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# A FIXED POINT METHOD IN QUASISTATIC RATE–TYPE VISCOPLASTICITY

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Two initial and boundary value problems describing the quasistatic evolution of semilinear rate–type viscoplastic models with/without internal state variables are considered. The existence and uniqueness of the solution is proved using only classical existence and uniqueness results for linear elasticity followed by a fixed point technique.

## 1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^N (N = 1, 2, 3)$ with a smooth boundary $\partial\Omega = \Gamma$ and let $\Gamma_1$ be an open subset of $\Gamma$ such that *meas* $\Gamma_1 > 0$. We denote by $\Gamma_2 = \Gamma - \overline{\Gamma}_1$, $\nu$ the outward unit normal vector on $\Gamma$ and by $S_N$ the set of second order symmetric tensors on $\mathbb{R}^N$. Let $T$ be a real positive constant. We consider the following mixed problem:

$$\dot{\sigma} = \mathcal{E}\varepsilon(\dot{u}) + F(\sigma, \varepsilon(u)) \quad \text{in } \Omega \times (0, T) \tag{1}$$

$$\text{Div}\,\sigma + f = 0 \quad \text{in } \Omega \times (0, T) \tag{2}$$

$$u = g \quad \text{on } \Gamma_1 \times (0, T) \tag{3}$$

$$\sigma\nu = h \quad \text{on } \Gamma_2 \times (0, T) \tag{4}$$

$$u(0) = u_0, \quad \sigma(0) = \sigma_0 \quad \text{in } \Omega \tag{5}$$

in which the unknowns are the functions $u : \Omega \times [0, T] \to \mathbb{R}^N$ and $\sigma : \Omega \times [0, T] \to S_N$ (for simplicity, in (1)–(5) the independent variables $x \in \overline{\Omega}$ and $t \in [0, T]$ were suppressed).

This problem represents a quasistatic problem for rate–type viscoplastic models of the form (1) in which $\sigma$ is the stress function, $u$ is the displacement function and $\varepsilon(u) : \Omega \times [0, T] \to S_N$ is the small strain tensor (i.e $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla^t u)$). In (1) $\mathcal{E}$ and $F$ are given constitutive functions which may depend on $x \in \Omega$ and, as well as everywhere in this paper, the dot above a quantity represents the derivate with respect to the time variable of that quantity. In (2) Div $\sigma$ represents

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the Divergence of vector–valued function $\sigma$ and $f$ is the given body force; the functions $g$ and $h$ in (3),(4) are the given boundary data and finally the functions $u_0$, $\sigma_0$ in (5) are the initial data.

Viscoplastic models of the form (1) are used in order to model the behaviour of real bodies like rubbers, metals, rocks and so on, for which the plastic rate of deformation depends on a full coupling in stress and strain. Various results and mechanical interpretations concerning models of the form (1) may be found for instance in the work of Cristescu and Suliciu (1982). In the case when $F$ depends only on $\sigma$, existence and uniqueness results for the problem of form (1)–(5) were obtained by Duvaut and Lions (1972), Suquet (1981a; 1981b), Djaoua and Suquet (1984). In the case when a full coupling in stress and strain is involved in $F$, the existence of the solution $(u, \sigma)$ of the problem (1)–(5) was obtained by Ionescu and Sofonea (1988) using Cauchy–Lipchitz arguments.

Let $M$ be a natural number; we also consider the following mixed problem:

$$\dot{\sigma} = \mathcal{E}\varepsilon(\dot{u}) + F(\sigma, \varepsilon(u), \kappa) \quad \text{in } \Omega \times (0,T) \tag{6}$$

$$\dot{\kappa} = \varphi(\sigma, \varepsilon(u), \kappa) \quad \text{in } \Omega \times (0,T) \tag{7}$$

$$\text{Div}\, \sigma + f = 0 \quad \text{in } \Omega \times (0,T) \tag{8}$$

$$u = g \quad \text{on } \Gamma_1 \times (0,T) \tag{9}$$

$$\sigma\nu = h \quad \text{on } \Gamma_2 \times (0,T) \tag{10}$$

$$u(0) = u_0 \ , \quad \sigma(0) = \sigma_0 \ , \quad \kappa(0) = \kappa_0 \quad \text{in } \Omega \tag{11}$$

in which the unknowns are the functions $u : \Omega \times [0,T] \to \mathbb{R}^N$, $\sigma : \Omega \times [0,T] \to S_N$ and $\kappa : \Omega \times [0,T] \to \mathbb{R}^M$ (for simplicity, as in the case of the problem (1)– (5), in (6)–(11) the independent variables $x \in \overline{\Omega}$ and $t \in [0,T]$ were suppressed).
This problem represents a quasistatic problem for rate–type viscoplastic models of the form (6), (7) in which $\kappa$ may be interpreted as an internal state variable and $\mathcal{E}$, $F$, $\varphi$ are given constitutive functions. In (6)–(11) we used similar notations as in the problem (1)–(5): $u$ represents the displacement function, $\sigma$ represents the stress function, $\varepsilon(u)$ denotes the small strain tensor, $f$ is the given body force, $g$ and $h$ are the given boundary data and finally $u_0$, $\sigma_0$, $\kappa_0$ are the initial data.

Viscoplastic models of the form (6), (7) are used in order to model the behaviour of real bodies for which the plastic rate of deformation depends also on an internal state variable. Some of the internal state variables considered by many authors are the plastic strain, a number of tensor variables that takes into account the spatial display of dislocations or the work–hardening of the material. A major and still remaining open problem in viscoplasticity concerns the way of establishing the evolution equation for the internal state variables. Here we suppose that $\kappa$ is a vector–valued function which satisfies (7) where $\varphi$ is a given function.

Concrete examples of viscoplastic models of the form (6), (7) were proposed by Cristescu (1987) for rock–like materials and, for more details in the field, we refer for instance to the book of Cristescu and Suliciu (1982).

Existence and uniqueness results in the study of elastic–viscoplastic materials with internal state variables for different forms of $F$ and $\varphi$ were given by Kratochvil and Necas (1973), John (1974), Laborde (1979), in the case when $F$ does not depend on $\varepsilon$. An existence result concerning the problem (6)–(11) was obtained by Sofonea (1989) using again Cauchy–Lipschitz arguments in a product Hilbert space.

The aim of this paper is to give two new demonstrations for the existence results obtained by Ionescu and Sofonea (1988), Sofonea (1989) in the study of the problem (1)–(5) and respectively in the study of the problem (6)–(11). These demonstrations are based only on classical existence and uniqueness results for linear elasticity followed by a fixed point technique. The paper is structured as follows: section 2 contains the basic notations and some preliminaries on the functional spaces used in the following, section 3 contains the proof of the existence result concerning the problem (1)–(5) (theorem 3.1) and finally section 4 contains the proof of the existence result concerning the problem (6)–(11) (theorem 4.1).

## 2. Notations and Preliminaries

We denote by "." the inner product on the spaces $\mathbb{R}^N$, $\mathbb{R}^M$ and $S_N$ and by $|\cdot|$ the Euclidean norms on these spaces. The following notations are also used:

$$H = \{ \ v = (v_i) \mid v_i \in L^2(\Omega), \ i = \overline{1,N} \ \}$$

$$H_1 = \{ \ v = (v_i) \mid v_i \in H^1(\Omega), \ i = \overline{1,N} \ \}$$

$$\mathcal{H} = \{ \ \tau = (\tau_{ij}) \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega), \ i,j = \overline{1,N} \ \}$$

$$\mathcal{H}_1 = \{ \ \tau = (\tau_{ij}) \mid \text{Div} \ \tau \in H \ \}$$

$$Y = \{ \ \kappa = (\kappa_i) \mid \kappa_i \in L^2(\Omega), \ \ i = \overline{1,M} \ \}$$

The spaces $H$, $H_1$, $\mathcal{H}$, $\mathcal{H}_1$ and $Y$ are real Hilbert spaces endowed with the canonical inner products denoted by $<\cdot,\cdot>_H$, $<\cdot,\cdot>_{H_1}$, $<\cdot,\cdot>_{\mathcal{H}}$, $<\cdot,\cdot>_{\mathcal{H}_1}$ and $<\cdot,\cdot>_Y$ respectively.

Let $H_\Gamma = [H^{\frac{1}{2}}(\Gamma)]^N$ and $\gamma : H_1 \to H_\Gamma$ be a trace map. We denote by

$$V = \{ \ u \in H_1 \mid \gamma u = 0 \ \ \text{on} \ \Gamma_1 \ \}$$

and let $E$ be the subspace of $H_\Gamma$ defined by

$$E = \gamma(V) = \{ \ \xi \in H_\Gamma \mid \xi = 0 \ \ \text{on} \ \Gamma_1 \} \ .$$

The deformation operator $\varepsilon : H_1 \to \mathcal{H}$ defined by:

$$\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla^t u)$$

is a linear and continuous operator. Morever, since *meas* $\Gamma_1 > 0$, Korn's inequality holds:

$$|\varepsilon(v)|_{\mathcal{H}} \geq C|v|_{H_1} \quad \text{for all } \ v \in V \tag{12}$$

where $C$ is a strictly positive constant which depends only on $\Omega$ and $\Gamma_1$. Let $H'_\Gamma = [H^{-\frac{1}{2}}(\Gamma)]^N$ be the strong dual of the space $H_\Gamma$ and let $<\cdot,\cdot>$ denote the duality between $H'_\Gamma$ and $H_\Gamma$. If $\tau \in \mathcal{H}_1$ there exists an element $\gamma_\nu \tau \in H'_\Gamma$ such that:

$$<\gamma_\nu \tau, \gamma v> = <\tau, \varepsilon(v)>_{\mathcal{H}} + <\text{Div } \tau, v>_H \qquad \text{for all } \ v \in H_1 \tag{13}$$

By $\tau\nu|_{\Gamma_2}$ we shall understand the element of $E'$ (the strong dual of $E$) that is the restriction of $\gamma_\nu \tau$ on $E$.

Let us now denote by $\mathcal{V}$ the following subspace of $\mathcal{H}_1$:

$$\mathcal{V} = \{ \ \tau \in \mathcal{H}_1 \ | \ \text{Div } \tau = 0 \ \text{ in } \ \Omega, \ \ \tau\nu = 0 \ \text{ on } \ \Gamma_2 \ \}$$

Using (13) it may be proved that $\varepsilon(V)$ is the orthogonal complement of $\mathcal{V}$ in $\mathcal{H}$, hence

$$<\tau, \varepsilon(v)>_{\mathcal{H}} = 0, \quad \text{for all } \ v \in V, \ \ \tau \in \mathcal{V} \tag{14}$$

Finally, for every real Hilbert space $X$ we denote by $|\cdot|_X$ the norm on $X$ and by $C^j(0,T,X)(j = 0,1)$ the spaces defined as follows:

$$C^0(0,T,X) = \{z : [0,T] \to X \ | \ z \text{ is continuous } \}$$

$$C^1(0,T,X) = \{z : [0,T] \to X \ | \ \text{there exists } \ \dot{z} \ \text{ the derivate of } \ z \ \text{ and } \ \dot{z} \in C^0(0,T,X) \ \}$$

In a similar way the spaces $C^0(\mathbb{R}_+, X)$ and $C^1(\mathbb{R}_+, X)$, where $\mathbb{R}_+ = [0, +\infty)$, can be defined.

Let as recall that $C^j(0,T,X)$ are real Banach spaces endowed with the norms

$$|z|_{0,T,X} = \max_{t \in [0,T]} |z(t)|_X \tag{15}$$

and

$$|z|_{1,T,X} = |z|_{0,T,X} + |\dot{z}|_{0,T,X} \ ,$$

respectively.

## 3. The First Existence and Uniqueness Result

In the study of the problem (1)–(5), we consider the following assumptions (see also Ionescu and Sofonea (1988)):

$$\begin{cases} \mathcal{E} : \Omega \times S_N \to S_N \quad \text{is a symmetric and positive definite tensor, i.e. :} \\ \text{(a) } \mathcal{E}_{ijkh} \in L^\infty(\Omega) \quad \text{for all } i,j,k,h, = \overline{1,N} \\ \text{(b) } \mathcal{E}\sigma \cdot \tau = \sigma \cdot \mathcal{E}\tau \quad \forall \sigma, \tau \in S_N, \text{a.e. in } \Omega \\ \text{(c) there exists } \alpha > 0 \text{ such that } \quad \mathcal{E}\sigma \cdot \sigma \geq \alpha |\sigma|^2 \quad \text{for all } \sigma \in S_N \end{cases} \tag{16}$$

$$\begin{cases} F : \Omega \times S_N \times S_N \to S_N \quad \text{and} \\ \text{(a) there exists } L > 0 \text{ such that} \\ \quad |F(x,\sigma_1,\varepsilon_1) - F(x,\sigma_2,\varepsilon_2)| \leq L(|\sigma_1 - \sigma_2| + |\varepsilon_1 - \varepsilon_2|) \\ \quad \text{for all } \sigma_1,\sigma_2,\varepsilon_1,\varepsilon_2 \in S_N, \text{ a.e. in } \Omega \\ \text{(b) } x \to F(x,\sigma,\varepsilon) \text{ is a measurable function in the} \\ \quad \text{Lebesgue sense, for all } \sigma,\varepsilon \in S_N \\ \text{(c) } x \to F(x,0,0) \in \mathcal{H} \end{cases} \tag{17}$$

$$f \in C^1(0,T,H), \quad g \in C^1(0,T,H_\Gamma), \quad h \in C^1(0,T,E') \tag{18}$$

$$u_0 \in H_1, \quad \sigma_0 \in \mathcal{H}_1 \tag{19}$$

$$\text{Div } \sigma_0 + f(0) = 0 \text{ in } \Omega, \ u_0 = g(0) \text{ on } \Gamma_1, \ \sigma_0\nu = h(0) \text{ on } \Gamma_2 \tag{20}$$

**Remark 3.1.** Using (16) and (17) we obtain that for all $\varepsilon \in \mathcal{H}$, $\sigma \in \mathcal{H}$ the functions $x \to \mathcal{E}(x)\varepsilon(x) \in \mathcal{H}$, $x \to F(x,\sigma(x),\varepsilon(x)) \in \mathcal{H}$ hence we may define the operators $\tilde{\mathcal{E}} : \mathcal{H} \to \mathcal{H}$, $\tilde{F} : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ by the equalities

$$(\tilde{\mathcal{E}}\varepsilon(x) = (\mathcal{E}_{ijkh}(x)\varepsilon(x)), \quad \tilde{F}(\sigma,\varepsilon)(x) = F(x,\sigma(x),\varepsilon(x)) \quad \text{a.e. } x \in \Omega, \varepsilon \in \mathcal{H}, \ \sigma \in \mathcal{H}$$

For simplicity, we use in the following the notations $\mathcal{E}$ and $F$ instead of $\tilde{\mathcal{E}}$ and $\tilde{F}$, respectively.

The main result of this section is the following:

**Theorem 3.1.** *Let* (16)–(20) *hold. Then there exists a unique solution* $u \in C^1(0,T,H_1)$, $\sigma \in C^1(0,T,\mathcal{H}_1)$ *of the problem* (1)–(5).

*Proof.* We start by the existence part. Let $\eta \in C^0(0,T,\mathcal{H})$ and let $z_\eta \in C^1(0,T,\mathcal{H})$ be the function defined by:

$$z_\eta(t) = \int_0^t \eta(s)\mathrm{d}s + z_0 \qquad \text{for all } t \in [0,T] \tag{21}$$

where

$$z_0 = \sigma_0 - \mathcal{E}\varepsilon(u_0) \tag{22}$$

Using standard arguments of linear elasticity, we obtain the existence and uniqueness of two functions $u_\eta \in C^1(0,T,H_1)$, $\sigma_\eta \in C^1(0,T,\mathcal{H}_1)$ such that:

$$\sigma_\eta = \mathcal{E}\varepsilon(u_\eta) + z_\eta \quad \text{in} \quad \Omega \times (0,T) \tag{23}$$

$$\text{Div}\, \sigma_\eta + f = 0 \quad \text{in} \quad \Omega \times (0,T) \tag{24}$$

$$u_\eta = g \quad \text{on} \quad \Gamma_1 \times (0,T) \tag{25}$$

$$\sigma_\eta \nu = h \quad \text{on} \quad \Gamma_2 \times (0,T) \tag{26}$$

Morever, by (19), (20), (24)–(26), we have $u_\eta(0) - u_0 \in V$, $\sigma_\eta(0) - \sigma_0 \in \mathcal{V}$ and by (21)–(23), it results:

$$\sigma_\eta(0) - \sigma_0 = \mathcal{E}\varepsilon(u_\eta(0)) - \mathcal{E}\varepsilon(u_0) \quad \text{in } \Omega$$

Using now (14), (16) and (12), it follows:

$$u_\eta(0) = u_0, \quad \sigma_\eta(0) = \sigma_0 \quad \text{in} \quad \Omega \tag{27}$$

Since by (17), we obtain that $t \to F(\sigma_\eta(t), \varepsilon(u_\eta(t)))$ is a continuous function on $[0,T]$ with values in $\mathcal{H}$, we can define the operator $\Lambda : C^0(0,T,\mathcal{H}) \to C^0(0,T,\mathcal{H})$ in the following way:

$$\Lambda\eta(t) = F(\sigma_\eta(t), \varepsilon(u_\eta(t))) \quad \text{for all} \quad t \in [0,T] \tag{28}$$

We shall prove that $\Lambda$ has a unique fixed point. Indeed, let $\eta_1, \eta_2 \in C^0(0,T,\mathcal{H})$; for simplicity we denote: $z_{\eta_1} = z_1$ , $z_{\eta_2} = z_2$, $u_{\eta_1} = u_1$, $u_{\eta_2} = u_2$, $\sigma_{\eta_1} = \sigma_1$, $\sigma_{\eta_2} = \sigma_2$. Using (24)–(26), we have $u_1 - u_2 \in V$, $\sigma_1 - \sigma_2 \in \mathcal{V}$ and by (23) it results

$$\sigma_1 - \sigma_2 = \mathcal{E}\varepsilon(u_1) - \mathcal{E}\varepsilon(u_2) + z_1 - z_2 \quad \text{in} \quad \Omega \times (0,T)$$

Using again (14), (16) and (12) it follows

$$|u_1(t) - u_2(t)|_{H_1} + |\sigma_1(t) - \sigma_2(t)|_{\mathcal{H}} \le C|z_1(t) - z_2(t)|_{\mathcal{H}} \quad \text{for all} \quad t \in [0,T] \tag{29}$$

where $C > 0$ depends only on $\Omega$, $\Gamma_1$ and $\mathcal{E}$. Using now (28), (17), (29) and (21) we get

$$|\Lambda\eta_1(t) - \Lambda\eta_2(t)|_{\mathcal{H}} \le CL \int_0^t |\eta_1(s) - \eta_2(s)|_{\mathcal{H}} \mathrm{d}s \quad \text{for all} \quad t \in [0,T] \tag{30}$$

By recurrence, denoting by $\Lambda^p$ the powers of the operator $\Lambda$, (30) implies

$$|\Lambda^p\eta_1(t) - \Lambda^p\eta_2(t)|_{\mathcal{H}} \le (CL)^p \underbrace{\int_0^t \int_0^s \cdots \int_0^q}_{p \text{ integrals}} |\eta_1(r) - \eta_2(r)|_H \mathrm{d}r \ldots \mathrm{d}s$$

for all $t \in [0,T]$ and $p \in \mathbb{N}$. Using (15) we obtain

$$|\Lambda^p\eta_1 - \Lambda^p\eta_2|_{0,T,H} \le \frac{(CLT)^p}{p!} |\eta_1 - \eta_2|_{0,T,H} \quad \text{for all} \quad p \in \mathbb{N} \tag{31}$$

and since $\lim_{p\to\infty} \dfrac{(CLT)^p}{p!} = 0$, (31) implies that for $p$ large enough the operator $\Lambda^p$ is a contraction in $C^0(0,T,\mathcal{H})$. Then, there exists a unique $\eta^* \in C^0(0,T,\mathcal{H})$ such that $\Lambda^p\eta^* = \eta^*$. Moreover, $\eta^*$ is the unique fixed point of $\Lambda$. Using now (21), (23)–(28) we obtain that $u_{\eta^*} \in C^1(0,T,H_1)$, $\sigma_{\eta^*} \in C^1(0,T,\mathcal{H}_1)$ is the solution of (1)–(5).

In order to prove the uniqueness part, let $(u_{\eta^*}, \sigma_{\eta^*})$ be the solution of (1)–(5) obtained above and let $(u,\sigma)$ be another solution of (1)–(5) having the same regularity, i.e $u \in C^1(0,T,H_1)$, $\sigma \in C^1(0,T,\mathcal{H}_1)$. We denote by $\eta \in C^0(0,T,\mathcal{H})$ the function defined by:

$$\eta(t) = F(\sigma(t), \varepsilon(u(t))) \quad \text{for all } t \in [0,T] \tag{32}$$

and let $z_\eta \in C^1(0,T,\mathcal{H})$ be defined by (21), (22). Since from (1)–(5) it results that $(u,\sigma)$ satisfy (23)–(26) and this problem has a unique solution $u_\eta \in C^1(0,T,H)$, $\sigma_\eta \in C^1(0,T,\mathcal{H}_1)$ it results

$$u = u_\eta, \quad \sigma = \sigma_\eta. \tag{33}$$

Using now (28), (33) and (32) we get $\Lambda\eta = \eta$ and by the uniqueness of the fixed point of $\Lambda$ it results:

$$\eta = \eta^* \tag{34}$$

The uniqueness part of theorem 3.1 is now a consequence of (33), (34).

**Remark 3.2.** Problem (1)–(5) may also be considered in the case of the infinite time interval $(0,+\infty)$ instead of $(0,T)$. In this case, if (16), (17), (19), (20) are fullfiled and

$$f \in C^1(\mathbb{R}_+, H), \quad g \in C^1(\mathbb{R}_+, H_\Gamma), \quad h \in C^1(\mathbb{R}_+, E'), \tag{35}$$

problem (1)–(5) has a unique solution $(u,\sigma)$ having the regularity $u \in C^1(\mathbb{R}_+, H_1)$, $\sigma \in C^1(\mathbb{R}_+, \mathcal{H}_1)$.

Indeed, with minor adjustments, we consider the operator $\Lambda : X \to X$ given by (28) where

$$X = \{\ \eta \in C^0(\mathbb{R}_+, \mathcal{H}) \mid \sup_{t\geq 0}\ e^{-kt}|\eta(t)|_{\mathcal{H}} < +\infty\ \}$$

and $k > 0$. $X$ is a Banach space for the norm

$$|\eta|_X = \sup_{t\geq 0}\ e^{-kt}|\eta(t)|_{\mathcal{H}}$$

and by (30) we get

$$|\Lambda\eta_1 - \Lambda\eta_2|_X \leq \frac{CL}{k}|\eta_1 - \eta_2|_X \quad \text{for all } \eta_1, \eta_2 \in X$$

Taking now $k > CL$ we get that $\Lambda$ is a contraction on $X$. Further on the same arguments as in the poof of theorem 4.1 can be used.

## 4. The Second Existence and Uniqueness Result

In the study of the problem (6)–(11), we consider the following assumptions (see also Sofonea (1989)):

$$\begin{cases} F : \Omega \times S_N \times S_N \times \mathbb{R}^M \to S_N \ \text{ and} \\ \text{(a) there exists } L > 0 \text{ such that } |F(x,\sigma_1,\varepsilon_1,\kappa_1) - F(x,\sigma_2,\varepsilon_2,\kappa_2)| \leq \\ \quad \leq L(|\sigma_1-\sigma_2| + |\varepsilon_1-\varepsilon_2| + |\kappa_1-\kappa_2|) \text{ for all } \sigma_1,\sigma_2,\varepsilon_1,\varepsilon_2 \in S_N, \\ \quad \kappa_1,\kappa_2 \in \mathbb{R}^M, \text{ a.e. in } \Omega \\ \text{(b) } x \to F(x,\sigma,\varepsilon,\kappa) \text{ is a measurable function in the} \\ \quad \text{Lebesgue sense, for all } \sigma,\varepsilon \in S_N, \kappa \in \mathbb{R}^M \\ \text{(c) } x \to F(x,0,0,0) \in \mathcal{H} \end{cases} \tag{36}$$

$$\begin{cases} \varphi : \Omega \times S_N \times S_N \times \mathbb{R}^M \to \mathbb{R}^M \ \text{ and} \\ \text{(a) there exists } L' > 0 \text{ such that } |\varphi(x,\sigma_1,\varepsilon_1,\kappa_1) - \varphi(x,\sigma_2,\varepsilon_2,\kappa_2)| \leq \\ \quad \leq L'(|\sigma_1-\sigma_2| + |\varepsilon_1-\varepsilon_2| + |\kappa_1-\kappa_2|) \text{ for all } \sigma_1,\sigma_2,\varepsilon_1,\varepsilon_2 \in S_N, \\ \quad \kappa_1,\kappa_2 \in \mathbb{R}^M, \text{ a.e. in } \Omega \\ \text{(b) } x \to \varphi(x,\sigma,\varepsilon,\kappa) \text{ is a measurable function in the} \\ \quad \text{Lebesgue sense, for all } \sigma,\varepsilon \in S_N, \kappa \in \mathbb{R}^M \\ \text{(c) } x \to \varphi(x,0,0,0) \in Y \end{cases} \tag{37}$$

$$\kappa_0 \in Y \tag{38}$$

**Remark 4.1.** Using (16), (36), (37) and similar arguments as in Remark 3.1 in the following we shall consider $\mathcal{E} : \mathcal{H} \to \mathcal{H}$, $F : \mathcal{H} \times \mathcal{H} \times Y \to \mathcal{H}$ and $\varphi : \mathcal{H} \times \mathcal{H} \times Y \to Y$.

The main result of this section is the following:

**Theorem 4.1.** *Let* (16), (18)–(20), (36)–(38) *hold. Then, there exists a unique solution* $u \in C^1(0,T,H_1)$, $\sigma \in C^1(0,T,\mathcal{H}_1)$, $\kappa \in C^1(0,T,Y)$ *of the problem* (6)–(11).

*Proof.* We use a similar technique as in the proof of Theorem 3.1, hence we start with the existence part. Let $X$ be the product Hilbert space $X = \mathcal{H} \times Y$ and let $\eta = (\eta^1,\eta^2) \in C^0(0,T,X)$. We define the function $z_\eta = (z_\eta^1, z_\eta^2) \in C^1(0,T,X)$ by:

$$z_\eta(t) = \int_0^t \eta(s)\mathrm{d}s + z_0 \quad \text{for all } t \in [0,T] \tag{39}$$

where

$$z_0 = (\sigma_0 - \mathcal{E}\varepsilon(u_0),\ \kappa_0) \tag{40}$$

Using standard arguments of linear elasticity we obtain the existence and uniqueness of two functions $u_\eta \in C^1(0,T,H_1)$, $\sigma_\eta \in C^1(0,T,\mathcal{H}_1)$ such that:

$$\sigma_\eta = \mathcal{E}\varepsilon(u_\eta) + z_\eta^1 \quad \text{in } \Omega \times (0,T) \tag{41}$$

$$\text{Div } \sigma_\eta + f = 0 \quad \text{in } \Omega \times (0,T) \tag{42}$$

$$u_\eta = g \quad \text{on } \Gamma_1 \times (0,T) \tag{43}$$

$$\sigma_\eta \nu = h \quad \text{on } \Gamma_2 \times (0,T) \tag{44}$$

and, using the same method as in the proof of Theorem 3.1, we have

$$u_\eta(0) = u_0, \quad \sigma_\eta(0) = \sigma_0 \quad \text{in } \Omega. \tag{45}$$

Let $\kappa_\eta \in C^1(0,T,Y)$ be the function defined by:

$$\kappa_\eta = z_\eta^2 \tag{46}$$

Using (36) and (37) we obtain that

$$t \to \big(F(\sigma_\eta(t), \varepsilon(u_\eta(t)), \kappa_\eta(t)),\ \varphi(\sigma_\eta(t), \varepsilon(u_\eta(t)), \kappa_\eta(t))\big)$$

is a continuous function on $[0,T]$ with values in $X$ hence we can define the operator $\Lambda : C^0(0,T,X) \to C^0(0,T,X)$ in the following way:

$$\Lambda\eta(t) = \big(F(\sigma_\eta(t), \varepsilon(u_\eta(t)), \kappa_\eta(t)),\ \varphi(\sigma_\eta(t), \varepsilon(u_\eta(t)), \kappa_\eta(t))\big) \quad \text{for all } t \in [0,T] \tag{47}$$

We shall prove that $\Lambda$ has a unique fixed point. Indeed, let $\eta_1 = (\eta_1^1, \eta_1^2), \eta_2 = (\eta_2^1, \eta_2^2) \in C^0(0,T,X)$; for simplicity, we denote: $z_{\eta_1} = z_1$, $z_{\eta_2} = z_2$, $u_{\eta_1} = u_1$, $u_{\eta_2} = u_2$, $\sigma_{\eta_1} = \sigma_1$, $\sigma_{\eta_2} = \sigma_2$, $\kappa_{\eta_1} = \kappa_1$, $\kappa_{\eta_2} = \kappa_2$. From (41)–(44) and (46) we get

$$\begin{cases} |u_1(t) - u_2(t)|_{H_1} + |\sigma_1(t) - \sigma_2(t)|_{\mathcal{H}} + |\kappa_1(t) - \kappa_2(t)|_Y \leq \\ \leq C|z_1(t) - z_2(t)|_X \qquad \text{for all } t \in [0,T] \end{cases} \tag{48}$$

where $C > 0$ depends only on $\Omega$, $\Gamma_1$ and $\mathcal{E}$. Using now (47), (36), (37), (48) and (39) we get

$$|\Lambda\eta_1(t) - \Lambda\eta_2(t)|_X \le \tilde{C}\int_0^t |\eta_1(s) - \eta_2(s)|_H \mathrm{d}s \qquad \text{for all } t \in [0,T] \quad (49)$$

where $\tilde{C}$ depends only on $\Omega$, $\Gamma_1$, $\mathcal{E}$, $F$ and $\varphi$.

Denoting now by $\Lambda^p$ the powers of the operator $\Lambda$, from (49) we get that for $p$ large enough the operator $\Lambda^p$ is a contraction in the space $C^0(0,T,X)$. In consequence, there exists a unique element $\eta^* \in C^0(0,T,X)$ such that $\Lambda^p\eta^* = \eta^*$ and morever $\eta^*$ is the unique fixed point of $\Lambda$. Using now (39)–(46) we obtain that $u_{\eta^*} \in C^1(0,T,H_1)$, $\sigma_{\eta^*} \in C^1(0,T,\mathcal{H}_1)$, $\kappa_{\eta^*} \in C^1(0,T,\mathcal{H}_1)$ is the solution of (6)–(11).

The uniqueness part of theorem follows from the uniqueness of the fixed point of $\Lambda$ using the same technique as in the proof of Theorem 3.1 or by standard arguments for evolution equations.

**Remark 4.2.** Problem (6)–(11) may also be considered in the case of the infinite time interval $(0,+\infty)$ instead of $(0,T)$. Similar arguments as in Remark 3.2. can be used in order to prove that if (16), (19), (20), (35), (36)–(38) hold, then this problem has a unique solution $u \in C^1(\mathbb{R}_+,H_1)$, $\sigma \in C^1(\mathbb{R}_+,\mathcal{H}_1)$, $\kappa \in C^1(\mathbb{R}_+,Y)$.

## 5. Conclusion

The paper deals with quasistatic processes for rate–type viscoplastic models used in order to model the behaviour of real bodies like rubbers, metals, rocks and so on, for which the plastic rate of deformation may depend on an internal state variable.Two new demonstrations concerning the existence and the uniqueness of the solution are presented.These demonstrations are based only on standard techniques for elliptic problems followed by a fixed point technique and they avoid any monotony or Cauchy–Lipschitz arguments.

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