FINITE HORIZON NONLINEAR PREDICTIVE CONTROL BY THE TAYLOR APPROXIMATION: APPLICATION TO ROBOT TRACKING TRAJECTORY

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In industrial control systems, practical interest is driven by the fact that today's processes need to be operated under tighter performance specifications. Often these demands can only be met when process nonlinearities are explicitly considered in the controller. Nonlinear predictive control, the extension of well-established linear predictive control to nonlinear systems, appears to be a well-suited approach for this kind of problems. In this paper, an optimal nonlinear predictive control structure, which provides asymptotic tracking of smooth reference trajectories, is presented. The controller is based on a finite–horizon continuous time minimization of nonlinear predicted tracking errors. A key feature of the control law is that its implementation does not need to perform on-line optimization, and asymptotic tracking of smooth reference signal is guaranteed. An integral action is used to increase the robustness of the closed-loop system with respect to uncertainties and parameters variations. The proposed control scheme is first applied to planning motions problem of a mobile robot and, afterwards, to the trajectory tracking problem of a rigid link manipulator. Simulation results are performed to validate the tracking performance of the proposed controller.

Keywords: nonlinear continuous time predictive control, Taylor approximation, tracking trajectory and robot

1. Introduction

Linear model predictive control (LMPC), or receding-horizon control of linear systems, has become an attractive feedback strategy (Boucher *et al.*, 1996). Generalized predictive control (GPC) of constrained multivariable systems has found successful applications, especially in process industries, due to its robustness to parameter uncertainties and to the fact that the constraints are incorporated directly into the associated open-loop optimal-control problem (Clarke *et al.*, 1987a; 1987b). Many systems are, however, inherently nonlinear and the LMPC is inadequate for highly nonlinear processes which have large operating regimes. This shortcoming coupled with increasingly stringent demands on product quality have spurred the development of nonlinear model predictive control (NMPC) (Henson, 1998). Thus, much effort has been made to extend LMPC to nonlinear systems where more accurate nonlinear models are used for process prediction and optimization. However, in nonlinear model predictive control a nonlinear optimization problem must be solved online, with high computational complexity (Henson *et al.*, 1997) at each sampling period to generate the control signal to be applied to the nonlinear process. This significant computation effort requires an appropriate (not too short) sampling time. Therefore, this kind of control scheme can be applied only to systems with slow dynamics (chemical processes). Three practical problems arise and are summarized as follows (Henson, 1998; Morari *et al.*, 1999):

- The optimization problem is generally nonconvex because the model equations are nonlinear. Consequently, the problem of the existence of an on-line solution of the nonlinear program is a crucial one.
- Nominal stability is insured only when the prediction horizon is infinite or a terminal constraint is imposed. However, these conditions are not suitable for practical implementation.
- Although NMPC has some degree of robustness to modelling errors, there is no rigorous theory that would allow the robustness of the closed loop system to be analyzed.

To avoid these problems, several nonlinear predictive laws have been developed (Ping, 1995; Singh *et al.*, 1995; Souroukh *et al.*, 1996), where the one-step ahead predictive error is obtained by expanding the output and reference signals in an r_i-th order Taylor series, where r_i is the relative degree of the i-th output. Then, the continuous minimization of the predicted tracking errors is used to derive the control law. Note that these nonlinear predictive controllers are given in a closed form, thus no online optimization is required. Stability is also guaranteed. Moreover, these methods can be applied to nonlinear systems with fast dynamics (robots, motors, etc.) provided that the dynamic zeros are stable and the relative degree is well defined.

This paper examines the nonlinear continuous-time generalized predictive control approach based on finite-horizon dynamic minimization of predicted tracking errors to achieve tracking objectives. The proposed control scheme takes advantages of the properties of robustness, good reference tracking and reducing the computation burden. Indeed, the optimization is performed offline and suitable Taylor series expansions are adopted for the prediction process. Thus, our contributions are as follows:

- The performance index is minimized along the interval [0,*h*] and this will increase the dynamics of the tracking error with regard to Ping's method (Ping, 1995), where the performance index is minimized over a fixed time horizon (one step ahead).
- To increase the robustness of the proposed control algorithm with respect to model uncertainties and parameters variations, we propose to introduce an integral action into the loop.

Note that this method can be viewed as an extension to nonlinear systems of continuous GPC developed for linear systems (Demircioglu *et al.*, 1991). Moreover, it will be shown that, when compared with input-output linearization methods, some advantages of this control scheme include good tracking performance, clear physical meaning of maximum and minimum control values when saturation occurs.

Two kinds of nonlinear systems are considered. First, we deal with a general multi-variable affine nonlinear system. The proposed nonlinear predictive controller is derived by minimizing a predictive cost function along a finite horizon. A mobile robot is used as an illustrative example to show the tracking performance achieved by this nonlinear predictive controller. Afterwards, a particular multi-variable affine nonlinear system is considered. To increase the robustness of the proposed control scheme to parameter variations and/or to uncertainties, an integral action is incorporated into the loop. The derived nonlinear predictive controller is applied to a rigid-link robot manipulator to achieve both position and speed angular tracking objectives in matched or mismatched cases. The nonlinear observer is used to estimate the speed angular joint of the robot. It is mentioned in (Lee *et al.*, 1997) that the feedback control algorithm with the sliding observer developed in (Canudas De Wit *et al.*, 1992) guarantees that the tracking error tends to zero exponentially but constrains the initial estimation errors in the joint positions to be zero. In this paper, this constraint is weakened by the proposed feedback nonlinear predictive control approach with Gauthier's observer.

The rest of the paper is organized as follows: In Section 2, the problem statement is given, and a control law is developed to minimize the difference between the predicted and desired responses. The properties of the control law are discussed, including stability and robustness. In Section 3, the proposed controller is first applied to the planning motion problem of the mobile robot. The second application of the proposed control approach deals with the trajectory tracking problem of the rigid link robot manipulator in matched and mismatched cases. Our results are summarized in Section 4, where we also provide some directions for related research.

2. Optimal Nonlinear Predictive Control

In the receding horizon control strategy, the following control problem is solved at each $t > 0$ and $\boldsymbol{x}(t)$:

$$\min\ J\big(\boldsymbol{x}(t), \boldsymbol{u}(t), t\big) = \min_{\boldsymbol{u}(t)} \frac{1}{2} \int_t^{t+h} \Big[\boldsymbol{x}(\tau)^T \mathbf{Q}\mathbf{x}(\tau) + \boldsymbol{u}(\tau)^T \mathbf{R}\boldsymbol{u}(\tau)\Big] \mathrm{d}\tau \quad (1)$$

subject to the state equation (2), where h >0 is the prediction horizon, $\mathbf{Q}$ is a positive-definite matrix and $\mathbf{R}$ a semi-positive-definite matrix. We denote by $\boldsymbol{u}^*(\tau)$, $\tau \in [t, t+h]$ the optimal control vector for the above problem. The currently applied control $\boldsymbol{u}(t)$ is set equal to $\boldsymbol{u}^*(\tau)$. This process is repeated for every t for the stabilization of the system at the origin. However, solving a nonlinear dynamic optimization problem is highly computationally intensive, and in many cases it is impossible to perform it within a reasonable time limit. Thus, the derived control law can be applied only to slow dynamic systems. Furthermore, the global optimization solution cannot be guaranteed in each optimization procedure since, in general, it is a nonconvex, constrained nonlinear optimization problem (Henson *et al.*, 1997).

2.1. General Multi-Variable Affine Nonlinear System

First, we consider a general multi-variable affine nonlinear system modelled by

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\,\boldsymbol{u}(t), \\ \boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}), \end{cases} \tag{2}$$

where $\boldsymbol{x}(t) \subset \boldsymbol{X} \in \mathbb{R}^n$ is the state vector, $\boldsymbol{u}(t) \subset \boldsymbol{U} \in \mathbb{R}^n$ represents the control vector and $\boldsymbol{y}(t) \subset \boldsymbol{Y} \in \mathbb{R}^m$ is the output. The functions $\boldsymbol{f}(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}^n$, $\boldsymbol{g}(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ and $\boldsymbol{h}(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}^m$ are sufficiently smooth. The desired trajectory is specified by a smooth vector function $\boldsymbol{x}_{\text{ref}}(t) \in \mathbb{R}^n$ for $t \in [t, t_f]$.

Assumptions:

(A1) The vector function $\boldsymbol{f}(\boldsymbol{x})$ is bounded, i.e., there exist two constants $f_{\max}$ and $f_{\min}$ that satisfy $f_{\min} \leq \| \boldsymbol{f}(\boldsymbol{x}) \| \leq f_{\max}, \forall\, \boldsymbol{x} \in \mathbb{R}^n$.

(A2) The matrix $\boldsymbol{g}(\boldsymbol{x})$ is symmetric, bounded and nonsingular.

(A3) The reference trajectories are bounded: $\| \boldsymbol{x}_{\text{ref}}(t) \| \leq r_1, \| \dot{\boldsymbol{x}}_{\text{ref}} \| \leq r_2$ and $\| \ddot{\boldsymbol{x}}_{\text{ref}} \| \leq r_3$.

(A4) From Assumptions (A1) and (A3), we can deduce that there exist a scalar function $\overline{\delta}(\boldsymbol{x}, \boldsymbol{x}_{\text{ref}})$ and a positive number δ that satisfy the following inequality: $\| \boldsymbol{f}(\boldsymbol{x}) - \dot{\boldsymbol{x}}_{\text{ref}} \| \leq \overline{\delta}(\boldsymbol{x}, \boldsymbol{x}_{\text{ref}}) < \delta$, where $\| \cdot \|$ is the Euclidean norm.

The problem consists in designing a control law $\boldsymbol{u}(\boldsymbol{x}, t)$ that will improve the tracking accuracy along the interval $[t, t+h]$, where h >0 is a prediction horizon, such that $\mathbf{x}(t+h)$ tracks $\boldsymbol{x}_{\text{ref}}(t+h)$. That is, the predicted tracking error is defined by

$$\boldsymbol{e}(t+h) = \boldsymbol{x}(t+h) - \boldsymbol{x}_{\text{ref}}(t+h). \tag{3}$$

A simple and efficient way to predict the influence of $\boldsymbol{u}(t)$ on $\boldsymbol{x}(t+h)$ is to expand it in the r_i-th order Taylor series, in such a way as to obtain

$$\boldsymbol{x}(t+h) = \boldsymbol{x}(t) + \mathbf{Z}(\boldsymbol{x}, h) + \boldsymbol{\Lambda}(h)\mathbf{W}(\boldsymbol{x})\boldsymbol{u}(t), \tag{4}$$

where

$$\mathbf{Z}(\boldsymbol{x}, h) = \begin{bmatrix} z_1(\boldsymbol{x}, h) & z_2(\boldsymbol{x}, h) & \dots & z_n(\boldsymbol{x}, h) \end{bmatrix}^T,$$
$$\boldsymbol{\Lambda}(h) = \operatorname{diag}\begin{pmatrix} h^{r_1}/r_1! & \dots & h^{r_n}/r_n! \end{pmatrix},$$
$$\mathbf{W}(\boldsymbol{x}) = \begin{pmatrix} \boldsymbol{w}_1 & \boldsymbol{w}_2 & \cdots & \boldsymbol{w}_n \end{pmatrix}^T$$

with

$$z_i(\boldsymbol{x}, h) = h f_i + \frac{h^2}{2} L_f f_i + \cdots + \frac{h^{r_i}}{r_i} L_f^{r_i - 1} f_i$$

and

$$\mathbf{w}_i = \left(L_{g_1} L_f^{r_i-2} f_i \, \dots \, L_{g_m} L_f^{r_i-2} f_i \right)$$

for $i = 1, \dots, n$. Note that $L_f(\cdot)$ or $L_f L_g(\cdot)$ represents the Lie-Derivative.

In both cases, we also expand each component of $\boldsymbol{x}_{\text{ref}}(t+h)$ in the r_i-th order Taylor series to have

$$\boldsymbol{x}_{\text{ref}}(t+h) = \boldsymbol{x}_{\text{ref}}(t) + \boldsymbol{d}(t, h),$$

where

$$\mathbf{d}(t, h) = \begin{pmatrix} d_1 & d_2 & \cdots & d_n \end{pmatrix}^T$$

with

$$d_i = h\, \dot{x}_{\text{ref}_i} + \frac{h^2}{2}\, \ddot{x}_{\text{ref}_i} + \cdots + \frac{h^{r_i}}{r_i}\, x_{\text{ref}_i}^{(r_i)}.$$

The tracking error at the next instant $(t+h)$ is then predicted as a function of $\mathbf{u}(t)$ by

$$\begin{aligned} \boldsymbol{e}(t+h) &= \boldsymbol{x}(t+h) - \boldsymbol{x}_{\text{ref}}(t+h) \\ &= \boldsymbol{e}(t) + \mathbf{Z}(\boldsymbol{x}, h) - \boldsymbol{d}(t, h) \\ &\quad + \boldsymbol{\Lambda}(h)\mathbf{W}(\mathbf{x})\mathbf{u}(t). \end{aligned} \tag{5}$$

In order to find current control $\mathbf{u}(t)$ that improves the tracking error along a fixed interval and to avoid the computational burden, the tracking error $\boldsymbol{e}(\tau)$ is used instead of the state vector $\boldsymbol{x}(\tau)$. Thus, the optimization problem can be reformulated as

$$\begin{aligned} \min\ & J_1(\boldsymbol{e}(t), \boldsymbol{u}(t), t) \\ &= \min \frac{1}{2} \int_0^h \boldsymbol{e}(t+T)^T \mathbf{Q}\, \boldsymbol{e}(t+T)\, \mathrm{d}T \\ &\quad + \frac{1}{2} \boldsymbol{u}(t)^T \mathbf{R}\, \boldsymbol{u}(t). \end{aligned} \tag{6}$$

Replace $\mathbf{e}(t+T)$ by the prediction equation (5). The cost function J_1 is quadratic in $\boldsymbol{u}(t)$, and hence the unique control signal $\boldsymbol{u}_{op}$ that minimizes J_1, obtained by setting $\partial J_1 / \partial \boldsymbol{u} = 0$, is

$$\begin{aligned} \mathbf{u}_{op} = &- \left(\mathbf{W}(\boldsymbol{x})^T \boldsymbol{\Gamma}(h) \mathbf{W}(\boldsymbol{x}) + \mathbf{R} \right)^{-1} \mathbf{W}(\boldsymbol{x})^T \big(\mathbf{K}(h) \boldsymbol{e}(t) \\ &+ \mathbf{V}(\boldsymbol{x}, \boldsymbol{x}_{\text{ref}}, h) \big), \end{aligned} \tag{7}$$

where

$$\boldsymbol{e}(t) = \boldsymbol{x}(t) - \boldsymbol{x}_{\text{ref}}(t),$$

$$\mathbf{K}(h) = \int_0^h \boldsymbol{\Lambda}(T) \mathbf{Q}\, \mathrm{d}T,$$

$$\boldsymbol{\Gamma}(h) = \int_0^h \boldsymbol{\Lambda}^T(T) \mathbf{Q} \boldsymbol{\Lambda}(T)\, \mathrm{d}T,$$

$$\begin{aligned} \mathbf{V}(\boldsymbol{x}, \boldsymbol{x}_{\text{ref}}, h) = \int_0^h & \boldsymbol{\Lambda}^T(T) \mathbf{Q} \big(\mathbf{Z}(\boldsymbol{x}, T) \\ & - \boldsymbol{d}(t, T) \big)\, \mathrm{d}T. \end{aligned}$$

Tracking performance. We will assume that the matrix $\mathbf{W}(\boldsymbol{x})$ has a full rank. This assumption is needed for the stability analysis, but is not necessary for the control law to be applicable, since one can always choose $\mathbf{R} > 0$, and then the inverse matrix in (7) will still exist. If $\mathbf{R} = 0$, then (7) becomes

$$\boldsymbol{u}_{\rm op} = -\mathbf{W}(\boldsymbol{x})^{-1}\boldsymbol{\Gamma}^{-1}(h)\left(\mathbf{K}(h)\boldsymbol{e}(t) + \mathbf{V}(\boldsymbol{x}, \boldsymbol{x}_{\rm ref}, h)\right).$$

This optimal control signal $\boldsymbol{u}_{\rm op}$, used in (2), leads to the closed-loop equation of the i-th component of the tracking error vector $\boldsymbol{e}(t)$ which is given by

$$\frac{h^{r_i}}{(2r_i+1)r_i!}\overset{(r_i)}{e_i} + \frac{h^{r_i-1}}{2r_i(r_i-1)!}\overset{(r_i-1)}{e_i} + \cdots + \frac{h^3}{3!(r_i+4)}\overset{(3)}{e_i} + \frac{h^2}{2!(r_i+3)}\ddot{e}_i + \frac{h}{(r_i+2)}\dot{e}_i + \frac{1}{(r_i+1)}e_i = 0,$$

or, in a compact form,

$$\sum_{j=0}^{r_i}\frac{h^j}{j!(r_i+j+1)}\overset{(j)}{e_i} = 0, \tag{8}$$

where

$$\overset{(j)}{e_i} = L_f^{j-1}f_i - \overset{(j)}{x}_{{\rm ref}_i}, \quad 0 < j \le r_i,$$

$$\overset{(0)}{e} = x_i - x_{{\rm ref}_i}, \quad j = 0.$$

The error dynamics (8) are linear and time-invariant. Thus, the proposed controller that minimizes the predicted tracking error naturally leads to a special case of input/state linearization. The advantage of this controller compared with the linearization method is a clear physical meaning of a maximum and a minimum when saturation occurs. Note that, by using the Routh-criterion, we can show that the tracking error dynamics (8) are stable only for systems with $r_i \le 4$.

For most mechanical systems with actuator dynamics neglected, the relative degree is $r_i = 1$ or $r_i = 2$. In this case, the eigenvalues of the characteristic equations of the error dynamics are:

- Ping's method (Ping, 1995):
  $s_1 = -\frac{1}{h}(r_i = 1)$
  or $s_{1,2} = -\frac{1}{h}(1 \pm j)\,(r_i = 2)$.
- the proposed method:
  $s_1 = -\frac{3}{2h}(r_i = 1)$
  or $s_{1,2} = -\frac{5}{4h}\left(1 \pm j\sqrt{\frac{17}{15}}\right)(r_i = 2)$.

Thus, the proposed controller achieves higher tracking error dynamics compared with Ping's method (Ping, 1995).

2.2. Particular Affine Nonlinear Systems

To overcome the stability restriction of the relative degree $r_i \le 4$, we will consider a special form of nonlinear systems that are modelled by the equations

$$\begin{cases} \dot{\boldsymbol{x}}_1 = \boldsymbol{x}_2, \\ \dot{\boldsymbol{x}}_2 = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\boldsymbol{u}(t), \end{cases} \tag{9}$$

where $\boldsymbol{x} = \left[\begin{array}{cc}\boldsymbol{x}_1 & \boldsymbol{x}_2\end{array}\right]^T \in \mathbb{R}^{2n}$ and $\boldsymbol{u}(t) \in \mathbb{R}^n$. We note that many physical systems can be modeled by the above equations. For example, in mechanical systems, $\boldsymbol{x}_1$ can represent a position vector and $\boldsymbol{x}_2$ a velocity vector.

In this case, the objective function to minimize is

$$J_2(\boldsymbol{e}, \boldsymbol{u}, t) = \frac{1}{2}\int_0^h \begin{pmatrix} \boldsymbol{e}_1(t+T) \\ \boldsymbol{e}_2(t+T) \end{pmatrix}^T \begin{pmatrix} \mathbf{Q}_1 & 0 \\ 0 & T^2\,\mathbf{Q}_2 \end{pmatrix} \times \begin{pmatrix} \boldsymbol{e}_1(t+T) \\ \boldsymbol{e}_2(t+T) \end{pmatrix} {\rm d}T + \frac{1}{2}\boldsymbol{u}^T(t)\mathbf{R}\,\boldsymbol{u}(t). \tag{10}$$

The tracking error is given by

$$\boldsymbol{e}(t) = \boldsymbol{x}(t) - \boldsymbol{x}_{\rm ref}(t) = \left|\begin{array}{c} \boldsymbol{e}_1(t) \\ \boldsymbol{e}_2(t) \end{array}\right| = \left|\begin{array}{c} \boldsymbol{x}_1 - \boldsymbol{x}_{\rm ref1} \\ \boldsymbol{x}_2 - \boldsymbol{x}_{\rm ref2} \end{array}\right|.$$

By using the Taylor approximation, the tracking error is then predicted as a function of $\boldsymbol{u}(t)$ by

$$\begin{cases} \boldsymbol{e}_1(t+T) = \boldsymbol{e}_1(t) + T\dot{\boldsymbol{e}}_1 + \frac{T^2}{2!}\left(\boldsymbol{f}(\boldsymbol{x}) - \ddot{\boldsymbol{x}}_{\rm ref1}\right) + \frac{T^2}{2!}\boldsymbol{g}(\boldsymbol{x})\,\boldsymbol{u}(t), \\ \boldsymbol{e}_2(t+T) = \boldsymbol{e}_2(t) + T\left(\boldsymbol{f}(\boldsymbol{x}) - \dot{\boldsymbol{x}}_{\rm ref2}\right) + T\boldsymbol{g}(\boldsymbol{x})\,\boldsymbol{u}(t), \end{cases}$$

and the minimization of the cost function $\mathbf{J}_2$ gives

$$\boldsymbol{u}(t) = -\mathbf{M}(\boldsymbol{x})\mathbf{P}^{-1}\Big(\frac{h^3}{6}\mathbf{Q}_1\boldsymbol{e}_1 + \frac{h^4}{8}(\mathbf{Q}_1 + 2\mathbf{Q}_2)\boldsymbol{e}_2 + \frac{h^5}{20}(\mathbf{Q}_1 + 4\mathbf{Q}_2)(\boldsymbol{f}(\boldsymbol{x}) - \dot{\boldsymbol{x}}_{\rm ref2})\Big), \tag{11}$$

where

$$\mathbf{P} = \frac{h^5}{20}(\mathbf{Q}_1 + 4\mathbf{Q}_2) + \mathbf{M}(\boldsymbol{x})\mathbf{R}\mathbf{M}(\boldsymbol{x})$$

is a positive-definite matrix, $\ddot{\boldsymbol{x}}_{{\rm ref}_1} = \dot{\boldsymbol{x}}_{{\rm ref}_2}$ and $\mathbf{M}(\boldsymbol{x}) = \boldsymbol{g}^{-1}(\boldsymbol{x})$.

2.3. Stability Issues

Dynamic performance. To obtain the tracking error dynamic, one substitutes the control signal (11) in (9), to have

$$\begin{cases} \dot{\boldsymbol{e}}_1 = \boldsymbol{e}_2, \\ \dot{\boldsymbol{e}}_2 = -\frac{h^3}{6}\mathbf{P}^{-1}\mathbf{Q}_1 e_1 - \frac{h^4}{8}\mathbf{P}^{-1}(\mathbf{Q}_1 + 2\mathbf{Q}_2)\boldsymbol{e}_2 \\ \quad +\mathbf{P}^{-1}\mathbf{M}(\boldsymbol{x})\mathbf{R}\,\mathbf{M}(\boldsymbol{x})(\boldsymbol{f}(\boldsymbol{x}) - \dot{\boldsymbol{x}}_{\text{ref2}}). \end{cases} \tag{12}$$

Let $\mathbf{Q}_1 = q_1\mathbf{I}_n$ and $\mathbf{Q}_2 = q_2\mathbf{I}_n$. The tracking error equation (12) can be written in a compact form:

$$\dot{\boldsymbol{e}} = \boldsymbol{\Lambda}(x,h)\boldsymbol{e} + \mathbf{B}\,\mathbf{S}_l(\boldsymbol{x}, \boldsymbol{x}_{\text{ref}}), \tag{13}$$

where

$$\boldsymbol{\Lambda}(x,h) = \begin{vmatrix} 0 & \mathbf{I}_n \\ -\frac{h^3 q_1}{6}\mathbf{P}^{-1} & -\frac{h^4(q_1+2q_2)}{8}\mathbf{P}^{-1} \end{vmatrix},$$

$$\mathbf{B} = \begin{vmatrix} 0 \\ \mathbf{I}_n \end{vmatrix},$$

and the perturbed term is given by

$$\mathbf{S}_l(\boldsymbol{x}, \boldsymbol{x}_{\text{ref}}) = \mathbf{P}^{-1}\mathbf{M}(\boldsymbol{x})\mathbf{R}\,\mathbf{M}(\boldsymbol{x})(\boldsymbol{f}(\boldsymbol{x}) - \dot{\boldsymbol{x}}_{\text{ref2}}).$$

Assumptions (A1)–(A4) insure the boundedness of this additional term.

Lemma 1. *The matrix* $\boldsymbol{\Lambda}(x,h)$ *is Hurwitz.*

Proof. Both the matrix $\mathbf{P}$ and its inverse are symmetric and positive definite. Let $\overline{x} \in \mathbb{R}^n$ and $\overline{\lambda} \in \mathbb{R}^+$ be the eigenvector and the correspondent eigenvalue of the matrix $\mathbf{P}^{-1}$, respectively. Thus, for $\lambda \in \mathbb{R}$ we have the equalities

$$\boldsymbol{\Lambda}(x,h)\begin{vmatrix} \overline{x} \\ \lambda\overline{x} \end{vmatrix} = \begin{vmatrix} \lambda\overline{x} \\ -\frac{h^3 q_1}{6}\overline{\lambda}\overline{x} - \frac{h^4(q_1+2q_2)}{8}\lambda\overline{\lambda}\overline{x} \end{vmatrix} = \begin{vmatrix} \lambda\overline{x} \\ \lambda^2\overline{x} \end{vmatrix} = \lambda\begin{vmatrix} \overline{x} \\ \lambda\overline{x} \end{vmatrix},$$

where λ is the solution of the equation

$$\lambda^2 + \frac{h^4(q_1 + 2q_2)}{8}\overline{\lambda}\lambda + \frac{h^3 q_1}{6}\overline{\lambda} = 0. \tag{14}$$

Therefore, λ is the eigenvalue of the matrix $\boldsymbol{\Lambda}(x,h)$ and $\left|\frac{\overline{x}}{\lambda\overline{x}}\right|$ is the correspondent eigenvector. Setting λ_1 and λ_2 as the solutions of (14), we have the relations

$$\lambda_1 + \lambda_2 = -\frac{h^4}{8}(q_1 + 2q_2)\overline{\lambda},$$

$$\lambda_1\lambda_2 = \frac{h^3}{6}q_1\overline{\lambda}.$$

Since the eigenvalue $\overline{\lambda}$ is positive, then both λ_1 and λ_2 have negative real parts. Thus, the matrix $\boldsymbol{\Lambda}(x,h)$ is Hurwitz. Consequently, for any symmetric positive-definite matrix $\mathbf{Q}_a(x,h)$, there exists a symmetric positive-definite matrix $\mathbf{P}_a(x,h)$ being a solution of the Lyapunov equation

$$\dot{\mathbf{P}}_a(x,h) + \boldsymbol{\Lambda}^T(x,h)\mathbf{P}_a(x,h) + \mathbf{P}_a(x,h)\boldsymbol{\Lambda}(x,h) = -\mathbf{Q}_a(x,h).$$

Theorem 1. *The solution* $\boldsymbol{e}(t)$ *of the system (12) is uniformly ultimately bounded (Khalil, 1992) for all* $t \geq t_0 > 0$.

Proof. Consider the Lyapunov function candidate

$$\mathbf{V}(\boldsymbol{e}) = \boldsymbol{e}^T\mathbf{P}_a\boldsymbol{e}. \tag{15}$$

The differentiation of $\mathbf{V}$ along the trajectories of the system (12) leads to

$$\dot{V}(\boldsymbol{e}) = -\boldsymbol{e}^T\mathbf{Q}_a\boldsymbol{e} + 2\mathbf{S}_l^T\,\mathbf{B}^T\mathbf{P}_a\boldsymbol{e}, \tag{16}$$

which can be bounded by using Assumptions (A1)–(A4) as

$$\dot{V}(\boldsymbol{e}) \leq -\lambda_{\min}(\mathbf{Q}_a) \parallel \boldsymbol{e} \parallel^2 + 2\lambda_{\max}(\mathbf{P}_a)r\delta m^2 \parallel \boldsymbol{e} \parallel,$$

where $r = \parallel \mathbf{R} \parallel$ and $m = \parallel \mathbf{M}(\boldsymbol{x}) \parallel$.

We will use the well-known inequality

$$ab \leq za^2 + \frac{b^2}{4z}$$

for any real a, b and $z > 0$. With $z = \theta\lambda_{\min}(\mathbf{Q}_a)$ and $0 < \theta < 1$, we obtain

$$\dot{V}(\boldsymbol{e}) \leq -(1-\theta)\lambda_{\min}(\mathbf{Q}_a)\|\boldsymbol{e}\|^2 + \frac{\lambda_{\max}(\mathbf{P}_a)^2 r^2\delta^2 m^4}{\theta\lambda_{\min}(\mathbf{Q}_a)}. \tag{17}$$

The solution of this inequality is

$$V(t) \leq \left[V(0) - \frac{\beta}{\alpha}\right]\exp\left(-\alpha t + \frac{\beta}{\alpha}\right),$$

where

$$\alpha = (1-\theta)\frac{\lambda_{\min}(\mathbf{Q}_a)}{\lambda_{\max}(\mathbf{P}_a)}$$

and

$$\beta = \frac{\lambda_{\max}(\mathbf{P}_a)^2 r^2\delta^2 m^4}{\theta\lambda_{\min}(\mathbf{Q}_a)}.$$

As $t \to \infty$, the tracking error is bounded by

$$\|\boldsymbol{e}\| \leq \frac{\lambda_{\max}(\mathbf{P}_a)}{\lambda_{\min}(\mathbf{Q}_a)}\frac{r\delta m^2}{\sqrt{\theta(1-\theta)}}\sqrt{\frac{\lambda_{\max}(\mathbf{P}_a)}{\lambda_{\min}(\mathbf{P}_a)}}. \tag{18}$$

It can be easily shown that, as $\mathbf{R}$ tends to a null matrix by reducing the penalty on control, the bound of the perturbed term decreases for large t and the equilibrium point tends to the origin. Setting $\mathbf{R} = 0$ in (12), the time derivative of the Lyapunov function becomes

$$\dot{V}(\boldsymbol{e}) = -\boldsymbol{e}^T \mathbf{Q}_a\, \boldsymbol{e},$$

which is negative definite for all $\boldsymbol{e}$. By LaSalle's invariance theorem, the solution $\boldsymbol{e}(t)$ of (12) tends to the invariant set $S = \{\boldsymbol{e} \mid \boldsymbol{e}_2 = 0, \mathbf{P}^{-1}\boldsymbol{e}_1 = 0\}$. Since the matrix $\mathbf{P}^{-1}$ has a full rank, we have that $\boldsymbol{e}_1 = 0$. So, the origin $\boldsymbol{e} = 0$ is globally asymptotically stable. ■

2.4. Robustness Issues

In the real world, model uncertainties are frequently encountered in nonlinear control systems. These model uncertainties may decrease significantly the performance of the method in terms of tracking accuracy. Therefore, one should inspect the robustness of the closed-loop system with respect to uncertainties. The model of the nonlinear system (9) with uncertainties can be written as

$$\begin{cases} \dot{\boldsymbol{x}}_1 = \boldsymbol{x}_2, \\ \dot{\boldsymbol{x}}_2 = \boldsymbol{f}(\boldsymbol{x}) + \Delta f(x) + (\boldsymbol{g}(\boldsymbol{x}) + \Delta g(x))\, \boldsymbol{u}(t). \end{cases} \tag{19}$$

To estimate the worst-case bound of the uncertainties, we make the following assumptions:

(A5) $\forall\, \boldsymbol{x}(t) \in X,\ \exists\, \kappa > 0, \|\Delta f(x)\| < \kappa.$

(A6) $\forall\, \mathbf{x}(t) \in X,\ \exists\, \mu = \max \frac{\|\Delta g(x)\|}{\|g(x)\|}$, the uncertainties in the matrix $\mathbf{g}(x)$ can be bounded by $\Delta g(x) = \mu \mathbf{g}(\boldsymbol{x})$ with $0 < \mu < 1$.

Let $\mathbf{R} = 0$. The dynamics of the tracking error in a mismatched case in a closed loop with the optimal control (11) is given by

$$\begin{cases} \dot{\boldsymbol{e}}_1 = \boldsymbol{e}_2, \\ \dot{\boldsymbol{e}}_2 = -\dfrac{h^3}{6}(1+\mu)\mathbf{P}^{-1}\mathbf{Q}_1 e_1 \\ \qquad - \dfrac{h^4}{8}(1+\mu)\mathbf{P}^{-1}(\mathbf{Q}_1 + 2\mathbf{Q}_2)e_2 \\ \qquad + \Delta f(\boldsymbol{x}) - \mu(\boldsymbol{f}(\boldsymbol{x}) - \dot{\boldsymbol{x}}_{\mathrm{ref2}}). \end{cases} \tag{20}$$

Note that here, even though $\mathbf{R} = 0$, the origin is not an equilibrium point of the system (20). However, we can use the steps of the Lemma 1 and Theorem 1 to show that the tracking error $\mathbf{e}(t)$ is ultimately bounded in this mismatched case and the equilibrium point is given by the set S= $\{\boldsymbol{e}/\boldsymbol{e}_1 \neq 0, \boldsymbol{e}_2 = 0\}$. Hence, the uncertainties will introduce only a short steady-state error in the tracking position error. The bound of this steady state error depends on the magnitude of the uncertainties.

Integral action. It is known in the literature that the integral action increases the robustness of the closed-loop system against low frequency disturbances as long as the closed-loop system is stable. In this part, we shall incorporate an integral action into the loop to enhance the robustness of the proposed control scheme with respect to model uncertainties and disturbances. The price to be paid is an increase in the system dimension. Thus, the nonlinear system (9) is augmented with the differential equation $\dot{x}_0 = x_1$ and the tracking error vector becomes

$$\mathbf{e}(t) = \left|\ \boldsymbol{e}_0(t) \quad \boldsymbol{e}_1(t) \quad \boldsymbol{e}_2(t)\ \right|^T,$$

with $\boldsymbol{e}_0 = \int_0^t \boldsymbol{e}_1(\tau)\,\mathrm{d}\tau$ and $\boldsymbol{e}_0(t+h)$ given by

$$\boldsymbol{e}_0(t+T) = \boldsymbol{e}_0 + T\boldsymbol{e}_1 + \frac{T^2}{2}\boldsymbol{e}_2 + \frac{T^3}{6}(\boldsymbol{f}(\boldsymbol{x}) - \dot{\boldsymbol{x}}_{\mathrm{ref2}}) + \frac{T^3}{6}\boldsymbol{g}(\boldsymbol{x})\boldsymbol{u}(t). \tag{21}$$

The cost function to be minimized becomes

$$J_3(\boldsymbol{e}, \boldsymbol{u}, t) = \frac{1}{2}\int_0^h \boldsymbol{e}(t+T)^T \mathbf{Q}\boldsymbol{e}(t+T)\,\mathrm{d}T + \frac{1}{2}\boldsymbol{u}(t)^T \mathbf{R}\,\boldsymbol{u}(t), \tag{22}$$

where $\mathbf{Q} = \mathrm{diag}\ (\mathbf{Q}_0, \mathbf{Q}_1, T^2\mathbf{Q}_2)$ and $\mathbf{Q}_0 \in \mathbb{R}^{n\times n}$ is a positive-definite matrix. Following the same steps as in the previous section, the optimal control $\boldsymbol{u}(t)$ that minimizes the cost function $J_3(\boldsymbol{e}, \boldsymbol{u}, t)$ is

$$\boldsymbol{u}(t) = -\mathbf{M}(\boldsymbol{x})\overline{\mathbf{P}}^{-1}\Big(\boldsymbol{\alpha}_0(h)\boldsymbol{e}_0 + \boldsymbol{\alpha}_1(h)\boldsymbol{e}_1 + \boldsymbol{\alpha}_2(h)\boldsymbol{e}_2 + \boldsymbol{\alpha}_3(h)(\boldsymbol{f}(\boldsymbol{x}) - \dot{\boldsymbol{x}}_{\mathrm{ref2}})\Big), \tag{23}$$

where

$$\boldsymbol{\alpha}_0(h) = \frac{h^4}{12}\mathbf{Q}_0,$$

$$\boldsymbol{\alpha}_1(h) = \frac{h^5}{15}\mathbf{Q}_0 + \frac{h^3}{6}\mathbf{Q}_1,$$

$$\boldsymbol{\alpha}_2(h) = \frac{h^6}{36}\mathbf{Q}_0 + \frac{h^4}{8}\mathbf{Q}_1 + \frac{h^4}{4}\mathbf{Q}_2,$$

$$\boldsymbol{\alpha}_3(h) = \frac{h^7}{63}\mathbf{Q}_0 + \frac{h^5}{20}\mathbf{Q}_1 + \frac{h^5}{5}\mathbf{Q}_2,$$

$$\overline{\mathbf{P}} = \alpha_3(h) + \mathbf{M}(\boldsymbol{x})\mathbf{R}\mathbf{M}(\boldsymbol{x}).$$

Dynamic performance. Let $\mathbf{R} = 0$. Then the dynamics of the tracking error are given in a compact form by

$$\dot{\boldsymbol{e}} = \boldsymbol{\Gamma}(x,h)\boldsymbol{e} + \overline{\mathbf{B}}\,\overline{\mathbf{S}}_l, \tag{24}$$

where

$$\boldsymbol{\Gamma}(x,h) = \begin{vmatrix} 0 & \mathbf{I}_n & 0 \\ 0 & 0 & \mathbf{I}_n \\ -(1+\mu)\overline{\mathbf{P}}^{-1}\alpha_0(h) & -(1+\mu)\overline{\mathbf{P}}^{-1}\alpha_1(h) & -(1+\mu)\overline{\mathbf{P}}^{-1}\alpha_2(h) \end{vmatrix},$$

$$\overline{\mathbf{S}_l} = \Delta f(\boldsymbol{x}) - \mu(\boldsymbol{f}(\boldsymbol{x}) - \dot{\boldsymbol{x}}_{\text{ref2}})$$

and

$$\overline{\mathbf{B}} = \begin{vmatrix} 0 \\ 0 \\ \mathbf{I}_n \end{vmatrix}.$$

Note that also the perturbed term $\overline{\mathbf{S}}_l$ is bounded.

Lemma 2. *Let the parameters $\boldsymbol{Q}_0$, $\boldsymbol{Q}_1$, $\boldsymbol{Q}_2$ and $h > 0$ satisfy the inequality*

$$\lambda_{\max}(\overline{\boldsymbol{P}}) < \frac{(1+\mu)\alpha_1(h)\alpha_2(h)}{\alpha_0(h)}.$$

Hence the matrix $\Gamma(\boldsymbol{x},h)$ is Hurwitz.

Proof. Let $\mathbf{Q}_0 = q_0\mathbf{I}_n$, $\mathbf{Q}_1 = q_1\mathbf{I}_n$, $\mathbf{Q}_2 = q_2\mathbf{I}_n$, and assume that $\overline{x}$ and $\overline{\lambda}$ are an eigenvector and the corresponding eigenvalue of the matrix $\overline{\mathbf{P}}^{-1}$, respectively. Hence we have

$$\boldsymbol{\Gamma}(\mathbf{x},h)\begin{vmatrix} \overline{x} \\ \lambda\overline{x} \\ \lambda^2\overline{x} \end{vmatrix} = \begin{vmatrix} \lambda\overline{x} \\ \lambda^2\overline{x} \\ \lambda^3\overline{x} \end{vmatrix} = \lambda\begin{vmatrix} \overline{x} \\ \lambda\overline{x} \\ \lambda^2\overline{x} \end{vmatrix},$$

where $\lambda \in \mathbb{R}$ is the solution of

$$\lambda^3 + (1+\mu)\alpha_2(h)\overline{\lambda}\lambda^2 + (1+\mu)\alpha_1(h)\overline{\lambda}\lambda + (1+\mu)\alpha_0(h)\overline{\lambda} = 0. \tag{25}$$

The roots of this equation are stable if

$$\overline{\lambda} > \frac{\alpha_0(h)}{(1+\mu)\alpha_1(h)\alpha_2(h)}.$$

Since $\overline{\lambda}$ is the eigenvalue of the matrix $\overline{\mathbf{P}}^{-1}$, the previous inequality becomes

$$\lambda_{\min}(\overline{\mathbf{P}}^{-1}) > \frac{\alpha_0(h)}{(1+\mu)\alpha_1(h)\alpha_2(h)}$$

or

$$\lambda_{\max}(\overline{\mathbf{P}}) < \frac{(1+\mu)\alpha_1(h)\alpha_2(h)}{\alpha_0(h)},$$

and this ensures that all poles of the matrix $\boldsymbol{\Gamma}(\boldsymbol{x},h)$ lie in the stable domain. It is to be noted that λ is the eigenvalue of the matrix $\boldsymbol{\Gamma}(\boldsymbol{x},h)$ and the vector $\begin{vmatrix} \overline{x} & \lambda\overline{x} & \lambda^2\overline{x} \end{vmatrix}^T$ is the corresponding eigenvector. ■

Theorem 2. *Under the assumptions of Lemma 2, the solution of the tracking error (24) is ultimately uniformly bounded.*

Proof. Since the matrix $\boldsymbol{\Gamma}(\boldsymbol{x},h)$ is Hurwitz, we know that for any symmetric positive-definite matrix $\overline{\mathbf{Q}}_a(x,h)$, the solution $\overline{\mathbf{P}}_a(x,h)$ of the Lyapunov equation

$$\dot{\overline{\mathbf{P}}}_a(\boldsymbol{x},h) + \boldsymbol{\Gamma}^T(x,h)\overline{\mathbf{P}}_a(x,h) + \overline{\mathbf{P}}_a(x,h)\boldsymbol{\Gamma}(x,h) = -\overline{\mathbf{Q}}_a(x,h) \tag{26}$$

is a positive-definite matrix. We use $V = \boldsymbol{e}^T\overline{\mathbf{P}}_a\boldsymbol{e}$ as a Lyapunov function candidate for the augmented nonlinear system (24). Following the same steps as in the proof of Theorem 1, we can show that the tracking error is bounded by

$$\|\boldsymbol{e}\| \le \frac{\lambda_{\max}(\overline{\mathbf{P}}_a)}{\lambda_{\min}(\overline{\mathbf{Q}}_a)}\frac{(\kappa+\mu\delta)}{\sqrt{\theta(1-\theta)}}\sqrt{\frac{\lambda_{\max}(\overline{\mathbf{P}}_a)}{\lambda_{\min}(\overline{\mathbf{P}}_a)}}. \tag{27}$$

The tracking error in the mismatched case with integral action is bounded. Here also the bound depends on the magnitude of the uncertainties. However, the equilibrium point of the augmented system is $S = \{\boldsymbol{e} \mid \boldsymbol{e}_0 \neq 0, \boldsymbol{e}_1 = 0, \boldsymbol{e}_2 = 0,\}$, the position tracking error in this case converges to zero. Consequently, the steady error induced by uncertainties is eliminated by the integral action. Note that the price to be paid is the control signal that will not vanish as the time *t* goes towards infinity.

3. Simulation Examples

In this section, the reference trajectory tracking problem is simulated to show the validity and the achieved performance of the proposed method.

3.1. Nonlinear Predictive Control of a Nonholonomic Mobile Robot

A kinematic model of a wheeled mobile robot with two degrees of freedom is given by (Kim *et al.*, 2003):

$$\begin{aligned} \dot{x} &= v\ \cos(\theta) - d\ \omega\ \sin(\theta), \\ \dot{y} &= v\ \sin(\theta) + d\ \omega\ \cos(\theta), \\ \dot{\theta} &= \omega, \end{aligned} \tag{28}$$

where the forward velocity v and the angular velocity ω are considered as the inputs, (x,y) is the centre of the rear axis of the vehicule, θ is the angle between the heading direction and the x-axis, and *d* is the distance from the coordinate of the origin of the mobile robot to the axis of the driving wheel.

The nonholonomic constraint is written as

$$\dot{y}\ \cos(\theta) - \dot{x}\ \sin(\theta) = d\ \dot{\theta}.$$

The nonlinear model of the mobile robot can be rewritten as

$$\dot{\mathbf{Z}} = \mathbf{G}(\theta)\mathbf{U},$$

where

$$\mathbf{Z} = \begin{bmatrix} x & y & \theta \end{bmatrix}^T,$$

$$\mathbf{G}(\theta) = \begin{vmatrix} \cos(\theta) & -d\ \sin(\theta) \\ \sin(\theta) & d\ \cos(\theta) \\ 0 & 1 \end{vmatrix},$$

$$\mathbf{U} = \begin{bmatrix} v & \omega \end{bmatrix}^T.$$

Note that the above model matches the first general multi-variable affine nonlinear system given by (2) with $\boldsymbol{f}(\boldsymbol{x}) = 0$.

Consider the problem of tracking a reference trajectory given by

$$\begin{aligned} \dot{x}_{\text{ref}} &= v_{\text{ref}}\ \cos(\theta_{\text{ref}}), \\ \dot{y}_{\text{ref}} &= v_{\text{ref}}\ \sin(\theta_{\text{ref}}), \\ \dot{\theta}_{\text{ref}} &= \omega_{\text{ref}}, \end{aligned} \tag{29}$$

or, in a compact form, $\dot{\mathbf{Z}}_{\text{ref}} = \overline{\mathbf{G}}(\theta_{\text{ref}})\mathbf{U}_{\text{ref}}$. The optimal control that minimizes the objective function (6) subject to (28) is

$$\begin{aligned} \mathbf{U} = &-\left[\frac{h^3}{3}\mathbf{G}^T(\theta)\mathbf{Q}\ \mathbf{G}(\theta) + \mathbf{R}\right]^{-1} \\ &\times \mathbf{G}^T(\theta)\mathbf{Q}\left(\frac{h^2}{2}\boldsymbol{e}(t) - \frac{h^3}{3}\overline{\mathbf{G}}(\theta_{\text{ref}})\mathbf{U}_{\text{ref}}\right), \end{aligned} \tag{30}$$

where $\boldsymbol{e}(t) = \mathbf{Z}(t) - \mathbf{Z}_{\text{ref}}$.

In the simulation, the control parameters are

$$h = 0.01, \quad \mathbf{Q} = 10^4\mathbf{I}_3, \quad \mathbf{R} = 10^{-7}\mathbf{I}_2.$$

The reference model and initial conditions are

$$\begin{aligned} \omega_{\text{ref}} &= 4\ [\text{rad/s}], & d &= 0.5\ [\text{m}], \\ v_{\text{ref}} &= 15\ [\text{m/s}], & x(0) &= 0, \\ y(0) &= 4\ [\text{m}], & \theta(0) &= -\pi\ [\text{rad}]. \end{aligned}$$

Figures 1 and 2 show the resulting trajectory and position tracking error

$$\boldsymbol{e}_r(t) = \sqrt{(\boldsymbol{x} - \boldsymbol{x}_{\text{ref}})^2 + (\boldsymbol{y} - \boldsymbol{y}_{\text{ref}})^2},$$

when the nonlinear predictive controller (30) is applied to the system (28). We can see that the mobile robot tracks the reference trajectory successfully. Figure 3 depicts the manipulated variables $v(t)$ and $w(t)$.

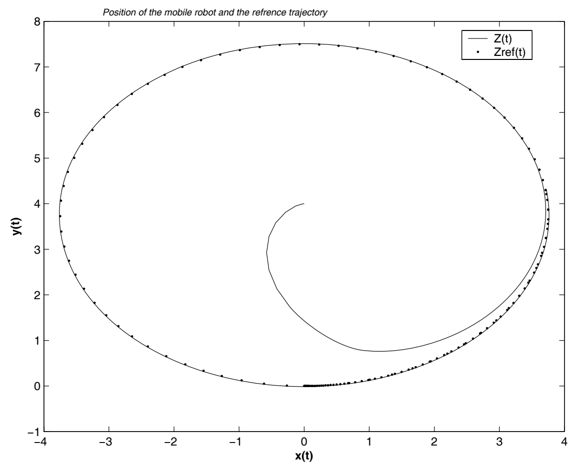

Fig. 1. Tracking performance.

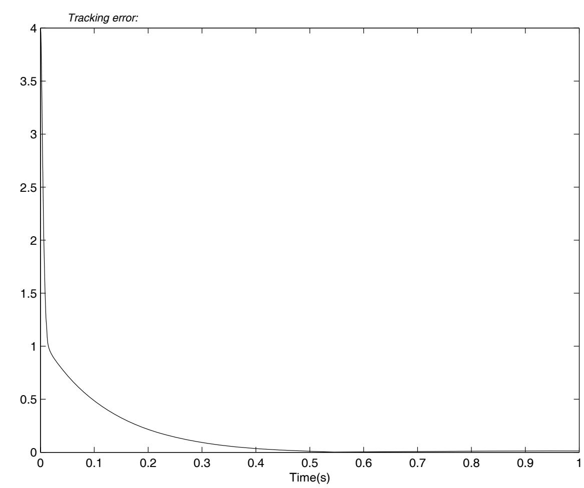

Fig. 2. Tracking error dynamics.

Consequently, the proposed approach, which can be viewed as an extension to nonlinear systems of the CGPC developed by Demircioglu *et al.* (1991), was successfully applied to control a nonlinear system with a nonholonomic constraint. On the other hand, the CGPC approach (Demircioglu *et al.*, 1991), can be applied only to linear systems. Moreover, with the proposed algorithm, the stability of the closed-loop system is guaranteed and asymptotic tracking performances are achieved.

3.2. Nonlinear Predictive Control of a Rigid-Link Robot

To illustrate the conclusions of this paper, we have simulated the nonlinear predictive scheme (11) on a two-link robot arm used in (Lee *et al.*, 1997; Spong *et al.*, 1992) with the parameters given in Table 1.

Table 1. Physical parameters of a two-link robot manipulator.

Link_1	$m_1 = 10\,\mathrm{kg}\ \ l_1 = 1\,\mathrm{m}\ \ l_{c1} = 0.5\,\mathrm{m}\ \ I_1 = \frac{10}{12}\,\mathrm{kgm}^2$
Link_2	$m_2 = 5\,\mathrm{kg}\ \ l_2 = 1\,\mathrm{m}\ \ l_{c2} = 0.5\,\mathrm{m}\ \ I_2 = \frac{5}{12}\,\mathrm{kgm}^2$

The kinetic energy of a robot manipulator with n degrees of freedom can be calculated as (Spong *et al.*, 1989):

$$\mathbf{K}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2}\dot{\boldsymbol{q}}^T(t)\mathbf{M}(\boldsymbol{q})\dot{\boldsymbol{q}}(t),$$

where $\boldsymbol{q}(t) \in \mathbb{R}^n$ is the link position vector, $\mathbf{M}(\boldsymbol{q})$ is the inertia matrix, and $\mathbf{U}(\boldsymbol{q})$ stands for the potential energy generating gravity forces. Applying Euler-Lagrange equations (Spong *et al.*, 1989), we obtain the model

$$\mathbf{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \mathbf{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \mathbf{G}(\boldsymbol{q}) + \mathbf{F}_r = \tau, \tag{31}$$

where

$$\mathbf{G}(q) = \frac{\partial \mathbf{U}(\boldsymbol{q})}{\partial \boldsymbol{q}} \in \mathbb{R}^n,$$

$\mathbf{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$ is the vector of the Coriolis and centripetal torques, $\tau \in \mathbb{R}^n$ stands for the applied torque, and $\mathbf{F}_r$ represents friction torques acting on the joints. These friction are unknown and are modeled by

$$\mathbf{F}_r = \boldsymbol{f}\ \boldsymbol{q}(t) + \overline{\boldsymbol{f}}\ \mathrm{sign}(\dot{\boldsymbol{q}}(t))$$

with $\boldsymbol{f} = \mathrm{diag}(f, f, \ldots, f) \in \mathbb{R}^{n\times n}$
and $\overline{\boldsymbol{f}} = \mathrm{diag}\,(\overline{f}, \overline{f}, \ldots, \overline{f}) \in \mathbb{R}^{n\times n}$.

A state representation. The dynamic equation of an n-link robot manipulator (31) can be written in the state space representation as

$$\begin{cases} \dot{\boldsymbol{x}}_1 = \boldsymbol{x}_2, \\ \dot{\boldsymbol{x}}_2 = \boldsymbol{f}(\boldsymbol{x}_1, \boldsymbol{x}_2) + \mathbf{P}(\boldsymbol{x}_1)\ \tau(t), \\ \boldsymbol{y} = \boldsymbol{x}_1, \end{cases} \tag{32}$$

where $x = \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 \end{bmatrix}^T = \begin{bmatrix} \boldsymbol{q} & \dot{\boldsymbol{q}} \end{bmatrix}^T \in \mathbb{R}^{2n}$ is the state vector, $\tau(t) \in \mathbb{R}^n$ represents the control torque vector and $\boldsymbol{y}(t)$ is the output vector (angular position).

Here $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}_1, \boldsymbol{x}_2) = -\mathbf{M}(\boldsymbol{q})^{-1}(\mathbf{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \mathbf{G}(\boldsymbol{q})) \in \mathbb{R}^n$ and $\mathbf{P}(\boldsymbol{x}_1) = \mathbf{M}(\boldsymbol{q})^{-1} \in \mathbb{R}^{n\times n}$ are a bounded vector under the assumption of the boundedness of joints velocities and a bounded matrix, respectively. We note that both the symmetric positive-definite matrix $\mathbf{M}(\boldsymbol{q})$ and its inverse are uniformly bounded with respect to the joint angular position $\boldsymbol{q}(t)$. Thus Assumptions (A1)–(A4) are satisfied by the nonlinear model of the robot given by (32), c.f. (Spong *et al.*, 1989).

The dynamic model is described by (31) with (see Lee *et al.*, 1997; Spong *et al.*, 1992):

$$\begin{aligned} M_{11}(\boldsymbol{q}) &= m_1 l_{c1}^2 + m_2 l_{c2}^2 + m_2 l_1^2 \\ &\quad + 2m_2 l_1 l_{c2}\cos(q_2) + I_1 + I_2, \\ M_{21}(\boldsymbol{q}) &= M_{12}(\boldsymbol{q}) = m_2 lc2^2 + m_2 L_1 l_{c2}\cos(q_2) + I_2, \\ M_{22}(\boldsymbol{q}) &= m_2 l_{c2}^2 + I_2, \\ C_{11}(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= -m_2 l_1 l_{c2}\ \sin(q_2)\dot{q}_2, \\ C_{12}(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= -m_2 l_1 l_{c2}\ \sin(q_2)(\dot{q}_1 + \dot{q}_2), \\ C_{21}(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= m_2 l_1 l_{c2}\ \sin(q_2)\dot{q}_1, \\ C_{22}(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= 0, \\ G_1(\boldsymbol{q}) &= (m_1 l_{c1} + m_2 l_1)\ g\ \cos(q_1) \\ &\quad + m_2 l_{c2}\ g\ \cos(q_1 + q_2), \\ G_2(\boldsymbol{q}) &= m_2 l_{c2}\ g\ \cos(q_1 + q_2). \end{aligned}$$

Nonlinear observer. A drawback of the previous nonlinear predictive controller is that it requires at least the measurement of the velocity on the link side. Therefore, a nonlinear observer proposed in (Gauthier *et al.*, 1992) is used in this paper. Define the new state vector as

$$\boldsymbol{z}(t) = \mathbf{T}\,\boldsymbol{x}(t) = \begin{bmatrix} \cdots & q_i(t) & \dot{q}_i(t) & \cdots \end{bmatrix} \in \mathbb{R}^{2n},$$

where $q_i(t)$and $\dot{q}_i(t)$ are the link position and the velocity of the i-th arm, respectively. $\mathbf{T} \in \mathbb{R}^{2n\times 2n}$ is the transformation matrix. With the assumption that the control torque $\tau(\mathrm{t})$ is uniformly bounded, the observer described in (Gauthier *et al.*, 1992) can be used to estimate the angular positions and angular velocities of the n-link rigid robot manipulator (32). The dynamic nonlinear observer is given by

$$\begin{cases} \dot{\hat{\boldsymbol{z}}} = \mathbf{A}\hat{\boldsymbol{z}} + \mathbf{H}\ \boldsymbol{f}(\boldsymbol{q}, \dot{\hat{\boldsymbol{q}}}) + \mathbf{H}\ \mathbf{P}(\boldsymbol{q})\tau(t) \\ \qquad - \mathbf{S}_\propto^{-1}\mathbf{C}^T(\boldsymbol{y} - \hat{\boldsymbol{y}}), \\ \hat{\boldsymbol{y}} = \mathbf{C}\ \hat{\boldsymbol{z}}, \\ \hat{\boldsymbol{x}} = \mathbf{T}^{-1}\ \hat{\boldsymbol{z}}, \end{cases} \tag{33}$$

where

$$\mathbf{A} = \mathrm{diag}(\mathbf{A}_i), \quad \mathbf{A}_i = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$\mathbf{C} = \mathrm{diag}(\mathbf{C}_i), \quad \mathbf{C}_i = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

and

$$\mathbf{H} = \mathrm{diag}(\mathbf{H}_i), \quad H_i = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

with $i = 1, n$.

The observer gain $\mathbf{S}_\propto(\theta) = \mathrm{diag}(\mathbf{S}_i(\theta))$ is given by the solution of the following Ricatti equation with the real positive factor θ:

$$\theta\mathbf{S}_i(\theta) + \mathbf{A}_i^T\mathbf{S}_i(\theta) + \mathbf{S}_i(\theta)\mathbf{A}_i = \mathbf{C}_i^T\mathbf{C}_i. \tag{34}$$

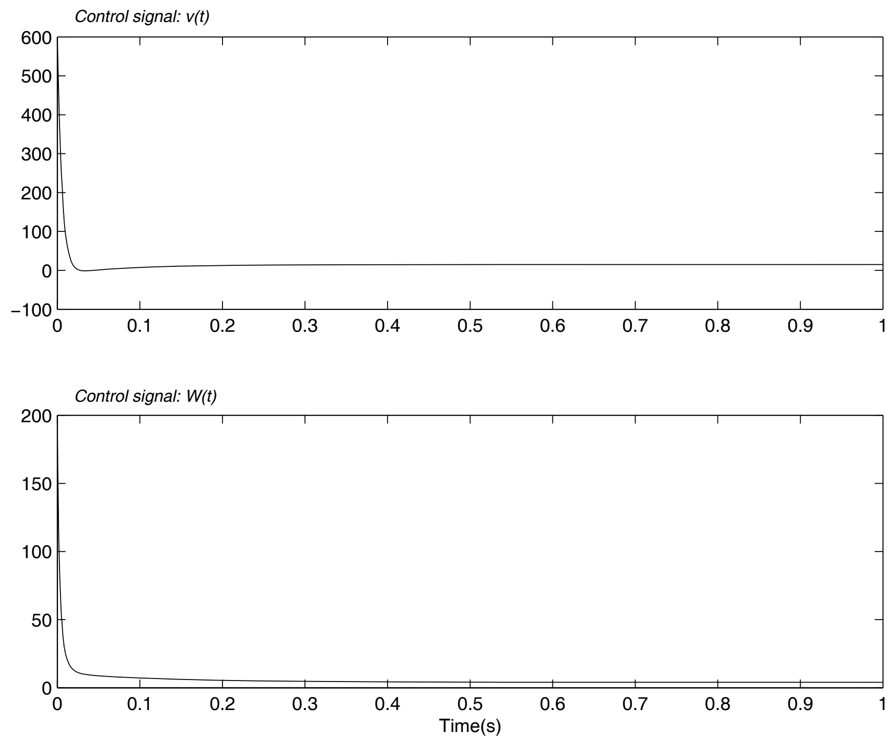

Fig. 3. Manipulated variables.

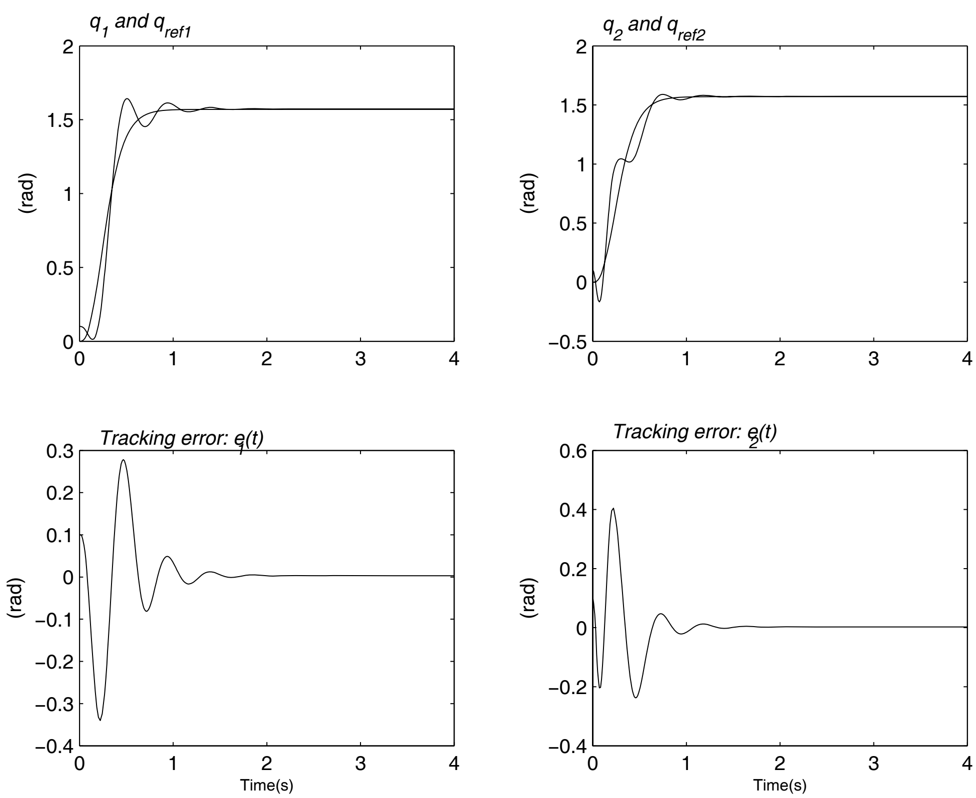

Fig. 4. Position tracking performance.

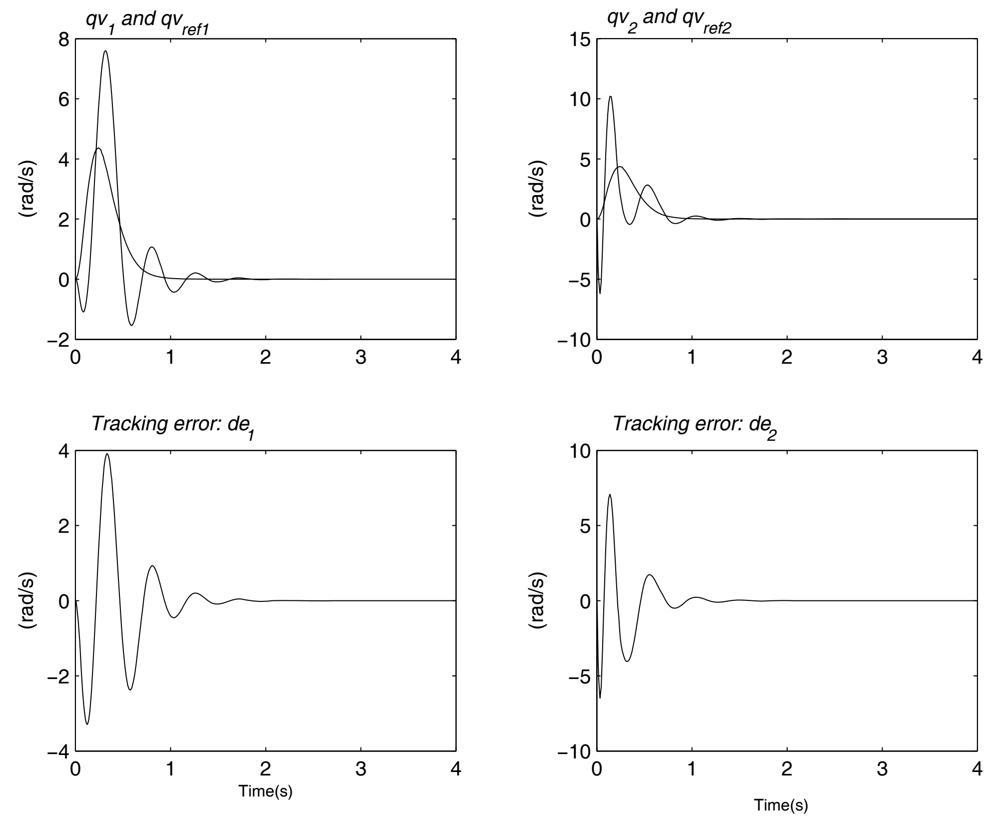

Fig. 5. Velocity tracking performance.

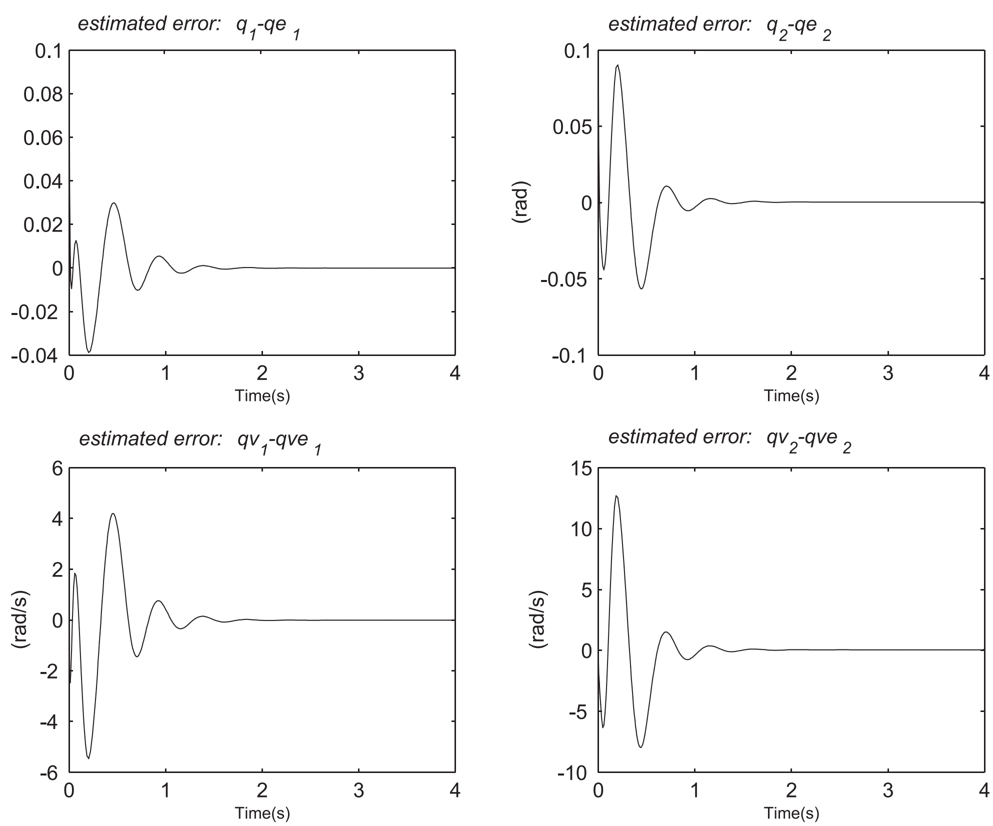

Fig. 6. Estimation error.

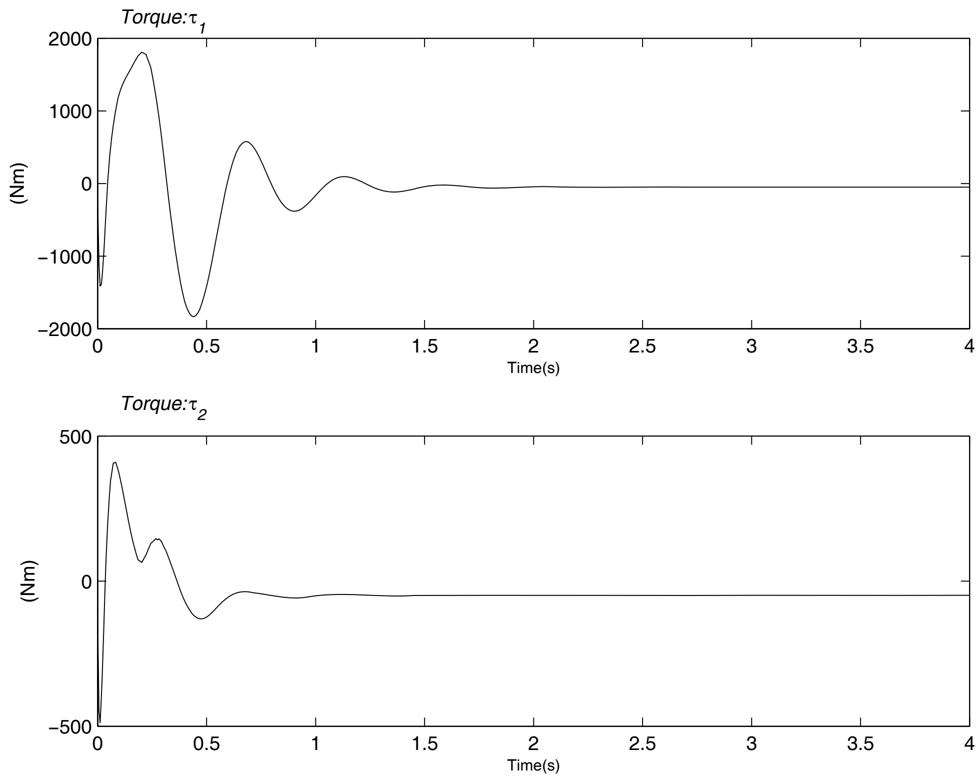

Fig. 7. Applied control signals.

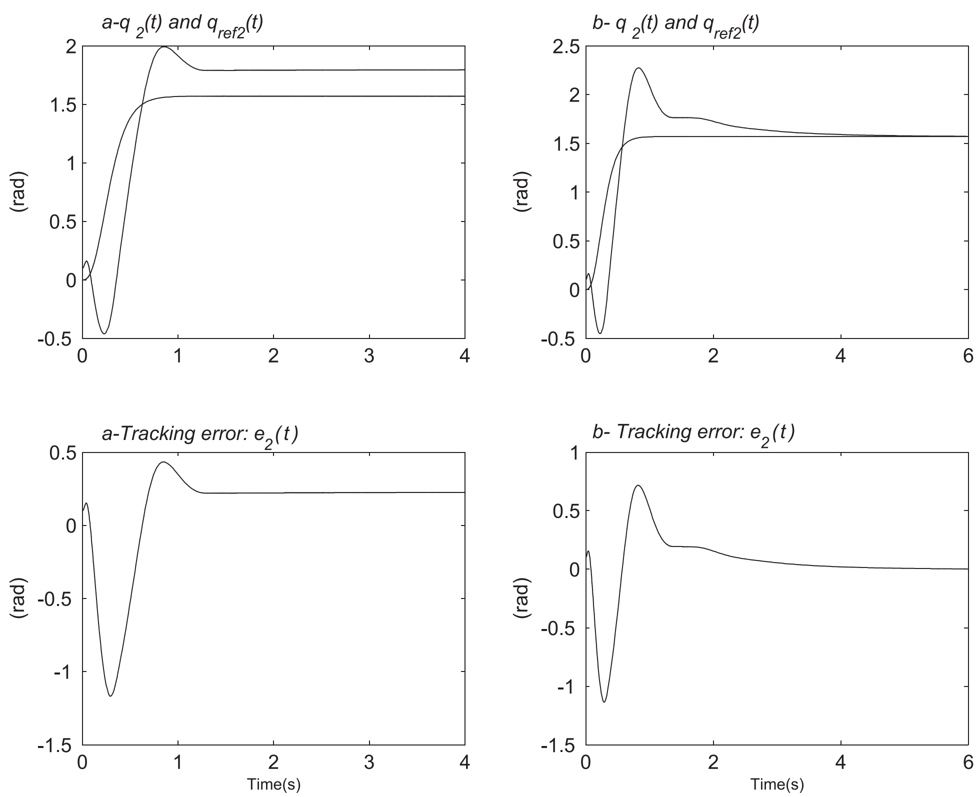

Fig. 8. Performance in the mismatched case.

According to (Gauthier *et al.*, 1992), a Lyapunov analysis shows that for a real positive factor θ and the uniform observability assumption of the nonlinear system (32), the solution (34) guarantees an exponential decay of the observation error.

The reference models chosen in continuous time are

$$\boldsymbol{x}_{\mathrm{ref1}} = \begin{vmatrix} \boldsymbol{q}_{\mathrm{ref1}} \\ \boldsymbol{q}_{\mathrm{ref2}} \end{vmatrix} = \begin{vmatrix} \boldsymbol{x}_{\mathrm{ref1}} \\ \boldsymbol{x}_{\mathrm{ref2}} \end{vmatrix}.$$

They are smoothed by means of second-order polynomials that are respectively given by

$$q_{\mathrm{ref1}}(s) = \frac{\omega_1^2}{s^2 + 2\xi\omega_1 s + \omega_1^2}\, r_1(s)$$

and

$$q_{\mathrm{ref2}}(s) = \frac{\omega_2^2}{s^2 + 2\xi\omega_2 s + \omega_2^2}\, r_2(s).$$

The nonlinear predictive controller (11) is used to force the joint positions to track the desired trajectory (Lee *et al.*, 1997):

$$r_1(t) = r_2(t) = 1.5(1 - \exp(-5t)(1+5t)) \text{ [rad]}.$$

For this simulation, the parameter values of the two reference models are chosen as $\xi = 1$ and $\omega_1 = \omega_2 = 10$ and the initial conditions are

$$\begin{aligned} \boldsymbol{x}(0) &= \begin{bmatrix} q_1(0) & q_2(0) & \dot{q}_1(0) & \dot{q}_2(0) \end{bmatrix} \\ &= \begin{bmatrix} 0.1 & 0.1 & 0 & 0 \end{bmatrix}, \\ \hat{\boldsymbol{x}}(0) &= \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Note that the initial estimation errors are different from zero. Thus, with the proposed feedback nonlinear predictive controller, one does not need to constrain the initial estimation errors in the joint position to be zero to ensure the convergence of the tracking error to zero as in (Canudas De Wit *et al.*, 1992).

The finite horizon predictive controller (11) has been tested by simulation with the following control parameters: $\mathbf{Q}_1 = 2\ 10^2\mathbf{I}_2$, $\mathbf{Q}_2 = 2\ 10^2\mathbf{I}_2$, $\mathbf{R} = 10^{-8}\mathbf{I}_2$ and $h = 0.01$. The resulting position and speed tracking error are depicted in Figs. 4 and 5. The behaviour of the state $\boldsymbol{x}(t)$ is close to the reference trajectory $\boldsymbol{x}_{\mathrm{ref}}(t)$. Observing these results, the state $\boldsymbol{x}(t)$ tracks tightly the reference trajectories $\boldsymbol{x}_{\mathrm{ref}}(t)$. Figure 6 displays the observation tracking error achieved by Gauthier's observer. Figure 7 illustrates the torque signals applied to the robot manipulator, which are inside the saturation limits (Lee *et al.*, 1997; Spong *et al.*, 1992).

In the mismatched case, the uncertainties used are

- Parameter variations in Link 2 due to an unknown load are $\Delta m_2 = 5\,\mathrm{kg}$; $\Delta l_2 = 0.3\,\mathrm{m}$ and $\Delta I_2 = 1/6$.
- The friction is added to the robot manipulator (31) with the values $f_1 = f_2 = 10\,\mathrm{Nm}$ and $\overline{f}_1 = \overline{f}_2 = 10\,\mathrm{N/ms^2}$.

Case (a) in Fig. 8 shows the tracking performance of Link 2 without integral action. It observed that the uncertainties induce a short steady-state error in the second link position $q_2(t)$, and this was expected by the robustness analysis. When the integral action is introduced in the control loop with $\mathbf{Q}_0 = 10^4\mathbf{I}_2$, it is observed from the same figure (Case (b)) that the position reference trajectory is closely tracked and the tracking error converges towards the origin. Thus the torque frictions and parameters uncertainties have no effect on the state tracking error.

4. Conclusion

In this paper, a finite-horizon nonlinear predictive controller using the Taylor approximation is presented and applied to two kinds of nonlinear systems. Minimizing a quadratic cost function of the predicted tracking error and the control input, we derived the control law. One of the main advantages of these control schemes is that they do not require on-line optimization and asymptotic tracking of the smooth reference signal is guaranteed. The stability was shown by using the Lyapunov method. According to a suitable choice of the control parameters, we showed that all variables of the state tracking error are bounded. The boundedness can be made small by reducing the penalty on the control torque signal. Moreover, to increase the robustness of the proposed scheme to variations and uncertainties in parameters, an integral action was incorporated into the loop. The proposed controllers are applied to both the planning motion problem of a mobile robot under nonholomonic constraints and the problem of tracking trajectories of a rigid link robot manipulator. Finally, we expect that the results presented here can be explored and extended to a discrete implementation of these continuous-time predictive controllers through either computers or specialized chips that can run at a higher speed.

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