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A STOCHASTIC CRITERION FOR THE OPTIMAL INVESTMENT PROCESS

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The paper describes an approach to the construction of a criterion suitable for optimization of an investment project. An income growth resulting from realization of the investment project is considered as a time sequence of random variables. Then an expression is derived which characterizes the project cost at a sequence of time intervals it and raises a possibility to find the real income as a random variable. Two situations are considered, namely when the investor possesses complete and incomplete information about the economic situation and the corresponding incomes. The obtained results can be used to construct an investment criterion in the case of a vague economic situation.

1. Introduction

The problem of estimating the efficiency of economic projects has been receiving particular attention recently. Clearly, the problem of construction of an appropriate model for the investment process is related to financial mathematics. In the literature, the following mathematical tools have been exploited to address this issue: classical statistical methods (Lukashin, 1995), mathematical-programming methods (Kobila, 1991; Pleshinsky, 1995; Voloshinov and Levitan, 1996), the theory of random processes (Leonenko *et al.*, 1995) and a martingale-based approach (Malliaris and Brock, 1982; Samuelson, 1965), as well as stochastic control (Cantor and Lippman, 1995) and stochastic game theory (Döge and Nollau, 1995; Pham-Gia, 1995).

The investment project is considered as an investment of money resources aiming at a definite income in the future. The initial expenditures which are necessary for the project realization are supposed to give rise to accumulation of money in future, from which the investor's income will be deducted.

Estimation of the economic efficiency of an investment project during its "life cycle" creates the problem of combining distributed (in time) incomes and expenditures to one specific moment which is called the calculation period. This follows from the fact that the value of financial resources which describes the profits and expenditures is different for different time periods. As a rule, the calculation is conducted on the basis of the period of investment realization, which is considered as the initial one. The period which follows the first one is considered as the period generating the first interests due to expenditures (Bagal, 1996).

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The use of the above principle yields the following expression concerning the profit realized within the period of the investment activities:

$$P = \sum_{i=1}^{T} \frac{x_i}{(1+r)^i} \tag{1}$$

where P is the current value of a profit flow for T periods, x_i denotes the growth of the profit during the i-th period due to realization of the investment project, r stands for the rate of discounting and is usually considered as a long-term interest rate. In other words, (1) describes the real total income due to an investment.

Consider the case when a capital I is invested at the initial moment of the investment cycle. Then we get the following expression for the index of the present net value of the project:

$$S = P - I = \sum_{i=1}^{T} \frac{x_i}{(1+r)^i} - I \tag{2}$$

2. Problem Statement

Consider a sequence of deterministic values $x_0, \ldots, x_t, \ldots, x_{t+i}, \ldots,\ i = 1, 2, \ldots, N$, $t \geq 0$, representing the values of profit gains and obtained from realization of the investment project by some economic agent. We may rewrite (2) in the following form:

$$S_t = \sum_{i=1}^{N} \frac{x_{t+i}}{(1+\alpha)^i} - R_t$$

where α is the capital interest rate which is used as the rate of discounting, N is the term of the project fulfilling, R_t stands for the size of the initial investment capital. Note that the size of the index of the present net value S_t is calculated with respect to some moment t which is accepted as the initial one. This parameter concerns the moment $t+1$ and, with the same investment expenditures $R_{t+1} = R_t$, it is defined by

$$S_{t+1} = \sum_{i=1}^{N} \frac{x_{t+1+i}}{(1+\alpha)^i} - R_t = \sum_{i=2}^{N+1} \frac{x_{t+i}}{(1+\alpha)^{i-1}} - R_t$$

Hence

$$S_t - \frac{S_{t+1}}{1+\alpha} = \frac{x_{t+1}}{1+\alpha} - \frac{x_{t+N+1}}{(1+\alpha)^{N+1}} - \frac{\alpha R_t}{1+\alpha}$$

or

$$S_{t+1} = (1+\alpha)S_t - x_{t+1} + \frac{x_{t+N+1}}{(1+\alpha)^N} + \alpha R_t$$

We may extend the obtained expressions to a stochastic case. Let $(\Omega, \Im, P)$ be a probability space, where Ω is a set consisting of elementary events ω with a system $\Im$ of its subsets (events) forming a σ-algebra, and P is a probability measure defined on the sets from $\Im$. In what follows, we let $x_0(\omega), \dots, x_t(\omega), x_{t+1}(\omega), \dots, x_{t+i}(\omega), \dots$ be a sequence of non-negative random values defined on this space and designating the sizes of the profit gains from the investment project realization, $i = 1, 2, \dots, N,\ t \geq 0$, $\omega \in \Omega$. A natural assumption from the economic point of view is the boundedness of the values $x_t = x_t(\omega)$.

In conformity with the sequence of values x_t, let us introduce the sequence of σ-algebras $\Im_t = \sigma(x_0, \dots, x_{t-1}, x_t)$. In the language of economy, it is possible to say that the economic agent has at his disposal information about economic conditions incorporated into $\Im_t$ up to and including the moment t, but he does not have similar information starting from the moment $t+i$, $i = 1, 2, \dots, N$.

Define the random variable

$$\xi_t = E\left[\sum_{i=1}^{N} \frac{x_{t+i}}{(1+\alpha)^i} - R_t \,|\Im_t\right] = E\left[\sum_{i=1}^{N} \frac{x_{t+i}}{(1+\alpha)^i} \,|\Im_t\right] - R_t$$

Similarly to the deterministic case, we have

$$\xi_{t+1} = E\left[\sum_{i=1}^{N} \frac{x_{t+i+1}}{(1+\alpha)^i} - R_t \,|\Im_{t+1}\right] = E\left[\sum_{i=2}^{N+1} \frac{x_{t+i}}{(1+\alpha)^{i-1}} \,|\Im_{t+1}\right] - R_t$$

Now we formulate the following problem: it is required to find a value of the index of the present net cost of the project ξ_{t+1}, with respect to the moment $t+1$ and at expenditures R_t, which is not less than ξ_t. In other words, "starting" the investment project with the delay of one period, but possessing a larger volume of information incorporated into $\Im_{t+1}$, the economic agent should not lose.

3. Formulation of the Criterion

Taking into consideration the foregoing statements, we conclude that

$$\xi_t \leq \xi_{t+1}$$

or

$$E\left[\sum_{i=1}^{N} \frac{x_{t+i}}{(1+\alpha)^i} \,|\Im_t\right] - R_t \leq E\left[\sum_{i=2}^{N+1} \frac{x_{t+i}}{(1+\alpha)^{i-1}} \,|\Im_{t+1}\right] - R_t \quad (P\text{-a.e.}) \qquad (3)$$

For example, if $x_t(\omega)$ is a narrow-sense stationary process, then $\xi_t = \xi_{t+1}$ (P-a.e.). Note that the term $x_{t+1}/(1+\alpha)$ on the left-hand side of (3) is a deterministic value. Actually, the investor who makes an investment at the time moment $t+1$ takes his decision based on the observations of the economic situation up to and including the moment $t+1$. Hence the size of the profit growth $x_{t+1}/(1+\alpha)$ gained by his competitor as a result of realization of the same investment scheme is known. Thus (3) can be written down as

$$\frac{x_{t+1}}{1+\alpha} + E\left[\sum_{i=2}^{N} \frac{x_{t+i}}{(1+\alpha)^i} \,|\Im_t\right] \le E\left[\sum_{i=2}^{N} \frac{x_{t+i}}{(1+\alpha)^{i-1}} \,|\Im_{t+i}\right] + E\left[\frac{x_{t+N+1}}{(1+\alpha)^N} \,|\Im_{t+1}\right]$$

or

$$\frac{x_{t+1}}{1+\alpha} - E\left[\frac{x_{t+N+1}}{(1+\alpha)^N} \,|\Im_{t+1}\right] \le \sum_{i=2}^{N} \left\{ E\left[\frac{x_{t+i}}{(1+\alpha)^{i-1}} \,|\Im_{t+i}\right] - E\left[\frac{x_{t+i}}{(1+\alpha)^i} \,|\Im_t\right]\right\} \quad (4)$$

It is possible to consider (4) as a property which the random process $x_t(\omega)$ should satisfy.

Let us find the conditional expectation with respect to $\Im_t$ for both parts of (3). We have

$$E[\xi_t \,|\Im_t] = \xi_t = E\left[\sum_{i=1}^{N} \frac{x_{t+i}}{(1+\alpha)^i} \,|\Im_t\right] - R_t$$

and

$$\begin{aligned} E[\xi_{t+1} \,|\Im_t] &= E\left[E\left[\sum_{i=2}^{N+1} \frac{x_{t+i}}{(1+\alpha)^{i-1}} \,|\Im_{t+1}\right] |\Im_t\right] - R_t \\ &= E\left[\sum_{i=2}^{N+1} \frac{x_{t+i}}{(1+\alpha)^{i-1}} \,|\Im_t\right] - R_t \\ &= E\left[\left(\pm x_{t+1} + \sum_{i=2}^{N} \frac{x_{t+i}}{(1+\alpha)^{i-1}} + \frac{x_{t+N+1}}{(1+\alpha)^N}\right) |\Im_t\right] - R_t \end{aligned}$$

Multiplying and dividing the obtained expression by $(1+\alpha)$, we get

$$\begin{aligned} E[\xi_{t+1} \,|\Im_t] &= (1+\alpha)E\left[\sum_{i=1}^{N} \frac{x_{t+i}}{(1+\alpha)^i} \,|\Im_t\right] - E[x_{t+1} \,|\Im_t] \\ &\quad + E\left[\frac{x_{t+N+1}}{(1+\alpha)^N} \,|\Im_t\right] - R_t \end{aligned}$$

Inequality (3) can then be rewritten as follows:

$$\begin{aligned} E\left[\sum_{i=1}^{N} \frac{x_{t+i}}{(1+\alpha)^i} \,|\Im_t\right] - R_t \le (1+\alpha)\, E\left[\sum_{i=1}^{N} \frac{x_{t+i}}{(1+\alpha)^i} \,|\Im_t\right] \\ - E[x_{t+1} \,|\Im_t] + E\left[\frac{x_{t+N+1}}{(1+\alpha)^N} \,|\Im_t\right] - R_t \end{aligned}$$

or

$$\frac{x_{t+1}}{1+\alpha} - E\left[\frac{x_{t+N+1}}{(1+\alpha)^{N+1}} |\Im_t\right] \leq \frac{\alpha}{1+\alpha} E\left[\sum_{i=1}^{N} \frac{x_{t+i}}{(1+\alpha)^i} |\Im_t\right] \tag{5}$$

Inequality (5) can be viewed as a consequence of (3).

Example 1. Let an initial investment be \$1000.0 million. Besides, let this investment represent some project of technological transforms, realization of which does not depend on economic pre-history (though in a real economic situation this is an over-idealized case). Then the conditional expectation in (4) can be replaced by the expectation. Assume that this innovation project has already been functioning for a year and due to this fact it brings on the average profit equal to 10% of return, i.e. \$100 million. If the innovation works in such an economic mode for the whole period of its life cycle (e.g., for 5 years), i.e. the mathematical expectation of the process $x_t(\omega)$ is constant, then (4) takes the form

$$\frac{100}{1+0,1} - \frac{100}{(1+0,1)^5} \leq \sum_{i=2}^{5} \left\{ \frac{100}{(1+0,1)^{i-1}} - \frac{100}{(1+0,1)^i} \right\}$$

or

$$0 \leq 0$$

Hence, (4) holds in this case. ♦

Without special knowledge about the structure of the process $x_t(\omega)$ it is difficult, however, to expect a fulfilment of the condition (3) which is too strong in the context of the postulated boundedness of initial assumptions. In fact, the requirement of a greater index of the present net value of the project which was started by an economic agent with the delay of one period, in comparison with the same index for the project started by his competitor in due time, would mean creation of ideally favourable conditions for this agent, which cannot be achieved in a real-life economy. For example, when an investment of a capital into state capital issues which have a fixed but low income is considered, it is necessary to know a structure of the process $x_t(\omega)$ so as to fulfil (4).

Therefore it would be more reasonable to use a less idealized criterion which could be satisfied in real economic conditions. For example, such a criterion could be represented as

$$P\Big\{\xi_{t+1} - \xi_t \geq 0\Big\} > \frac{1}{2} \tag{6}$$

or an average criterion like

$$E\left(\int_0^t (\xi_{\tau+1} - \xi_\tau) \mathrm{d}\tau\right) > 0, \quad \tau \in (0,t) \tag{7}$$

4. Conclusion

The idea proposed in this work deals with construction of a version of the criterion for optimization of the investment project. Further research can be performed in the direction of a search for some properties and structure of the random process $x_t(\omega)$, which would allow us to formulate more specific optimal criteria and to put forward conditions of their realization. Martingale theory can be applied to solve this problem which has found a wide application in financial mathematics. Finally, an interesting result can be obtained when using this technique for estimating specific types of investments with application of criteria (6) and (7) to them.

References

Bagal Y.M. (1996): *Economic Theory of Technological Transforms.* — Kiev: Zapovit, p.240 (in Ukrainian).

Cantor David G. and Lippman Steven A. (1995): *Optimal investment selection with a multitude of projects.* — Econometrica, Vol.63, No.5, pp.1231–1240.

Döge G. and Nollau V. (1995): *Semi-Markovian decision models with vector-valued reward.* — Optimization, Vol.34, No.3, pp.247–259.

Kobila T.O. (1991): *Partial investment under uncertainty,* In: Stochastic Models and Option Values (Elsevier, Ed.). — North-Holland, pp.167–185.

Leonenko M.M., Mishura J.S., Parhomenko V.M. and Yadrenko M.I. (1995): *Theoretical-probability and Statistical Methods in Econometrics and Financial Mathematics.* — Kiev: Informtechnique, p.380 (in Ukrainian).

Lukashin Y.P. (1995): *Optimization of the structure of investment security portfolio.* — Economics and Math. Methods, Vol.31, No.1, pp.138–150 (in Russian).

Malliaris A.G. and Brock W.A. (1982): *Stochastic methods in economics and finance.* — North-Holland: Elsevier, pp.16–29.

Pham-Gia T. (1995): *Some applications of the Lorenz curve in decision analysis.* — Amer. J. Math. and Manag. Sci., Vol.15, No.1–2, pp.1–34.

Pleshinsky A.S. (1995): *Optimization of enterprise investment projects in the market economy conditions.* — Economics and Math. Methods, Vol.31, No.2, pp.81–90 (in Russian).

Samuelson P.A. (1965): *Rational theory of warrant pricing.* — Industrial Management Review, Vol.6 (Spring), pp.13–31.

Voloshinov V.V. and Levitan E.S. (1996): *Extremal restrictions in the models of investment programs with financial mechanism of forthcoming discharges' collateral.* — Economics and Math. Methods, Vol.31, No.2, pp.117–127 (in Russian).

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