robust Systems: Theory and Applications*, Wiley, New York, NY.

Sturm, J.F. (1999). Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones, *Optimization Methods and Software* **11–12**(1–4): 625–653.

Takagi, T. and Sugeno, M. (1985). Fuzzy identification of systems and its applications to modeling and control, *IEEE Transactions on Systems, Man, and Cybernetics* **15**(1): 116–132.

Tanaka, K., Ikeda, T. and Wang, H. (1998). Fuzzy regulators and fuzzy observers: Relaxed stability conditions and LMI-based designs, *IEEE Transactions on Fuzzy Systems* **6**(2): 250–265.

Tanaka, K. and Wang, H.O. (2001). *Fuzzy Control Systems. Design and Analysis. A Linear Matrix Inequality Approach*, Wiley, New York, NY.

Teixeira, M.C.M., Assunçao, E. and Avellar, R.G. (2003). On relaxed LMI-based designs for fuzzy regulators and fuzzy observers, *IEEE Transactions on Fuzzy Systems* **11**(5): 613–623.

Tognetti, E.S. and Oliveira, V.A. (2010). Fuzzy pole placement based on piecewise Lyapunov functions, *International Journal of Robust and Nonlinear Control* **20**(5): 571–578.

Tuan, H., Apkarian, P., Narikiyo, T. and Yamamoto, Y. (2001). Parameterized linear matrix inequality techniques in fuzzy control system design, *IEEE Transactions on Fuzzy Systems* **9**(2): 324–332.

Vafamand, N., Asemani, M.H. and Khayatian, A. (2017). Robust l_1 observer-based non-PDC controller design for persistent bounded disturbed TS fuzzy systems, *IEEE Transactions on Fuzzy Systems* **26**(3): 1401–1413.

Vafamand, N., Asemani, M.H. and Khayatian, A. (2017b). TS fuzzy robust L_1 control for nonlinear systems with persistent bounded disturbances, *Journal of The Franklin Institute* **354**(14): 5854–5876.

Vafamand, N., Asemani, M.H. and Khayatiyan, A. (2016). A robust L_1 controller design for continuous-time TS systems with persistent bounded disturbance and actuator saturation, *Engineering Applications of Artificial Intelligence* **56**: 212–221.

Yang, W. and Tong, S. (2015). Output feedback robust stabilization of switched fuzzy systems with time-delay and actuator saturation, *Neurocomputing* **164**: 173–181.

Zhao, Y. and Gao, H. (2012). Fuzzy-model-based control of an overhead crane with input delay and actuator saturation, *IEEE Transactions on Fuzzy Systems* **20**(1): 181–186.

José V. Salcedo received his BSc (1997) and PhD (2005) degrees in industrial engineering from the Polytechnic University of Valencia, Spain. He is currently a professor at the Department of Systems Engineering and Control there. He has published numerous papers in conference proceedings and journals. His research interests include model predictive control, fuzzy systems, and evolutionary optimization applied to identification and process control.

Miguel Martínez is a professor in the Department of Systems Engineering and Control of the Polytechnic University of Valencia, Spain. He holds BSc and PhD degrees from the same university. He has three years of industrial experience in control engineering in cement plants. He has published several chapters of control books, and numerous papers in conference proceedings and journals. His research interests are applied adaptive control, model predictive control, and evolutionary optimization applied to identification and process control.

Sergio García-Nieto received the BSc degree in electronic engineering and the PhD degree in control and automation, both honored with valedictorian awards in 2004 and 2010, respectively. Since 2007, he has been working as an associate professor at the Systems Engineering and Control Department, Polytechnic University of Valencia. His research interests are in non-linear modelling, predictive control, fuzzy logic, robust control and genetic algorithms.

Adolfo Hilario received his MSc degree in automation and electronics from the Spanish University for Distance Education in 1999. He is currently a professor at the Department of Systems Engineering and Control of the Polytechnic University of Valencia. His research interests include non-linear modelling and control, and control of high performance robotic actuators.

Appendix

Proof. (Theorem 2) Condition (20) can be transformed into (23) using page 83 of the work of Boyd *et al.* (1994). By the congruence transformation with $\operatorname{diag}\left(\mu^{-1/2}\boldsymbol{I}, \mu^{-1/2}\boldsymbol{I}, \mu^{1/2}\boldsymbol{I}\right)$, (21) is equivalent to

$$\begin{pmatrix} \sigma'\bar{\boldsymbol{P}} & \boldsymbol{0} & {\boldsymbol{C}_z^{CL}}^T \\ \boldsymbol{0} & (\mu-\sigma')\boldsymbol{I} & \delta {\boldsymbol{D}_z^{CL}}^T \\ \boldsymbol{C}_z^{CL} & \delta\boldsymbol{D}_z^{CL} & \mu\boldsymbol{I} \end{pmatrix} \geq 0, \tag{A1}$$

where $\sigma' = \mu^{-1}\sigma$. First, it is shown that if (23) and (A1) have a common solution, $(\alpha_0, \beta_0, \sigma'_0, \mu_0, \bar{\boldsymbol{P}}_0)$, then (23) and (24) also do with the same value for μ. Introducing

$$\sigma'_0\bar{\boldsymbol{P}}_0 = \alpha_0\frac{\sigma'_0}{\alpha_0}\bar{\boldsymbol{P}}_0 = \alpha_0\bar{\boldsymbol{P}}_1, \quad \bar{\boldsymbol{P}}_1 \triangleq \frac{\sigma'_0}{\alpha_0}\bar{\boldsymbol{P}}_0,$$

(A1) is transformed into

$$\begin{pmatrix} \alpha_0\bar{\boldsymbol{P}}_1 & \boldsymbol{0} & {\boldsymbol{C}_z^{CL}}^T \\ \boldsymbol{0} & (\mu_0-\sigma'_0)\boldsymbol{I} & \delta {\boldsymbol{D}_z^{CL}}^T \\ \boldsymbol{C}_z^{CL} & \delta\boldsymbol{D}_z^{CL} & \mu_0\boldsymbol{I} \end{pmatrix} \geq 0.$$

Multiplying (23) by σ'_0/α_0, we get

$$\begin{pmatrix} {\boldsymbol{A}_z^{CL}}^T\bar{\boldsymbol{P}}_1 + \bar{\boldsymbol{P}}_1\boldsymbol{A}_z^{CL} + \alpha_0\bar{\boldsymbol{P}}_1 & \delta\bar{\boldsymbol{P}}_1\boldsymbol{B}_z^{CL} \\ \delta{\boldsymbol{B}_z^{CL}}^T\bar{\boldsymbol{P}}_1 & -\beta_0\frac{\sigma'_0}{\alpha_0}\boldsymbol{I} \end{pmatrix} \leq 0. \tag{A2}$$

From $\beta_0 \leq \alpha_0$ it follows that

$$\beta_0\frac{\sigma'_0}{\alpha_0} \leq \sigma'_0,$$

which yields

$$\begin{pmatrix} {\boldsymbol{A}_z^{CL}}^T\bar{\boldsymbol{P}}_1 + \bar{\boldsymbol{P}}_1\boldsymbol{A}_z^{CL} + \alpha_0\bar{\boldsymbol{P}}_1 & \delta\bar{\boldsymbol{P}}_1\boldsymbol{B}_z^{CL} \\ \delta{\boldsymbol{B}_z^{CL}}^T\bar{\boldsymbol{P}}_1 & -\sigma'_0\boldsymbol{I} \end{pmatrix} \leq 0.$$

Thus, a common solution for (23) and (24) is obtained:

$$(\alpha_1 = \alpha_0, \beta_1 = \sigma_0', \mu_1 = \mu_0, \bar{\boldsymbol{P}}_1).$$

Since β_1 must satisfy $\beta_1 \leq \alpha_1$, we get $\sigma_0' \leq \alpha_0$. This condition will be verified at the end of the proof. Secondly, it is shown that if (23) and (24) have a common solution, $(\alpha_1, \beta_1, \mu_1, \bar{\boldsymbol{P}}_1)$, then (23) and (A1) also do with the same value for μ. Introducing

$$\alpha_1 \bar{\boldsymbol{P}}_1 = \beta_1 \frac{\alpha_1}{\beta_1} \bar{\boldsymbol{P}}_1 = \beta_1 \bar{\boldsymbol{P}}_0, \quad \bar{\boldsymbol{P}}_0 \triangleq \frac{\alpha_1}{\beta_1} \bar{\boldsymbol{P}}_1,$$

(24) is transformed into

$$\begin{pmatrix} \beta_1 \bar{\boldsymbol{P}}_0 & \mathbf{0} & {\boldsymbol{C}_z^{CL}}^T \\ \mathbf{0} & (\mu_1 - \beta_1)\boldsymbol{I} & \delta {\boldsymbol{D}_z^{CL}}^T \\ \boldsymbol{C}_z^{CL} & \delta \boldsymbol{D}_z^{CL} & \mu_1 \boldsymbol{I} \end{pmatrix} \geq 0.$$

Multiplying (23) by α_1/β_1, we have

$$\begin{pmatrix} {\boldsymbol{A}_z^{CL}}^T \bar{\boldsymbol{P}}_0 + \bar{\boldsymbol{P}}_0 \boldsymbol{A}_z^{CL} + \alpha_1 \bar{\boldsymbol{P}}_0 & \delta \bar{\boldsymbol{P}}_0 \boldsymbol{B}_z^{CL} \\ \delta {\boldsymbol{B}_z^{CL}}^T \bar{\boldsymbol{P}}_0 & -\alpha_1 \boldsymbol{I} \end{pmatrix} \leq 0.$$

Thus, a common solution for (23) and (A1) is obtained:

$$(\alpha_0 = \alpha_1, \beta_0 = \alpha_1, \sigma_0' = \beta_1, \mu_0 = \mu_1, \bar{\boldsymbol{P}}_0)$$

As $\beta_1 \leq \alpha_1$, we have $\rightarrow$ $\sigma_0' \leq \alpha_0$. ■

Proof. (Theorem 3) Let us show that conditions (33) imply $\mathcal{E}(\boldsymbol{x}^{-1}) \subset \mathcal{P}_u$. By the congruence transformation with $\operatorname{diag}(\boldsymbol{X}^{-1}, \boldsymbol{I},)$, where $\boldsymbol{P} = \boldsymbol{X}^{-1}$ and $\boldsymbol{W}_z = \boldsymbol{Z}_z \boldsymbol{X}^{-1}$, we get

$$\begin{bmatrix} \boldsymbol{P} & \boldsymbol{F}_{i,l}^T - \boldsymbol{W}_{i,l}^T \\ * & u_{\max,l}^2 \end{bmatrix} \geq 0,$$

$$i = 1, \ldots, r, \quad l = 1, \ldots, n_u,$$

which yields

$$\begin{bmatrix} \boldsymbol{P} & \boldsymbol{F}_{z,l}^T - \boldsymbol{W}_{z,l}^T \\ * & u_{\max,l}^2 \end{bmatrix} \geq 0, \quad l = 1, \ldots, n_u.$$

Applying the Schur complement, we obtain

$$\boldsymbol{P} \geq \frac{(\boldsymbol{F}_{z,l} - \boldsymbol{W}_{z,l})^T (\boldsymbol{F}_{z,l} - \boldsymbol{W}_{z,l})}{u_{\max,l}^2}, \quad l = 1, \ldots, n_u. \tag{A3}$$

Consequently, if $\boldsymbol{x} \in \mathcal{E}(\boldsymbol{P})$ then $\boldsymbol{x} \in \mathcal{P}_u$.

Following a similar argument, conditions (34) imply $\mathcal{E}(\boldsymbol{P}) \subset \mathcal{P}_x$. Applying Lemma 2 to conditions (36) and (37), we get

$$\begin{bmatrix} \boldsymbol{X}\boldsymbol{A}_z^T + \boldsymbol{A}_z \boldsymbol{X} + \boldsymbol{B}_{1z} \boldsymbol{Y}_z + \boldsymbol{Y}_z^T \boldsymbol{B}_{1z}^T + \alpha \boldsymbol{X} & * & * \\ -\boldsymbol{S}_z \boldsymbol{B}_{1z}^T + \boldsymbol{Z}_z & -2\boldsymbol{S}_z & * \\ \delta \boldsymbol{B}_{2z}^T & \mathbf{0} & -\beta \mathbf{I} \end{bmatrix} \leq 0.$$

The congruence transformation with $\operatorname{diag}(\boldsymbol{X}^{-1}, \boldsymbol{I}, \boldsymbol{I})$ yields

$$\begin{bmatrix} (\boldsymbol{A}_z + \boldsymbol{B}_{1z}\boldsymbol{F}_z)^T \boldsymbol{P} + \boldsymbol{P}(\boldsymbol{A}_z + \boldsymbol{B}_{1z}\boldsymbol{F}_z) + \alpha \boldsymbol{P} & * & * \\ -\boldsymbol{S}_z \boldsymbol{B}_{1z}^T \boldsymbol{P} + \boldsymbol{W}_z & -2\boldsymbol{S}_z & * \\ \delta \boldsymbol{B}_{2z}^T \boldsymbol{P} & \mathbf{0} & -\beta \mathbf{I} \end{bmatrix} \leq 0.$$

By the congruence transformation with $\operatorname{diag}(\boldsymbol{I}, \boldsymbol{S}_z^{-1}, \boldsymbol{I})$,

$$\begin{bmatrix} (\boldsymbol{A}_z + \boldsymbol{B}_{1z}\boldsymbol{F}_z)^T \boldsymbol{P} + \boldsymbol{P}(\boldsymbol{A}_z + \boldsymbol{B}_{1z}\boldsymbol{F}_z) + \alpha \boldsymbol{P} & * & * \\ -\boldsymbol{S}_z \boldsymbol{B}_{1z}^T \boldsymbol{P} + \boldsymbol{W}_z & -2\boldsymbol{S}_z^{-1} & * \\ \delta \boldsymbol{B}_{2z}^T \boldsymbol{P} & \mathbf{0} & -\beta \mathbf{I} \end{bmatrix} \leq 0. \tag{A4}$$

Condition (35) implies

$$\begin{bmatrix} \alpha \boldsymbol{X} & * & * \\ \mathbf{0} & (\mu - \beta)\mathbf{I} & * \\ \boldsymbol{C}_z \boldsymbol{X} & \delta \boldsymbol{D}_z & \mu \mathbf{I} \end{bmatrix} \geq 0.$$

By the congruence transformation with $\operatorname{diag}(\boldsymbol{X}^{-1}, \boldsymbol{I}, \boldsymbol{I})$,

$$\begin{bmatrix} \alpha \boldsymbol{P} & * & * \\ \mathbf{0} & (\mu - \beta)\mathbf{I} & * \\ \boldsymbol{C}_z & \delta \boldsymbol{D}_z & \mu \mathbf{I} \end{bmatrix} \geq 0. \tag{A5}$$

Following the proof of Theorem 2, conditions (A4) and (A5) are equivalent to (A4), and

$$\begin{bmatrix} \sigma \boldsymbol{P} & * & * \\ \mathbf{0} & (\mu^2 - \sigma)\mathbf{I} & * \\ \boldsymbol{C}_z & \delta \boldsymbol{D}_z & \mathbf{I} \end{bmatrix} \geq 0, \quad \sigma > 0. \tag{A6}$$

Next, if (A4) is pre- and post-multiplied by vector $\begin{bmatrix} \boldsymbol{x}^T & \boldsymbol{\psi}^T & \boldsymbol{\phi}^T \end{bmatrix}$ and its transpose, respectively, the following inequality can be obtained after some algebraic manipulations using (26) and $V(x) \triangleq \boldsymbol{x}^T \boldsymbol{P} \boldsymbol{x}$:

$$\dot{V}(x) + \alpha(\boldsymbol{x}^T \boldsymbol{P} \boldsymbol{x} - 1) + \beta \left(1 - \frac{\boldsymbol{\phi}^T \boldsymbol{\phi}}{\delta^2}\right) + \boldsymbol{\psi}^T \boldsymbol{S}_z^{-1} \boldsymbol{W}_z \boldsymbol{x} + \boldsymbol{x}^T \boldsymbol{W}_z^T \boldsymbol{S}_z^{-1} \boldsymbol{\psi} - 2\boldsymbol{\psi}^T \boldsymbol{S}_z^{-1} \boldsymbol{\psi} + \alpha - \beta \leq 0.$$

As $\boldsymbol{S}_z$ is a diagonal matrix, we get

$$\dot{V}(x) + \alpha(\boldsymbol{x}^T \boldsymbol{P} \boldsymbol{x} - 1) + \beta \left(1 - \frac{\boldsymbol{\phi}^T \boldsymbol{\phi}}{\delta^2}\right) - 2\boldsymbol{\psi}^T \boldsymbol{S}_z^{-1} (\boldsymbol{\psi} - \boldsymbol{W}_z \boldsymbol{x}) + \alpha - \beta \leq 0.$$

Applying Lemma 1, we obtain

$$\dot{V}(x) + \alpha\left(\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x} - 1\right) + \beta\left(1 - \frac{\boldsymbol{\phi}^T\boldsymbol{\phi}}{\delta^2}\right) + \alpha - \beta \le 0.$$

Since $\beta \le \alpha$,

$$\dot{V}(x) + \alpha\left(\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x} - 1\right) + \beta\left(1 - \frac{\boldsymbol{\phi}^T\boldsymbol{\phi}}{\delta^2}\right) \le 0. \quad \text{(A7)}$$

As $\beta \ge 0$ and $\alpha \ge 0$, applying the S-procedure (Boyd *et al.*, 1994), we get

$$\dot{V}(x) \le 0 \quad \text{(A8)}$$

when $\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x} \ge 1$ and $\boldsymbol{\phi}^T\boldsymbol{\phi} \le \delta^2$. This condition implies that $\mathcal{E}(\boldsymbol{P})$ is an inescapable ellipsoid (Salcedo and Martinez, 2008). Finally, it is shown that condition (A6) implies that $\boldsymbol{y}^T\boldsymbol{y}$ is bounded by μ^2. By the congruence transformation with $\operatorname{diag}\left(\boldsymbol{I}, \delta^{-1}\boldsymbol{I}, \boldsymbol{I}\right)$, we have

$$\begin{bmatrix} \sigma\boldsymbol{P} & * & * \\ \boldsymbol{0} & \frac{(\mu^2-\sigma)}{\delta^2}\mathbf{I} & * \\ \boldsymbol{C}_z & \boldsymbol{D}_z & \mathbf{I} \end{bmatrix} \ge 0.$$

Applying the Schur complement, we get

$$\begin{bmatrix} \sigma\boldsymbol{P} - \boldsymbol{C}_z^T\boldsymbol{C}_z & -\boldsymbol{C}_z^T\boldsymbol{D}_z \\ -\boldsymbol{D}_z^T\boldsymbol{C}_z & \frac{(\mu^2-\sigma)}{\delta^2}\mathbf{I} - \boldsymbol{D}_z^T\boldsymbol{D}_z \end{bmatrix} \ge 0.$$

Pre- and post-multiplying by vector $\begin{bmatrix}\boldsymbol{x}^T & \boldsymbol{\phi}^T\end{bmatrix}$ and its transpose, respectively, the following inequality can be obtained after some algebraic manipulations using (26):

$$\left(\mu^2 - \boldsymbol{y}^T\boldsymbol{y}\right) - \sigma\left(1 - \boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x}\right) - \left(\mu^2 - \sigma\right)^2\left(1 - \frac{\boldsymbol{\phi}^T\boldsymbol{\phi}}{\delta^2}\right) \ge 0.$$

By the S-procedure, we have

$$\boldsymbol{y}^T\boldsymbol{y} \le \mu^2$$

when $\boldsymbol{x} \in \mathcal{E}(\boldsymbol{P})$ and $\boldsymbol{\phi}^T\boldsymbol{\phi} \le \delta^2$. This implies that

$$\inf_{\alpha>0} N(\alpha)$$

yields the minimum $\star$-norm of (26) among all the inescapable ellipsoids. ∎

Proof. (Theorem 4) This proof is quite similar to that of Theorem 3. The only difference is in how the volume of ellipsoid $\mathcal{E}(\boldsymbol{X}^{-1})$ is computed. According to Boyd *et al.* (1994) the volume of $\mathcal{E}(\boldsymbol{X}^{-1})$ is proportional to $\sqrt{\det(\boldsymbol{X})}$. This function is monotonic but not convex. However, $\log\det$ (Boyd *et al.*, 1994) is also a convex function. Consequently, the problem of maximizing the volume of ellipsoid $\mathcal{E}(\boldsymbol{X}^{-1})$ is equivalent to

$$\max \log\det(\boldsymbol{X}) = -\min\left(-\log\det(\boldsymbol{X})\right).$$

∎

Received: 23 August 2017
Revised: 11 January 2018
Accepted: 28 March 2018