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ON CONVEX COMBINATIONS OF HURWITZ POLYNOMIALS†

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The zero set of a convex hull of two polynomials is investigated with special attention to the class of Hurwitz polynomials. New necessary and sufficient conditions are given for a convex hull of these polynomials to belong to the same class. An algorithm of construction of Hurwitz interval polynomials containing a given Hurwitz polynomial is formulated.

1. Introduction

Location of roots of a polynomial is an age-old problem of mathematics. System-theoretical considerations focused attention on qualitative decisions on the set of zeros of a given polynomial. The result of Hurwitz more than a century old is certainly one of the most important mathematical achievements of this type.

The classical setting of the Hurwitz problem has been solved by three equivalent approaches. Let p be a polynomial; the validity of the implication

$$p(z_0) = 0 \Longrightarrow \mathrm{Re} z_0 < 0$$

can be checked in terms of the coefficients of p or in terms of the behaviour of $\arg p(j\omega)$ or in terms of the zeros of the polynomials a and b in $p(s) = a(s^2) + s\, b(s^2)$. All these found their appropriate field of applications.

A renewed interest in Hurwitz polynomials during the last years has been invoked mainly by studies of sensitivity of linear systems to small perturbations of their parameters. First of all Kharitonov's well-known result (Kharitonov, 1979) has to be mentioned here together with an ever growing number of subsequent results. Robust stability has become an exciting and important field of research for both mathematicians and engineers. Recently the book (Barmish, 1993) has given an up-to-date summary of these efforts.

One of the important achievements is the so-called Edge theorem, given in (Huang *et al.*, 1987). Its application demands an answer to the following question: Given two Hurwitz polynomials p and q, under which conditions can it be guaranteed that their convex hull consists of Hurwitz polynomials only? The solution to this

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problem has been given in terms of the coefficients in (Białas, 1985) and in terms of $\arg p(j\omega)$, $\arg q(j\omega)$ in (Rantzer, 1992).

In this paper, we solve the convex hull problem in terms of the even and odd parts of p and q. Our results are based on a generalization of the well-known interlacing property of zeros of polynomials possessing real and simple zeros only (Theorem 2). For convex hulls of polynomials we prove the necessary and sufficient conditions in terms of the even and odd parts of the polynomials involved. Our Theorems 3 and 4 complete the results of (Białas, 1985) and (Rantzer, 1992) in the above-mentioned sense and, similarly as in the classical setting, enable us to obtain constructive results in some robustness problems, e.g. construction of stable interval polynomials from a fixed given polynomial. Our theorems also strengthen some results of (Bose, 1985) and generalize those in (Białas, 1985). In our opinion, they also make the solution to some decision problems in this context easier.

We summarize the notation and basic facts in Section 2. Here the relevant subclasses of Hurwitz polynomials are considered and the concept of a convex pair of Hurwitz polynomials is discussed. In Section 3 new results on polynomials with real and simple zeros are given. These results include the necessary and sufficient conditions for convex combinations of these polynomials to belong to the same class. Section 4 contains new results on convex combinations of Hurwitz polynomials and Section 5 shows some applications to problems of robust system theory.

2. Notation and Basic Facts

Notation and basic facts will be presented here mainly for future references.

In what follows, $p, q, \dots$ will denote polynomials with real coefficients. We denote the classes of Hurwitz and related polynomials as follows:

$$p \in \mathcal{H} \iff \Big(p(z_0) = 0 \Rightarrow \operatorname{Re} z_0 \le 0\Big)$$

$$p \in \mathcal{H}^{\mathrm{S}} \iff \Big(p(z_0) = 0 \Rightarrow \operatorname{Re} z_0 < 0\Big)$$

$$p \in \mathcal{M} \iff \Big(p(z_0) = 0 \Rightarrow \big(\operatorname{Re} z_0 = 0,\ p'(z_0) \neq 0\big)\Big)$$

Note that any polynomial $p \in \mathcal{M}$ has either only even or only odd powers.

If p and q are polynomials, then for the function $f = p/q$ we shall deal with the following classes of positive real functions:

$$f \in \mathcal{B} \iff \Big(\operatorname{Re} z > 0 \Rightarrow \operatorname{Re} f(z) > 0\Big)$$

$$f \in \mathcal{R} \iff \Big(f \in \mathcal{B},\ f(z) + f(-z) = 0\Big)$$

Functions from $\mathcal{B}$ ($\mathcal{R}$) are often called Brune (reactance) ones. It is well-known that both classes are closed under addition and multiplication by a positive constant. Moreover, $f \in \mathcal{B}$ ($\mathcal{R}$) iff $1/f \in \mathcal{B}$ ($\mathcal{R}$).

With $e_p(z) = \frac{1}{2}(p(z) + p(-z))$ and $o_p(z) = \frac{1}{2}(p(z) - p(-z))$ we will summarize some known facts:

$$p \in \mathcal{H}(\mathcal{H}^S) \Longrightarrow p' \in \mathcal{H}(\mathcal{H}^S) \tag{1}$$

$$p \in \mathcal{H}^S \Longleftrightarrow \frac{e_p}{o_p} \in \mathcal{R} \tag{2}$$

$$f \in \mathcal{B}(\mathcal{R}), f = p/q \Longrightarrow p, q \in \mathcal{H}(\mathcal{M}) \tag{3}$$

$$p \in \mathcal{H}(\mathcal{M}^R) \Longleftrightarrow \frac{p'}{p} \in \mathcal{B}(\mathcal{R}) \tag{4}$$

Here, (1) is a specialization of the Lucas theorem (see Householder, 1970), while in (2) e_p and o_p are supposed to be coprime. The remaining relations follow from the principle of the argument. These relations are important tools in investigating Hurwitz polynomials.

Our main concern is with pairs of polynomials. Since Hurwitz polynomials have coefficients of equal sign, only linear combinations with positive multipliers have to be considered.

Definition 1. If $p, r \in \mathcal{H}(\mathcal{H}^S)$, then we call (p, r) a convex pair if $\lambda p + (1-\lambda) r \in \mathcal{H}(\mathcal{H}^S)$ for all $0 \leq \lambda \leq 1$.

If the corresponding convex combination belongs to a larger class $\mathcal{C}$, we shall say that (p, r) forms a convex pair in $\mathcal{C}$.

Combining (1)–(4) above, we easily obtain the following facts.

Fact 1. *If $p \in \mathcal{H}^S$, then (p, p') and (p, p'') are convex pairs.*

Fact 2. *Let $p, r \in \mathcal{H}^S$. Then (p, r) is a convex pair iff (ap, br) is a convex pair for all positive real a and b.*

To see the last statement, observe that for any $\lambda \in [0, 1]$

$$\lambda ap + (1-\lambda)br = \Big(\lambda a + (1-\lambda)b\Big)\left(\frac{\lambda a}{\lambda a + (1-\lambda)b} p + \frac{(1-\lambda b)}{\lambda a + (1-\lambda)b} r\right) \in \mathcal{H}^S$$

Fact 3. i) *Any two polynomials in $\mathcal{H}^S$ of degree less than or equal to two form a convex pair.*

ii) *If $p/r \in \mathcal{B}$, then (p, r) is a convex pair.*

Note that (ii) can be deduced as follows: for any $\lambda > 0$

$$\frac{p}{r} + \lambda = \frac{p + \lambda r}{r} \in \mathcal{B}$$

By (3) $p + \lambda r \in \mathcal{H}$, and using Fact 2 we obtain the desired result.

More interesting is the following known result:

Theorem 1. *If* (p, r) *is a convex pair in* $\mathcal{H}^S$, *then*

$$|\deg p - \deg r| \leq 2$$

A simple proof of this statement is a corollary of Theorem 2.

Example 1. i) For $p(z) = 4z^3 + 4z^2 + 5z + 4, p \in \mathcal{H}^S$ none of its indefinite integrals belongs to $\mathcal{H}^S$. (cf. (1))

ii) Two polynomials in $\mathcal{H}^S$ may form a convex pair in $\mathcal{H}$ but not in $\mathcal{H}^S$. Take $p(z) = z^3 + 6z^2 + 11z + 6$ and $r(z) = z^2 + z + 17 + 4\sqrt{15}$. For $q = p + \lambda r$ we obtain $q \in \mathcal{H}$ for all $\lambda \geq 0$ but $q \notin \mathcal{H}^S$ for $\lambda = 2\sqrt{15}$. Indeed, $p + 2r\sqrt{15} = z^3 + (6 + 2\sqrt{15})z^2 + (11 + 2\sqrt{15})z + 126 + 34\sqrt{15}$ has a pure imaginary zero z_0 with $z_0^2 = -11 - 2\sqrt{15}$.

Some polynomials of equal degree form convex pairs. The corresponding necessary and sufficient conditions have been formulated in (Białas, 1985) as follows. If $p, r \in \mathcal{H}^S, \deg p = \deg r = n$, then (p, r) is a convex pair iff all the real eigenvalues of the matrix $W = -H_n(p)\, H_n^{-1}(r)$ are negative. Here $H_n(p)$ stands for the square matrix forming the Hurwitz determinant of the polynomial p.

3. Convex Pairs

Denote by $\mathcal{Q}$ the set of even polynomials with real and simple zeros only and with a positive value at the point 0. The class $\mathcal{Q}$ consists of those polynomials that can be expressed as

$$d_0(d_1 - \omega^2)(d_2 - \omega^2) \dots (d_n - \omega^2)$$

for some positive mutually different $d_0, d_1, \dots, d_n$. If $p \in \mathcal{H}$, then $p = e_p + o_p$, where e_p and o_p are even and odd parts of p, respectively. It is well-known that $e_p(j\omega) \in \mathcal{Q}$ and $o_p(j\omega)/j\omega \in \mathcal{Q}$.

Definition 2. Two finite subsets $A, B \subset \mathbb{R}$ are called *interlacing* if for any two points in A there is a point in B lying between them and vice versa. The sets A and B are called *complementary* if there exists a finite set C that is interlacing with both A and B.

Note that the cardinality of two complementary sets A and B can differ by at most two. Also, the positioning of the sets A and B on the real axis is considered to be of no importance: a shift does not change the complementarity, nor the interlacing of the configuration. The notion of complementarity is similar to Weinberg's *double alternating* configuration as introduced in (Weinberg, 1962). Two polynomials in $\mathcal{Q}$ will be called interlacing (complementary) iff their zero sets are interlacing (complementary).

The following theorem motivates the concept of complementarity.

Theorem 2. *Let p and r be polynomials belonging to $\mathcal{Q}$. Then they form a convex pair in $\mathcal{Q}$ if and only if they are complementary.*

To prove this theorem, some specialized notation and a lemma will be used. Because of symmetry we shall consider the behaviour of the polynomials on $[0, \infty)$. The positive roots of p and r are denoted by a_i, $i = 1, 2, \ldots, n$, and $b_i, i = 1, 2, \ldots, m$, respectively, in the increasing way. We may also assume that $\deg p = 2n \leq 2m = \deg r$.

For $\lambda \geq 0$ and $i = 0, 1, \ldots, n$ we define the function $N_i(\lambda)$ as the number of roots of $\lambda p + r$ lying in the interval (a_i, a_{i+1}), where $a_0 = 0$ and $a_{n+1} = \infty$. Similarly, we can define $M_j(\lambda)$, $j = 0, 1, \ldots m$, to be the number of roots of $\lambda p + r$ in (b_j, b_{j+1}). Then the following lemma holds.

Lemma 1. *If p, $r \in \mathcal{Q}$ form a convex pair, then the functions N_i and M_j, $i = 0, 1, \ldots n$, $j = 0, 1, \ldots m$ are constant in $(0, \infty)$.*

Proof. We prove that all N_i's are constant. The same argument can be applied to M_j's. For a given i consider the sets $\{\lambda > 0 \mid N_i(\lambda) = k\}$ for $k = 0, 1, \ldots, \deg p/2$. They are clearly disjoint and since $\lambda p + r \in \mathcal{Q}$, the derivative of this polynomial at any of its root is non-zero. It follows that the sets defined above are also open (possibly empty). Further,

$$\bigcup_{k=0}^{\deg p/2} \left\{\lambda > 0 \mid N_i(\lambda) = k\right\} = (0, \infty)$$

Since $(0, \infty)$ is connected and the sets $\{\lambda > 0 \mid N_i(\lambda) = k\}$ form a disjoint open cover, there is one k_0 such that

$$(0, \infty) = \left\{\lambda > 0 \mid N_i(\lambda) = k_0\right\}$$

and the lemma follows. ∎

Now we are ready for the proof of Theorem 2.

Proof of Theorem 2. Let $A = \{a_i \mid i = 1, \ldots, n\}$ and $B = \{b_j \mid j = 1, \ldots, m\}$ denote the positive roots of p and r, respectively. We shall construct a set C interlacing both A and B. Put $0 \in C$. Look at the two smallest elements of $A \cup B$. They cannot belong to the same set A or B. Indeed, suppose that $0 < a_1 < a_2 \leq b_1$. It follows that $M_0(\lambda) \geq 2$ for some $\lambda > 0$. But for a sufficiently small λ all but one roots of $\lambda p + r$ in $(0, b_1)$ disappear. This contradicts Lemma 1. The same argument applies to the case where $0 < b_1 < b_2 \leq a_1$. Hence there is a $c_1 > 0$ such that it separates $\{a_1, b_1\}$ and the rest of $A \cup B$. Now we continue in a similar way. Look at the two smallest elements of $A \cup B \setminus \{a_1, b_1\}$. It cannot happen again that they belong to the same set A or B. If so, then either $0 < c_1 < a_2 < a_3 \leq b_2$ or $0 < c_1 < b_2 < b_3 \leq a_2$. Both p and r have the same sign at c_1. In the former case, p and r have opposite signs on (a_2, a_3). So for λ decreasing to 0 the polynomial

$\lambda p + r$ loses at least two roots in (b_1, b_2), which is a contradiction to Lemma 1. In the latter case, the polynomials p and r have opposite signs on (b_2, b_3) and for $\lambda \to \infty$ the polynomial $\lambda p + r$ loses at least two roots in (a_1, a_2). We can therefore conclude that there is a $c_2 > c_1$ such that it separates $\{a_2, b_2\}$ from all larger elements of $A \cup B$. Proceeding in the above manner we get a set $c_1 < c_2 < \cdots < c_n$ with the property

$$\begin{aligned} \max(a_1, b_1) &< c_1 < \min(a_2, b_2), \\ \max(a_2, b_2) &< c_2 < \min(a_3, b_3), \\ \vdots \quad &\quad \vdots \quad\quad \vdots \\ \max(a_n, b_n) &< c_n < b_{n+1} \end{aligned}$$

(Recall that if $\deg r = n$, then $b_{n+1} = \infty$.) At this point all positive roots of p have been used. Moreover, we notice that both p and r have the same sign at every $c_1, \dots, c_n$. Owing to a_n the polynomial p does not change sign. Our already familiar argument reveals that there cannot be more than one remaining element of B. Thus $C \cup (-C)$ is the required set interlacing the roots of both p and r.

To prove the converse we assume that C with the elements $0 = c_0 < c_1 < \cdots < c_n$ is the set interlacing positive roots of both p and r. The values of the polynomials p and r at the points of C have the same sign, namely

$$\operatorname{sign} p(c_i) = \operatorname{sign} r(c_i) = (-1)^i$$

for $i = 0, 1, \dots, n$. It follows that for any combination $\lambda p + r$ we have

$$\operatorname{sign}(\lambda p + r)(c_i) = (-1)^i$$

for $i = 0, 1, \dots, n$. If $\deg p = \deg r = 2n$, the proof is complete because the above combination has exactly $2n$ (symmetrical) roots. It remains to consider the case when $\deg r = 2n + 2$. Since the interval $[0, c_n]$ contains already n roots of the combination $\lambda p + r$, it suffices to show that there is another root greater than c_n. On $[c_n, \infty)$ the polynomial p has the sign $(-1)^n$ while

$$\lim_{x \to \infty} \operatorname{sign} r(x) = (-1)^{n+1}$$

Since $\deg r$ is greater than $\deg p$,

$$\lim_{x \to \infty} \operatorname{sign}(\lambda p(x) + r(x)) = (-1)^{n+1}$$

But $\operatorname{sign}(\lambda p(c_n) + r(c_n)) = (-1)^n$. Thus, the polynomial $\lambda p + r$ has to possess a root in $[c_n, \infty)$. ∎

The proof of Theorem 1 follows now from Theorem 2. Indeed, if two polynomials form a convex pair in $\mathcal{H}^S$, then their even (odd) parts are complementary (Theorem 2). Hence the degrees of even (odd) parts may differ by at most two.

4. Convex Pairs of Hurwitz Polynomials

For polynomials in $\mathcal{H}^S$ the following necessary and sufficient condition has been proved (Huang *et al.*, 1987):

Theorem 3. *Polynomials $p,\ r \in \mathcal{H}^S$ form a convex pair if and only if*

$$\arg\Big[p(j\omega)/r(j\omega)\Big] \neq \pi$$

for all $\omega \geq 0$.

Verification of this condition might be computationally inconvenient.

Some necessary conditions follow easily from Theorem 2. As already mentioned before Definition 2, if $p \in \mathcal{H}^S$, then $e_p(j\omega), o_p(j\omega)/j\omega \in \mathcal{Q}$. Therefore we shall call a pair $(e_p,\ e_r)$ (or $(o_p,\ o_r)$) complementary iff $e_p(j\omega)$ and $e_r(j\omega)$ (or $o_p(j\omega)/j\omega$ and $o_r(j\omega)/j\omega$) are complementary. Similarly, we shall use the term interlacing.

Lemma 2. *If $p,\ r \in \mathcal{H}^S$ form a convex pair, then the pairs of polynomials (e_p, e_r) and (o_p, o_r) are both complementary.*

This condition is not sufficient, as shown by the following example.

Example 2. Put $p(z) = 5z^3 + 10z^2 + 11z + 20, r(z) = 10z^3 + 10z^2 + 6z + 3$. We have $p,\ r \in \mathcal{H}^S$ and the conditions of Lemma 2 are satisfied, which can readily be verified. On the other hand, $2p + r \notin \mathcal{H}^S$.

To examine the sufficient conditions we shall reformulate one of the implications of Theorem 3, introducing

$$D(\omega) = (e_p o_r - e_r o_p)(j\omega)$$

Lemma 3. *If $p, r \in \mathcal{H}^S$ form a convex pair, then $(e_p e_r + o_p o_r)(j\omega_0) > 0$ for all $\omega_0 > 0$ such that $D(\omega_0) = 0$.*

Proof. From Theorem 3, it follows that $p(j\omega) + \lambda r(j\omega) \neq 0$ for all $\lambda > 0$ and for all $\omega \geq 0$, since $p, r \in \mathcal{H}^S$. Rewrite this condition as $(e_p + \lambda e_r + o_p + \lambda o_r)(j\omega) \neq 0$. Since the first two summands are real and the remaining two are pure imaginary, it follows that if $(p,\ r)$ is a convex pair, then $e_p + \lambda e_r$ and $o_p + \lambda o_r$ have no common roots for any $\lambda > 0$. Consider some $\omega_0 > 0$ with $D(\omega_0) = 0$, and the system

$$e_p + \lambda e_r = 0, \qquad o_p + \lambda o_r = 0$$

at the point $j\omega_0$. Since $p,\ r \in \mathcal{H}^S$, this system has no positive solution for λ. If $e_p = e_r = 0$ at $j\omega_0$, then both o_p and o_r are non-zero. Now the impossibility of solution yields $o_p o_r > 0$. The case when exactly one of the numbers e_p and e_r equals zero is excluded by the condition $D(j\omega_0) = 0$. It remains to consider the case where both e_p and e_r are nonzero at $j\omega_0$. If $e_p e_r > 0$, then $D(j\omega) = 0$ implies that also $o_p o_r \geq 0$. The last possibility $e_p e_r < 0$ is not allowed since it would imply the existence of a positive solution of the system above. Finally, we get $e_p e_r + o_p o_r > 0$ at $j\omega_0$. ■

Theorem 4. *Polynomials $p, r \in \mathcal{H}^S$ form a convex pair if and only if both (e_p, e_r) and (o_p, o_r) are complementary pairs and $e_p e_r + o_p o_r > 0$ for all pure imaginary solutions of the equation $e_p o_r - e_r o_p = 0$.*

Proof. The conditions are necessary, as has been proved in Lemma 2 and Lemma 3. We shall prove that they are sufficient. If (e_p, e_r) and (o_p, o_r) are complementary pairs, then according to Theorem 2 they form a convex pair in $\mathcal{Q}$. Let us introduce the polynomials $P(\lambda, \omega)$ and $R(\lambda, \omega)$ of two real variables by the requirements

$$P(\lambda, \omega) = \lambda\, e_p(j\omega) + (1 - \lambda)\, e_r(j\omega)$$

$$R(\lambda, \omega) = \lambda\, \hat{o}_p(j\omega) + (1 - \lambda)\, \hat{o}_r(j\omega)$$

where $\hat{o}(z) = o(z)/z$. Then for all $\lambda \in [0, 1]$ both $P(\lambda, \omega)$ and $R(\lambda, \omega)$ have simple positive zeros. We show that for all $\lambda \in (0, 1)$ the polynomials $P(\lambda, \omega)$ and $R(\lambda, \omega)$ are interlacing. Each of the sets $\{[(\lambda, \omega)] : P(\lambda, \omega) = 0\}$ and $\{[(\lambda, \omega)] : R(\lambda, \omega) = 0\}$ consists of finitely many curves that start for $\lambda = 0$ at the interlacing positions. If for some $\lambda \in (0, 1)$ the polynomials $P(\lambda, \omega)$ and $R(\lambda, \omega)$ are not interlacing, then there must exist such $[\lambda_0, \omega_0]$ that $0 = P(\lambda_0, \omega_0) = R(\lambda_0, \omega_0)$, i.e.

$$\begin{aligned} \lambda_0 \Big[e_p(j\omega_0) - e_r(j\omega_0)\Big] + e_r(j\omega_0) = 0 \\ \lambda_0 \Big[o_p(j\omega_0) - o_r(j\omega_0)\Big] + o_r(j\omega_0) = 0 \end{aligned} \tag{5}$$

If we consider the following system of equations

$$u\Big[e_p(j\omega_0) - e_r(j\omega_0)\Big] + v e_r(j\omega_0) = 0$$

$$u\Big[o_p(j\omega_0) - o_r(j\omega_0)\Big] + v o_r(j\omega_0) = 0$$

we see from (5) that the pair $(\lambda_0, 1)$ is one of its solutions. Therefore the determinant of this system, namely $e_p\, o_r - e_r\, o_p$, equals zero at the point $j\omega_0$. Also, we have $e_p(j\omega_0)\, e_r(j\omega_0) + o_p(j\omega_0)\, o_r(j\omega_0) > 0$. Hence at least one summand is positive. We may assume that it is the first one. Then from (5)

$$\lambda_0 = \frac{e_r(j\omega_0)}{e_r(j\omega_0) - e_p(j\omega_0)} \notin [0, 1]$$

which is a contradiction. Since for any $\lambda \in [0, 1]$ the polynomials $P(\lambda, \omega)$ and $R(\lambda, \omega)$ are interlacing and, according to Theorem 2, both of them also belong to $\mathcal{Q}$, they compose a polynomial in $\mathcal{H}^S$. ■

The positions of the zeros of the even and odd parts of two polynomials may not be sufficient for the characterization of the Hurwitz property of their convex combination. But the following stronger notion is completely characterized by the positions of the roots.

Definition 3. Let

$$p(s) = e_p(s) + o_p(s) \in \mathcal{H}(\mathcal{H}^S)$$
$$r(s) = e_r(s) + o_r(s) \in \mathcal{H}(\mathcal{H}^S)$$

Denote for $\lambda,\ \mu \in [0,1]$

$$p_0(s) = \lambda e_p(s) + \mu o_p(s)$$
$$r_0(s) = (1-\lambda)e_r(s) + (1-\mu)o_r(s)$$

Then $(p,\ r)$ will be called a *strongly convex pair* if $p_0+r_0 \in \mathcal{H}(\mathcal{H}^S)$ for all $\lambda,\mu \in [0,1]$.

Note that a convex pair need not be strongly convex. When reducing the requirements of Definition 3 so that $\lambda = \mu$ the strongly convex pairs become convex. The concept of strong convexity, although not explicitly, has been used in (Basu, 1990).

Strong convexity is characterized by the zeros of the even and odd polynomials involved, as shown by the following result.

Theorem 5. *Let polynomials $p, r \in \mathcal{H}^S$ be as in Definition 3. They form a strongly convex pair if and only if $(e_p,\ o_r)$ and $(e_r,\ o_p)$ are interlacing pairs.*

Proof. For the necessity of the condition of strong convexity put $\lambda = 1$ and $\mu = 0$. According to Definition 3 we have $e_p+o_r = p_0+r_0 \in \mathcal{H}^S$. Recall now that formula (2) yields that $(e_p,\ o_r)$ is an interlacing pair. Similarly for $(e_r,\ o_p)$.

The converse statement is easy to prove. Due to the interlacing property all the functions $o_r/e_p,\ o_p/e_p,\ o_p/e_r,\ o_r/e_r$ are reactance functions and remain in this class when multiplied by arbitrary positive constants. Since $f \in \mathcal{R}$ iff $1/f \in \mathcal{R}$, we also have

$$\frac{e_p}{\mu o_p + (1-\mu)o_r} = \frac{1}{\mu\frac{o_p}{e_p} + (1-\mu)\frac{o_r}{e_p}} \in \mathcal{R}$$

Similarly

$$\frac{e_r}{\mu o_p + (1-\mu)o_r} \in \mathcal{R}$$

Therefore

$$\frac{\lambda e_p + (1-\lambda)e_r}{\mu o_p + (1-\mu)o_r} \in \mathcal{R}$$

Again with equivalence (2), $p_0 + r_0 = \lambda e_p + (1-\lambda)e_r + \mu o_p + (1-\mu)o_r \in \mathcal{H}^S$, i.e. the polynomials $(p,\ r)$ form a strongly convex pair. ■

Example 3. The polynomials $p(s) = s^3+s^2+16\,s+7.29$ and $q(s) = s^3+s^2+1.69\,s+1$ in $\mathcal{H}^S$ form a convex pair which is not strongly convex.

Theorem 4 improves the result of Proposition 4 in (Bose, 1985), since we give an equivalent condition for polynomials to form a convex pair. Also Theorem 5 is an improvement of Theorem 2 in (Basu, 1990), when specialized to polynomials of one variable.

5. An Application to Interval Polynomials

Probably one of the most important examples of strongly convex Hurwitz pairs are the pairs of polynomials derived from the four polynomials known as Kharitonov polynomials.

The set

$$\mathcal{P} = \left\{ p(z) : \alpha_0 z^n + \alpha_1 z^{n-1} + \ldots + \alpha_n, a_i \le \alpha_i \le b_i,\, i = 0, 1, \ldots, n \right\}$$

is called an interval polynomial. Note that we also allow $a_i = b_i$. The following statement is known as the famous Kharitonov theorem (Kharitonov, 1979):

All polynomials $p \in \mathcal{P}$ belong to the class $\mathcal{H}^S$ iff all four so-called Kharitonov polynomials $k_i \in \mathcal{H}^S, i = 1, 2, 3, 4$, where

$$\begin{aligned} k_1(z) &= a_0 + a_1 z + b_2 z^2 + b_3 z^3 + a_4 z^4 + \ldots \\ k_2(z) &= a_0 + b_1 z + b_2 z^2 + a_3 z^3 + a_4 z^4 + \ldots \\ k_3(z) &= b_0 + a_1 z + a_2 z^2 + b_3 z^3 + b_4 z^4 + \ldots \\ k_4(z) &= b_0 + b_1 z + a_2 z^2 + a_3 z^3 + b_4 z^4 + \ldots \end{aligned} \tag{6}$$

Denoting the even and odd parts of these polynomials by e_i and o_i, respectively, we can see immediately that

$$e_1 = e_2, \qquad o_1 = o_3, \qquad e_3 = e_4, \qquad o_2 = o_4, \tag{7}$$

and therefore the pairs (e_1, o_1) and (e_4, o_4) determine uniquely the four Kharitonov polynomials. Recalling Theorem 5 it is easy to see that the conditions of Kharitonov's theorem will be satisfied if and only if two of the Kharitonov polynomials k_1 and k_4 form a strongly convex pair. The converse statement needs some additional conditions.

Lemma 4. *A strongly convex pair of Hurwitz polynomials* $(q, r), q(z) = q_0 + q_1 z + q_2 z^2 + \ldots, r(z) = r_0 + r_1 z + r_2 z^2 + \ldots$, *generates an interval polynomial* $\mathcal{P} \subset \mathcal{H}^S$ *iff*

$$(-1)^i q_{2i} \le (-1)^i r_{2i} \quad \text{and} \quad (-1)^i q_{2i+1} \le (-1)^i r_{2i+1}, \quad i = 0, 1, \ldots$$

Proof. Writing $r = e_r + o_r$, $q = e_q + o_q$ we set

$$p_1 = e_r + o_r, \quad p_2 = e_r + o_q, \quad p_3 = e_q + o_r, \quad p_4 = e_q + o_q \tag{8}$$

These four polynomials are in $\mathcal{H}^S$ since all (even and odd) pairs are interlacing. Comparing the definition of p_i with he four Kharitonov polynomials (6) it becomes clear that the assumptions in Lemma are exactly those, that make the intervals for the coefficients nonempty. ■

The condition in Lemma 4 is evidently equivalent to the following inequalities:

$$\begin{aligned} &e_q(j\omega) \le e_r(j\omega) && \text{for all real } \omega \\ &o_q(j\omega)/j\omega \le o_r(j\omega)/j\omega && \text{for all real } \omega \ne 0 \end{aligned} \tag{9}$$

Since zeros of the even and odd parts of polynomials in $\mathcal{H}^S$ are simple, there is only one possible configuration of the zeros of e_q, e_r, o_q, o_r if they are supposed to compose a quadruple of Kharitonov polynomials. Denoting the positive imaginary part of these zeros by er_i, eq_i, or_i, oq_i this configuration must be

$$0 < eq_1 < er_1 < oq_1 < or_1 < er_2 < eq_2 < or_2 < oq_2 < \dots \tag{10}$$

Indeed, in view of (9), $eq_1 \leq er_1, er_2 \leq eq_2, \dots, oq_1 \leq or_1, or_2 \leq oq_2, \dots$. Also the pairs (e_r, o_r) and (e_q, o_q) are interlacing since p and r are in $\mathcal{H}^S$. Moreover, (e_r, o_q) and (e_q, o_r) are interlacing by Theorem 5. So (10) is the only possible arrangement of zeros.

This reasoning implies also a procedure for the construction of an interval polynomial for a given Hurwitz polynomial, i.e. to find intervals, in which the coefficients of a given polynomial may vary while remaining in the class $\mathcal{H}^S$. The continuity argument obviously gives for any $p \in \mathcal{H}^S$ an interval polynomial $\mathcal{P}$ such that $p \in \mathcal{P} \subset \mathcal{H}^S$. We would like, however, to obtain at least some quantitative estimates concerning $\mathcal{P}$.

The corresponding algorithm can be described as follows:

Step 1. Let a polynomial $p(z) = a_0 + a_1 z + \ldots + a_n z^n \in \mathcal{H}^S$ be given. Denote by s_i and l_k the imaginary parts of the zeros of its even and odd parts on the imaginary axis, respectively, with ordering $0 < s_i < s_{i+1}, i = 1, \dots, m$, $0 < l_k < l_{k+1}, k = 1, \dots, M$, and find the value

$$d = \min_{i,k} |s_i - l_k|$$

Fix a real value λ such that $0 < \lambda < \min(s_1, d/2)$.

Step 2. Set $F_0(\underline{s}, \underline{d}) = \prod_{i=1}^m (s_i - (-1)^i d_i)$, $G_0(\underline{l}, \underline{c}) = \prod_{k=1}^M (l_k + (-1)^k c_k)$ with the values $\underline{d} = (d_i), \underline{c} = (c_k)$ to be determined. Define

$$F_{k+1} = \sum_{i=1}^n \frac{\partial}{\partial s_i} F_k$$

and similarly

$$G_{k+1} = \sum_{i=1}^n \frac{\partial}{\partial l_i} G_k$$

for $k = 0, 1, \dots, n$. Solve for d_i and c_i the following two sets of inequalities

$$\begin{aligned} F_0(\underline{s}, \underline{d}) &\geq F_0(\underline{s}, -\underline{d}) \qquad & G_0(\underline{l}, \underline{c}) &\geq G_0(\underline{l}, -\underline{c}) \\ F_1(\underline{s}, \underline{d}) &\leq F_1(\underline{s}, -\underline{d}) & G_1(\underline{l}, \underline{c}) &\leq G_1(\underline{l}, -\underline{c}) \\ F_2(\underline{s}, \underline{d}) &\geq F_2(\underline{s}, -\underline{d}) & G_2(\underline{l}, \underline{c}) &\geq G_2(\underline{l}, -\underline{c}) \\ &\vdots & &\vdots \end{aligned}$$

so that $0 \leq d_i \leq \lambda$; $0 \leq c_i \leq \lambda$.

Step 3. Form the polynomials e_r, e_q, o_r, o_q as follows:

$$e_r(\omega) = (-1)^m \prod_{j=1}^{m} \left(\omega^2 - \left(s_j - (-1)^j d_j\right)^2\right)$$

$$e_q(\omega) = (-1)^m \prod_{j=1}^{m} \left(\omega^2 - \left(s_j + (-1)^j d_j\right)^2\right)$$

$$o_r(\omega) = (-1)^m \omega \prod_{j=1}^{M} \left(\omega^2 - \left(l_j - (-1)^j c_j\right)^2\right)$$

$$o_q(\omega) = (-1)^m \omega \prod_{j=1}^{M} \left(\omega^2 - \left(l_j + (-1)^j c_j\right)^2\right)$$

Note that the inequalities in Step 2 guarantee that the inequalities among coefficients, as formulated in Lemma 4, will be satisfied.

Step 4. The four normalized Kharitonov polynomials can now be obtained as

$$p_4(z) = (\alpha e_r + \beta o_r)(-jz)$$

$$p_1(z) = (\alpha e_q + \beta o_q)(-jz)$$

$$p_2(z) = (\alpha e_q + \beta o_r)(-jz)$$

$$p_3(z) = (\alpha e_r + \beta o_q)(-jz)$$

where α and β are the leading coefficients of the even and odd parts of the given polynomial, respectively.

We include an example as an illustration.

Example 4. For the polynomial

$$p(z) = z(z^2+4)(z^2+16)(z^2+36) + 2(z^2+1)(z^2+9)(z^2+25)$$
$$= z^7 + 2z^6 + 56z^5 + 70z^4 + 784z^3 + 518z^2 + 2304z + 450$$

the first set of inequalities reduces to

$$d_1 + d_3 \geq d_2$$

$$8d_1 - 6d_2 + 4d_3 \leq 0$$

$$15d_1 - 5d_2 + 3d_3 \geq d_1 d_2 d_3$$

and the second one to

$$c_1 + c_3 \geq c_2$$

$$10c_1 - 8c_2 + 6c_3 \leq 0$$

$$24c_1 - 12c_2 + 6c_3 \geq c_1 c_2 c_3$$

Since $d = 1$, we have to find $d_i, c_j \in (0, \frac{1}{2})$. Some simple algebra suggests that e.g.

$$\begin{array}{lll} c_1 = 0.16, & c_2 = 0.35, & c_3 = 0.2 \\ d_1 = 0.14, & d_2 = 0.31, & d_3 = 0.18 \end{array}$$

Hence we obtain

$$e_r = z^6 + 35.3681z^4 + 238.4374z^2 + 252.3328$$

$$e_q = z^6 + 34.9281z^4 + 279.8223z^2 + 188.2552$$

$$o_r = z^7 + 56.4z^5 + 753.6z^3 + 2389.3z$$

$$o_q = z^7 + 55.9z^5 + 814.5z^3 + 2155.1z$$

The originally given polynomial p belongs to the interval polynomial $\mathcal{P}$ (degenerate interval coefficients are allowed)

$$\begin{aligned} \mathcal{P} = \Big\{ & z^7 + 2z^6 + [55.9,\ 56.4]z^5 + [69.8562,\ 70.7362]z^4 + [753.6,\ 814.5]z^3 \\ & +[476.8748,\ 559.6446]z^2 + [2155.1,\ 2389.3]z + [376.5104,\ 504.6656] \Big\} \end{aligned}$$

With another possible choice of c_i and d_i,

$$\begin{array}{lll} c_1 = 0.16, & c_2 = 0.4, & c_3 = 0.25 \\ d_1 = 0.1, & d_2 = 0.42, & d_3 = 0.42 \end{array}$$

we obtain the following interval polynomial

$$\begin{aligned} \mathcal{P} = \Big\{ & z^7 + 2z^6 + [55.8081,\ 56.6881]z^5 + [66.9656,\ 74.4856]z^4 \\ & +[748.9662,\ 817.5716]z^3 + [478.2815,\ 543.6267]z^2 \\ & +[2167.0887,\ 2361.96]z + [397.4644,\ 473.2094] \Big\} \end{aligned}$$

Robustness considerations might prefer intervals of maximal length. No decision of this type can be made between the two polynomials given here. However, the algorithm implies that among the admissible solutions of the inequalities formulated in Step 2 there exists one solution yielding the interval polynomial with maximal sum of lengths of intervals. The interval polynomial with such a maximal sum of lengths could be considered as an optimal solution.

In our case this sum equals 507.405 and 412.9669 for the first and second choices of d_i and c_i, respectively.

In case we want to enlarge the interval corresponding to the $(2k)$-th power in the polynomials obtained above, we may proceed as follows. Since multiplication with positive constants of either the odd or the even part of a Hurwitz polynomial retains the polynomial in the Hurwitz class, we may design a procedure which changes the length of any interval in the definition of the interval Hurwitz polynomial.

With e_1 and e_4 multiplied by α and β, respectively, the length of the $(2k)$-th interval of the even part of the interval polynomial $\mathcal{P}$ will be

$$\sigma = \begin{cases} \alpha b_{2k} - \beta a_{2k} & \text{for odd } k \\ \beta b_{2k} - \alpha a_{2k} & \text{for even } k \end{cases} \tag{11}$$

We want to find values α and β such that σ attains its maximum value under the following constraints

$$\begin{aligned} \alpha b_{2j} - \beta a_{2j} \geq 0 \qquad & \text{for odd } j \text{ and } j \neq k \\ \beta b_{2j} - \alpha a_{2j} \geq 0 \qquad & \text{for even } j \text{ and } j \neq k \end{aligned} \tag{12}$$

Evidently, if such α and β exist, then the pair αe_p, βe_r exhibits the same pattern of coefficients as required for Kharitonov polynomials in Lemma 4. If we set $\max_{j \neq k} a_{2j}/b_{2j} = \mu$, then inequalities (12) are satisfied iff

$$\frac{1}{\mu} \geq \frac{\alpha}{\beta} \geq \mu$$

To obtain an interval $[\beta\, a_{2k}, \alpha b_{2k}]$ (resp. $[\alpha\, a_{2k}, \beta b_{2k}]$) if k is even (resp. odd) of maximal length, we choose $\beta = \alpha\, \mu$ $(\beta = \alpha/\mu)$. The remaining free parameter can be used to locate suitably the centre of the k-th interval.

A similar procedure can be applied to the odd parts of the polynomials.

Example 5. The first interval polynomial in the previous example has intervals of zero length as coefficients with two highest powers. This restriction can be removed as follows. For its even part we have $k = 3$ and

$$\begin{aligned} \mu &= \max(69.8562/70.7362,\ 476.8748/559.6446,\ 376.5104/504.6656) \\ &= 0.98756 \end{aligned}$$

With $\alpha = 1, \beta = 1/\mu = 1.10126$ we obtain a 'new' even part of the polynomial and in a similar manner also a 'new' odd part (here $\mu = 0.991135$). Finally,

$$\begin{aligned} \mathcal{P} \;=\; & \Big\{[1,\ 1.0089]z^7 + [2,\ 2.0252]z^6 + 56.4z^5 + 70.7362z^4 + [760.31,\ 814.5]z^3 \\ & + [482.88,\ 559.6446]z^2 + [2174.28,\ 2389.3]z + [381.25,\ 504.6656]\Big\} \end{aligned}$$

A similar technique has been used by (Zeheb, 1988). The concept of strong convexity seems to give more transparency to his procedure.

6. Conclusions

It has been shown that complementarity and interlacing of zeros as defined here is an effective tool in characterizing properties of polynomials important in system theory. The core of the paper is formulated as Theorem 2. The results obtained here emphasize synthetical approaches. They enable us to construct models (polynomials and rational functions) with some prescribed robustness properties rather than decide on given ones. Examples seem to motivate further investigations in this direction.

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