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THE USE OF SLIDING MODES TO SIMPLIFY THE BACKSTEPPING CONTROL METHOD†

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A simple sliding mode based controller for nonlinear systems with mismatched uncertainties is proposed. The design methodology is similar to a backstepping and multiple surface control method, but with the inclusion of sliding mode filters for estimating the derivatives of the plant output.

1. Introduction

Advanced industrial applications require simple algorithms to realize accurate tracking in the presence of mismatched uncertainties. Recently the elegant backstepping design methodology (Kannelakopoulous *et al.*, 1991) has been proposed to deal with mismatched uncertainties. However, the integrator backstepping has a problem of an "explosion of terms" which makes the backstepping controller difficult for implementation. A procedure similar to backstepping, called Multiple Surface Sliding control (MSS) was developed to simplify the controller design (Gerdes *et al.*, 1997; Green and Hedrick, 1990). This approach is based on approximation of differentiation of the desired trajectories by finite differences and worked well in many experimental applications. In the first versions of the MSS method first order finite differences were used to obtain the derivatives of the desired trajectories (Green and Hedrick, 1990), Later numerical differentiation was replaced by first order low pass filters and a complete stability analysis was performed for systems with Lipschitz nonlinearities (Gerdes *et al.*, 1997).

In this paper, we suggest the use of sliding mode filters to obtain the derivatives of the desired trajectories for the system with mismatched uncertainties and non-Lipschitz nonlinearities. A similar idea has been applied recently for stabilization of rotational motion of a vertical shaft magnetic bearing (DeCarlo *et al.*, 1996), see also the tutorial (Drakunov and Utkin, 1995).

We demonstrate our approach on a simple second order system example. Let the plant be in the form:

$$\dot{x}_1 = x_2 + \theta_* x_1^2 \tag{1}$$

$$\dot{x}_2 = u \tag{2}$$

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where θ_* is an unknown constant parameter, $|\theta_*| \leq l_1$, $l_1 > 0$ is a known constant, x_1 and x_2 are measurable states.

Our aim is to regulate x_1 and x_2 to zero. The main idea for controller design is to ensure a sliding motion on the surface $\dot{x}_1 + cx_1 = 0$, where $c > 0$, and $\dot{x}_1$ is estimated through the estimation of $\theta_* x_1^2$ in the sliding mode. Further, we use differential equations with discontinuous right-hand sides and we understand their solutions in the sense of (Gelig *et al.*, 1978).

The design procedure proposed here can be divided into the following steps:

Step 1. Estimation of $\dot{x}_1$.

Let us introduce the following filter:

$$\dot{\varepsilon} = \alpha_0(x_1 - \varepsilon) + x_2 + \gamma \,\mathrm{sign}(x_1 - \varepsilon) \tag{3}$$

where $\alpha_0 > 0$, $\gamma = l_1 x_1^2 + \beta$, $\beta > 0$. Subtracting (3) from (1), we get the error model

$$\dot{x}_1 - \dot{\varepsilon} = -\alpha_0(x_1 - \varepsilon) + \theta_* x_1^2 - \gamma \,\mathrm{sign}(x_1 - \varepsilon) \tag{4}$$

Our first substep is to organize a sliding motion on the surface $(x_1 - \varepsilon) = 0$. Taking the Lyapunov function candidate $V = (x_1 - \varepsilon)^2$, we evaluate its derivative along the solutions of the system (4):

$$\begin{aligned} \dot{V} &= 2(x_1 - \varepsilon)\{-\alpha_0(x_1 - \varepsilon) + \theta_* x_1^2 - \gamma \,\mathrm{sign}(x_1 - \varepsilon)\} \\ &\leq 2|x_1 - \varepsilon|\{l_1 x_1^2 - \gamma\} \\ &\leq -\beta\sqrt{V} \end{aligned} \tag{5}$$

and

$$\sqrt{V(t)} \leq \sqrt{V(0)} - \frac{\beta}{2}t \tag{6}$$

It is easy to see that the sliding surface $(x_1 - \varepsilon) = 0$ is reached in a finite time and in the sliding mode $\theta_* x_1^2$ is equal to $\gamma \,\mathrm{sign}(x_1 - \varepsilon)$, i.e.,

$$\theta_* x_1^2 = \gamma \,\mathrm{sign}(x_1 - \varepsilon) \tag{7}$$

where $\gamma \,\mathrm{sign}(x_1 - \varepsilon)$ is understood as the nonlinearity defined in the sense of (Gelig *et al.*, 1978) and determined after closing the system. From a practical point of view, $\gamma \,\mathrm{sign}(x_1 - \varepsilon)$ is the observable output of the nonlinear block.

We introduce the following filter to get the "equivalent control" (Utkin, 1978):

$$\tau\dot{z} = -z + \gamma \,\mathrm{sign}(x_1 - \varepsilon) \tag{8}$$

or

$$\tau\dot{z} = -\left(z - \theta_* x_1^2\right) \tag{9}$$

where τ is a positive constant which should be chosen "large enough" to reduce the high frequency component of the signal, but "small enough" so as not to alter the low frequency component which is, in fact, the "equivalent control" that we need. As an estimate of $\dot{x}_1$ we take

$$\hat{\dot{x}}_1 = x_2 + z \tag{10}$$

Step 2. Controller design.
Define the sliding surface

$$s = \hat{\dot{x}}_1 + cx_1 \tag{11}$$

where $c > 0$ is an algorithm parameter. Evaluating $\dot{s}$, we have

$$\dot{s} = u + \dot{z} + c\big(x_2 + \theta_* x_1^2\big) \tag{12}$$

Selecting the control action as $u = -\dot{z} - cx_2 - \gamma_1 \operatorname{sign}(s)$, where $\gamma_1 = cl_1 x_1^2 + \beta_1$, $\beta_1 > 0$, we see that s converges to zero in a finite time.

Step 3. Stability analysis of the overall system.
Our aim is to prove that the convergence of s to zero implies the convergence of x_1 and x_2 to zero. First of all we present the error model of the system. Notice that

$$s = e + \dot{x}_1 + cx_1 \tag{13}$$

where $e = z - \theta_* x_1^2$. Rewriting (9), we have

$$\dot{e} = -\frac{1}{\tau}e - 2\theta_* x_1 x_2 - 2\theta_*^2 x_1^3 \tag{14}$$

Taking into account that

$$\dot{x}_1 = -cx_1 - e \tag{15}$$

where we neglected the reaching phase, we substitute

$$x_2 = \dot{x}_1 - \theta_* x_1^2 = -cx_1 - e - \theta_* x_1^2 \tag{16}$$

into (14). After simple calculations, we get the error model in terms of e and x_1:

$$\begin{aligned} \dot{e} &= -\frac{1}{\tau}e + 2c\theta_* x_1^2 + 2\theta_* x_1 e \\ \dot{x}_1 &= -cx_1 - e \end{aligned} \tag{17}$$

Now we select the Lyapunov function candidate in the following form:

$$V = \frac{1}{2}e^2 + \frac{1}{2}x_1^2 \tag{18}$$

Evaluating its derivative along the solutions to (17), we have

$$\begin{aligned} \dot{V} &= e\Big\{-\frac{1}{\tau}e + 2c\theta_* x_1^2 + 2\theta_* x_1 e\Big\} - cx_1^2 - x_1 e \\ &\le e^2\Big(-\frac{1}{\tau} + \frac{1}{2} + 2l_1|x_1| + l_1^2 c^2\Big) + x_1^2\Big(-c + \frac{1}{2} + x_1^2\Big) \end{aligned} \tag{19}$$

If we choose the algorithm parameters c and τ to satisfy the constraints

$$c > \frac{1}{2} + 2V(0) \tag{20}$$

$$\frac{1}{\tau} > \frac{1}{2} + 2l_1\sqrt{2}\sqrt{V(0)} + l_1^2c^2 \tag{21}$$

where $V(0) = \frac{1}{2}e^2(0) + \frac{1}{2}x_1^2(0)$, then there exists a positive number κ such that

$$\dot{V} \leq -\kappa V \tag{22}$$

and e with x_1 converge exponentially to zero. It is easy to see that $\dot{x}_1$ and x_2 are also exponentially convergent, the system is semiglobally stable and the region of attraction can be enlarged by amplifying the design parameter c and reducing τ.

Notice that the right-hand sides of (20),(21) depend on the unknown parameter θ_*. Using the inequality $|\theta_*| \leq l_1$ one can easily verify that if we choose the parameters c and τ such that they satisfy the constraints

$$c > \frac{1}{2} + z(0)^2 + 2l_1|z(0)|x_1(0)^2 + l_1^2x_1^4(0) + x_1^2(0)$$

$$\frac{1}{\tau} > \frac{1}{2} + 2l_1\left\{|x_1(0)| + \sqrt{z(0)^2 + 2l_1|z(0)|x_1(0)^2 + l_1^2x_1^4(0)}\right\} + l_1^2c^2$$

then (20) and (21) are valid.

The controller described above prevents the explosion of terms and treats non-Lipschitz uncertainties. Now we are in a position to generalize this approach.

2. Problem Statement

The following cascade nonlinear system is considered in this paper:

$$\dot{x}_1 = x_2 + f_1(x_1, t) \tag{23}$$

$$\dot{x}_2 = x_3 + f_2(x_1, x_2, t) \tag{24}$$

$$\vdots$$

$$\dot{x}_{n-1} = x_n + f_{n-1}(x_1, x_2, \ldots, x_{n-1}, t) \tag{25}$$

$$\dot{x}_n = u \tag{26}$$

where $x_i \in \mathbb{R}^1$, $i = 1, \ldots, n-1$, and the functions $f_i(\cdot)$ are unknown. We only assume that the functions are bounded together with their derivatives and the bounds are known, i.e., $|f_1(x_1,t)| \leq \bar{f}_1(x_1,t)$, $|f_2(x_1,x_2,t)| \leq \bar{f}_2(x_1,x_2,t)$, $|\partial f_1(x_1,t)/\partial t| \leq \bar{f}_{11}(x_1,t)$, $|\partial f_1(x_1,t)/\partial x_1| \leq \bar{f}_{12}(x_1,t)$, etc. The functions $\bar{f}_1(x_1,t)$, $\bar{f}_2(x_1,x_2,t)$, $\bar{f}_{11}(x_1,t)$, $\bar{f}_{12}(x_1,t)$ are known. We suppose as well that x_i are measurable states and $f_i(0,0,0,\ldots,0) = 0$.

Notice that the functions $f_i(\cdot)$ depend explicitly on t and therefore there can be unknown time-varying disturbances such that upper bounds of disturbances together with upper bounds of several derivatives of the disturbances are known.

Our problem is to find a control action u which depends on measurable states x_i such that the following control aim is achieved:

$$\lim_{t\to\infty} |y_i(t)| = 0, \qquad i = 1, \dots, n \tag{27}$$

where $y_1 = x_1$ is the plant output and $y_{i+1} = \mathrm{d}^i y_1/\mathrm{d}t^i$. It should be emphasized that the states x_i are available for measurements, but not the derivatives of the output $y_{i+1} = \mathrm{d}^i y_1/\mathrm{d}t^i$.

The cascade systems (23)–(26) are very often met in many practical applications such as automotive power train systems, aircraft control, etc., see (Hedrick, 1993) and references therein.

A similar cascade system was considered in (Kannelakopoulous *et al.*, 1991), but functions $f_i(\cdot)$ depend explicitly on t here and, as was already mentioned, bounded unmeasurable disturbances act on the plant.

3. Outline of the Solution to the Problem

As the first step in the controller design we differentiate (23) $(n-1)$ times and rewrite (23)–(26) in the following canonical form:

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= y_3 \\ &\vdots \\ \dot{y}_{n-1} &= y_n \\ \dot{y}_n &= u + f_0(x_1, \dots, x_n, t) \end{aligned} \tag{28}$$

where $f_0(x_1, \dots, x_n, t)$ is an unknown function with a known upper bound, i.e., $|f_0(x_1, \dots, x_n, t)| \le \bar{f}_0(x_1, \dots, x_n, t)$ and $\bar{f}_0(x_1, \dots, x_n, t)$ is known.

In order to outline the solution to the problem stated above, let us suppose at a moment that all $(n-1)$ derivatives of the output $y_1, \dots, y_n$ are measurable. Let us try to find u such that any solution to the system (23)–(26), (28) has the property $s(y) = 0$ with $t \ge t_*$, where t_* is some constant which depends on the solution and

$$s(y) = \delta_1 y_1 + \delta_2 y_2 + \cdots + \delta_{n-1} y_{n-1} + y_n \tag{29}$$

where δ_i, $i = 1, \dots, n-1$ are constants to be chosen. Further, we use differential equations with discontinuous right-hand sides and we understand their solutions in the sense of (Gelig *et al.*, 1978).

Taking the Lyapunov function candidate

$$V = s^2 \tag{30}$$

we evaluate its derivative along the solutions of (28)

$$\dot{V} = 2s\big(\delta_1 y_2 + \delta_2 y_3 + \cdots + \delta_{n-1} y_n + u + f_0(x_1, \dots, x_{n-1}, t)\big) \tag{31}$$

Choosing the control action as

$$u = -\delta_1 y_2 - \delta_2 y_3 - \cdots - \delta_{n-1} y_n - \gamma \operatorname{sign}(s) \tag{32}$$

we get

$$\dot{V} \leq -\lambda \sqrt{V} \tag{33}$$

if $\gamma = \bar{f}_0(x_1, \ldots, x_n, t) + \lambda/2,\ \lambda > 0$. Hence s converges to zero in a finite time and for all $t \geq t_*$, $t_* = (2/\lambda)\sqrt{V(0)}$ we have

$$y_n = -\delta_1 y_1 - \delta_2 y_2 - \cdots - \delta_{n-1} y_{n-1} \tag{34}$$

Substituting (34) in (28) we see that

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= y_3 \\ &\vdots \\ \dot{y}_{n-1} &= -\delta_1 y_1 - \delta_2 y_2 - \cdots - \delta_{n-1} y_{n-1} \end{aligned} \tag{35}$$

so the dynamics of the system in the sliding mode $(t \geq t_*)$ is determined by the coefficients δ_i, $i = 1, \ldots, n-1$ only and they can be arbitrarily chosen.

Unfortunately, the above controller is not implementable, since the derivatives y_i of the output are not measurable.

Our next step is to develop a procedure for estimating the derivatives of the output. Since

$$\dot{x}_1 = x_2 + f_1(x_1, t) \tag{36}$$

we see that in order to find $\dot{x}_1$ we have to measure $f_1(x_1, t)$. Introduce the filter

$$\dot{\varepsilon}_1 = \alpha_0(x_1 - \varepsilon_1) + x_2 + \gamma_1 \operatorname{sign}(x_1 - \varepsilon_1) \tag{37}$$

where $\alpha_0 > 0$, $\gamma_1(x_1, t) = \bar{f}_1(x_1, t) + \lambda_1/2$, $\lambda_1 > 0$. Then the error dynamics is given by the following equation:

$$\dot{x}_1 - \dot{\varepsilon}_1 = -\alpha_0(x_1 - \varepsilon_1) + f_1(x_1, t) - \gamma_1 \operatorname{sign}(x_1 - \varepsilon_1) \tag{38}$$

Taking the Lyapunov function candidate

$$V_1 = (x_1 - \varepsilon_1)^2 \tag{39}$$

and evaluating its derivative along the solutions to (38), we obtain

$$\dot{V}_1 \leq -\lambda_1 \sqrt{V_1} \tag{40}$$

Therefore the surface $(x_1 - \varepsilon_1) = 0$ is reached in a finite time t_* and for all $t \geq t_*$ $\gamma_1(x_1(t), t) \operatorname{sign}(x_1(t) - \varepsilon_1(t))$ is equivalent to $f_1(x_1(t), t)$, i.e.,

$$\gamma_1(x_1(t), t) \operatorname{sign}(x_1(t) - \varepsilon_1(t)) = f_1(x_1(t), t) \tag{41}$$

Notice that the choice of initial value $\varepsilon_1(0)$ as $\varepsilon_1(0) = x_1(0)$ eliminates the reaching phase, i.e., $t_* = 0$. Now $\dot{x}_1$ can be found from the following relationship:

$$\dot{x}_1 = x_2 + \gamma_1 \operatorname{sign}(x_1 - \varepsilon_1) \tag{42}$$

where $\gamma_1 \operatorname{sign}(x_1 - \varepsilon_1)$ is understood as the nonlinearity defined in the sense of (Gelig *et al.*, 1978) and determined after closing the system. From a practical point of view $\gamma_1 \operatorname{sign}(x_1 - \varepsilon_1)$ is the observable output of the nonlinear block.

Our next step is to estimate $\ddot{x}_1$. Differentiating (23), we have

$$\ddot{x}_1 = x_3 + d(x_1, x_2, t) \tag{43}$$

with

$$d(x_1, x_2, t) = f_2(x_1, x_2, t) + \frac{\partial f_1(x_1, t)}{\partial t} + \frac{\partial f_1(x_1, t)}{\partial x_1}\big(x_2 + f_1(x_1, t)\big)$$

and

$$|d(x_1, x_2, t)| \le \bar{d}(x_1, x_2, t)$$

where $\bar{d}(x_1, x_2, t)$ is known.

Introduce the following filter:

$$\dot{\varepsilon}_2 = \alpha_0(\dot{x}_1 - \varepsilon_2) + x_3 + \gamma_2 \operatorname{sign}(\dot{x}_1 - \varepsilon_2), \qquad \varepsilon_2(0) = \dot{x}_1(0) \tag{44}$$

where $\alpha_0 > 0$. Notice that ε_2 is implementable since $\dot{x}_1$ is known (see (42)). Subtracting (44) from (43), we get the error model

$$\ddot{x}_1 - \dot{\varepsilon}_2 = -\alpha_0(\dot{x}_1 - \varepsilon_2) + d(x_1, x_2, t) - \gamma_2 \operatorname{sign}(\dot{x}_1 - \varepsilon_2) \tag{45}$$

It is easy to show that the sliding surface $(\dot{x}_1 - \varepsilon_2) = 0$ is reached in a finite time if $\gamma_2 = \bar{d} + \lambda_2$ where $\lambda_2 > 0$. In the sliding mode we have

$$d(x_1, x_2, t) = \gamma_2 \operatorname{sign}(\dot{x}_1 - \varepsilon_2) \tag{46}$$

and $\ddot{x}_1$ can be estimated as

$$\ddot{x}_1 = x_3 + \gamma_2 \operatorname{sign}(\dot{x}_1 - \varepsilon_2) \tag{47}$$

where $\gamma_2 \operatorname{sign}(\dot{x}_1 - \varepsilon_2)$ is again the output of some nonlinear block. We continue this procedure until we find all the $(n-1)$ derivatives of x_1.

Substituting all the estimated derivatives in the control action (32), we get a nested signum function controller. For convenience, we recall our control action:

$$u = -\delta_1 y_2 - \delta_2 y_3 - \cdots - \delta_{n-1} y_n - \gamma \operatorname{sign}(s) \tag{48}$$

where

$$y_1 = x_1 \tag{49}$$

$$y_2 = x_2 + \gamma_1 \operatorname{sign}(x_1 - \varepsilon_1) \tag{50}$$

$$y_3 = x_3 + \gamma_2 \operatorname{sign}(y_2 - \varepsilon_2) \tag{51}$$

$$\vdots$$

ε_1 and ε_2 satisfy (37), (44) and

$$s(y) = \delta_1 y_1 + \delta_2 y_2 + \cdots + \delta_{n-1} y_{n-1} + y_n \tag{52}$$

In the closed-loop system (23)–(26), (28), (48)–(52) sign(z) is understood as a function which has the range $[-1, 1]$ when $z = 0$ and the differential equations according to (Gelig *et al.*, 1978) are understood as differential inclusions. The solutions to the system exist according to Theorem 2.2.1 of the book (Gelig *et al.*, 1978).

Now we are in a position to formulate our main result.

Theorem 1. *Consider the system (23)–(25), (28) with the control action (48)–(52). Let the parameters δ_i be chosen such that the polynomial*

$$\delta_1 + \delta_2 p + \delta_3 p^2 + \cdots + \delta_{n-1} p^{n-2} + p^{n-1} = 0 \tag{53}$$

is Hurwitz. Then the control aim (27) is achieved.

The proof of this theorem is outlined above.

Remark 1. The controller (48)–(52) is universal in the sense that it guarantees that the control aim (27) is reached for all the plants (23)–(26) from a given class.

Remark 2. The convergence of state variables x_i to zero can be easily established for some special cases of the upper bounds $\bar{f}(x_1, t)$, $\bar{d}(x_1, x_2, t)$. For instance, let $\bar{f}(x_1) = k_1|x_1| + k_2 x_1^2$ and $\bar{d}(x_1, x_2) = k_3|x_1| + k_4 x_1^2 + k_5|x_2| + k_6 x_2^2$, $k_i > 0$, $i = 1, \ldots, 6$. Then from (23) it follows that $x_2(t) \to 0$ as $t \to \infty$, since $\dot{x}_1(t) \to 0$ and $x_1(t) \to 0$ as $t \to \infty$. Then (43) yields $x_3(t) \to 0$ as $t \to \infty$, since $\ddot{x}_1(t) \to 0$, $x_1(t) \to 0$ and $x_2(t) \to 0$ as $t \to \infty$. We continue this procedure and conclude that all the states x_i converge to zero.

Remark 3. The implementation of a nested signum function controller is difficult in practice, and sometimes even impossible due to the discontinuous nature of the sliding surfaces. In order to make the above controller suitable for implementation, we should use an "equivalent control" instead of signum functions in the controller. To get the "equivalent control", we use the theory of approximability developed by Utkin (Utkin *et al.*, 1978). According to this theory, the "equivalent control" coincides with the average value of the appropriate signum function and is ***physically realizable*** as the output of a first order filter. Applying this concept to the system presented above, we get the controller presented below. The relationship (42) can be implemented as

$$\begin{aligned}
\hat{\dot{x}}_1 &= x_2 + z_1 \\
\tau_1 \dot{z}_1 &= -z_1 + \gamma_1 \operatorname{sign}(x_1 - \varepsilon_1) \\
\dot{\varepsilon}_1 &= \alpha_0 (x_1 - \varepsilon_1) + x_2 + \gamma_1 \operatorname{sign}(x_1 - \varepsilon_1)
\end{aligned} \tag{54}$$

and (47) in form

$$\begin{aligned}
\hat{\ddot{x}}_1 &= x_3 + z_2 \\
\tau_2 \dot{z}_2 &= -z_2 + \gamma_2 \operatorname{sign}(\hat{\dot{x}}_1 - \varepsilon_2) \\
\dot{\varepsilon}_2 &= \alpha_0 (\hat{\dot{x}}_1 - \varepsilon_2) + x_3 + \gamma_2 \operatorname{sign}(\hat{\dot{x}}_1 - \varepsilon_2)
\end{aligned} \tag{55}$$

where τ_1 and τ_2 are positive constants which should be chosen so as to filter out the high-frequency component of the signal on the one hand and to make the averaging isolate the slow–changing component on the other hand.

Finally, the resulting control action can be written as

$$u = -\delta_1 \hat{\dot{x}}_1 - \delta_2 \hat{\ddot{x}}_1 - \cdots - \delta_{n-1} \hat{x}^{n-1} - \gamma \operatorname{sign}(\hat{s}) \tag{56}$$

where $\hat{s} = \delta_1 x_1 + \delta_2 \hat{\dot{x}}_1 + \cdots + \delta_{n-1} \hat{x}^{n-2} + \hat{x}^{n-1}$.

Notice that one can also prove the stability of the overall system with filters. The full proof for the second-order system is presented in the Introduction. Solutions to the regularized system contain again discontinuous right-hand sides and are understood in the sense of (Gelig *et al.*, 1978). We also remark that there exist some other ways of regularization. For instance, $\operatorname{sign}(x)$ can be replaced with $x/(|x| + \sigma)$, where $\sigma > 0$ is "small enough".

4. Numerical Example

Consider the following example:

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1 \sin(x_2) \\ \dot{x}_2 &= x_3 + x_2 \cos(x_1) + x_1 \sin(x_2) \\ \dot{x}_3 &= u \\ y &= x_1 \end{aligned} \tag{57}$$

The control objective is to synthesize a state feedback law for u to regulate the output y at 0. Using the notation introduced in the previous section, the bounds on the nonlinearities are given by

$$|f_1(x_1)| = |x_1 \sin(x_1)| \leq \bar{f}_1(x_1) = |x_1| \tag{58}$$

$$\begin{aligned} |f_0(x_1, \ldots, x_3)| &\leq \bar{f}_0(x_1, \ldots, x_3) \\ &= 2(|x_1| + 1)(|x_1| + |x_2| + |x_3|) \\ &\quad + (|x_1| + 3|x_2| + |x_1 x_2| + 2 + |x_1|^2)(|x_1| + |x_2|) \end{aligned} \tag{59}$$

$$|d(x_1, x_2)| \leq \bar{d}(x_1, x_2) = |x_1| + |x_2| + (1 + |x_1|)(|x_1| + |x_2|) \tag{60}$$

The equations implementing the estimator for $\dot{x}_1$ and $\ddot{x}_1$ are as follows:

$$\left\{ \begin{aligned} \hat{\dot{x}}_1 &= x_2 + z_1 \\ \tau_1 \dot{z}_1 &= -z_1 + \gamma_1 \operatorname{sign}(x_1 - \varepsilon_1) \\ \dot{\varepsilon}_1 &= \alpha_0 (x_1 - \varepsilon_1) + x_2 + \gamma_1 \operatorname{sign}(x_1 - \varepsilon_1) \end{aligned} \right. \tag{61}$$

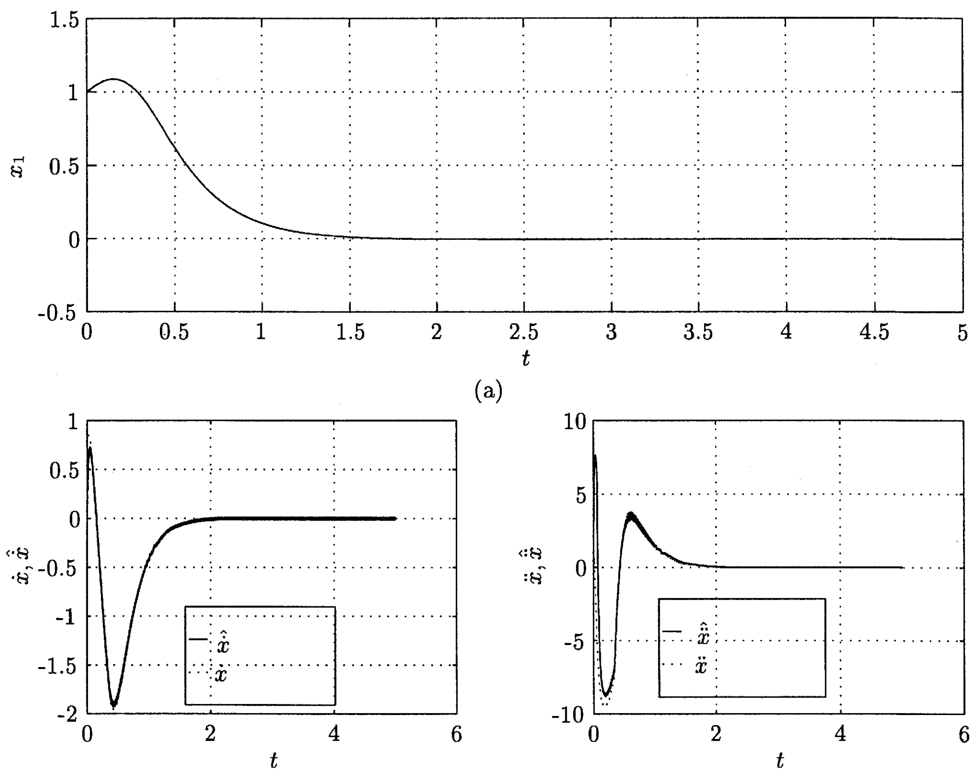
$$\begin{cases} \hat{\ddot{x}}_1 = x_3 + z_2 \\ \tau_2 \dot{z}_2 = -z_2 + \gamma_2 \operatorname{sign}(\hat{\dot{x}}_1 - \varepsilon_2) \\ \dot{\varepsilon}_2 = \alpha_0(\hat{\dot{x}}_1 - \varepsilon_2) + x_3 + \gamma_2 \operatorname{sign}(\hat{\dot{x}}_1 - \varepsilon_2) \end{cases} \tag{62}$$

Finally, the control action is implemented as

$$u = -\delta_1 \hat{\dot{x}}_1 - \delta_2 \hat{\ddot{x}}_1 - \gamma \operatorname{sign}(\hat{s}) \tag{63}$$

where $\hat{s} = \delta_1 x_1 + \delta_2 \hat{\dot{x}}_1 + \hat{\ddot{x}}_1$.

Choosing $\delta_1 = 25$, $\delta_2 = 10$, $\lambda = 10$, $\lambda_1 = \lambda_2 = 20$, $\alpha_0 = 10$, $\tau_1 = \tau_2 = 20 \times 10^{-3}$, the simulation results are shown in Fig. 1 with initial conditions $x_1(0) = 1$, $x_2(0) = 0$, $x_3(0) = 0$. Figure 1(a) shows that the output x converges nicely to 0. Figures 1(b) and 1(c) show the estimates of $\dot{x}_1$ and $\ddot{x}_1$ versus the true $\dot{x}_1$ and $\ddot{x}_1$.



(b) (c)

Fig. 1. Simulation results.

5. Conclusion

In this paper, we presented a new controller for uncertain dynamical systems, which is based on a procedure for estimation of the plant derivatives. The key idea consists in the introduction of sliding mode filters which estimate the uncertain nonlinear functions on the right-hand sides of the plant differential equations during the sliding motion. Within this framework we use first-order filters to get the average values of signum functions or an "equivalent control". The result extends the Multiple Sliding Surface Control concept for systems with non-Lipschitz drift and mismatched uncertainties.

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