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CONSTRAINED CONTROLLABILITY OF RETARDED DYNAMICAL SYSTEMS

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Recent years have witnessed a good deal of research focused on abstract control dynamical systems defined in infinite-dimensional linear spaces. The main purpose of the present paper is to study the concept of constrained approximate relative and approximate absolute controllability for linear stationary abstract retarded dynamical systems defined in infinite-dimensional Hilbert spaces. First, using the methods of functional analysis, the brief and compact theory of such dynamical systems is recalled and the general integral form of solution is presented. It is generally assumed that the admissible controls are non-negative square integrable functions. Using the methods taken from the spectral theory of linear unbounded operators, the necessary and sufficient conditions for constrained approximate relative controllability are formulated and proved. These conditions are a generalization for infinite-dimensional retarded dynamical systems of the results derived recently for finite-dimensional dynamical systems with delays. Moreover, some additional remarks and comments on the relationships between different concepts of controllability are given. Finally, as simple illustrative examples, the necessary and sufficient conditions for constrained approximate relative controllability with non-negative controls for retarded distributed parameter parabolic-type dynamical systems with one constant delay and with homogeneous Dirichlet boundary conditions are presented.

1. Introduction

In recent years controllability problems for different kinds of dynamical systems have been considered in many publications. An extensive list of publications containing more than 500 positions can be found in the monograph (Klamka, 1991). However, most literature has been concerned with the so-called unconstrained controllability problems. Only a few papers deal with the so-called constrained controllability problems, i.e. with the case when the control functions are restricted to take their values in a prescribed admissible set (Brammer, 1972; Carja, 1988; Chukwu, 1979; 1987; Nakagiri and Yamamoto, 1989; Narukawa, 1982; Peichl and Schappacher, 1986; Saperstone, 1973; Saperstone and Yorke, 1971; Schmittendorf and Barmish, 1980; 1981; Son, 1990). Moreover, it should also be stressed that up to now constrained controllability problems for abstract retarded dynamical systems defined in infinite-dimensional Hilbert spaces have not been considered in the literature besides the paper (Klamka, 1993). In order to fill this gap, the present paper studies in detail the constrained

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controllability problems for some special kind of linear abstract stationary retarded dynamical systems with one lumped constant delay in the state variable.

The main purpose of the present paper is to formulate and prove the necessary and sufficient conditions for the so-called constrained approximate relative controllability using some general results given recently in the paper (Son, 1990). Moreover, it will be pointed out that in the special case of finite-dimensional retarded stationary dynamical systems it is easily to obtain from general results the computable constrained relative controllability criteria. Finally, simple numerical examples which illustrate the general theory will be presented. In these examples the computable necessary and sufficent conditions for constrained approximate relative controllability of linear retarded distributed parameter dynamical systems described by partial differential equations of parabolic type are given.

1. Abstract Retarded Dynamical Systems

Let us consider a linear abstract stationary retarded dynamical system with one constant delay in the state variables described by the following differential equation

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + \sum_{j=1}^{r} b_j u_j(t) \tag{1}$$

The initial conditions for eqn. (1) are as follows (Nakagiri, 1981; Nakagiri and Yamamoto, 1989; Webb, 1976):

$$x(0) = g^0 \in X, \quad x(s) = g^1(s) \quad \text{for} \quad s \in [-h, 0] \tag{2}$$

where $x(t) \in X$, X is a Hilbert space, $g^1(s) \in L_2([-h,0],X)$, $h > 0$ is a constant delay, $A_1 : X \to X$ is a bounded linear operator, $b_j \in X$ for $j = 1,2,3,...,r$, $A_0 : X \supset D(A_0) \to X$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t) : X \to X$, $t \geq 0$. Moreover, it is generally assumed that this semigroup is compact.

The scalar controls $u_j(t) \in \mathbb{R}$, $j = 1,2,3,...,r$ are assumed to be square integrable and non-negative, i.e. $u_j(t) \geq 0$ for all $t \geq 0$ and $j = 1,2,3,....,r$. Hence, the admissible values of the controls form a non-negative, convex, closed cone $V \in \mathbb{R}^r$ with the vertex at zero.

Let $B : \mathbb{R}^r \to X$ be a linear bounded operator defined as follows:

$$Bu(t) = \sum_{j=0}^{r} b_j u_j(t)$$

Let $BV \subset X$ denote the image of the cone V under the linear transformation represented by the linear operator B. Of course, BV is a closed convex cone in the Hilbert space X. For the cone BV we define in the Hilbert space X the so-called polar cone:

$$(BV)^0 = \{x \in X : \langle x, v \rangle \leq 0 \ \text{for all} \ v \in BV\}$$

Now, let us introduce some fundamental notations which will be usefull in the next sections of the present paper. For a Hilbert space X we denote by $L_2([-h,0],X)$ the usual Hilbert space of X-valued square integrable functions defined on the finite interval $[-h,0]$. Let $M_2([-h,0],X) = X \times L_2([-h,0],X)$ be a Hilbert space with the usual norm (see e.g. (Nakagiri, 1981; Nakagiri and Yamamoto, 1989; Webb, 1976) for more details). This space is often denoted shortly as M_2.

The fundamental theory of general abstract retarded dynamical systems is studied in detail in the papers (Nakagiri, 1981; Nakagiri and Yamamoto, 1989; Webb, 1976), where different formulae for the solution are formulated and proved. Dynamical system (1) is a special case of general linear abstract stationary retarded dynamical systems given e.g. in the papers (Nakagiri, 1981; Nakagiri and Yamamoto, 1989). Therefore, it is possible to express the solution $x(t,g^0,g^1,u) \in X$ of the eqn. (1) with the initial conditions (2) and the control function $u \in V_t = L_2([0,t],V)$ in the following compact form

$$x(t,g^0,g^1,u) = \begin{cases} W(t)g^0(t) + \int_{-h}^{0} W(t-s-h)A_1 g^1(s)\,\mathrm{d}s \\ +\int_0^t W(t-s)\sum_{j=1}^{r} b_j u_j(s)\,\mathrm{d}s \quad \text{for } t \geq 0 \\ \\ g^1(t) \quad \text{for} \quad t \in [-h,0] \end{cases} \tag{3}$$

where the linear bounded operator $W(t): X \to X$, $t \geq 0$ is the unique solution of the following abstract integral equation

$$W(t) = \begin{cases} T(t) + \int_0^t T(t-s)A_1 W(s-h)\,\mathrm{d}s \quad \text{for } t \geq 0 \\ \\ 0 \quad \text{for } t < 0 \end{cases} \tag{4}$$

Using the general response formula (3) we may associate with the differential equation (1) the so-called solution operator $S(t): M_2 \to M_2, t \geq 0$ defined as follows (Nakagiri, 1981; Nakagiri and Yamamoto, 1989; Webb, 1976):

$$S(t)g = (x(t,g,0),\ x_t(s,g,0)) \in M_2 \quad \text{for} \quad g \in M_2 \tag{5}$$

where $x_t(s,g,0) = x(t+s,g,0)$ for $s \in [-h,0)$.

The operators $S(t), t \geq 0$ are linear and bounded in the Hilbert space M_2. Moreover, the family of the operators $S(t)$ forms a strongly continuous semigroup on the Hilbert space M_2 with the infinitesimal generator $A : M_2 \supset D(A) \to M_2$ defined as follows (Nakagiri, 1981; Nakagiri and Yamamoto, 1989; Webb, 1976):

$$\begin{aligned} D(A) \;=\; & \{g = (g^0,g^1) \in M_2([-h,0],X): \\ & g^1(0) = g^0 \in D(A_0),\ \ g^1 \in W_2^{(1)}([-h,0],X)\Big\} \end{aligned}$$

$$Ag = \left(A_0 g^0 + A_1 g^1(-h), \frac{\mathrm{d}g^1(s)}{\mathrm{d}s} \right) \quad \text{for} \quad (g^0, g^1) \in D(A) \tag{6}$$

The spectrum $\sigma(A)$ of the operator A plays an important role in the investigation of controllability for the retarded dynamical system (1)

3. Basic Definitions

In this section we shall recall some basic definitions concerning different types of controllability for retarded dynamical system (Fattorini, 1966; 1967; Nakagiri and Yamamoto, 1989; Son, 1990; Triggiani, 1975; 1976).

Let V be a non-negative cone in the space $\mathbb{R}^r$. We define the so-called attainable set $C_t(V)$ at time $t > 0$ in the space M_2 as follows

$$C_t(V) = \{(x(t,0,u),\ x_t(s,0,u)) \in M_2 : u \in V_t\} \tag{7}$$

Moreover, we denote

$$C_\infty(V) = \bigcup_{t \geq 0} C_t(V)$$

Similarly, we define the relative attainable set $K_t(V)$ at time $t > 0$ in the space X as follows

$$K_t(V) = \{x(t,0,u) \in X : u \in V_t\} \tag{8}$$

Moreover, we denote

$$K_\infty(V) = \bigcup_{t \geq 0} K_t(V)$$

Now, we are in a position to give formal definitions for constrained exact and approximate absolute controllability, and for constrained exact and approximate relative controllability of the retarded dynamical system (1).

Definition 1. Dynamical system (1) is said to be V-exactly absolutely controllable if $C_\infty(V) = M_2$

Definition 2. Dynamical system (1) is said to be V-approximately absolutely controllable if $\mathrm{cl}(C_\infty(V)) = M_2$, where the symbol cl stands for the closure operation.

Definition 3. Dynamical system (1) is said to be V-exactly relatively controllable if $K_\infty(V) = X$.

Definition 4. Dynamical system (1) is said to be V-approximately relatively controllable if $\mathrm{cl}(K_\infty(V)) = X$.

From the above definitions it follows immediately that V-exact absolute controllability always implies V-exact relative controllability. Similarly, it is obvious that

V-approximate absolute controllability always implies V-approximate relative controllability. Moreover, V-exact absolute (relative) controllability is always a stronger property than V-approximate absolute (relative) controllability

Since the conditions for constrained exact absolute and relative controllability are very restrictive, in the sequel we shall entirely concentrate on the investigation of constrained approximate absolute and relative controllability.

4. Constrained Approximate Controllability

In this section we shall formulate the necessary and sufficent conditions for V-approximate absolute controllability and V-approximate relative controllability of the dynamical system (1). In order to do that, let us formulate the so-called spectral decomposition property for the linear operator A.

Assumption 1. (Spectral decomposition property) For every $\alpha \in \mathbb{R}$ the spectral set σ_α consists of a finite number of eigenvalues of the operator A with finite multiplicity, where $\sigma(A)$ denotes the spectrum of the operator A and the spectral sets are defined as follows: $\sigma_\alpha = \sigma(A) \cap \{z \in C : \mathrm{Re}\ z > \alpha\}$.

Theorem 1. *Suppose that*

i) *the operator A_0 is the infinitesimal generator of a compact semigroup $T(t)$,*
ii) *the operator A_1 is invertible, i.e.* $\mathrm{Ker}A_1 = 0$,
iii) *the operator A satisfies spectral decomposition property,*
iv) *the set V is a non-negative cone in the space $\mathbb{R}^r$,*
v) *the dynamical system (1) is approximately absolutely controllable without any constraints.*

Then the dynamical system (1) is V-approximately absolutely controllable if and only if

$$\mathrm{Ker}(sI - A_0^* - \exp(-sh)A_1^*) \cap (BV)^0 = 0 \ \text{ for all } \ s \in \mathbb{R} \tag{9}$$

Proof. The proof of Theorem 1 is based on Theorem 4.1 presented in the paper (Klamka, 1993). It is easy to verify by simple inspection that all the assumptions of the cited theorem are satisfied and hence our theorem is valid. ∎

Let us observe that assumption (ii) means that the linear retarded dynamical system (1) is spectrally complete in the Hilbert space M_2. Furthermore, it should be pointed out that for the finite-dimensional case, i.e. when $X = \mathbb{R}^n$, from Theorem 1 as a corollary we can obtain immediately the well-known result about V-approximate absolute controllability.

Corollary 1. *Suppose that*

i) $X = \mathbb{R}^n$
ii) $\mathrm{rank}\ A_1 = n$
iii) $\mathrm{rank}\ [sI - A_0 - \exp(-sh)A_1, B] = n$ *for* $s \in C$

Then the dynamical system (1) is V-approximately absolutely controllable if and only if condition (9) holds.

Proof. Since in this case the Hilbert space X is finite-dimensional, then the operator A generates the compact semigroup $T(t)$, for $t > 0$ (see e.g. (Klamka, 1993) for more details). Moreover, condition (ii) implies that the matrix A_1 is invertible. Finally, condition (iii) means that our retarded dynamical system is approximately controllable without any constraints (Fattorini, 1966; 1967; Nakagiri and Yamamoto, 1989; Triggiani, 1975; 1976). Therefore, all the assumptions of Theorem 1 are satisfied and, in consequence, equality (8) is the necessary and the sufficient condition for V-approximate controllability of the dynamical system (1). Hence our corollary follows. ■

Similarly as in the infinite-dimensional case, assumption (ii) means that our retarded dynamical system is spectrally complete.

Now, let us concentrate on V-approximate relative controllability of the dynamical system (1). In order to formulate the verifiable necessary and sufficient conditions for V-approximate relative controllability we shall consider only special but in practice a very popular case of the dynamical system (1). Namely, we shall consider the case, when the linear operator A_0 is self-adjoint with simple eigenvalues and the operator $A_1 = a_h I$, $a_h \in \mathbb{R}$ and I is the identity operator in the Hilbert space X. This situation often arises in the case of distributed parameter dynamical systems described by retarded partial differential equations of parabolic type. Moreover, for simplicity of notation, we shall assume that the number of admissible controls is equal to two, i.e. $r = 2$.

In order to formulate a constrained approximate relative controllability condition let us introduce some additional notation. Namely, let us denote: $s_i \in R$, $i = 1, 2, \ldots$ as the real eigenvalues of the self-adjoint operator A_0 and $x_i \in X$, $i = 1, 2, \ldots$ as the eigenvectors of the operator A_0 corresponding to the eigenvalues s_i, $i = 1, 2, \ldots$.

Theorem 2. *Suppose that*

i) *the operator A_0 is self-adjoint and generates a compact semigroup,*
ii) *the operator $A_1 = a_h I$, where $a_h \in \mathbb{R}$ and I is the identity operator in the Hilbert space X,*
iii) *the eigenvectors $x_i \in X$, $i = 1, 2, \ldots$ form a complete orthonormal set in the Hilbert space X,*
iv) *$r = 2$, i.e. $u(t) = [u_1(t), u_2(t)]^T$.*

Then the dynamical system (1) is V-approximate relative controllable if and only if

$$\langle b_1, x_i\rangle\langle b_2, x_i\rangle < 0 \quad \text{for} \quad i = 1, 2, 3 \ldots \tag{10}$$

Proof. Theorem 2 is a direct consequence of the results recently presented in Section 5 of the paper (Klamka, 1993) and in the paper (Nakagiri and Yamamoto, 1989). Hence, the detailed proof is omitted. ■

5. Parabolic-Type Retarded Dynamical Systems

In this section we shall consider approximate relative controllability problems for a very important class of retarded distributed parameter systems, namely the dynamical systems described by linear partial differential equations of parabolic type with homogeneous boundary conditions of Dirichlet type and a constant delay in the state variables.

Let $\Omega \supset \mathbb{R}^n$ be an n-dimensional closed rectangle in the space $\mathbb{R}^n$ with the boundary Γ and let $\sum = \Gamma \times [0, \infty)$. Let $y = (y_1, y_2, , ..., y_k, ..., y_n) \in \mathbb{R}^n$. Then the rectangle Ω is defined as follows

$$\Omega = \{y \in \mathbb{R}^n : 0 \leq y_k \leq d_k \ \text{ for } \ k = 1, 2..., n\}$$

We shall concentrate on a retarded dynamical system described by the following linear partial differential equation of parabolic type

$$\frac{\partial w(t,y)}{\partial t} = a_0 \sum_{k=1}^{n} \frac{\partial^2 w(t,y)}{\partial y_k^2} + a_h w(t-h, y) + b_1(y) u_1(t) + b_2(y) u_2(t) \tag{11}$$

defined in the domain $\Omega \times [0, \infty)$, where $a_0 \in \mathbb{R}^+$, $a_h \in \mathbb{R}$ are constant coefficients and $h > 0$ is a constant delay. Furthermore, $b_j(y) \in L_2(\Omega)$ for $j = 1, 2$.

The homogeneous Dirichlet type boundary conditions for eqn. (11) are as follows

$$w(t,y) = 0 \ \text{ on } \ \Sigma \tag{12}$$

The initial conditions for the delayed equation (11) are of the following form

$$\begin{aligned} w(0,y) &= w^0(y) \in L_2(\Omega) = X \\ w(\tau,y) &= w_0(\tau,y) \in L_2([-h,0],X) = L_2([-h,0],\ L_2(\Omega)) \end{aligned} \tag{13}$$

where $w^0(y)$ and $w_0(\tau, y)$, $\tau \in [-h, 0]$ are given functions.

Moreover, it is assumed that admissible controls $u_j \in L_2([0,\infty), \mathbb{R}^+)$ for $j = 1, 2$, i.e. they are non-negative and square integrable. Therefore, the cone $V = \{v \subset \mathbb{R}^2 : v_1 \geq 0, v_2 \geq 0\}$ is closed, convex and has non-empty interior in the space $\mathbb{R}^2$.

It is well known (see e.g. Nakagiri, 1981; Nakagiri and Yamamoto, 1989; Webb, 1976) for details) that eqn. (11) with the boundary condition (12) and the initial conditions (13) has the unique solution $w(t, y)$. Moreover, the complete state z_t at time $t \geq 0$ is defined as follows $z_t = \{w(t,y),\ w_t(\tau,y)\} \in M_2([-h,0],X) = L_2(\Omega) \times L_2([-h,0],\ L_2(\Omega))$.

Now, let us express the delayed partial differential equation (11) in the following abstract form (Fattorini, 1966; Klamka, 1991; Nakagiri, 1981; Nakagiri and Yamamoto, 1989; Triggiani, 1975; 1976; Webb, 1976).

$$\dot{x}(t) = A_0 x(t) + a_h x(t-h) + b_1 u_1(t) + b_2 u_2(t) \tag{14}$$

where $x(t) = w(t, y) \in X = L_2(\Omega)$ for $t \geq 0$, $b_j \in X$, for $j = 1, 2$. The linear differential unbounded operator $A_0 : X \supset D(A_0) \to X$ is defined in the following way

$$D(A_0) = \{w(y) = x \in X : A_0 w(y) \in X \text{ and } w(y) = 0 \text{ for } y \in \Gamma\}$$

$$A_0 x = A_0 w(y) = a_0 \sum_{k=1}^{n} \frac{\mathrm{d}^2 w(t,y)}{\mathrm{d}y_k^2}$$

It is well-known (Klamka, 1991) that the operator A_0 is self- adjoint and has a pure discrete-point spectrum consisting entirely of an infinite sequence of real eigenvalues $\{-s_\alpha\}$, where

$$S_\alpha = a_0 \left(\left(\frac{\pi\alpha_1}{d_1}\right)^2 + \left(\frac{\pi\alpha_2}{d_2}\right)^2 + \ldots + \left(\frac{\pi\alpha_k}{d_k}\right)^2 + \ldots + \left(\frac{\pi\alpha_n}{d_n}\right)^2 \right)$$

and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots, \alpha_n)$ is an arbitrary n-dimensional vector of positive integers. It is clear, that the multiplicities of the eigenvalues depend on the parameters d_k, $k = 1, 2, \ldots, n$ and generally they may be greater than one. It should be mentioned that this fact plays an important role in the controllability investigations.

The normalized eigenvectors $x_\alpha(y) \subset D(A_0)$ corresponding to the eigenvalue s_α form a complete ortonormal set in the space X and are given by the following formula

$$x_\alpha(y) = 2^{\frac{n}{2}} (d_1 d_2 \ldots d_k \ldots d_n)^{-\frac{1}{2}} \sin\left(\frac{\pi\alpha_1}{d_1} y_1\right) \sin\left(\frac{\pi\alpha_2}{d_2} y_2\right) \ldots \sin\left(\frac{\pi\alpha_k}{d_k} y_k\right) \ldots \sin\left(\frac{\pi\alpha_n}{d_n} y_n\right)$$

Moreover, the linear unbounded differential operator A_0 is the infinitesimal generator of a strongly continuous semigroup of linear bounded and compact operators.

Now, we shall formulate the necessary and sufficient conditions for approximate relative controllability of the dynamical system (11).

Theorem 3. *Suppose, that all the eigenvalues s_α of the operator A_0 are simple. Then the dynamical system (11) is approximately relatively controllable with non-negative controls if and only if*

$$\langle b_1, w_\alpha \rangle \langle b_2, w_\alpha \rangle < 0 \quad \textit{for every index } \alpha \tag{15}$$

Proof. In order to prove Theorem 3 it is sufficient to verify the assumptions of Theorem 2. First of all, let us observe that the operator A_0 is self-adjoint and generates a compact semigroup, so assumption (i) is satisfied. Next, from the form of eqn. (11) it follows immediately that assumption (ii) holds. Moreover, since the operator A_0 is self-adjoint and has a simple spectrum, then the eigenvectors $w_i(y)$, $i = 1, 2, 3, \ldots$ form a complete ortonormal system in the Hilbert space $X = L_2(\Omega)$. Hence assumption (iii) follows. Assumption (iv) is evidently satisfied. Hence Theorem 3 follows. ■

6. Examples

We illustrate the theory we have developed in the previous sections based on simple examples of linear retarded distributed parameter dynamical systems. We shall consider constrained approximate relative controllability of a dynamical system described by a linear partial differential equation of parabolic type with zero Dirichlet boundary conditions and with one lumped constant delay.

Example 1. Let us consider the following linear retarded partial parabolic differential equation with one constant delay

$$\frac{\partial w(t,y)}{\partial t} = a_0 \frac{\partial^2 w(t,y)}{\partial y^2} + cw(t,y) + a_h w(t-h,y) + b_1(y)u_1(t) + b_2(y)u_2(t) \quad (16)$$

defined for $y \in [0,d]$ and $t \in [0,\infty)$, and satisfying the homogeneous Dirichlet boundary conditions

$$w(t,0) = w(t,d) = 0 \ \text{ for } \ t \in [0,\infty) \quad (17)$$

The initial conditions for the delayed equation (16) are as follows

$$w(0,y) = w^0(y) \in L_2([0,d],\mathbb{R}) = X \quad (18)$$

$$w(t,y) = w_0(\tau,y) \in L_2([-h,0]\times[0,d],\mathbb{R}) \quad (19)$$

Moreover, it is assumed that a_0, a_h, and c are real coefficients, and $h > 0$ is a constant delay, $b_j(y) \in L_2([0,d],\mathbb{R})$ for $j = 1,2$. The admissible controls are square integrable and non-negative, i.e. $u_j(t) \geq 0$ for $t \geq 0$, $j = 1,2$.

For the retarded dynamical system (16) the complete state at time $t \geq 0$, denoted as z_t, is a pair of the functions $z_t = \{w(t,y), w_t(\tau,y)\} \in M_2([-h,0],X) = X \times L_2([-h,0],X)$, where $X = L_2([0,d],\mathbb{R})$.

In the abstract setting, the linear retarded partial differential equation (16) can be represented by the following retarded abstract ordinary differential equation:

$$\dot{x}(t) = A_0 x(t) + a_h x(t-h) + b_1 u_1(t) + b_2 u_2(t) \quad (20)$$

where $x(t) \in X = L_2([0,d],\mathbb{R})$ and the linear unbounded differential operator A_0 is defined as follows

$$A_0 x = A_0 w(y) = a \frac{\partial^2 w(y)}{\partial y^2} + cw(y) \quad (21)$$

The domain $D(A_0) \subset X = L_2([0,d],\mathbb{R})$ is defined as follows

$$D(A_0) = \{w(y) = X \in X : A_0 w(y) \in X, \ \ w(0) = w(d) = 0\} \quad (22)$$

It is well-known (see e.g. Triggiani, 1975; 1976) that the linear unbounded differential operator A_0 is self-adjoint and generates a compact semigroup. Moreover, its real single eigenvalues are $s_i = -i^2\pi^2/d^2 + c$ for $i = 1,2,3,\ldots$ The corresponding orthonormal eigenvectors are $x_i(y) = 2^{0.5}\sin(i\pi y/d)$, for $i = 1, 2, 3, \ldots$ The eigenvectors $x_i(y) \in D(A_0)$, $i = 1, 2, 3, \ldots$ form a complete orthonormal set in the Hilbert space X.

Now, we shall verify the constrained approximate relative controllability of the retarded dynamical system (11). In order to do that, let us observe that assumptions (i), (ii), (iii) and (iv) of Theorem 2 are satisfied. Therefore, the dynamical system (11) is constrained approximately relatively controllable if and only if inequalities (10) are satisfied. Taking into account the form of the inner product in the Hilbert space $L_2([0,d],\mathbb{R})$, from relation (10) we immediately obtain the following inequalities

$$\left(\int_0^d b_1(y)\sin\left(\frac{i\pi y}{d}\right)\mathrm{d}y\right)\left(\int_0^d b_2(y)\sin\left(\frac{i\pi y}{d}\right)\mathrm{d}y\right) < 0 \quad \text{for } i = 1, 2, 3, \ldots \tag{23}$$

Hence, the dynamical system (16) is approximately relatively controllable with non-negative controls if and only if inequalities (23) hold.

However, it should be stressed that for unconstrained controls the condition for approximate relative controllability of the retarded dynamical system (16) is less restrictive. Namely, the dynamical system (16) is approximately relatively controllable if and only if

$$\left(\int_0^d b_1(y)\sin\left(\frac{i\pi y}{d}\right)\mathrm{d}y\right)^2 + \left(\int_0^d b_2(y)\sin\left(\frac{i\pi y}{d}\right)\mathrm{d}y\right)^2 \neq 0 \quad \text{for } i = 1, 2, 3, \ldots \tag{24}$$

From formulae (23) and (24) it follows immediately that for the retarded dynamical system (1) constrained approximate relative controllability always implies approximate relative controllability without any constraints.

Example 2. As the second example we shall consider a linear partial differential equation of parabolic type, defined on an infinite interval. This dynamical system is described by the following partial differential equation with one constant delay:

$$\frac{\partial w(t,y)}{\partial t} = \frac{\partial^2 w(t,y)}{\partial y^2} + (k - y^2)w(t,y) + a_h w(t-h,y) + b_1(y)u_1(t) + b_2(y)u_2(t) \tag{25}$$

defined for $y \in \mathbb{R}$ and $t \in \mathbb{R}^+$, and with the initial condition

$$w(0,y) = w^0(y) \in L_2(\mathbb{R}) = X$$

The initial conditions for the delayed equation (25) are as follows

$$w(t,y) = w_0(\tau,y) \in L_2([-h,0],\ L_2(\mathbb{R}))$$

Moreover, $h > 0$ is a constant delay, $a_h \in \mathbb{R}$ is a constant coefficient, k is an integer and $b_1(y) \in L_2(\mathbb{R})$, $b_2(y) \in L_2(\mathbb{R})$. The admissible controls are non-negative and square integrable, i.e. $u_j(t) \in L_2([0,\infty),\mathbb{R}^+)$ for $j = 1, 2$.

For the retarded dynamical system (25) the complete state at time $t \geq 0$ denoted as z_t is a pair of functions

$$z_t = \{w(t,y), w_t(\tau,y)\} \in M_2([-h,0],X) = X \times L_2([-h,0],X)$$

where $X = L_2(\mathbb{R})$.
In an abstract setting, the retarded partial differential equation (25) can be represented by the following retarded abstract ordinary differential equation

$$\dot{x}(t) = A_{0k}x(t) + ax(t-h) + b_1u_1(t) + b_2u_2(t) \tag{26}$$

where $w(y) = x \in X = L_2(\mathbb{R})$, and the linear unbounded differential operator A_{0k} is defined as follows:

$$\begin{aligned} &A_{0k} : X \supset D(A_{0k}) \to X \\ &D(A_{0k}) = \{w(y) = x \in L_2(\mathbb{R}) : A_{0k}w(y) \in L_2(\mathbb{R})\} \\ &A_{0k}x = A_{0k}w(y) = \frac{\mathrm{d}^2 w(y)}{\mathrm{d}y^2} + (k - y^2)w(y) \end{aligned} \tag{27}$$

Now, let us collect some well-known facts about spectral properties of the operator A_{0k}. First of all, let us observe that the linear unbounded operator A_{0k} is self-adjoint and has the compact resolvent for all integers k. Moreover, it is an infinitesimal generator of a compact analytic semigroup of linear bounded operators. The operator A_{0k} has only a discrete pure point spectrum consisting entirely of single eigenvalues $s_{ki} = -2i + k - 1$, for $i = 0,1,2,\ldots$. The corresponding eigenfunctions $x_i(y) \in D(A_{0k})$ for $i = 0,1,2,\ldots$ have the following form:

$$x_i(y) = (2^i i!)^{-0.5}\pi^{-0.25}H_i(y)\exp(-0.5y^2),\ i = 0,1,2,\ldots$$

where $H_i(y)$, $i = 0,1,2,\ldots$ are Hermite polynomials defined as follows

$$H_i(y) = (-1)^i \exp(y^2)\frac{\mathrm{d}^i}{\mathrm{d}y^i}\exp(-y^2),\quad i = 0,1,2\ldots$$

The eigenfunctions $x_i(y)$, $i = 0,1,2,\ldots$ form a complete orthonormal system in the Hilbert space $X = L_2(\mathbb{R})$.

Now, let us investigate the constrained approximate relative controllability for the retarded dynamical system (25). First of all, similarly as in Example 1, let us observe that all the assumptions of Theorem 2 are satisfied. Therefore, the retarded dynamical system (25) is constrained approximately relatively controllable if and only if inequalities (10) are satisfied. Taking into account the form of the inner product in the Hilbert space $L_2(\mathbb{R})$ from relations (10) we obtain immediately the following inequalities

$$\left(\int_{-\infty}^{+\infty} b_1(y)x_i(y)\mathrm{d}y\right)\left(\int_{-\infty}^{+\infty} b_2(y)x_i(y)\mathrm{d}y\right) < 0 \ \text{ for } i = 0,1,2,\ldots \tag{28}$$

Hence, the retarded dynamical system (25) is approximately relatively controllable with non-negative controls if and only if inequalities (28) are satisfied.

Example 3. Let us consider a two-dimensional linear parabolic partial differential equation with Dirichlet type boundary conditions and a constant delay in the state variables.

$$\frac{\partial w(t,y,z)}{\partial t} = \frac{\partial^2 w(t,y,z)}{\partial y^2} + \frac{\partial^2 w(t,y,z)}{\partial z^2}$$
$$+a_h w(t-h,y,z) + b_1(y,z)u_1(t) + b_2(y,z)u_2(t) \qquad (29)$$

defined in a two-dimensional rectangular domain $\Omega \in \{(y,z) \in \mathbb{R}^2; y([0,d], z \in [0,p]\}$ with the boundary Γ.

The homogeneous Dirichlet boundary conditions are of the following form

$$w(t,y,z) = 0 \text{ for } (y,z) \in \Gamma \text{ and } t \geq 0$$

The initial conditions for the equation are given by the following formulae

$$w(0,y,z) = w^0(y,z) \in L_2(\Omega,\mathbb{R}) = X$$
$$w(t,y,z) = w_0(\tau,y,z) \in L_2([-h,0],X) \text{ for } t \in [-h,0]$$

Hence, the complete state for the retarded dynamical system (29) is given by the following relation

$$z_t = \{w(t,y,z),\ w_t(\tau,y,z)\} \in X \times L_2([-h,0],X) = M_2([-h,0],X)$$

Similarly as in the previous examples, it is assumed that the admissible controls are square integrable and non-negative, i.e. $u_j(t) \in L_2([0,\infty),\mathbb{R}^+)$, for $j = 1,2$.

In an abstract form, the retarded partial differential equation (29) can be represented by the following ordinary abstract differential equation

$$\dot{x}(t) = A_0 x(t) + x(t-h) + b_1 u_1(t) + b_2 u_2(t) \qquad (30)$$

where $x(t) = w(t,y,z) \in X = L_2(\Omega,\mathbb{R})$, $A_0 : X \supset D(A_0) \to X$ is a linear unbounded differential operator defined as follows

$$D(A_0) = \{w(y,z) = x \in X :$$
$$A_0 w(y,z) \in X \text{ and } w(y,z) = 0 \text{ for } (y,z) \in \Gamma\}$$
$$A_0 w(y,z) = \frac{\partial^2 w(y,z)}{\partial y^2} + \frac{\partial^2 w(y,z)}{\partial z^2}$$

It is well-known that the operator A_0 is self-adjoint and has a pure discrete point spectrum consisting of entirely the real eigenvalues $\sigma = \{s_{ik} = (i\pi/d)^2 + (k\pi/p)^2;\ i = 1,2,...,\ k = 1,2,...\}$. Moreover, the corresponding eigenfunctions have the following form

$$x_{ik}(y,z) = \frac{2}{\sqrt{dp}} \sin\left(\frac{i\pi}{d}y\right) \sin\left(\frac{k\pi}{p}z\right) \quad \text{for } i = 1,2...,\ k = 1,2...$$

Now, let us verify the constrained controllability conditions for the dynamical system under consideration. First of all let us assume that the parameters d and p of the rectangle Ω are such that all the eigenvalues s_{ik} are simple. Hence, taking into account the form of the inner product in the Hilbert space $L_2(\Omega)$ and inequality (10) we obtain the following relations:

$$\langle b_1(y,z), x_{ik}(y,z)\rangle \langle b_2(y,z), x_{ik}(y,z)\rangle$$

$$= \left(\iint_{\Omega} b_1(y,z), x_{ik}(y,z)\mathrm{d}y\mathrm{d}z \right)$$

$$\cdot \left(\iint_{\Omega} b_2(y,z), x_{ik}(y,z)\mathrm{d}y\mathrm{d}z \right) < 0 \ \text{ for } \ i=1,2,\dots,k=1,2,\dots \tag{31}$$

Therefore, the retarded dynamical system (29) is approximately relatively controllable with non-negative controls if and only if relation (31) holds.

6. Conclusions

In the present paper, constrained controllability problems for linear abstract retarded dynamical systems have been investigated. Using some very general results taken from the paper (Son, 1990) the necessary and sufficient conditions for constrained approximate relative controllability with non-negative controls have been formulated and proved. Moreover, the relationships between different types of controllability for abstract retarded dynamical systems have been explained and discussed. Finally, simple illustrative examples have been studied in detail. These examples represent a linear retarded dynamical system with distributed parameters described by a partial differential equation of parabolic type with homogeneous Dirichlet boundary conditions and a constant delay in the state variables.

Moreover, it should be pointed out that the results given in the present paper can be easily extended to the case of abstract retarded dynamical systems with many lumped constant delays in the state variables and also to other types of control constraints.

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