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A NEW APPROACH TO CONVERGENCE ANALYSIS OF RLS-BASED SELF-TUNING STOCHASTIC CONTROL

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It is demonstrated that a recently derived bound on the error convergence rate in (open-loop) RLS estimation is applicable to RLS-based stochastic self-tuning control as well. This leads to a generalized upper bound for the estimation error convergence rate in stochastic self-tuning control. The bound is shown to converge to zero under some assumptions regarding the model structure. The result is used to formulate two *principles of self-tuning* stating sufficient conditions under which self-tuning to stability and self-tuning to parameter consistency may occur.

1. Introduction

The development of stochastic self-tuning control started with the RLS-based minimum-variance self-tuning regulator, presented by Åström and Wittenmark (1973). It enjoys an uncountable number of descendants, differing more or less from their ancestor but having as a rule one thing in common with this ancestor, namely a stubborn resistance to theories trying to provide some down-to-earth but sound and complete explanation of why the self-tuner self-tunes at all. There are two issues involved in this problem: (1) When and why does the self-tuner self-tune to a stable control system? (2) If stability is achieved, when and why does the self-tuner self-tune to consistent parameter estimates of the target controller? By the *target* control system (*target* controller) we mean a constant-parameter control system (constant-parameter controller), towards which the self-tuning control system (self-tuning controller) is designed to tune.

Existing attempts to solve this problem may be roughly classified into the following broad categories:

1. Assume closed-loop stability of the self-tuning control system and calculate the properties of local convergence points. This approach uses various theoretical tools, the most popular being the *ODE-Approach* (Ordinary Differential Equation), which replaces the analysis of a time-discrete stochastic system by an averaged non-stochastic time-continuous system using a compressed time scale, see (Ljung, 1987). A critical appraisal of the ODE approach has been presented by Wellstead and Zarrop (1991).

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2. Modify the self-tuning controller so that the self-tuning control system becomes amenable to some kind of theory, usually based on martingales, which proves stability and parameter consistency at the same time. Examples of this approach are given e.g. by swapping the RLS algorithm for the stochastic gradient algorithm or some *modified* RLS algorithm in the Åström-Wittenmark self-tuner (see e.g. Goodwin *et al.*, 1981; Goodwin and Sin, 1984).

3. Use the technique of *Bayesian embedding* and assume the white noise to be Gaussian (Kumar, 1990) to arrive at a convergence condition which holds provided the true parameter vector is contained in some exceptional set.

4. Use limit martingale theory to provide upper bounds on the rate of change of the norm of parameter estimate errors and show that these bounds converge to zero. Technically the approach is based on various martingale extensions of the law of iterated logarithm. Pioneering work done in this area is attributed to Lai and Wei (1982; 1986), as well as to Chen and Guo (1991; 1995).

The present paper is in tune with the last approach. It aims at demonstrating that a recently developed upper bound for the estimation error convergence rate in RLS estimation for SISO ARX open-loop plants (Niederliński, 1995) is also applicable to RLS-based self-tuning stochastic control without any prior assumptions regarding its stability or parameter convergence. The paper is organized as followss: (1) The upper bound and its underlying assumptions are summarized. (2) To demonstrate that this bound is applicable to RLS-based self-tuning stochastic control, two technical lemmas are presented: they establish asymptotic properties of the bound numerator and denominator for the case of a plant driven by nonstationary output feedback for the case of stochastically disturbed output. (3) As a result, a new, more general version of the upper bound from (Niederliński, 1995) has been derived. It holds for any RLS estimation for stochastically disturbed plants excited via linear nonstationary output feedback. From this new upper bound it follows that any unbounded increase in control system signals is accelerating the bound convergence to zero. (4) This property is used to formulate two *principles of self-tuning* giving sufficient conditions for stability and estimate consistency in RLS-based stochastic self-tuning control systems, for which the estimation error is described by the upper bound. The applicability of the upper bound to such self-tuning control systems is illustrated by an example of direct RLS-based self-tuning minimum-variance control.

2. A Bound for RLS Convergence Rate

Consider RLS parameter estimation for the open-loop stable plant

$$A(z^{-1})y(i) = B(z^{-1})u(i-1) + e(i) \tag{1}$$

with $A(z^{-1}) = 1 + a_1 z^{-1} + \cdots + a_{dA} z^{-dA}$ and $B(z^{-1}) = b_0 + b_1 z^{-1} + \cdots + b_{dB} z^{-dB}$, z^{-1} denoting the unit delay operator and $A(z^{-1})$ being Hurwitz stable. The plant equation may also be put into a linear form

$$y(i) = \varphi^T(i-1)\theta + e(i) \tag{2}$$

with the *data vector*

$$\varphi^T(i-1) = \Big[-y(i-1)\ -y(i-2)\ldots -y(i-dA)\ u(i-1)\ u(i-2)\ldots u(i-1-dB)\Big] \quad (3)$$

and unknown *parameter vector*

$$\theta^T = \Big[a_1\ a_2\ \ldots\ a_{dA}\ b_0\ b_1\ \ldots\ b_{dB}\Big] \quad (4)$$

It is assumed that $e(i)$ is a martingale difference sequence with respect to a properly defined σ-field $\mathcal{F}_i$, which might be considered to be the Borel field generated by the sequence $,\ldots,e(i-2),e(i-1),e(i)$. Technically it is assumed that $E\{e(i)|\mathcal{F}_{i-1}\}=0$ and $E\{e^2(i)\}=\sigma^2$. It is further assumed that the model is given by

$$\hat{y}(i) = \varphi^T(i-1)\hat{\theta}(i) \quad (5)$$

with the same data vector as the plant and the model parameter vector

$$\hat{\theta}^T(i) = \Big[\hat{a}_1(i)\ \hat{a}_2(i)\ldots\hat{a}_{dA}(i)\ \hat{b}_0(i)\ \hat{b}(i)_1\ldots\hat{b}_{dB}(i)\Big] \quad (6)$$

The RLS estimates $\hat{\theta}(i)$ are computed by

$$\hat{\theta}(i) = \hat{\theta}(i-1) + \boldsymbol{P}(i)\varphi(i-1)\Big[y(i) - \varphi^T(i-1)\hat{\theta}(i-1)\Big] \quad (7)$$

$$\boldsymbol{P}^{-1}(i) = \boldsymbol{P}^{-1}(i-1) + \varphi(i-1)\varphi^T(i-1)$$

$$= \sum_{j=1}^{i}\varphi(j-1)\varphi^T(j-1) + \mathbf{1}\alpha^i\lambda_{\min}(0) \quad (8)$$

The initial conditions are given by $\hat{\theta}(0)$ and $\boldsymbol{P}(0) = \mathbf{1}/\lambda_{\min}(0)$, $0<\lambda_{\min}(0)\ll 1$. The estimation error is denoted by $\tilde{\theta}(i) = \hat{\theta}(i) - \theta$ and the minimum eigenvalue of $\boldsymbol{P}^{-1}(i)$ by $\lambda_{\min}(i)$. The number of estimated parameters is $s = dA + dB + 1$.

With the above assumptions and notations, the following result has been proven recently, see (Niederliński, 1995):

Theorem 1. *The upper bound for the Euclidean norm of the RLS estimation error is given by*

$$\|\tilde{\theta}(i)\|_2 = O\left(\frac{s\,\sigma\,\sup_{h\leq i}\|\varphi(h-1)\|_2\sqrt{i\log\log i}}{\lambda_{\min}(i)}\right) \quad a.s. \quad (9)$$

The mechanism of getting estimate consistency can thus be interpreted (Niederliński, 1995) in terms of a *race* between the *accumulated disturbance* (which increases with the rate $\sqrt{i\log\log i}$) and the *accumulated excitation* (which increases with the rate given by $\lambda_{\min}(i)$): for consistency $\lambda_{\min}(i)$ must increase at a rate faster than $\sqrt{i\log\log i}$. This is achieved in open-loop identification e.g. when driving the plant input with white noise.

In what follows it will be shown that the bound (9) may be used to derive sufficient conditions for stability and parameter consistency for RLS-based self-tuning control systems for stochastically disturbed plants. To justify that without presupposing control system stability, it must be demonstrated first that the bound (9) is also applicable when $\sup_{h\leq i}\|\varphi(h-1)\|_2$ is not assumed to be bounded, as it may happen when $u(i)$ is generated by output feedback. This follows from two additional lemmas. They exploit the following general idea: because the plant input during self-tuning may be considered to be a (not necessarily stable) nonstationary ARMA (or MA(∞)) time series, asymptotic expressions for $\lambda_{\min}(i)$ and $\sup_{h\leq i}\|\varphi(h-1)\|_2$ for this type of excitation may be derived. They demonstrate that the rate of increase for $\lambda_{\min}(i)$ is always greater than that for $\sup_{h\leq i}\|\varphi(h-1)\|_2$; those rates together with the *law of iterated logarithm* rate given by the term $\sqrt{i \log\log i}$, result in a bounding function (see Theorem 2, (30)), which always converges to zero. The result explains e.g. the well-known phenomena of RLS-based stochastic self-tuning control systems regaining stability after outbursts caused by a wrong choice of initial controller parameters. To emphasize the intuitive insight provided by the rather technical result, it has been put into the form of two *principles of self-tuning*.

3. $\lambda_{\min}(i)$ for Self-Tuning Stochastic Control

Asymptotic expressions for the minimum eigenvalue may be computed for plants excited by nonstationary ARMA series (nonstationary MA(∞) series), which corresponds exactly to what is happening during self-tuning for minimum-variance control.

Lemma 1. *Assume that the plant input $u(i)$ is given by a nonstationary ARMA (nonstationary MA(∞)) time series generated by plant white noise $e(i)$. Let the n-th Markov parameter at time i for the nonstationary channel between $e(i)$ (considered to be input) and $u(i)$ (considered to be output) be given by $h_{i,n}$ and let $\overline{h}_n$ be an upper bound for this Markov parameter for all $0 \leq i < \infty$, i.e. $h_{i,n} \leq \overline{h}_n$ for all i. Then for some $d > 0$ and $i \to \infty$:*

$$\lambda_{\min}(i) = d\sigma^2\, i \sum_{n=0}^{i-1} \overline{h}_n^2 \tag{10}$$

Proof. Lemma 1 is based on Lemma 3 in (Niederliński, 1995), which assures that $\lambda_{\min}(i)$ is tracking asymptotically $\lambda_{u,\min}(i)$, which is the minimum eigenvalue of the matrix $\sum_{j=1}^{i}\varphi_u(j-1)\varphi_u^T(j-1)$, where

$$\varphi_u^T(i-1) = \Big[u(i-1)\;\; u(i-2)\;\; \ldots\;\; u(i-1-dB)\Big] \tag{11}$$

Consider

$$u(k) = \sum_{n=0}^{k} h_{k,n}e(k-n), \quad k \to \infty$$

$$\varphi_u(k-1) = \begin{bmatrix} \sum_{n=0}^{k-1} h_{k-1,n} e(k-1-n) \\ \sum_{n=0}^{k-2} h_{k-2,n} e(k-2-n) \\ \vdots \\ \sum_{n=0}^{k-s} h_{k-s,n} e(k-s-n) \end{bmatrix}$$

with $\sum_{n=0}^{\infty} |h_{\infty,n}|$ not necessarily finite. It follows that

$$\sum_{k=1}^{i} \varphi_u(k-1)\varphi_u^T(k-1)$$
$$= i\frac{1}{i}\sum_{k=1}^{i} \begin{bmatrix} \sum_{n=0}^{k-1} h_{k-1,n} e(k-1-n) \\ \sum_{n=0}^{k-2} h_{k-2,n} e(k-2-n) \\ \vdots \\ \sum_{n=0}^{k-s} h_{k-s,n} e(k-s-n) \end{bmatrix} \begin{bmatrix} \sum_{n=0}^{k-1} h_{k-1,n} e(k-1-n) \\ \sum_{n=0}^{k-2} h_{k-2,n} e(k-2-n) \\ \vdots \\ \sum_{n=0}^{k-s} h_{k-s,n} e(k-s-n) \end{bmatrix}^T \tag{12}$$

There exist real numbers $\overline{h}_n$ bounding from above all $h_{k,n}$ for any k, $\overline{h}_n > h_{k,n}$, $k = 1, 2, \ldots, j$ and such that

$$\sum_{k=1}^{i} \begin{bmatrix} \sum_{n=0}^{k-1} h_{k-1,n} e(k-1-n) \\ \sum_{n=0}^{k-2} h_{k-2,n} e(k-2-n) \\ \vdots \\ \sum_{n=0}^{k-s} h_{k-s,n} e(k-s-n) \end{bmatrix} \begin{bmatrix} \sum_{n=0}^{k-1} h_{k-1,n} e(k-1-n) \\ \sum_{n=0}^{k-2} h_{k-2,n} e(k-2-n) \\ \vdots \\ \sum_{n=0}^{k-s} h_{k-s,n} e(k-s-n) \end{bmatrix}^T$$
$$< \sum_{k=1}^{i} \begin{bmatrix} \left(\sum_{n=0}^{k-1} \overline{h}_n e(k-1-n)\right)^2 & 0 & \ldots & 0 \\ 0 & \left(\sum_{n=0}^{k-2} \overline{h}_n e(k-2-n)\right)^2 \ldots & & 0 \\ \ldots & \ldots & \ddots & \ldots \\ 0 & 0 & \ldots & \left(\sum_{n=0}^{k-s} \overline{h}_n e(k-s-n)\right)^2 \end{bmatrix} \tag{13}$$

where $A<B$ means that $B-A$ is positive-definite. Since

$$i\frac{1}{i}\sum_{k=1}^{i}\Big(e(k-1-n)\Big)^2 \to i\,\sigma^2$$

we have

$$\lambda_{u,\min}(i) < \sigma^2\, i\sum_{n=0}^{i-1}\overline{h}_n^2 \tag{14}$$

Therefore for some $d>0$

$$\lambda_{\min}(i) = d\sigma^2\, i\sum_{n=0}^{i-1}\overline{h}_n^2$$

■

Obviously, linear nonstationary filtering of white noise preserves the excitation properties of white noise as reflected in the rate of change of $\lambda_{\min}(i)$ (Niederliński, 1995) and in the case of unstable linear nonstationary filters even enhancing them.

4. $\sup_{h\le i}\|\varphi(h-1)\|_2$ for Self-Tuning Stochastic Control

Asymptotic expressions for $\sup_{h\le i}\|\varphi(h-1)\|_2$ may also be computed for plants excited by nonstationary ARMA series (nonstationary MA(∞) series).

Lemma 2. *Assume that the plant input $u(i)$ is given by a nonstationary ARMA (nonstationary MA(∞)) time series generated by plant white noise $e(i)$. Let the n-th Markov parameter at time i for the nonstationary channel between $e(i)$ (considered to be input) and $u(i)$ (considered to be output) be given by $h_{i,n}$ and let $\overline{h}_n$ be an upper bound for this Markov parameter for all $0\le i<\infty$, i.e. $h_{i,n}\le\overline{h}_n$ for all i, as in Lemma 1. Then*

$$\sup_{j\le i}\|\varphi(j-1)\|_2 = O\Big(\sigma\sup_{j\le i}|\overline{h}_{j-1}|\Big) \tag{15}$$

for some real $M>0$ and $N>0$.

Proof. From (1) it follows that

$$\begin{aligned}\|\varphi(j-1)\|^2 &= \left\|\frac{B(z^{-1})}{A(z^{-1})}\varphi_{u,dA}(j-1)+\frac{1}{A(z^{-1})}\varphi_e(j-1)\right\|^2+\|\varphi_{u,dB}(j-1)\|^2\\ &\le \left\|\frac{B(z^{-1})}{A(z^{-1})}\varphi_{u,dA}(j-1)\right\|^2+\left\|\frac{1}{A(z^{-1})}\varphi_e(j-1)\right\|^2\\ &\quad+2\left\|\frac{B(z^{-1})}{A(z^{-1})}\varphi_{u,dA}(j-1)\right\|\left\|\frac{1}{A(z^{-1})}\varphi_e(j-1)\right\|+\|\varphi_{u,dB}(j-1)\|^2 \end{aligned}\tag{16}$$

where

$$\varphi_{u,dA}(i-1) = \begin{bmatrix} \sum_{m=0}^{i-1} h_{i-1,m} e(i-1-m) \\ \sum_{m=0}^{i-2} h_{i-2,m} e(i-2-m) \\ \vdots \\ \sum_{m=0}^{i-dA} h_{i-dA,m} e(i-dA-m) \end{bmatrix} \tag{17}$$

$$\varphi_{u,dB}(i-1) = \begin{bmatrix} \sum_{m=0}^{i-1} h_{i-1,m} e(i-1-m) \\ \sum_{m=0}^{i-2} h_{i-2,m} e(i-2-m) \\ \vdots \\ \sum_{m=0}^{i-dB} h_{i-dB,m} e(i-dB-m) \end{bmatrix} \tag{18}$$

and

$$\varphi_e(i-1) = \begin{bmatrix} e(i-1) \\ e(i-2) \\ \vdots \\ e(i-dA) \end{bmatrix} \tag{19}$$

Obviously, there exists an M, $0 < M < \infty$, such that

$$\|\varphi_{u,dA}(i-1)\|^2 = \left(\sum_{m=0}^{i-1} h_{i-1,m} e(i-1-m)\right)^2 + \left(\sum_{m=0}^{i-2} h_{i-2,m} e(i-2-m)\right)^2 + \cdots$$

$$+ \left(\sum_{m=0}^{i-dA} h_{i-dA,m} e(i-dA-m)\right)^2 \leq M \left(\sum_{m=0}^{i-1} h_{i-1,m} e(i-1-m)\right)^2 \tag{20}$$

Further

$$M\left(\sum_{m=0}^{i-1} h_{i-1,m}e(i-1-m)\right)^2$$

$$= M\left[\sum_{k=1}^{i}\left(\sum_{m=0}^{k-1} h_{k-1,m}e(k-1-m)\right)^2 - \sum_{k=1}^{i-1}\left(\sum_{m=0}^{k-1} h_{k-1,m}e(k-1-m)\right)^2\right]$$

$$= M\left[i\frac{1}{i}\sum_{k=1}^{i}\left(\sum_{m=0}^{k-1} h_{k-1,m}e(k-1-m)\right)^2 - (i-1)\frac{1}{i-1}\sum_{k=1}^{i-1}\left(\sum_{m=0}^{k-1} h_{k-1,m}e(k-1-m)\right)^2\right]$$

$$\leq M\left[i\frac{1}{i}\sum_{k=1}^{i}\left(\sum_{m=0}^{k-1} \overline{h}_m e(k-1-m)\right)^2 - (i-1)\frac{1}{i-1}\sum_{k=1}^{i-1}\left(\sum_{m=0}^{k-1} \overline{h}_m e(k-1-m)\right)^2\right]$$

$$= O\left(\sigma^2\, \overline{h}_{i-1}^2\right) \tag{21}$$

where the bounds $\overline{h}_i$ may obviously be exactly like in Lemma 1. It follows that

$$\sup_{j\leq i} \|\varphi_{u,dA}(j-1)\|_2 = O\Big(\sigma \sup_{j\leq i} |\overline{h}_{j-1}|\Big) \tag{22}$$

Similarly

$$\|\varphi_{u,dB}(i-1)\|^2 = O\left(\sigma^2\, \overline{h}_{i-1}^2\right) \tag{23}$$

$$\sup_{j\leq i} \|\varphi_{u,dB}(j-1)\|_2 = O\Big(\sigma \sup_{j\leq i} |\overline{h}_{j-1}|\Big) \tag{24}$$

Also

$$\|\varphi_e(i-1)\|^2 = O\left(dA\, \sigma^2\right) \tag{25}$$

Write

$$\frac{B(z^{-1})}{A(z^{-1})}\varphi_{u,dA}(i-1) = \sum_{n=0}^{i-1} g_n \varphi_{u,dA}(i-1-n)$$

$$\frac{1}{A(z^{-1})}\varphi_e(i-1) = \sum_{n=0}^{i-1} k_n \varphi_e(i-1-n)$$

Then

$$\left\|\frac{B(z^{-1})}{A(z^{-1})}\varphi_{u,dA}(i-1)\right\| \leq \sup_{m\leq i-1} \|\varphi_{u,dA}(m)\| \sum_{n=0}^{i-1} |g_n| \tag{26}$$

with $\sum_{n=0}^{\infty} |g_n|$ being finite. Therefore

$$\left\| \frac{B(z^{-1})}{A(z^{-1})} \varphi_{u,dA}(i-1) \right\| = O\Big(\sigma \sup_{j \le i} |\overline{h}_{j-1}|\Big) \tag{27}$$

Also

$$\left\| \frac{1}{A(z^{-1})} \varphi_e(i-1) \right\| \le \sup_{m \le i-1} \|\varphi_e(m)\| \sum_{n=0}^{i-1} |k_n| \tag{28}$$

with $\sum_{n=0}^{\infty} |k_n|$ being finite. Therefore

$$\left\| \frac{1}{A(z^{-1})} \varphi_{u,dA}(i-1) \right\| = O\Big(\sigma \sup_{j \le i} |\overline{h}_{j-1}|\Big) \tag{29}$$

From (27) and (29) we deduce (15), which completes the proof. ■

5. The Main Result

Now we are in a position to show that *if (9) holds for the estimation error in an RLS-based self-tuning control system and* $\tilde{\theta}(\infty) = 0$ *corresponds to parameter estimates guaranteeing stable performance of the target control system, then any signal outburst due to loss of stability is transient, because it accelerates estimate convergence and restores stability of the self-tuning control system at the same time.*

The above-mentioned upper-bound property is formulated as a separate theorem:

Theorem 2. *Assume that the plant input is generated by linear nonstationary output feedback in an RLS-based self-tuning arrangement, for which the bound (9) holds. Then the bound may be expressed as*

$$\|\tilde{\theta}(i)\|_2 = O\left(\frac{s \sup_{j \le i} |\overline{h}_{j-1}|}{\sigma \sum_{j=1}^{i} \overline{h}_{j-1}^2} \sqrt{\frac{\log\log i}{i}} \right) \tag{30}$$

Proof. From Lemmas 1 and 2 we have

$$\frac{s\,\sigma \sup_{j \le i} \|\varphi(j-1)\|_2 \sqrt{i \log\log i}}{\lambda_{\min}(i)} < \frac{s \sup_{j \le i} |\overline{h}_{j-1}|}{\sigma \sum_{j=1}^{i} \overline{h}_{j-1}^2} \sqrt{\frac{\log\log i}{i}} \tag{31}$$

This, together with (9), gives (30). ■

It follows that any outburst of $\overline{h}_j$ (which is in fact modelling an outburst of control system variables) accelerates the convergence of the bounding function $\frac{s \sup_{j \le i} |\overline{h}_{j-1}|}{\sigma \sum_{j=1}^{i} \overline{h}_{j-1}^2} \sqrt{\frac{\log\log i}{i}}$ to zero by making its denominator grow at a faster rate than its numerator.

This result may be used to formulate two *self-tuning principles*, being as a matter of fact *sufficient conditions* of self-tuning stabilization and parameter consistency:

Theorem 3. *Assume a self-tuning control system with RLS parameter estimation for which the upper bound (9) holds and for which the asymptotic estimation error* $\tilde{\theta}(\infty) = 0$ *exists and corresponds to parameter estimates guaranteeing some kind of stability of the target control system. Then the following two self-tuning principles hold:*

1. *The principle of self-tuning stabilization:*
   Any outburst of control system variables, typical for loss of stability and caused e.g. by improper initial controller parameters, results in additional excitation which accelerates the convergence to zero of the upper bound for the estimation error, in the process of computing RLS estimates that converge towards values assuring for the self-tuning control system the kind of stability retained by the target control system. As a result, stability is self-restored.

2. *The principle of self-tuning consistency:*
   For a self-tuning control system stabilized by the mechanism described in the principle of self-tuning stabilization holds:

$$\frac{s \, \sup_{j\le i} |\overline{h}_{j-1}|}{\sigma \sum_{j=1}^{i} \overline{h}_{j-1}^{2}} \sqrt{\frac{\log\log i}{i}} \to 0$$

   which is sufficient for the a.s. convergence of parameter estimates to their true values.

The proof of both principles is immediate.

Discussion:

1. For self-tuning to occur two things are sufficient:
   - The estimation error is limited by the upper bound (9).
   - The *true* values of estimated parameters correspond to a stable target control system.
2. The essence of self-tuning stabilization is a mechanism, by which any loss of stability or serious control deterioration, as bad as to improve considerably the persistency of excitation measured by the accumulated excitation $\lambda_{\min}(i)$, results in restoring stability by bringing the parameter estimates to such a neighborhood of the true values, that assures stability. The term *accumulated excitation* refers to the fact that $\lambda_{\min}(i)$ values form a non-decreasing series, see (Niederliński, 1995).
3. The essence of self-tuning consistency is a mechanism, by which eventual disturbances or specially introduced excitation increase further the accumulated excitation $\lambda_{\min}(i)$ of the already stabilized control system, thereby bringing the parameter estimates to such a neighborhood of the true values, that assures consistency.

4. Both properties of interest: *stability* (coming first) and *estimate consistency* (coming next) are explained by the same simple mechanism of accumulated excitation as given by $\lambda_{\min}(i)$ and its role in bounding the estimation error.
5. The explanation is done using a rather small set of simple concepts, (remember Occam's *Entia non sunt multiplicanda praeter necessitatem!*), the main one being asymptotic properties of $\lambda_{\min}(i)$ and $\sup_{j\leq i}|\bar{h}_{j-1}|$.
6. Both principles highlight the strong connection between self-tuning for stability/consistency and persistence of excitation in its *accumulated* form.

The self-tuning principle will be illustrated by means of an example.

Example. (The Åström-Wittenmark minimum variance self-tuner, known b_0, a minimum-phase system.) Consider the minimum-phase plant given by

$$y(i) = b_0 u(i-1) + \varphi^T(i-1)\theta + e(i) \tag{32}$$

with b_0 known,

$$\varphi^T(i-1) = \Big[-y(i-1)\;\; -y(i-2)\ldots -y(i-dA)\;\; u(i-2)\;\; u(i-3)\ldots u(i-1-dB)\Big] \tag{33}$$

and

$$\theta^T = \Big[a_1\;\; a_2\;\ldots\; a_{dA}\;\; b_1\;\; b_2\;\ldots\; b_{dB}\Big] \tag{34}$$

The minimum-variance target controller for this plant is obviously given by

$$u(i-1) = -\frac{1}{b_0}\varphi^T(i-1)\theta \tag{35}$$

and the resulting minimum-variance target control system is stochastically stable in the sense that the variance of the controlled variable is finite and given by the white-noise variance σ^2.

Consider the plant model given by

$$\hat{y}(i) = b_0 u(i-1) + \varphi^T(i-1)\hat{\theta}(i-1) \tag{36}$$

with usual RLS estimation:

$$\hat{\theta}(i) = \hat{\theta}(i-1) + \boldsymbol{P}(i)\varphi(i-1)\Big[y(i) - b_0 u(i-1) - \varphi^T(i-1)\hat{\theta}(i-1)\Big] \tag{37}$$

$$\boldsymbol{P}^{-1}(i) = \boldsymbol{P}^{-1}(i-1) + \varphi(i-1)\varphi^T(i-1) \tag{38}$$

and certainty-equivalence minimum-variance controller

$$u(i-1) = -\frac{1}{b_0}\varphi^T(i-1)\hat{\theta}(i-1) \tag{39}$$

From (37), (32), (38) and the definition of $\tilde{\theta}(i)$ we have

$$\tilde{\theta}(i) = \boldsymbol{P}(i)\boldsymbol{P}^{-1}(i-1)\tilde{\theta}(i-1) + \boldsymbol{P}(i)\varphi(i-1)e(i) \tag{40}$$

Iterating (40) results in

$$\tilde{\theta}(i) = \boldsymbol{P}(i)\boldsymbol{P}^{-1}(0)\tilde{\theta}(0) + \boldsymbol{P}(i)\sum_{j=1}^{i}\varphi(j-1)e(j) \tag{41}$$

Normalizing variables $f(j-1) = \varphi(j-1)/\sup_{h\le j}\|\varphi(h-1)\|_2$ and $w(j) = e(j)/\sigma$ and applying basic norm definitions results in

$$\|\tilde{\theta}(i)\|_2 \le \frac{\lambda_{\min}(0)}{\lambda_{\min}(i)}\|\tilde{\theta}(0)\|_2 + \frac{\sigma \sup_{h\le i}\|\varphi(h-1)\|_2}{\lambda_{\min}(i)}\Big\|\sum_{j=1}^{i} f(j-1)w(j)\Big\|_2 \tag{42}$$

Obviously, for any element $f_k(j-1)$ of the vector $f(j-1)$ (measurable with respect to $\mathcal{F}_{j-1}$) we have

$$E\Big\{f_k(j-1)w(j)\,|\,\mathcal{F}_{j-1}\Big\} = f_k(j-1)E\Big\{w(j)\,|\,\mathcal{F}_{j-1}\Big\} = 0 \tag{43}$$

$$E\Big\{f_k^2(j-1)w^2(j)\,|\,\mathcal{F}_{j-1}\Big\} = f_k^2(j-1) \le 1 \tag{44}$$

$$E\Big\{f_k^2(j-1)w^2(j)\Big\} = E\Big\{f_k^2(j-1)\Big\} \le 1 \tag{45}$$

It follows that $(f_k(j-1)w(j), \mathcal{F}_{j-1})$ is a uniformly bounded martingale difference sequence for which, on the set $s_i \to \infty$, where

$$s_i = \sum_{j=1}^{i} E\Big\{f_k^2(j-1)w^2(j)\,|\,\mathcal{F}_{j-1}\Big\} = \sum_{j=1}^{i} f_k^2(j-1) \le i \tag{46}$$

the following generalization of the *law of iterated logarithm* holds (see e.g. Hall and Heyde, 1980):

$$\sum_{j=1}^{i} f_k(j-1)w(j) = O\left(\sqrt{s_i \log\log s_i}\right) = O\left(\sqrt{i \log\log i}\right) \tag{47}$$

From (47) and (42) we deduce (9). Therefore the bound (9) holds for the direct Åström-Wittenmark minimum-variance self-tuning control system for known b_0 and the minimum-phase plant. Because while self-tuning the plant is excited by a nonstationary ARMA (nonstationary MA(∞)) series, Lemma 1 and Lemma 2 hold as well. We see that for this control system the bound (30) holds and that the parameter estimate vector $\theta(i)$ converges *a.s.* to the true target controller parameter vector.

6. Conclusions

It has been shown that a recently published upper bound for open-loop RLS estimation error convergence rate may be safely and profitably used to analyze stability and parameter consistency for some RLS-based stochastic self-tuning control systems as well. This has been done by deriving some more detailed results on the asymptotic behavior of this bound in a self-tuning environment.

The results clearly demonstrate an acceleration of the estimation error convergence rate due to the presence of signal outbursts, e.g. those occurring in control systems loosing stability. This property is used to provide a unified explanation of the mechanism of achieving stability and consistency in all those self-tuning control systems, for which the bound holds and for which the asymptotic values of estimates correspond to stable target controller settings. This insight into self-tuning provided by the upper bound has been summarized by two sufficient conditions in the form of self-tuning principles and illustrated by an example.

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