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SOME PROPERTIES OF THE SPECTRAL RADIUS OF A SET OF MATRICES

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In this paper we show new formulas for the spectral radius and the spectral subradius of a set of matrices. The advantage of our results is that we express the spectral radius of any set of matrices by the spectral radius of a set of symmetric positive definite matrices. In particular, in one of our formulas the spectral radius is expressed by singular eigenvalues of matrices, whereas in the existing results it is expressed by eigenvalues.

Keywords: spectral radius, spectral subradius, symmetric matrices

1. Introduction

The idea of the spectral radius of a set of matrices was introduced in the seminal paper (Rota and Strang, 1960). For two square matrices A and B, the authors defined

$$\rho(A,B) = \overline{\lim_{n\to\infty}} \Big[\text{largest norm of any product with } n \text{ factors}\Big]^{\frac{1}{n}}.$$

The product can have As and Bs in any order (and the $\overline{\lim}_{n\to\infty}$ is actually a limit). For a single matrix it equals the largest magnitude of the eigenvalues. But the products of A and B can produce norms and eigenvalues that are very hard to estimate (as n increases) from the two matrices. The Lyapunov exponent is a similar number, using averages over products of the length n instead of maxima, and it suffers from the same difficulty in actual computation. The definitions extend directly to sets of more than two matrices, and an l_p norm joint spectral radius has also proved useful (Jia, 1995). For a long time the generalized spectral radius has not found applications. However, Daubechies and Lagarias (1992a; 1992b) pointed an application in solving the key equation in wavelet theory, i.e., the refinement equation (or the dilation equation) for the scaling function. In (Michelli and Prautzsch, 1989), the idea of the spectral radius of a set of matrices was used in the subdivision algorithm for computer aided design.

Further applications of the generalized spectral radius were possible due the great result of Berger and Wang (1992), which provides the option of estimating ρ from the eigenvalues of the products, instead of their norms. A simpler proof of this fact was given by Elsner (1995). The absolute value of eigenvalues approaches from below and the norms from above. Further applications were found in the stability theory of time varying linear systems and linear inclusions (Czornik, 2005; Gurvits, 1995; Shih, 1999). The latter application is explained by Theorem 1 below.

Let Σ denote a nonempty set of real $l \times l$ matrices. For $m \geq 1$, Σ^m is the set of all products of matrices in Σ of the length m,

$$\Sigma^m = \Big\{A_1 A_2 \dots A_m : A_i \in \Sigma, i = 1, \dots, m\Big\}.$$

Denote by $\rho(A)$ the spectral radius and by $\|A\|$ a matrix norm of the matrix A. By the matrix norm we understand a norm that satisfies the submultiplicative property, i.e., $\|AB\| \leq \|A\| \, \|B\|$. The common spectral radius is defined as

$$\hat{\rho}(\Sigma) = \overline{\lim_{n\to\infty}} \Big[\sup\big\{\rho(A) : A \in \Sigma^n\big\}\Big]^{\frac{1}{n}}, \tag{1}$$

and the generalized spectral radius as

$$\widetilde{\rho}(\Sigma) = \overline{\lim_{n\to\infty}} \Big[\sup\big\{\|A\| : A \in \Sigma^n\big\}\Big]^{\frac{1}{n}}. \tag{2}$$

In (Berger and Yang, 1992; Elsner, 1995), it was shown that for a bounded set Σ the limit in (2) exists and we have

$$\hat{\rho}(\Sigma) = \widetilde{\rho}(\Sigma) =: \rho(\Sigma),$$

from which it follows that

$$\rho(\Sigma) = \inf_{n\in N} \sup_{A\in\Sigma^n} \|A\|^{\frac{1}{n}} = \sup_{n\in N} \sup_{A\in\Sigma^n} \rho(A)^{\frac{1}{n}}.$$

For a bounded set Σ, the common value of $\hat{\rho}(\Sigma)$, $\widetilde{\rho}(\Sigma)$ is called the spectral radius of Σ.

In (Gurvits, 1995), the ideas of a joint spectral subradius and a generalized spectral subradius of the set of matrices were introduced. They were further investigated in

(Czornik, 2005). The results were used in (Gurvits, 1995) to present conditions for the Markov asymptotic stability of a discrete linear inclusion. The definitions are as follows: The common spectral subradius is defined as

$$\underline{\hat{\rho}}(\Sigma) = \varliminf_{n\to\infty} \left[\sup\left\{\rho(A) : A \in \Sigma^n\right\}\right]^{\frac{1}{n}}, \tag{3}$$

and the generalized spectral subradius as

$$\underline{\tilde{\rho}}(\Sigma) = \varliminf_{n\to\infty} \left[\sup\left\{\|A\| : A \in \Sigma^n\right\}\right]^{\frac{1}{n}}.$$

In (Czornik, 2005), it was shown that for any nonempty set Σ we have

$$\underline{\tilde{\rho}}(\Sigma) = \inf_{n\in\mathbb{N}} \inf_{A\in\Sigma^n} \|A\|^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} \inf_{A\in\Sigma^n} \rho(A)^{\frac{1}{n}} = \underline{\hat{\rho}}(\Sigma).$$

The common value of $\underline{\hat{\rho}}(\Sigma)$ and $\underline{\tilde{\rho}}(\Sigma)$ is called the spectral subradius of Σ and it is denoted by $\underline{\rho}(\Sigma)$.

The relationship between the generalized spectral radii and the stability of discrete time-varying linear systems is explained by the following theorem (the proof can by found in (Czornik, 2005; Gurvits, 1995)):

Theorem 1. *Consider a discrete time-varying linear system*

$$x(t+1) = d(t)x(t), \quad x(0) = x_0,$$

where d is a sequence of matrices taken from Σ. Then

1. *for any sequence d and any $x_0 \in \mathbb{R}^l$ we have $\lim_{t\to\infty} x(t) = 0$ if and only if $\rho(\Sigma) < 1$,*
2. *there exists a sequence d such that for any $x_0 \in \mathbb{R}^l$ we have $\lim_{t\to\infty} x(t) = 0$ if and only if $\underline{\rho}(\Sigma) < 1$.*

The purpose of this paper is to present new formulas for the spectral radius and subradius of a set of matrices. In those formulas, the spectral radius is expressed by singular eigenvalues of matrices, unlike in (1), where we have to compute eigenvalues. It is well known that computing singular eigenvalues is much simpler and, for some algorithms, faster than computing eigenvalues. Therefore, our results can be used to simplify numerical algorithms for calculating the estimates of the spectral radius of a set of matrices. This is demonstrated by examples.

2. Main Results

The main idea of this paper is to express the spectral radius and subradius of a set of any matrices with a set of symmetric nonnegative definite matrices.

Write

$$\Sigma^n = \left\{ \prod_{i=1}^{n} A_i : A_i \in \Sigma \right\}$$

and

$$\Sigma_s^n = \left\{ A \cdot A^T : A \in \Sigma^n \right\}.$$

Now we formulate our first main result.

Theorem 2. *For any nonempty and bounded set Σ of real $l \times l$ matrices, we have*

$$\rho(\Sigma) = \inf_{n\in\mathbb{N}} \sup_{A\in\Sigma_s^n} \|A\|^{\frac{1}{2n}}. \tag{4}$$

The idea of the proof of the above theorem rests on the fact that the value of (4) does not depend on the choice of the matrix norm (Gripenberg, 1996). The proof of that theorem uses some properties of special cases of matrix norms and the properties of the suprema and infima of some sets. Let us start with the definitions of norms used in the lemmas and the proof of Theorem 2.

Define the following Euclidean vector norm:

$$\|x\|_w = \sqrt{\sum_{i=1}^{n} |x_i|^2},$$

and the matrix norms (Golub and Loan, 1996):

$$\|A\|_1 = \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|,$$

$$\|A\|_2 = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|^2},$$

$$\|A\|_3 = \max_i \sum_{j=1}^{n} |a_{ij}|,$$

$$\|A\|_4 = \max_j \sum_{i=1}^{n} |a_{ij}|,$$

$$\|A\|_5 = \sup_{x\neq 0} \frac{\|Ax\|_w}{\|x\|_w}.$$

In (Golub and Loan, 1996), it was shown that there is a simple relation between the norms $\|\cdot\|_1$ and $\|\cdot\|_2$, namely, for any matrix A we have $\sqrt{\|AA^T\|_1} \geq \|A\|_2$.

We will also use the following lemma that connects the spectral radius of a matrix with some special case of the matrix norm. The proof of Parts 1, 2 and 3 of the lemma can be found in (Golub and Loan, 1996; Guglielmi and Zennaro, 2001; Horn and Johnson, 1985), respectively.

Lemma 1. *For any matrix A, we have*

1. $\|A^T\|_5^2 = \rho(AA^T)$,
2. $\rho(A) = \inf_{\|\cdot\|} \|A\|$,

3. $\rho(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}$.

Now we are ready to prove Theorem 2. We have

$$\rho(\Sigma) = \inf_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \|A\|^{\frac{1}{n}}.$$

It is well known that the above expression does not depend on the choice of the matrix norm (Gripenberg, 1996). Thus

$$\inf_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \|A\|^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \|A\|_2^{\frac{1}{n}}.$$

Now we can write

$$\inf_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \|A\|_2^{\frac{1}{n}} \le \inf_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \left(\sqrt{\|AA^T\|_1}\right)^{\frac{1}{n}}$$
$$= \inf_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \|AA^T\|_1^{\frac{1}{2n}}.$$

Using the definition of the set Σ_s^n, we can write

$$\inf_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \|AA^T\|_1^{\frac{1}{2n}} = \inf_{n\in\mathbb{N}} \sup_{A\in\Sigma_S^n} \|A\|_1^{\frac{1}{2n}},$$

and because the value of the above does not depend on the choice of the matrix norm, we get

$$\inf_{n\in\mathbb{N}} \sup_{A\in\Sigma_S^n} \|A\|_1^{\frac{1}{2n}} = \inf_{n\in\mathbb{N}} \sup_{A\in\Sigma_S^n} \|A\|^{\frac{1}{2n}}.$$

Thus,

$$\rho(\Sigma) \le \inf_{n\in\mathbb{N}} \sup_{A\in\Sigma_S^n} \|A\|^{\frac{1}{2n}}. \tag{5}$$

On the other hand,

$$\rho(\Sigma) = \inf_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \|A\|^{\frac{1}{n}},$$

which does not depend on the choice of the matrix norm, either, so

$$\inf_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \|A\|^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \|A\|_3^{\frac{1}{n}}$$
$$= \inf_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \left(\|A\|_3^{\frac{1}{2n}}\right)^2.$$

Therefore,

$$\inf_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \left(\|A\|_3^{\frac{1}{2n}}\right)^2 = \left(\inf_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \|A\|_3^{\frac{1}{2n}}\right)^2$$
$$= \left(\inf_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \|A\|_3^{\frac{1}{2n}}\right)\left(\inf_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \|A\|_3^{\frac{1}{2n}}\right).$$

The norm $\|\cdot\|_3$ is defined as the maximum of the sums of absolute values of elements in individual rows of some matrix. The norm $\|\cdot\|_4$ is defined as the maximum of the sums of absolute values of elements in individual columns of some matrix. Therefore, it is clear that

$$\|A\|_3 = \|A^T\|_4,$$

and thus we obtain

$$\left(\inf_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \|A\|_3^{\frac{1}{2n}}\right)\left(\inf_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \|A\|_3^{\frac{1}{2n}}\right)$$
$$= \left(\inf_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \|A\|_3^{\frac{1}{2n}}\right)\left(\inf_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \|A^T\|_4^{\frac{1}{2n}}\right).$$

But the value of the expression containing the norm $\|\cdot\|_4$ does not depend on the choice of the matrix norm, whence

$$\left(\inf_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \|A\|_3^{\frac{1}{2n}}\right)\left(\inf_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \|A^T\|_4^{\frac{1}{2n}}\right)$$
$$= \left(\inf_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \|A\|_3^{\frac{1}{2n}}\right)\left(\inf_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \|A^T\|_3^{\frac{1}{2n}}\right).$$

Now we have

$$\left(\inf_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \|A\|_3^{\frac{1}{2n}}\right)\left(\inf_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \|A^T\|_3^{\frac{1}{2n}}\right)$$
$$= \inf_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \left(\|A\|_3^{\frac{1}{2n}} \|A^T\|_3^{\frac{1}{2n}}\right),$$

which implies

$$\inf_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \left(\|A\|_3^{\frac{1}{2n}} \|A^T\|_3^{\frac{1}{2n}}\right) \ge \inf_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \|AA^T\|_3^{\frac{1}{2n}}.$$

Once again, the fact that the value of the above expression does not depend on the choice of the matrix norm yields

$$\inf_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \|AA^T\|_3^{\frac{1}{2n}} = \inf_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \|AA^T\|^{\frac{1}{2n}}.$$

Using the definition of the set Σ_s^n, we get

$$\inf_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \|AA^T\|^{\frac{1}{2n}} = \inf_{n\in\mathbb{N}} \sup_{A\in\Sigma_s^n} \|A\|^{\frac{1}{2n}}.$$

Thus

$$\rho(\Sigma) \ge \inf_{n\in\mathbb{N}} \sup_{A\in\Sigma_s^n} \|A\|^{\frac{1}{2n}}. \tag{6}$$

From (5) and (6), we conclude that

$$\rho(\Sigma) = \inf_{n\in\mathbb{N}} \sup_{A\in\Sigma_s^n} \|A\|^{\frac{1}{2n}}. \tag{7}$$

From (Gripenberg, 1996), we know that

$$\rho(\Sigma) = \sup_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \rho(A)^{\frac{1}{n}}.$$

Now we show that there exists a simple dependence between the spectral radius of the set Σ and a positive symmetric set of matrices that is constructed from the set Σ. ■

Theorem 3. *For any nonempty and bounded set Σ of real $l \times l$ matrices, we have*

$$\rho(\Sigma) = \sup_{n\in\mathbb{N}} \sup_{A\in\Sigma_s^n} \rho(A)^{\frac{1}{2n}}.$$

Proof. By Part (2) of Lemma 1 and the definition of the set Σ_s^n, we have

$$\begin{aligned}\sup_{n\in\mathbb{N}} \sup_{A\in\Sigma_s^n} \rho(A)^{\frac{1}{2n}} &= \sup_{n\in\mathbb{N}} \sup_{A\in\Sigma_s^n} \left(\inf_{\|\cdot\|} \|A\|\right)^{\frac{1}{2n}}\\ &= \sup_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \left(\inf_{\|\cdot\|} \|AA^T\|\right)^{\frac{1}{2n}}.\end{aligned}$$

By the property of the matrix norms we obtain

$$\begin{aligned}&\sup_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \left(\inf_{\|\cdot\|} \|AA^T\|\right)^{\frac{1}{2n}}\\ &\qquad\le \sup_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \left(\inf_{\|\cdot\|} \left(\|A\|\|A^T\|\right)\right)^{\frac{1}{2n}}.\end{aligned}$$

It is then clear that

$$\begin{aligned}&\sup_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \left(\inf_{\|\cdot\|} \left(\|A\|\|A^T\|\right)\right)^{\frac{1}{2n}}\\ &\qquad= \sup_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \left(\inf_{\|\cdot\|} \left(\|A\|^{\frac{1}{2n}}\|A^T\|^{\frac{1}{2n}}\right)\right).\end{aligned}$$

Now, we can write

$$\begin{aligned}&\sup_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \left(\inf_{\|\cdot\|} \left(\|A\|^{\frac{1}{2n}}\|A^T\|^{\frac{1}{2n}}\right)\right)\\ &= \sup_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \left(\inf_{\|\cdot\|} \|A\|^{\frac{1}{2n}}\right) \sup_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \left(\inf_{\|\cdot\|} \|A^T\|^{\frac{1}{2n}}\right).\end{aligned}$$

It is obvious that

$$\rho(A) = \rho(A^T).$$

Hence

$$\inf_{\|\cdot\|} \|A\| = \inf_{\|\cdot\|} \|A^T\|,$$

and therefore

$$\begin{aligned}&\sup_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \left(\inf_{\|\cdot\|} \|A\|^{\frac{1}{2n}}\right) \sup_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \left(\inf_{\|\cdot\|} \|A^T\|^{\frac{1}{2n}}\right)\\ &\quad= \sup_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \left(\inf_{\|\cdot\|} \|A\|^{\frac{1}{2n}}\right) \sup_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \left(\inf_{\|\cdot\|} \|A\|^{\frac{1}{2n}}\right)\\ &\quad= \sup_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \left(\inf_{\|\cdot\|} \|A\|^{\frac{1}{2n}}\right)^2\\ &\quad= \sup_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \inf_{\|\cdot\|} \|A\|^{\frac{1}{n}} = \sup_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \rho(A)^{\frac{1}{n}} = \rho(\Sigma).\end{aligned}$$

Thus,

$$\sup_{n\in\mathbb{N}} \sup_{A\in\Sigma_s^n} \rho(A)^{\frac{1}{2n}} \le \rho(\Sigma). \tag{8}$$

Now we can use Lemma 1 and write

$$\begin{aligned}\sup_{n\in\mathbb{N}} \sup_{A\in\Sigma_s^n} \rho(A)^{\frac{1}{2n}} &= \sup_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \rho(AA^T)^{\frac{1}{2n}}\\ &= \sup_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \left(\|A^T\|_5^2\right)^{\frac{1}{2n}}\\ &= \sup_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \left(\|A^T\|_5\right)^{\frac{1}{n}}\\ &\ge \sup_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \left(\lim_{k\to\infty} \|\left(A^T\right)^k\|_5^{\frac{1}{k}}\right)^{\frac{1}{n}}.\end{aligned}$$

By the definition of the spectral radius of a single matrix, the last expression is equal to

$$\begin{aligned}\sup_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \rho(A^T)^{\frac{1}{n}} &= \sup_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \rho(A^T)^{\frac{1}{n}}\\ &= \sup_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \rho(A)^{\frac{1}{n}} = \rho(\Sigma).\end{aligned}$$

Thus,

$$\sup_{n\in\mathbb{N}} \sup_{A\in\Sigma_s^n} \rho(A)^{\frac{1}{2n}} \ge \rho(\Sigma). \tag{9}$$

From (8) and (9), we conclude that

$$\sup_{n\in\mathbb{N}} \sup_{A\in\Sigma_s^n} \rho(A)^{\frac{1}{2n}} = \rho(\Sigma). \tag{10}$$

■

One can find in the literature two other formulas for the spectral radius of the set Σ:

$$\rho(\Sigma) = \overline{\lim_{n\to\infty}} \sup_{A\in\Sigma^n} \rho(A)^{\frac{1}{n}},$$

and

$$\rho(\Sigma) = \lim_{n\to\infty} \sup_{A\in\Sigma^n} \|A\|^{\frac{1}{n}}.$$

Let us formulate a similar result that will let us express the spectral radius by the spectral radius and norm of a set of symmetric nonnegative matrices.

Theorem 4. *For any nonempty and bounded set $\sum$ of real $l \times l$ matrices, we have*

$$\rho(\Sigma) = \overline{\lim_{n\to\infty}} \sup_{A\in\Sigma_s^n} \rho(A)^{\frac{1}{2n}} = \lim_{n\to\infty} \sup_{A\in\Sigma^n} \|A\|^{\frac{1}{2n}}.$$

This theorem can be proved in the same way as Theorems 1 and 2. Now, we can gather the results of Theorems 2–4 to write

$$\begin{aligned}\inf_{n\in\mathbb{N}} \sup_{A\in\Sigma_s^n} \|A\|^{\frac{1}{2n}} &= \sup_{n\in\mathbb{N}} \sup_{A\in\Sigma_s^n} \rho(A)^{\frac{1}{2n}}\\ &= \overline{\lim_{n\to\infty}} \sup_{A\in\Sigma_s^n} \rho(A)^{\frac{1}{2n}}\\ &= \lim_{n\to\infty} \sup_{A\in\Sigma^n} \|A\|^{\frac{1}{2n}} = \rho(\Sigma).\end{aligned}$$

In much the same way, one can prove the following version of (11) for the spectral subradius:

Theorem 5. *For any nonempty set $\sum$ of real $l \times l$ matrices, we have*

$$\rho(\Sigma) = \inf_{n\in\mathbb{N}} \inf_{A\in\Sigma_s^n} \|A\|^{\frac{1}{2n}} = \inf_{n\in\mathbb{N}} \inf_{A\in\Sigma_s^n} \rho(A)^{\frac{1}{2n}}$$

$$= \varliminf_{n\to\infty} \inf_{A\in\Sigma_s^n} \rho(A)^{\frac{1}{2n}} = \lim_{n\to\infty} \inf_{A\in\Sigma^n} \|A\|^{\frac{1}{2n}}.$$

3. Examples

3.1. Theoretical Example. Consider the finite set

$$\Sigma = \left\{ A = \left[\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right] : a, b \in \mathbb{R}, a \neq 0, b \neq 0, \det A = 1 \right\}.$$

It is easy to show that, for an arbitrary matrix $A \in \Sigma$, we have

$$AA^T = I.$$

For

$$B = \prod_{i=1}^{n} A_i, \qquad (A_i \in \Sigma),$$

we obtain

$$B = \begin{bmatrix} a_n & b_n \\ -b_n & a_n \end{bmatrix}, \qquad (a_n, b_n \in \mathbb{R}, \det B = 1).$$

Thus for an arbitrary product B, we get

$$BB^T = I.$$

Let choose the matrix norm

$$\|A\|_3 = \max_{1\le i\le n} \sum_{j=1}^{2} |a_{ij}|.$$

For all $n \in \mathbb{N}$,

$$\sup_{A\in\Sigma_s^n} \|A\|_3^{\frac{1}{2n}} = \sup_{A\in\Sigma_s^n} \|I\|_3^{\frac{1}{2n}} = 1^{\frac{1}{2n}} = 1.$$

Therefore,

$$\rho(\Sigma) = \inf_{n\in\mathbb{N}} \sup_{A\in\Sigma_s^n} \|A\|_3^{\frac{1}{2n}} = \inf_{n\in\mathbb{N}} 1 = 1.$$

On the other hand,

$$\rho(\Sigma) = \inf_{n\in N} \sup_{A\in\Sigma^n} \|A\|_3^{\frac{1}{n}},$$

but the only thing that we know about the value of $\|A\|_*$ in this case is

$$\|A\|_3 \geq 1.$$

Thus,

$$\sup_{A\in\Sigma^n} \|A\|_3^{1/n} \geq 1,$$

too, and it complicates the process of finding the spectral radius of Σ according to the formula

$$\rho(\Sigma) = \inf_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \|A\|^{\frac{1}{n}}.$$

3.2. Practical Example. It is well known that any discrete time-varying linear system that is connected with the set Σ is unstable when $\rho(\Sigma) > 1$. Thus, when we want to check if it is unstable, we do not have to know the exact value of $\rho(\Sigma)$. All we have to know is that this value is greater than one. Therefore, we do not have to calculate the value of $\rho(\Sigma)$ using the formulas

$$\rho(\Sigma) = \sup_{n\in\mathbb{N}} \sup_{A\in\Sigma_s^n} \rho(A)^{\frac{1}{2n}}$$

and

$$\rho(\Sigma) = \sup_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \rho(A)^{\frac{1}{n}}.$$

The only thing we have to do is to try to find numbers p and q such that

$$\sup_{A\in\Sigma_s^p} \rho(A)^{\frac{1}{2p}} > 1$$

or

$$\sup_{A\in\Sigma^q} \rho(A)^{\frac{1}{q}} > 1.$$

If this can be done, then we can be sure that the given system is unstable.

A natural consequence of the above is that if we define two functions

$$f(n) = \sup_{k\in\{1,2,\ldots,n\}} \sup_{A\in\Sigma^k} \rho(A)^{\frac{1}{k}}$$

and

$$g(n) = \sup_{k\in\{1,2,\ldots,n\}} \sup_{A\in\Sigma_s^k} \rho(A)^{\frac{1}{2k}},$$

then the better function is that which grows faster than the other. Therefore, we decided to write a computer program to calculate the estimates of the spectral radius by using both functions described above. In the first phase, the program formed the set Σ that consisted of two matrices with random entries. In the second phase, the program created text files that included the products of all sequences of matrices taken from the set Σ such that their length varied from 1 to m ($m \geq 1$). Next, the suprema of functions $f(n)$ and $g(n)$ were calculated. The above test was repeated 1000 times for a set Σ that included two 10×10 random matrices with real elements. The results show that estimate of the spectral radius that was calculated with the procedure that uses symmetric matrices grows faster than the one calculated by the procedure that uses nonsymmetric matrices. Table 1 shows some estimates of the spectral radius of Σ for $m = 15$.

As the procedure that uses symmetric matrices to calculate the estimates has to multiply the matrices and their transpositions, and then has to find their tridiagonal forms, the resulting time of computations is very similar to that

Table 1. Results of numerical simulations.

‘Non symmetrical’ estimates of $\rho(\Sigma)$	‘Symmetric’ estimates of $\rho(\Sigma)$	Difference
5.0759532436	5.4977026931	8.3087733324%
5.2080473674	5.6922435597	9.2970773538%
5.7268488324	5.8506126114	2.1611148223%
5.0375119183	5.6316129380	11.7935407267%
6.2237710148	6.4388735096	3.4561440998%
4.8632082624	5.1817106903	6.5492245199%
5.8747584083	6.2546162919	6.4659319955%
5.0924269191	5.3351337170	4.7660339911%
5.3789582337	5.6207988577	4.4960494870%
5.0864498206	5.3297698549	4.7836908437%

obtained in the case when we use nonsymmetric matrices, and it depends on the chosen numerical algorithms. Therefore, the main advantage of this method in this case is that it gives at the same time better estimates of the value that we look for.

4. Conclusions

In this paper we propose new formulas for the generalized spectral radius of a set of matrices. The main advantage of our formulas is that they express the generalized spectral radius of any set of matrices in terms of a set of symmetric nonnegative definite matrices. It allows us to compute better estimates of the spectral radius of a finite nonempty set of matrices by using any faster or much simpler algorithm that was designed to calculate the eigenvalues of symmetric matrices. The only price we must pay for this is that we have to multiply the given matrix and its transposition. Finally, let us notice that in the theory of stochastic linear systems, the substitution of any matrices by symmetric ones allows us to obtain a very elegant proof of the Oseledets ergodic theorem, cf. (Gol’dsheid and Margulis, 1989). Generalizations for linear time-varying inclusions constitute the subject of our further investigations.

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