Appl. Math. and Comp. Sci., *1995*, *vol.5*, *No.2*, *373–389*

LYAPUNOV CONTROL IN ROBOTIC SYSTEMS: TRACKING REGULAR AND CHAOTIC DYNAMICS

HENK NIJMEIJER*, HARRY BERGHUIS**

Lyapunov-type controllers for trajectory tracking of rigid robot manipulators are described. The so-called passivity-based controllers exploit the desired physical energy of the robot system. A discussion about tracking control in the absence of complete state information and model knowledge then leads to practical stability. It is shown that these Lyapunov-type controllers can also be used in other mechanical systems as e.g. the controlled Duffing equation. In that case the controller can be used as a tool for creation or annihilation of chaotic dynamics.

1. Introduction

Over the last decade a lot of research has been done on designing sophisticated control strategies for rigid robot manipulators, see e.g. (Berghuis, 1993; Ortega and Spong, 1989; Spong and Vidyasagar, 1989) and references therein. In particular, for the tracking control problem of a rigid robot manipulator one may distinguish several controller schemes. Perhaps the best known method is the so-called computed torque controller (see Spong and Vidyasagar, 1989), which is essentially based upon feedback linearization of the robot model (cf. Nijmeijer and van der Schaft, 1990). Despite its mathematical elegance and simplicity the computed torque controller in robotics does not incorporate the physical nature of the manipulator involved. Therefore, more recently tracking controllers that are using the robot's physical structure, have been developed. These so-called *passivity-based* controllers are constructed by the idea of reshaping the energy of the manipulator in such a way so as to fulfil the control objective, see (Ortega and Spong, 1989; Takegaki and Arimoto, 1981) and others. Essentially, this energy-shaping philosophy induces a Lyapunov-type of controller design and in particular the stable tracking performance of the system in closed-loop is shown via an often tedious but in itself direct Lyapunov-function analysis. An interesting feature of these passivity-based schemes is that they are all built with a linear state feedback (Proportional-Derivative (PD)-feedback in the manipulator's position) which makes these schemes quite attractive in practice, see (Spong and Vidyasagar, 1989). In practical situations the inherent velocity-feedback – the derivative-feedback of the manipulator's position – may not be desirable because velocity-measurements are impossible (or corrupted by noise). In that case the tracking controller needs to

* Dept. of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, the Netherlands, *e-mail*: h.nijmeijer@math.utwente.nl

** Hollandse Signaalapparaten B.V., Dept. RDT-R&S-MFE-SER, P.O. Box 42, 7550 GD Hengelo, the Netherlands, *e-mail*: h_berghuis@hgl.signaal.nl

be fed with a velocity-estimate obtained from a velocity-observer. This methodology has been developed using Lyapunov-type arguments in (Berghuis and Nijmeijer, 1993) and will be described in Section 2. Since in practice there will always be uncertainty in the dynamic model for the robot manipulator, a further question studied in Section 2 is how our analysis would extend to the case where model uncertainties are incorporated. A solution to this problem without using *any* model knowledge is given that yields practical stability of the closed-loop system, see also (Berghuis and Nijmeijer, 1994).

Given the physical basis for the aformentioned tracking controller (and the tracking controller-observer combination) it should not be surprising that this sort of controller is of use in many other physical systems. In this paper we focuss on the Duffing equation

$$\ddot{x} + p_0\dot{x} + p_1x + p_2x^3 = a\cos(\omega t) \tag{1}$$

The above equation was introduced in 1918 by Duffing to describe a certain nonlinear oscillator with a cubic stiffness term. We note that we concentrate here on (1) but we could have treated the well-know driven van der Pol equation in a similar way. The interest of (1) – or the driven van der Pol equation – is that for certain parameter values of p_0, p_1, p_2, a and ω the dynamics (1) is complex and may include chaotic motion, see e.g. (Guckenheimer and Holmes, 1983). Equation (1) in itself is not controlled, but we will study its controlled version, in that we add in the right-hand side of (1) a control function u (which in principle can be physically realized in the system). The reason to do so is that in this way we are able to study the tracking control of the Duffing equation. We will show in Section 3 that the tracking controller-observer design for the controlled equation (1) parallels the developments of Section 2. Even more, like we have seen for the manipulator dynamics, also in this case we obtain practical stability of the closed-loop system without model knowledge (and thus, for instance, with unknown parameters p_0, p_1, p_2 and a). In particular, the given tracking control strategies that are again of Lyapunov-type, enable us to annihilate *any* complex or chaotic dynamics of (1) while tracking towards any desired trajectory. The latter has recently been studied extensively by e.g. Chen and Dong (1993a; 1993b) and Nijmeijer and Berghuis (1994).

It should be noted that the strong analogy between the tracking control schemes in robot manipulators and the controlled Duffing equation is not very surprising from a mathematical point of view since both systems are feedback equivalent to a second order linear system (cf. Nijmeijer and van der Schaft, 1990). The surprising point is that this also holds without using model knowledge. One consequence of what we have done is that a robot system may track any trajectory of the Duffing eqn. (1) and thus we may track periodic, complex and chaotic signals. In other words, the Lyapunov-type controller methods used in the tracking control of robot manipulators can also be employed as "a route towards chaos" and conversely as a methodology in "controlling chaos". Both subjects have received a lot of attention in the literature, see e.g. the survey-paper by Chen and Dong (1993b) and references therein, Ott *et al.* (1990) and Singer *et al.* (1991). It is certainly not the purpose of the present paper to repeat much on the control creation of chaos. It is remarkable in our opinion that the physically motivated Lyapunov-type control schemes used in robotics are also of relevance in the area of creation and annihilation of chaotic motion. In Section 4 we illustrate explicitly how chaotic motion tracking can be established by injecting a chaotic trajectory of the Duffing equation as the desired robot trajectory. In fact, it follows what kind of desired trajectory the overall system will follow (periodic, chaotic ....) depending on the initial condition of the Duffing equation. To illustrate our ideas we give a few simulation examples where the Lyapunov-type controller-observer with or without model knowledge is used to create a periodic, or chaotic motion for the robot system. Of course, in the case when model knowledge is not used the creation of periodic or chaotic motion is again up to any prescribed degree of accuracy.

We conclude this paper with some concluding remarks in Section 5.

2. Motion Control of Rigid Robots

Consider the dynamics of an n degrees-of-freedom (DOF) revolute joint rigid robot system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau \tag{2}$$

where $M(q)$ is the positive definite inertia matrix $[n \times n]$, $C(q,\dot{q})\dot{q}$ is the Coriolis and centripetal torques $[n \times 1]$, $G(q)$ is the gravitational torque $[n \times 1]$, and τ is the control input $[n \times 1]$. The matrix $C(q,\dot{q})$ is defined via the Christoffel symbols (see Ortega and Spong, 1989), which implies that $\dot{M}(q) - 2C(q,\dot{q})$ is skew symmetric. For the revolute joint system we have (e.g. Spong and Vidyasagar, 1989)

$$0 < M_m \leq ||M(q)|| \leq M_M \quad \forall\, q \in \mathbb{R}^n \tag{3}$$

$$||C(q,x)|| \leq C_M ||x|| \quad \forall\, q, x \in \mathbb{R}^n \tag{4}$$

$$||G(q)|| \leq G_M \quad \forall\, q \in \mathbb{R}^n \tag{5}$$

where in (3), (4), (5) and in the sequel the norm of a vector x is defined as

$$||x|| = \sqrt{x^T x} \tag{6}$$

and the norm of a matrix A as

$$||A|| = \sqrt{\lambda_{\max}(A^T A)} \tag{7}$$

with $\lambda_{\max}(\cdot)$ denoting the maximum eigenvalue. Moreover, similarly to (3), for any symmetric positive definite matrix $A(x)$ and for all x, A_m and A_M denote the minimum and maximum eigenvalue of $A(x)$, respectively, if they exist.

To let system (2) follow an arbitrary smooth reference trajectory, various model-based control methods have been developed. Among many references we mention (Craig, 1988; Kelly and Salgado, 1994; Khosla and Kanade, 1988; Koditschek, 1989; Ortega and Spong, 1989; Paden and Panja, 1988; Sadegh and Horowitz, 1990; Slotine and Li, 1987; Wen and Bayard, 1988). One of the most useful controllers is given by the *Desired Compensation Control Law* (DCCL). This controller belongs to the class of passivity-based controllers, and is described by (cf. Wen and Bayard, 1988)

$$\tau = M(q_d)\ddot{q}_d + C(q_d, \dot{q}_d)\dot{q}_d + G(q_d) - K_d\dot{e} - K_p e \tag{8}$$

where $q_d(t)$ represents the desired motion, $e \equiv q - q_d$ is the tracking error, and K_d and K_p are positive definite diagonal matrices $[n \times n]$. The DCCL consists of two parts: a linear state feedback part (also known as PD feedback) and a model-based compensation part that is computed along the reference trajectory. This way of model compensation is attractive for two reasons. Firstly, it permits off-line calculation of the computationally expensive model-based components, and, consequently, it preserves the simplicity of the PD controller. Secondly, the use of clean reference signals instead of noise-corrupted sensor data in the model compensation allows us to enhance the tracking accuracy, as can be concluded from several experimental studies, see e.g. (Leahy and Whalen, 1991; Whitcomb *et al.*, 1993; Berghuis, 1993).

Let us make the following assumption on the reference input.

Assumption 1. *The desired trajectory signal is bounded, i.e.*

$$V_M = \sup_t ||\dot{q}_d(t)|| < \infty \tag{9}$$

$$A_M = \sup_t ||\ddot{q}_d(t)|| < \infty \tag{10}$$

In addition, we assume

Assumption 2. *The controller gains K_p and K_d in* (8) *are related as*

$$K_p = \lambda K_d \tag{11}$$

where λ is a positive scalar.

In the sequel, we exploit the robot model properties (e.g. Kelly and Salgado, 1994).

$$||M(x)z - M(y)z|| \leq k_M ||x - y|| \, ||z|| \tag{12}$$

$$||C(x, v)w - C(y, z)w|| \leq k_C ||x - y|| \, ||z|| \, ||w|| + C_M ||v - z|| \, ||w|| \tag{13}$$

$$||G(x) - G(y)|| \leq k_G ||x - y|| \tag{14}$$

Then Proposition 1 can be proved (see also Kelly and Salgado, 1994).

Proposition 1. *Consider the closed-loop system* (2), (8) *under Assumptions* 1 *and* 2. *Define*

$$x_1^T = \left[\dot{e}^T, (\lambda e)^T\right] \tag{15}$$

and assume that $||x_1(0)||$ represents an upper bound on the initial error state $x_1(0)$. Then under the condition

$$K_{d,m} > \lambda M_M + 2C_M V_M + 2\lambda^{-1} k_1 \tag{16}$$

where $k_1 \equiv k_M A_M + k_C V_M^2 + k_G$*, the closed-loop system* (2), (8) *is asymptotically stable with guaranteed region of attraction*

$$B = \left\{ x_1 \in \mathbb{R}^{2n} \mid \ ||x_1|| < \sqrt{\frac{\lambda M_m}{18 K_{d,M}}} \left[\frac{K_{d,m} - \lambda M_M - 2C_M V_M - 2\lambda^{-1} k_1}{C_M} \right] \right\} \tag{17}$$

Moreover, region (17) *can be enlarged arbitrarily by increasing* K_d*; i.e. system* (2), (8) *is semi-globally asymptotically stable.*

Proof. The closed-loop dynamics (2), (8) are equal to

$$M(q)\ddot{e} + C(q,\dot{q})\dot{e} + K_d s_1 = \Delta Y(\cdot) \tag{18}$$

where

$$\begin{aligned} \Delta Y(\cdot) \ &= \Big(M(q_d)\ddot{q}_d - M(q)\ddot{q}_d\Big) + \Big(C(q_d,\dot{q}_d)\dot{q}_d \\ &\quad -C(q,\dot{q})\dot{q}_d\Big) + \Big(G(q_d) - G(q)\Big) \end{aligned} \tag{19}$$

$$s_1 = \dot{e} + \lambda e \tag{20}$$

Using (3), (4), (5), (9), (10), (12), (13), (14) we obtain

$$\begin{aligned} ||\Delta Y(\cdot)|| \ &\le (k_M A_M + k_C V_M^2 + k_G)||e|| + C_M V_M ||\dot{e}|| \\ &= k_1 ||e|| + C_M V_M ||\dot{e}|| \end{aligned} \tag{21}$$

Let us consider the candidate Lyapunov function

$$V_1(e,\dot{e}) = \frac{1}{2} s_1^T M(q) s_1 + \frac{1}{2} e^T (2\lambda K_d - \lambda^2 M(q)) e \tag{22}$$

As shown in Appendix, $V_1(e,\dot{e})$ satisfies

$$\frac{1}{2} P_m ||x_1||^2 \le V_1(e,\dot{e}) \le \frac{1}{2} P_M ||x_1||^2 \tag{23}$$

where $P_m = \frac{1}{3} M_m$ and $P_M = 6\lambda^{-1} K_{d,M}$.

The time-derivative of $V_1(e,\dot{e})$ along (18) satisfies

$$\begin{aligned} \dot{V}_1(e,\dot{e}) \ &= -\dot{e}^T \Big(K_d - \lambda M(q)\Big)\dot{e} - (\lambda e)^T K_d (\lambda e) \\ &\quad + (\dot{e} + \lambda e)^T \Delta Y(\cdot) + \dot{e}^T C(q,\dot{q})(\lambda e) \end{aligned} \tag{24}$$

where Assumption 2 and the skew symmetry of $\dot{M}(q) - 2C(q,\dot{q})$ has been used. Employing (21) and using the sum of perfect squares, an upper bound on $\dot{V}_1(e,\dot{e})$ is given by

$$\begin{aligned} \dot{V}_1(e,\dot{e}) \le \ & -\Big(K_{d,m} - \lambda M_M - 2C_M V_M - \frac{1}{2}\lambda^{-1} k_1 - C_M ||\lambda e||\Big) ||\dot{e}||^2 \\ & -(K_{d,m} - C_M V_M - \frac{3}{2}\lambda^{-1} k_1) ||\lambda e||^2 \end{aligned} \tag{25}$$

This implies

$$\dot{V}_1(e,\dot{e}) \leq -(K_{d,m} - \lambda M_M - 2C_M V_M - 2\lambda^{-1}k_1)||x_1||^2 + C_M||x_1||^3 \quad (26)$$

Hence, from (26) $\dot{V}_1(e,\dot{e})$ is locally negative definite. This, together with the positive definiteness of $V_1(e,\dot{e})$, implies that the closed-loop system (2), (8) is locally asymptotically stable (cf. Khalil, 1992). Finally, because the region of attraction (17) can be arbitrarily enlarged by increasing K_d we obtain semi-global asymptotic stability (see (Teel and Praly, 1994) for a similar definition). ■

The DCCL in (8) yields semi-global asymptotic stability. By introducing a *nonlinear PD action* $-K_f||e||^2(\dot{e}+\lambda e)$ in the control loop it is possible to establish global asymptotic stability, see (Sadegh and Horowitz, 1990).

In practice, there will always be uncertainty in the dynamical model (2). This leads us to the natural question: what can be said about the stability issue in the presence of model errors? To this end, consider a simple case in which the robot is controlled by just linear PD feedback, i.e. (8) without model knowledge. So

$$\tau = -K_d\dot{e} - K_p e \quad (27)$$

Then the next proposition can be proven (Qu and Dorsey, 1991), which essentially states that linear high-gain state-feedback guarantees the robot system to follow any reference trajectory with bounded error.

Proposition 2. *Consider the PD-feedback controller* (27) *under Assumptions* 1 *and* 2. *Then the closed-loop system* (2), (27) *is semi-globally uniformly ultimately bounded or practically stable under some suitably selected* (high-gain) *condition on the derivative controller gain* K_d.

Proof (Main steps). The closed-loop dynamics (2), (27) are equal to those of (18) except for an additional term at the right-hand side representing the model-based feedforward component in (8), i.e.

$$\Delta Z = M(q_d)\ddot{q}_d + C(q_d,\dot{q}_d)\dot{q}_d + G(q_d) \quad (28)$$

According to (3), (4), (5), (9), (10) and (28) can be bounded as

$$\begin{aligned} ||\Delta Z|| &\leq M_M A_M + C_M V_M^2 + G_M \\ &\equiv k_2 \end{aligned} \quad (29)$$

Consequently, the bound on $\dot{V}_1(e,\dot{e})$ in (26) changes into

$$\dot{V}_1(e,\dot{e}) \leq 2k_2||x_1|| - (K_{d,m} - \lambda M_M - 2C_M V_M - 2\lambda^{-1}k_1)||x_1||^2 + C_M||x_1||^3 \quad (30)$$

Hence, the time-derivative of the Lyapunov function is negative definite in an annulus of a certain width around the origin. Therefore, the closed-loop system is locally uniformly ultimately bounded (cf. Chen and Leitmann, 1987; Qu and Dorsey, 1991). ■

One characteristic feature of the above state-feedback controllers is that they require both position *and* velocity measurements. In practice, however, this requirement is generally not fulfilled. Hence, it makes sense to develop a controller that preserves the attractive implementation properties of the DCCL but which only employs position feedback. For this purpose, consider the following modification to (8) (cf. Kaneko and Horowitz, 1994; Berghuis and Nijmeijer, 1993):

$$\text{Controller}\left\{\tau = M(q_d)\ddot{q}_d + C(q_d,\dot{q}_d)\dot{q}_d + G(q_d) - K_d\dot{\hat{e}} - K_p\hat{e}\right. \tag{31}$$

$$\text{Observer}\left\{\begin{array}{rcl} \dot{\hat{e}} & = & w + L_d(e-\hat{e}) \\ \dot{w} & = & L_p(e-\hat{e}) \end{array}\right. \tag{32}$$

where $\hat{e}$ represents the estimated tracking error, $\left[\dot{\hat{e}}^T\ \hat{e}^T\right]$ is the estimated error state, and L_p and L_d are positive definite diagonal matrices $[n \times n]$. The estimated error state is generated by a *linear* observer, which only requires the position error e as input. Next, the state estimate is injected in the control loop.

Let us make the following assumption on the structure of the observer gains:

Assumption 3. *L_d and L_p can be written as*

$$L_d = (\ell_d + \lambda)I \tag{33}$$

$$L_p = \lambda \ell_d I \tag{34}$$

where $\ell_d > 0$ is a scalar and λ is as before (see (11)).

Then we have

Proposition 3. *Consider the closed loop (2), (31), (32) under Assumptions 1–3. Define*

$$x_2^T = \left[\dot{e}^T, (\lambda e)^T, \dot{\tilde{e}}, (\lambda\tilde{e})^T\right] \tag{35}$$

where $\tilde{e} \equiv e - \hat{e}$, and assume that $||x_2(0)||$ is an upper bound on $x_2(0)$. If

$$K_{d,m} > \lambda M_M + 4C_M V_M + 3\lambda^{-1}k_1 \tag{36}$$

$$\ell_d > 2M_m^{-1}K_{d,M} \tag{37}$$

then the closed loop system is semi-globally asymptotically stable with guaranteed region of attraction

$$B = \left\{x_2 \in \mathbb{R}^{4n} \mid \ ||x_2|| < \sqrt{\frac{\lambda M_m}{450 K_{d,M}}}\left[\frac{K_{d,m} - \lambda M_M - 4C_M V_M - 3\lambda^{-1}k_1}{C_M}\right]\right\} \tag{38}$$

Proof. Introduce

$$s_2 = \dot{\tilde{e}} + \lambda \tilde{e} \tag{39}$$

then the closed-loop dynamics (2), (31), (32) are described by

$$M(q)\ddot{e} + C(q,\dot{q})\dot{e} + K_d(s_1 - s_2) = \Delta Y(\cdot) \tag{40}$$

$$M(q)\dot{s}_2 + C(q,\dot{q})s_2 + \ell_d M(q)s_2 + K_d(s_1 - s_2) = \Delta Y(\cdot) + C(q,\dot{q})(s_2 - \dot{e}) \tag{41}$$

where $\Delta Y(\cdot)$ is defined in (19). Consider a candidate Lyapunov function

$$V(e,\dot{e},\tilde{e},\dot{\tilde{e}}) = V_1(e,\dot{e}) + V_2(\tilde{e},\dot{\tilde{e}}) \tag{42}$$

with $V_1(e,\dot{e})$ as in (22), and $V_2(\tilde{e},\dot{\tilde{e}})$ equal to

$$V_2(\tilde{e},\dot{\tilde{e}}) = \frac{1}{2}s_2^T M(q)s_2 + \frac{1}{2}\tilde{e}^T(2\lambda K_d)\tilde{e} \tag{43}$$

As before, $V(e,\dot{e},\tilde{e},\dot{\tilde{e}})$ satisfies

$$\frac{1}{2}P_m\|x_2\|^2 \le V(e,\dot{e},\tilde{e},\dot{\tilde{e}}) \le \frac{1}{2}P_M\|x_2\|^2 \tag{44}$$

where $P_m = \frac{1}{3}M_m$ and $P_M = 6\lambda^{-1}K_{d,M}$.

Evaluating the time-derivative of $V(\cdot)$ along (40), (41), using bound (21) on $\Delta Y(\cdot)$, and employing the sum of perfect squares, the proof can be completed along the lines of the proof of Proposition 1. ■

As can be seen by comparing Propositions 1 and 3, both the state- and output-feedback solution to DCCL can be treated in a unified way. Using the same mathematical machinery essentially the same stability result is established. As a natural consequence, we also obtain the next proposition, in analogy to Proposition 2. For the proof we refer to Berghuis and Nijmeijer (1994).

Proposition 4. *Consider the linear estimated state-feedback controller*

$$\tau = K_d\dot{\hat{e}} - K_p\hat{e} \tag{45}$$

where the error state is determined with the linear observer proposed in (32)*. Suppose Assumptions* 1–3 *are satisfied. Then the closed-loop system* (2), (45) *is semi-globally uniformly ultimately bounded under suitably selected* (high-gain) *conditions on* K_d *and* ℓ_d.

Remark 1. Proposition 4 was given under the condition that both Assumption 2 and Assumption 3 hold, which in particular implies that in (11) and (33) the same parameter λ appears. This is in fact not at all necessary in that we may replace (11) and (33) as $K_p = \lambda_1 K_d$, $L_d = (\ell_d + \lambda_2)I$ and $L_p = \lambda_2\ell_d I$ for different parameters λ_1 and λ_2, and the result of Proposition 4 is still true, but the proof becomes slightly more involved. This extra freedom of selecting the controller-observer gains will be used in the simulations of Section 4.

Hence, the highly non-linear robot dynamics can be stabilized around any bounded reference trajectory by a linear output-feedback controller. In the next section it is shown that the presented results in robot control can also be used in other physical systems, in particular the Duffing dynamics.

3. Feedback Control of Duffing Equation

The Duffing equation describes a specific non-linear circuit or a pendulum moving in a viscous medium, and is given as (see (1))

$$\ddot{x} + p_0\dot{x} + p_1 x + p_2 x^3 = a\cos(\omega t) \tag{46}$$

where $p_0 > 0$, $p_1 > 0$, p_2, a and ω are known constants. Depending on the choice of these constants it is known that solutions of (46) exhibit periodic, almost periodic and chaotic behaviour cf. (Chen and Dong, 1993a; 1993b; Guckenheimer and Holmes, 1983).

It is our purpose to discuss a controlled version of (46). For this we consider

$$\ddot{x} + p_0\dot{x} + p_1 x + p_2 x^3 = u + a\cos(\omega t) \tag{47}$$

where $u(\cdot)$ is the *physical control input*. The general problem that we want to study is whether we are able to find a suitable feedback controller

$$u = k(x, \dot{x}, x_d, t) \tag{48}$$

such that for the closed-loop system (47), (48) the solution $x(t)$ asymptotically converges to a desired trajectory $x_d(t), t \geq 0$. Here $x_d(t)$ may represent *any* smooth and bounded time-function, including fixed points or periodic orbits.

The crucial observation is that from a control point of view the dynamics of the controlled Duffing equation (47) is essentially the same as that of a one-DOF robot system. The main difference are the linear and cubic term in x, but these terms do not cause any problems in both the control design and the stability analysis, as will be shown below.

Assume we want the system to follow the reference $x_d(t)$. For this purpose, we define in analogy with (8) the control input as

$$u = \ddot{x}_d + p_0\dot{x}_d + p_1 x_d + p_2 x_d^3 - a\cos(\omega t) - K_d\dot{e} - K_p e + v \tag{49}$$

$$v = 3p_2 x x_d e \tag{50}$$

with $e \equiv x - x_d$, and $K_d > 0, K_p > 0$ scalar. The compensation term v is introduced in order to deal with the cubic term in (47). As before, let us take (cf. (11))

$$K_p = \lambda K_d \tag{51}$$

and $\lambda > 0$ scalar. Then we can prove:

Proposition 5. *Let K_p satisfy* (51). *Then under the condition*

$$K_d > \max(\lambda, -p_1\lambda^{-1}) \tag{52}$$

the closed-loop system (47), (49), (50) *is globally asymptotically stable.*

Proof. The closed-loop system (47), (49), (50) is described by

$$\ddot{e} + (p_0 + K_d)\dot{e} + (p_1 + K_p)e + p_2 e^3 = 0. \tag{53}$$

In correspondence to (22), consider the candidate Lyapunov function

$$V_1(e, \dot{e}) = \frac{1}{2}s_1^2 + \frac{1}{2}((p_1 + \lambda K_d) + \lambda(p_0 + K_d) - \lambda^2)e^2 + \frac{1}{4}p_2 e^4 \tag{54}$$

Because of (52) and $p_0 > 0, V_1(e, \dot{e})$ is positive definite. Along (53), $\dot{V}_1(e, \dot{e})$ equals

$$\dot{V}_1(e, \dot{e}) = -(p_0 + K_d - \lambda)\dot{e}^2 - \lambda(p_1 + \lambda K_d)e^2 - \lambda p_2 e^4 \tag{55}$$

Under (52) we have $\dot{V}_1(e, \dot{e})$ globally negative definite in $(e, \dot{e})$. Consequently, the closed-loop system (53) is globally asymptotically stable (cf. Khalil, 1992). ∎

Controller (49), (50) allows us to steer the Duffing equation towards an arbitrary reference trajectory $x_d(t)$. Hence, the chaotic behaviour the uncontrolled Duffing dynamics may display is completely annihilated by feedback control. Now, suppose that $x_d(t)$ represents a (stable or unstable) equilibrium motion of the uncontrolled dynamics, i.e.

$$\ddot{x}_d + p_0\dot{x}_d + p_1 x_d + p_2 x_d^3 = a\cos(\omega t) \tag{56}$$

Then by combining (49), (50) and (56) we have:

Corollary 1. *If the desired motion $x_d(t)$ satisfies* (56), *then the controller*

$$u = -K_d\dot{e} - K_p e + 3p_2 x \;\; x_d e \tag{57}$$

guarantees that the Duffing equation asymptotically converges under assumption (51) *and condition* (52) *towards* (56) *in a global sense.*

It is rather straightforward to show that PD-feedback (27) allows the controlled Duffing dynamics (47) to follow any bounded reference trajectory with bounded error. In particular, under high-gain PD-control semi-global stability of the closed-loop can be shown, in analogy to Proposition 2. As discussed in (Nijmeijer and Berghuis, 1994), for (47) also output-feedback type of controllers can be developed that yield global asymptotic stability. Here we will concentrate on the model-independent linear estimated state-feedback controller (45), and analyze its stability properties when it is used to control the Duffing equation. In particular, consider

$$\text{Controller}\left\{ u = -K_d\dot{\hat{e}} - K_p\hat{e} \right. \tag{58}$$

$$\text{Observer}\left\{ \begin{array}{rcl} \dot{\hat{e}} & = & w + 2K_d(e-\hat{e}) \\ \dot{w} & = & 2K_p(e-\hat{e}) \end{array} \right. \tag{59}$$

Since measuring $\dot{x}$ in the controlled Duffing equation (47) might be difficult and noise-sensitive, the controller-observer combination (58), (59) seems very attractive. We can prove the following result.

Proposition 6. *Consider Duffing equation* (47) *under robust linear output-feedback control* (51), (58), (59). *Then the closed-loop dynamics is locally uniformly ultimately bounded for* K_d *sufficiently large.*

Proof. The error dynamics (47, (51), (58), (59) is given by

$$\ddot{e} + p_0\dot{e} + p_1 e + p_2 e^3 + K_d s_1 = K_d s_2 + \Delta W - 3p_2 x x_d e \tag{60}$$

$$\ddot{\tilde{e}} + K_d s_2 = -K_d s_1 + \Delta W - 3p_2 x x_d e - p_0\dot{e} - p_1 e - p_2 e^3 \tag{61}$$

where

$$\Delta W(\cdot) = a\cos(\omega t) - \ddot{x}_d - p_0\dot{x}_d - p_1 x_d - p_2 x_d^3 \tag{62}$$

Take the Lyapunov function candidate

$$V(e,\dot{e},\tilde{e},\dot{\tilde{e}}) = V_1(e,\dot{e}) + V_2(\tilde{e},\dot{\tilde{e}}) \tag{63}$$

where $V_1(e,\dot{e})$ as in (54) and

$$V_2(\tilde{e},\dot{\tilde{e}}) = \frac{1}{2}s_2^2 + \frac{1}{2}(2\lambda K_d - \lambda^2)\tilde{e}^2 \tag{64}$$

The time-derivative of $V(\cdot)$ along (60), (61) equals to

$$\begin{aligned} \dot{V}(e,\dot{e},\tilde{e},\dot{\tilde{e}}) &= -(p_0 + K_d - \lambda)\dot{e}^2 - (\lambda^{-1}p_1 + K_d)(\lambda e)^2 - \lambda p_2 e^4 \\ &\quad -(K_d - \lambda)\dot{\tilde{e}}^2 - K_d(\lambda\tilde{e})^2 \\ &\quad +(\dot{e} + \lambda e + \dot{\tilde{e}} + \lambda\tilde{e})(\Delta W - 3p_2 x_d^2 e - 3p_2 x_d e^2) \\ &\quad -(\dot{\tilde{e}} + \lambda\tilde{e})(p_0\dot{e} + p_1 e + p_2 e^3) \end{aligned} \tag{65}$$

For the bounded reference trajectory we define

$$P_M = \sup_t \|x_d(t)\|, \qquad V_M = \sup_t \|\dot{x}_d(t)\|, \qquad A_M = \sup_t \|\ddot{x}_d(t)\| \tag{66}$$

Then a bound on $\dot{V}(\cdot)$ is given by

$$\dot{V}(\cdot) \leq a_1||x_2|| - (K_d - a_2)||x_2||^2 + a_4||x_2||^4 \tag{67}$$

where x_2 as in (35), and the constants a_i, $i = 1, 2, 4$, are given by

$$a_1 \equiv 4(| a | + A_M + p_0 V_M + | p_1 | P_M + p_2 P_M^3) \tag{68}$$

$$a_2 \equiv \max(\lambda, -\lambda^{-1} p_1) + 2p_0 + 2\lambda^{-1} | p_1 | + 18\lambda^{-1} V_M^2 p_2) \tag{69}$$

$$a_4 \equiv 8\lambda^{-3} p_2 \tag{70}$$

Thus, $\dot{V}(\cdot)$ is negative definite in an annulus around the origin, whose width can be enlarged with K_d. As shown in (Chen and Leitmann, 1987) (see also Qu and Dorsey, 1991), this implies that the closed-loop system is locally uniformly ultimately bounded. This completes the proof. ■

4. A Route to Chaos

To generate chaos in robot systems, we consider the simplest case, the dynamics of a one-DOF robot, i.e.

$$m\ell^2 \ddot{q} + mg\ell \sin(q) = \tau \tag{71}$$

where $m > 0$ and $\ell > 0$ scalar. Let us take the robust linear output-feedback controller (cf. (32), (45))

$$\text{Controller} \left\{ \tau = -K_d \dot{\hat{e}} - K_p \hat{e} \right. \tag{72}$$

$$\text{Observer} \left\{ \begin{array}{rcl} \dot{\hat{e}} & = & w + L_d(e - \hat{e}) \\ \dot{w} & = & L_p(e - \hat{e}) \end{array} \right. \tag{73}$$

The reference trajectory is assumed to satisfy the Duffing characteristics

$$\ddot{q}_d + p_0 \dot{q}_d + p_1 q_d + p_2 q_d^3 = a \cos(\omega t) \tag{74}$$

Now, by selecting different sets of parameters in (74) and initial conditions for (74) the controlled robot system can be forced to display periodic, almost periodic and chaotic behavior. Because we consider the robust controller, the actual state trajectory of the robot system follows the prespecified reference with bounded error. This error bound can be arbitrarily decreased by enlarging the controller gains K_d.

To illustrate our results we simulated the one-DOF robot (71) to track a periodic and chaotic trajectory of the Duffing equation (74). This was done by using the controller-observer combination (72), (73). The robot characteristics were chosen as

$$m = 0.1kg, \quad \ell = 2m \tag{75}$$

The Duffing parameters from (74) were selected either as

$$p_0 = 0.4, \quad p_1 = -1,1, \quad p_2 = 1.0, \quad \omega = 1.8, \quad a = 0.62 \tag{76}$$

for generating a periodic orbit or

$$p_0 = 0.4, \quad p_1 = -1.1, \quad p_2 = 1.0, \quad \omega = 1.8, \quad a = 1.8 \tag{77}$$

for generating a chaotic trajectory (cf. Chen and Dong, 1993). The controller-observer gains in (72), (73) were chosen in both cases as (see Remark 1)

$$K_d = 10, \quad \lambda = 5 \tag{78}$$

$$\ell_d = 30, \quad \lambda_2 = 0.3 \tag{79}$$

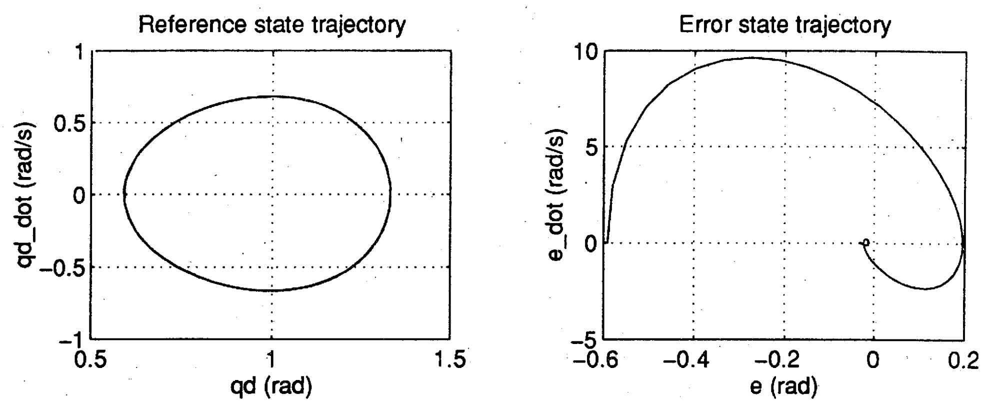

Fig. 1. Results for period-one solution of Duffing equation (74).

The resulting Duffing trajectories in the $(q_d, \dot{q}_d)$-plane are given in Figs. 1 and 2. Additionally we visualize in these figures the error-trajectories of the closed-loop system (71), (72), (73), (74) with the above parameter selections (75), (76), (77).

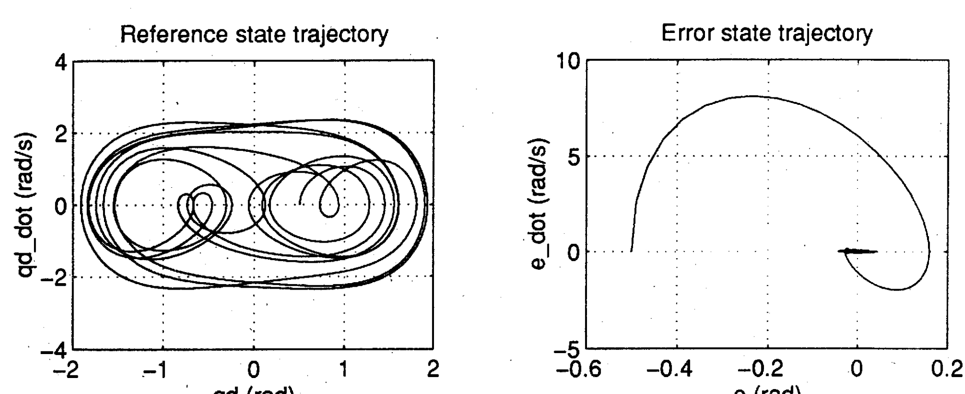

Fig. 2. Results for chaotic solution of Duffing equation (74).

The error-trajectories in closed loop were initialized at, respectively, $(e(t_0), \dot{e}(t_0)) = (-0.5909, 0)$ for the periodic orbit (Fig. 1) and $(e(t_0), \dot{e}(t_0)) = (-0.5, 0)$ for the chaotic trajectory (Fig. 2). Notice that in order to eliminate the transient effects we have initialized the error-trajectories at some time $t_0 > 0$. Inspection of both simulations clearly shows that the error trajectories readily converge to a neighbourhood of $(0, 0)$ despite the fact that our controller-observer combination (72), (73) does not use model information nor velocity measurements. By increasing the gains this neighborhood of stability can be made smaller, cf. Proposition 6.

5. Conclusions

We have shown in this paper how physically based controllers and controller-observer combinations can be designed for solving the tracking control problem for rigid robot manipulators. These so-called passivity-based techniques centre around the idea of reshaping the energy of the manipulator in such a way so as to fulfil the control objective, see (Takegaki and Arimoto, 1981). The modified energy is then used as a Lyapunov-function for the closed-loop system. We show that this type of Lyapunov control can also be used when no model information is used; in this case the closed-loop system is shown to be practically stable (instead of asymptotically stable). The Lyapunov control we use in robot manipulators can also be used in other physical control systems. In particular we illustrate this fact here for the controlled Duffing equation. The nice feature of our techniques is that, no matter how the initial dynamical system behaves (stable, periodic, chaotic .....), the controlled system can follow *any* desired trajectory with any degree of accuracy. In this way this work complements ongoing research on the control of chaotic dynamics, cf. (Chen and Dong, 1993a; 1993b).

Appendix

Function (22) can be written as

$$V_1(\cdot) = \frac{1}{2} y^T R(q) y \tag{A1}$$

where

$$y^T = [s_1^T, (\lambda e)^T] \tag{A2}$$

and

$$R(q) = \begin{bmatrix} M(q) & 0 \\ 0 & 2\lambda^{-1} K_d - M(q) \end{bmatrix} \tag{A3}$$

According to (16), we have

$$K_{d,m} > \lambda M_M \tag{A4}$$

Hence, we obtain

$$\frac{1}{2}R_m\|y\|^2 \leq V_1(\cdot) \leq \frac{1}{2}R_M\|y\|^2 \tag{A5}$$

where

$$R_m = M_m, \quad R_M = 2\lambda^{-1}K_{d,M} \tag{A6}$$

By definition,

$$y = Tx_1 \tag{A7}$$

with

$$T = \begin{bmatrix} \begin{bmatrix} I & I \\ 0 & I \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} I & I \\ 0 & I \end{bmatrix} \end{bmatrix} \tag{A8}$$

Note from (A7)

$$\frac{1}{3}\|x_1\|^2 \leq \|y\|^2 \leq 3\|x_1\|^2 \tag{A9}$$

Together with (A5) and (A6) this implies (23).

References

Berghuis H. (1993): *Model-based Robot Control - from Theory to Practice.* — PhD-thesis. Dept. of Applied Mathematics and Electrical Eng., Univ. of Twente, Enschede, The Netherlands.

Berghuis H. and Nijmeijer H. (1993): *A passivity approach to controller-observer design for robots.* — IEEE Trans. Robotics and Automation, v.9, No.6, pp.740–754.

Berghuis H. and Nijmeijer H. (1994): *Robust control of robots via linear estimated state feedback.* — IEEE Trans. Automatic Control, v.39, No.10, pp.2159–2162.

Chen G. and Dong X. (1993a): *On feedback control of chaotic continuous-time systems.* — IEEE Trans. Circuits and Systems, v.40, No.9, pp.591–601.

Chen G. and Dong X. (1993b): *From chaos to order-perspective and methodologies in controlling chaotic non-linear dynamical systems.* — Int. J. Bifurcation and Chaos, v.3, No.6, pp.1363–1409.

Chen Y.H. and Leitmann G. (1987): *Robustness of uncertain systems in the absence of matching assumptions.* — Int. J. Control, v.45, No.5, pp.1527–1542.

Craig J.J. (1988): *Adaptive Control of Mechanical Manipulators.* — New York: Addison Wesley.

Guckenheimer J. and Holmes P. (1983): *Non-linear Oscillations, Dynamical Systems and Bifurcations of Vector Fields.* — New York: Springer Verlag.

Kaneko K. and Horowitz R. (1994): *Repetitive and adaptive control of robot manipulators with velocity estimation.* — to appear in IEEE Trans. Robotics and Automation.

Kelly R. and Salgado R. (1994): *PD Control with computed feedforward of robot manipulators: a design procedure.* — IEEE Trans. Robotics and Automation, v.10, No.4, pp.566–571.

Khalil H.K. (1992): *Non-linear Systems.* — New York: MacMillan.

Khosla P.K. and Kanade T. (1988): *Experimental evaluation of non-linear feedback and feedforward control schemes for manipulators.* — Int. J. Robotics Research, v.7, No.1, pp.18–28.

Klafter R.D., Chmielewski T.A. and Negin M. (1989): *Robotic Engineering - An Integrated Approach.* — New York: Englewood Cliffs, Prentice-Hall International Editions.

Koditschek D. (1989): *Robot planning and control via potential functions.* — In: The Robotics Review I (O. Khatib, J.J. Craig and T. Lozano-Perez, Eds.), Boston: The MIT Press, pp.349–367.

Leahy M.B.Jr. and Whalen P.V. (1991): *Direct adaptive control for industrial manipulators.* — Proc. IEEE Conf. *Robotics and Automation*, Sacramento, U.S.A, pp.1666–1672.

Nijmeijer H. and Berghuis H. (1994): *On the Relation Between Controlling Certain Robotic and Chaotic Systems.* — Memo 1225, Faculty of Applied Mathematics, University of Twente.

Nijmeijer H. and van der Schaft A.J. (1990): *Non-linear Dynamical Control Systems.* — Berlin: Springer Verlag.

Ortega R. and Spong M.W. (1989): *Adaptive motion control of rigid robots: a tutorial.* — Automatica, v.25, No.6, pp.877–888.

Ott E., Grebogi C. and Yorke J.A. (1990): *Controlling chaos.* — Phys. Rev. Letters, v.64, No.11, pp.1196–1199.

Paden B. and Panja R. (1988): *Globally asymptotically stable 'PD+' controller for robot manipulators.* — Int. J. on Control, v.47, No.6, pp.1697–1712.

Qu Z. and Dorsey J.F. (1991): *Robust tracking control of robots by a linear feedback law.* — IEEE Trans. Automatic Control, v.36, No.9, pp.1081–1084.

Sadegh N. and Horowitz R. (1990): *Stability and robustness analysis of a class of adaptive controllers for robotic manipulators.* — Int. J. of Robotics Research, v.9, No.3, pp.74–94.

Singer J., Wang Y-Z. and Bau H.H. (1991): *Controlling a chaotic system.* — Physical Rev. Letters, v.66, No.9, pp.1123–1125.

Slotine J.J.-E. and Li W. (1987): *On the Adaptive control of robot manipulators.* — Int. J. of Robotics Research, v.6, No.3, pp.49–59.

Spong M.W. and Vidyasagar M. (1989): *Robot Dynamics and Control.* — New York: Wiley.

Takegaki M. and Arimoto S. (1981): *A new feedback method for dynamic control of manipulators.* — ASME J. Dynamic Systems, Measurement and Control, v.102, No.2, pp.119–125.

Teel A. and Praly L. (1994): *Global stabilizability and observability imply semi-global stabilizability by output-feedback.* — Systems Control Letters, v.22, No.5, pp.313–325.

Wen J.T. and Bayard D.S. (1988): *New class of control laws for robotic manipulators: part I. non-adaptive case.* — Int. J. Control, v.47, No.5, pp.1361–1386.

Whitcomb L.L., Rizzi A.A. and Koditschek D.E. (1993): *Comparative experiments with a new adaptive controller for robot arms.* — IEEE Trans. Robotics and Automation, v.9, No.1, pp.54–70.

Received: December 5, 1994