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# TIME-DOMAIN SYNTHESIS OF LINEAR CIRCUITS WITH PERIODICALLY VARIABLE PARAMETERS

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A novel synthesis method of linear electric networks with periodically variable parameters is presented. These types of networks can be implemented as resistance-parametric, (conductance-capacitance)-parametric or (resistance-inductance)-parametric multiports. The time-varying parameters are determined so as to extremize some specific optimisation criteria. Consequently, the synthesis task has a unique solution. The starting point of the solution method is the original theorem proved in the paper, which deals with the minimum-energy networks.

**Keywords:** parametric circuits, synthesis, optimisation

## 1. Introduction

The synthesis of linear time-varying networks seems a desert field. This results from the fact that even the analysis of these networks is difficult. Although it is not possible to use well-known analysis methods, complex analytic-numerical procedures can still be applied.

Parametric networks can be implemented as four-terminal networks or, more generally, as correction multi-port networks for co-operation with non-linear systems or as parametric amplifiers. In the present paper, a time-domain synthesis is performed based on some optimisation criteria.

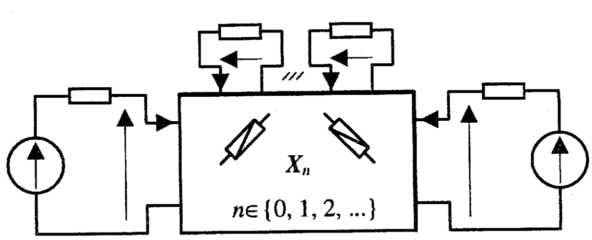


Fig. 1. Multi-port network with periodically time-varying elements.

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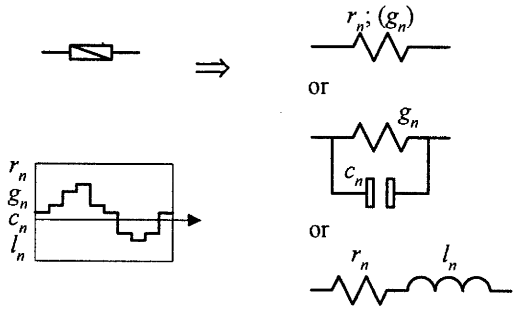


Fig. 2. Classes of time-varying circuit elements and their approximate wave forms in discrete time.

Within a synthetized multi-port (Fig. 1) there are resistance-parametric elements or (conductance-capacitance)-parametric or dual (resistance-inductance)-parametric ones (elements of class $R_n$, $G_n$, $(GC)_n$ or $(RL)_n$, respectively, where $n \in \{0, 1, 2, \ldots\}$, i.e. the time is discrete):

- $r_n$, $(g_n)$ – resistance-parametric class $(R \text{ or } G)_n$,
- $g_n$, $(c_n)$ – (conductance-capacitance)-parametric class $(GC)_n$,
- $r_n$, $l_n$ – (resistance-inductance)-parametric class $(RL)_n$.

Each component is a separate branch. The branches are connected into the network and form an optionally connected structure.

The synthesis of a multi-port consists in finding a discrete-time function $x_n =$ vector$(r_n, g_n, c_n, l_n)$ (usually periodic), which expresses the variation of a vector of elements inside the network, in such a way as to ensure some specified signal values (usually also periodic) of currents and voltages of all the ports to which the sources and receivers of the system are connected. The set of input signals constitutes the so-called *conditions on the edge*. It is assumed that the conditions on the edge and the vector $x_n$ have a common period and, consequently, a stable steady state.

In general, the search for $x_n$ does not result in a unique solution. The theorem concerning minimum-energy networks given in Section 2 makes it possible to uniquely spread the current and voltage signals on particular branches inside the multi-port.

## 2. Theorem on Minimum-Energy Networks

Assume that some actual voltage vector $v$ and an actual current vector $j$ at the edge (at ports) of a multi-port (Fig. 3) are given.

**Theorem 1.** *For the assumed structure inside the multi-port there exists a unique distribution of branch voltages such that the sum of the squared instantaneous inner voltages (Euclidean norm) is minimal and there exists a unique distribution of the*

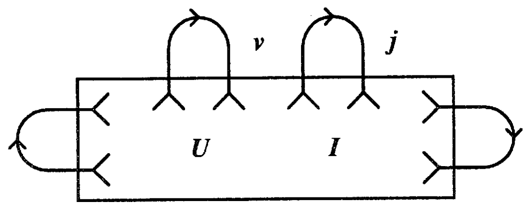


Fig. 3. The assumed set of branches ↔ external signals $\boldsymbol{v}$, $\boldsymbol{j}$ and unknown instantaneous internal voltages and currents $\boldsymbol{U}$ and $\boldsymbol{I}$, respectively.

*inner branch currents which satisfies the rule of the minimum of the sum of their squared actual values.*

*Proof.* Inside the system there exists a vector (isomorphically a set) of free voltages (branches) $\boldsymbol{u}$ and a vector of free currents $\boldsymbol{i}$. Either of them or both can be empty.

Calculation of the free voltages is performed according to the following scheme:

1. Select a tree beginning from the input (port) branch $\boldsymbol{v}$.
2. Find the set of the free voltages:

$$\boldsymbol{u} =: \left\{ \begin{array}{c} \text{TREE} \\ \text{BRANCHES} \end{array} \right\} \setminus \left\{ \begin{array}{c} \text{OUTER} \\ \text{BRANCHES} \end{array} \right\}.$$

The free currents are calculated using strings:

1. The input branches (ports $\boldsymbol{j}$) are strings. Add the other strings in such a way that the complement is a tree.
2. Calculate the set of free currents:

$$\boldsymbol{i} =: \{\text{STRINGS}\} \setminus \left\{ \begin{array}{c} \text{OUTER} \\ \text{STRINGS} \end{array} \right\}.$$

The instantaneous vector (set) of voltages of the inner branches is defined by the formula

$$\boldsymbol{U} = \boldsymbol{P} \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{u} \end{bmatrix},$$

where $\boldsymbol{P}$ stands for the section matrix. In turn, the instantaneous vector (set) of currents of the inner branches is defined by

$$\boldsymbol{I} = \boldsymbol{C} \begin{bmatrix} \boldsymbol{j} \\ \boldsymbol{i} \end{bmatrix},$$

where $\boldsymbol{C}$ is the cycle matrix.

We select $\boldsymbol{i}$ in such a way as to minimize the scalar product

$$\boldsymbol{I}^*\boldsymbol{I} \to \min,$$

or

$$[\boldsymbol{j}^*\ \boldsymbol{i}^*]\,\boldsymbol{C}^T\boldsymbol{C}\begin{bmatrix} \boldsymbol{j} \\ \boldsymbol{i} \end{bmatrix} \to \min .$$

Setting

$$\boldsymbol{C}^T\boldsymbol{C} = \begin{bmatrix} \boldsymbol{B}_{11} & \boldsymbol{B}_{12} \\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22} \end{bmatrix}$$

we obtain the following condition:

$$\boldsymbol{j}^*\boldsymbol{B}_{12}\boldsymbol{i} + \boldsymbol{i}^*\boldsymbol{B}_{21}\boldsymbol{j} + \boldsymbol{i}^*\boldsymbol{B}_{22}\boldsymbol{i} \to \min .$$

The minimum is attained when, for any variation $\delta\boldsymbol{i}$, the following inequality is satisfied:

$$\boldsymbol{j}^*\boldsymbol{B}_{12}\delta\boldsymbol{i} + \delta\boldsymbol{i}^*\boldsymbol{B}_{21}\boldsymbol{j} + 2\delta\boldsymbol{i}^*\boldsymbol{B}_{22}\boldsymbol{i} + \delta\boldsymbol{i}^*\boldsymbol{B}_{22}\delta\boldsymbol{i} > 0$$

The matrix $\boldsymbol{B}_{22}$ is positive definite, so for each $\delta\boldsymbol{i}$ we must have

$$\delta\boldsymbol{i}^*\left[2\boldsymbol{B}_{22}\boldsymbol{i} + \left(\boldsymbol{B}_{21} + \boldsymbol{B}_{12}^T\right)\boldsymbol{j}\right] = 0$$

which amounts to

$$\boldsymbol{B}_{22}\boldsymbol{i} + \boldsymbol{B}_{21}\boldsymbol{j} = 0.$$

The obtained set of equations has only one solution

$$\boldsymbol{i} = -\boldsymbol{B}_{22}^{-1}\boldsymbol{B}_{21}\boldsymbol{j}$$

because the matrix $\boldsymbol{B}_{22}$ is non-singular.

We can give an analogous proof for the voltages and obtain

$$\boldsymbol{u} = -\boldsymbol{B}_{22}^{-1}\boldsymbol{B}_{21}\boldsymbol{v},$$

where the $\boldsymbol{B}_{ij}$'s result from the matrix decomposition

$$[\boldsymbol{v}^*\ \boldsymbol{u}^*]\,\boldsymbol{P}^T\boldsymbol{P}\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{u} \end{bmatrix} = [\boldsymbol{v}^*\ \boldsymbol{u}^*]\begin{bmatrix} \boldsymbol{B}_{11} & \boldsymbol{B}_{12} \\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22} \end{bmatrix}\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{u} \end{bmatrix}$$

This completes the proof. ∎

**Remark 1.** Simultaneous minimization of the instantaneous norms of the vectors of branch voltages and currents amounts to minimization of the instantaneous index

$$\sigma = \sqrt{\boldsymbol{I}^* \boldsymbol{I}} \sqrt{\boldsymbol{U}^* \boldsymbol{U}} \to \min .$$

This index can be called the instantaneous apparent power of the network inside the multi-port.

**Remark 2.** It is easy to prove that the theorem on minimum-energy networks being true for the instantaneous functionals

$$\forall\, n \in \{0,1,2,\ldots\},\ \boldsymbol{I}_n^* \boldsymbol{I}_n \to \min \quad \text{and} \quad \forall\, n,\ \boldsymbol{U}_n^* \boldsymbol{U}_n \to \min,$$

remains valid for the functionals averaging along time, i.e.

$$\sum_{n \in \{0,1,2,\ldots\}} \boldsymbol{I}_n^* \boldsymbol{I}_n \to \min \quad \text{and} \quad \sum_n \boldsymbol{U}_n^* \boldsymbol{U}_n \to \min .$$

**Remark 3.** The theorem is also true for the complex spectrum of the voltage and current signals.

## 3. Minimization of the Voltage-Current Functionals

This method consists in distributing the voltage and current signals on the branches inside the network in such a manner that at each moment the following functional is minimized:

$$f(\boldsymbol{u}, \boldsymbol{i}) = f(\boldsymbol{y}) \to \min,$$

where $\boldsymbol{y} = \left[\begin{smallmatrix} \boldsymbol{u} \\ \boldsymbol{i} \end{smallmatrix}\right]$ denotes the vector of free voltages and currents.

The Taylor expansion gives

$$f(\boldsymbol{y} + \delta \boldsymbol{y}) = \delta \boldsymbol{y}^T \frac{\partial f}{\partial \boldsymbol{y}} + \delta \boldsymbol{y}^T \frac{\partial}{\partial \boldsymbol{y}} \left( \frac{\partial f}{\partial \boldsymbol{y}} \right) \partial \boldsymbol{y} > 0$$

for each $\partial \boldsymbol{y}$. The sufficient condition for optimality is the vanishing of the gradient, i.e.

$$\frac{\partial f}{\partial \boldsymbol{y}} = \boldsymbol{0},$$

and positive definiteness of the Hessian, i.e.

$$\frac{\partial}{\partial \boldsymbol{y}} \left( \frac{\partial f}{\partial \boldsymbol{y}} \right) > 0,$$

where

$$\left[ \frac{\partial}{\partial \boldsymbol{y}} \left( \frac{\partial f}{\partial \boldsymbol{y}} \right) \right]_{nm} = \frac{\partial^2 f}{\partial y_n \partial y_m} .$$

The above static equation for the gradient can be solved using the continuous version of the steepest-descent method, which involves solving the equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{\partial f}{\partial y} \tag{1}$$

Then the condition for the Hessian coincides with the stability of a stationary point of the system (1).

Accordingly, in practice the most important is the functional

$$\boldsymbol{U}^T (\operatorname{diag} \boldsymbol{I})^2 \boldsymbol{U} = \boldsymbol{I}^T (\operatorname{diag} \boldsymbol{U})^2 \boldsymbol{I} \to \min,$$

i.e. the sum of the squared instantaneous powers inside the multi-port.

Minimizing the functional for a fixed vector of currents $\boldsymbol{I}$, we obtain the problem

$$\underset{\boldsymbol{U}}{\mathrm{VAR}}\ \boldsymbol{U}^T \boldsymbol{Q} \boldsymbol{U} \to \min,$$

where

$$\boldsymbol{Q} = (\operatorname{diag} \boldsymbol{I})^2, \quad \boldsymbol{U} = \boldsymbol{P} \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{u} \end{bmatrix}.$$

From this we have the differential

$$\mathrm{d}\boldsymbol{U} = \boldsymbol{P} \begin{bmatrix} \boldsymbol{0} \\ \mathrm{d}\boldsymbol{u} \end{bmatrix}.$$

Since

$$\left(\boldsymbol{U}^T + \mathrm{d}\boldsymbol{U}^T\right) \boldsymbol{Q} (\boldsymbol{U} + \mathrm{d}\boldsymbol{U}) = \boldsymbol{U}^T \boldsymbol{Q} \boldsymbol{U} + \boldsymbol{U}^T \boldsymbol{Q}\, \mathrm{d}\boldsymbol{U} + \mathrm{d}\boldsymbol{U}^T \boldsymbol{Q} \boldsymbol{U} + \mathrm{d}\boldsymbol{U}^T \boldsymbol{Q}\, \mathrm{d}\boldsymbol{U},$$

the optimality condition has the form

$$[\boldsymbol{v}^T\ \boldsymbol{u}^T]\, \boldsymbol{P}^T \boldsymbol{Q} \boldsymbol{P} \begin{bmatrix} \boldsymbol{0} \\ \mathrm{d}\boldsymbol{u} \end{bmatrix} + [\boldsymbol{0}^T\ \mathrm{d}\boldsymbol{u}^T]\, \boldsymbol{P}^T \boldsymbol{Q} \boldsymbol{P} \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{u} \end{bmatrix} = 0$$

or

$$[\boldsymbol{v}^T\ \boldsymbol{u}^T] \begin{bmatrix} \boldsymbol{B}_{11} & \boldsymbol{B}_{12} \\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{0} \\ \mathrm{d}\boldsymbol{u} \end{bmatrix} + [\boldsymbol{0}^T\ \mathrm{d}\boldsymbol{u}^T] \begin{bmatrix} \boldsymbol{B}_{11} & \boldsymbol{B}_{12} \\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{u} \end{bmatrix} = 0.$$

Thus

$$[\boldsymbol{v}^T\ \boldsymbol{u}^T] \begin{bmatrix} \boldsymbol{B}_{12}\, \mathrm{d}\boldsymbol{u} \\ \boldsymbol{B}_{22}\, \mathrm{d}\boldsymbol{u} \end{bmatrix} + [\mathrm{d}\boldsymbol{u}^T \boldsymbol{B}_{21}\ \mathrm{d}\boldsymbol{u}^T \boldsymbol{B}_{22}] \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{u} \end{bmatrix} = 0$$

or

$$2\mathrm{d}\boldsymbol{u}^T \boldsymbol{B}_{22} \boldsymbol{u} + \mathrm{d}\boldsymbol{u}^T \left(\boldsymbol{B}_{21} + \boldsymbol{B}_{12}^T\right) \boldsymbol{v} = 0.$$

Consequently, the optimality condition can be rewritten as

$$\forall\, \mathrm{d}\boldsymbol{u}, \quad \mathrm{d}\boldsymbol{u}^T (\boldsymbol{B}_{22}\boldsymbol{u} + \boldsymbol{B}_{21}\boldsymbol{v}) = 0.$$

The corresponding optimal vector of free voltages is obtained by solving the system of linear equations

$$\boldsymbol{B}_{22}\boldsymbol{u} = -\boldsymbol{B}_{21}\boldsymbol{v}, \tag{2}$$

where

$$\begin{bmatrix} \boldsymbol{B}_{11} & \boldsymbol{B}_{12} \\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22} \end{bmatrix}_{nm} = \left[\boldsymbol{P}^T (\operatorname{diag} \boldsymbol{I})^2 \boldsymbol{P}\right]_{nm} = \sum_{p\in\{n\}\cap\{m\}} \operatorname{sgn}(p) I_p^2,$$

$n$ and $m$ denote tree branches of the network, $\{n\}$ stands for the section of the $n$-th tree branch, $\{n\}\cap\{m\}$ signifies the algebraic intersection of sets $\{n\}$ and $\{m\}$ (the set of common elements and, at the same time, if an element has the same sign in $\{n\}$ and $\{m\}$, then it is included into the intersection with the '+' sign, otherwise it enters it with the '−' sign), $\operatorname{sgn}(p)$ is the sign of $p$. Hence

$$[\boldsymbol{B}_{22}]_{nm} = \sum_{\substack{p\in\{n\}\cap\{m\} \\ \text{internal sections}}} \operatorname{sgn}(p)\, I_p^2, \quad [\boldsymbol{B}_{21}]_{nm} = \sum_{\substack{p\in\{n\}\cap\{m\} \\ \text{internal sections} \quad \text{outer sections}}} \operatorname{sgn}(p)\, I_p^2.$$

The analogous problem

$$\operatorname*{VAR}_{\boldsymbol{I}}\ \boldsymbol{I}^T (\operatorname{diag} \boldsymbol{U})^2 \boldsymbol{I} \to \min$$

has the solution

$$\boldsymbol{B}_{22}\boldsymbol{i} = -\boldsymbol{B}_{21}\boldsymbol{j}, \tag{3}$$

where

$$\begin{bmatrix} \boldsymbol{B}_{11} & \boldsymbol{B}_{12} \\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22} \end{bmatrix}_{nm} = \left[\boldsymbol{C}^T (\operatorname{diag} \boldsymbol{U})^2 \boldsymbol{C}\right]_{nm} = \sum_{p\in\{n\}\cap\{m\}} \operatorname{sgn}(p)\, U_p^2,$$

and $\{n\}$, $\{m\}$ are cycles (strings). Consequently,

$$[\boldsymbol{B}_{22}]_{nm} = \sum_{\substack{p\in\{n\}\cap\{m\} \\ \text{internal sections}}} \operatorname{sgn}(p)\, U_p^2, \quad [\boldsymbol{B}_{21}]_{nm} = \sum_{\substack{p\in\{n\}\cap\{m\} \\ \text{internal sections} \quad \text{outer sections}}} \operatorname{sgn}(p)\, U_p^2.$$

The search for a minimum point of the voltage-current functional is conducted by means of the differential

$$\mathrm{d}\boldsymbol{U}^T (\operatorname{diag} \boldsymbol{I})^2 \boldsymbol{U} + \boldsymbol{U}^T (\operatorname{diag} \boldsymbol{I})^2 \mathrm{d}\boldsymbol{U} + \mathrm{d}\boldsymbol{I}^T (\operatorname{diag} \boldsymbol{U})^2 \boldsymbol{I} + \boldsymbol{I}^T (\operatorname{diag} \boldsymbol{U})^2 \mathrm{d}\boldsymbol{I} = 0. \tag{4}$$

But this equation can be split into two parts, namely

$$\mathrm{d}\boldsymbol{U}^T\left(\operatorname{diag}\boldsymbol{I}^k\right)^2\boldsymbol{U}+\boldsymbol{U}^T\left(\operatorname{diag}\boldsymbol{I}^k\right)^2\mathrm{d}\boldsymbol{U}=0\Rightarrow\boldsymbol{U}^k,$$

$$\mathrm{d}\boldsymbol{I}^T\left(\operatorname{diag}\boldsymbol{U}^k\right)^2\boldsymbol{I}+\boldsymbol{I}^T\left(\operatorname{diag}\boldsymbol{U}^k\right)^2\mathrm{d}\boldsymbol{I}=0\Rightarrow\boldsymbol{I}^{k+1}.$$

Therefore the following iterative algorithm is obtained:

$$\begin{aligned}
&\boldsymbol{U}^T\boldsymbol{U}\to\min\Rightarrow\boldsymbol{U}^0,\quad\text{(initial distribution)}\\
&\boldsymbol{I}^T\left(\operatorname{diag}\boldsymbol{U}^0\right)^2\boldsymbol{I}\to\min\Rightarrow\boldsymbol{I}^0,\\
&\boldsymbol{U}^T\left(\operatorname{diag}\boldsymbol{I}^0\right)^2\boldsymbol{U}\to\min\Rightarrow\boldsymbol{U}^1,\\
&\boldsymbol{I}^T\left(\operatorname{diag}\boldsymbol{U}^1\right)^2\boldsymbol{I}\to\min\Rightarrow\boldsymbol{I}^1,\\
&\boldsymbol{U}^T\left(\operatorname{diag}\boldsymbol{I}^1\right)^2\boldsymbol{U}\to\min\Rightarrow\boldsymbol{U}^2,\\
&\qquad\vdots\\
&\boldsymbol{I}^T\left(\operatorname{diag}\boldsymbol{U}^k\right)^2\boldsymbol{I}\to\min\Rightarrow\boldsymbol{I}^k,\\
&\boldsymbol{U}^T\left(\operatorname{diag}\boldsymbol{I}^k\right)^2\boldsymbol{U}\to\min\Rightarrow\boldsymbol{U}^{k+1},\\
&\boldsymbol{I}^T\left(\operatorname{diag}\boldsymbol{U}^{k+1}\right)^2\boldsymbol{I}\to\min\Rightarrow\boldsymbol{I}^{k+1},\\
&\qquad\vdots
\end{aligned}$$

or

$$\begin{aligned}
&\boldsymbol{I}^T\boldsymbol{I}\to\min\Rightarrow\boldsymbol{I}^0,\\
&\boldsymbol{U}^T\left(\operatorname{diag}\boldsymbol{I}^0\right)^2\boldsymbol{U}\to\min\Rightarrow\boldsymbol{U}^0,\\
&\boldsymbol{I}^T\left(\operatorname{diag}\boldsymbol{U}^0\right)^2\boldsymbol{I}\to\min\Rightarrow\boldsymbol{I}^1,\\
&\boldsymbol{U}^T\left(\operatorname{diag}\boldsymbol{I}^1\right)^2\boldsymbol{U}\to\min\Rightarrow\boldsymbol{U}^1,\\
&\boldsymbol{I}^T\left(\operatorname{diag}\boldsymbol{U}^1\right)^2\boldsymbol{I}\to\min\Rightarrow\boldsymbol{I}^2,\\
&\qquad\vdots
\end{aligned}$$

The circuit interpretation of the iterative algorithm is shown in Fig. 4.

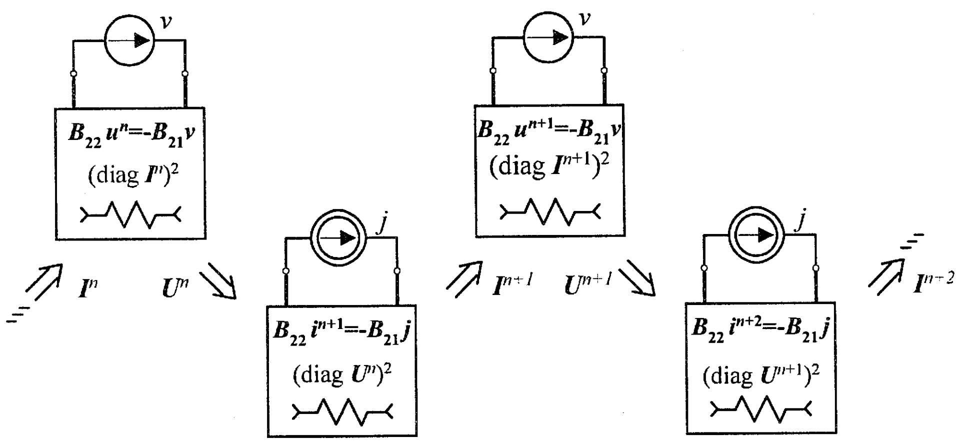


Fig. 4. Circuit interpretation of the iterative algorithm of finding the minimum of the power function.

In each step, by means of the nodal (loop) method, we obtain the distribution of the voltages (currents) in the network for which the conductances (resistances) are equal to the squared voltages (currents) from the previous step.

## 4. Calculation of Parametric Elements in the Inner Branches of the Network

The theorem on minimum-energy networks or minimization of voltage-current functionals makes possible the calculation of instantaneous free voltage signals $\boldsymbol{u}_n$ and/or currents $\boldsymbol{i}_n$ (in discrete time $n$). In order to calculate the vector of parameter changes $\boldsymbol{x}_n$, we have the cyclic equations

$$\boldsymbol{A}_n^T \boldsymbol{x}_n = \begin{bmatrix} \boldsymbol{v}_n \\ \mathbf{0} \end{bmatrix} \tag{5}$$

or section (nodal) equations

$$\boldsymbol{A}_n^T \boldsymbol{x}_n = \begin{bmatrix} \boldsymbol{j}_n \\ \mathbf{0} \end{bmatrix} \tag{6}$$

of the parametric network in discrete time $n \in \{0, 1, 2, \dots\}$.

In each cyclic equation the elements of $\boldsymbol{A}_n$ are formed by appropriate linear combinations (with coefficients 0 or $\pm 1$) of coordinate vectors $\boldsymbol{j}_n$, $\boldsymbol{i}_n$ for the $R_n$-class or by such combinations of coordinates $\boldsymbol{j}_n$, $\boldsymbol{i}_n$, $\Delta \boldsymbol{j}_n$, $\Delta \boldsymbol{i}_n$ for the $(RL)_n$-class $(\Delta \boldsymbol{i}_n = \boldsymbol{i}_n - \boldsymbol{i}_{n-1})$.

In the section equations the elements of $\boldsymbol{A}_n$ set up the same combinations of the coordinate vectors $\boldsymbol{v}_n$, $\boldsymbol{u}_n$ for the $G_n$-class or the coordinates $\boldsymbol{v}_n$, $\boldsymbol{u}_n$, $\Delta\boldsymbol{v}_n$, $\Delta\boldsymbol{u}_n$ for the $(GC)_n$-class.

The number of equations, both cyclic and section ones, is less than the number of the calculated components (the dimension of the $\boldsymbol{x}_n$ vector). In order to guarantee an explicit solution, we need some extra conditions. According to the designer's decision, we can have e.g. the criterion of parameter smoothness:

$$\Delta\boldsymbol{x}_n^T \Delta\boldsymbol{x}_n \to \min \tag{7}$$

subject to

$$\boldsymbol{x}_n^T \boldsymbol{Q}_n \boldsymbol{x}_n = q_n, \tag{8}$$

where $q_n$ is the input function of discrete time, $\boldsymbol{Q}_n$ stands for a positive-definite matrix (in most cases the identity matrix) and $\Delta\boldsymbol{x}_n = \boldsymbol{x}_n - \boldsymbol{x}_{n-1}$.

The problems (5), (7), (8) and (6)–(8) have in general unique solutions which are found by means of numerical methods. However, in rare cases the solutions may not exist.

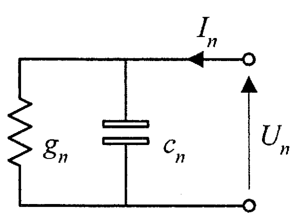


Fig. 5. Elementary parametric two-terminal network of the $(CG)_n$-class.

For the elementary two-terminal network of the $(GC)_n$-class shown in Fig. 5, we get

$$\boldsymbol{a}_n = \begin{bmatrix} U_n \\ \Delta U_n \end{bmatrix}^{(k)}, \quad \boldsymbol{x}_n = \begin{bmatrix} g_n \\ c_n \end{bmatrix}^{(k)}.$$

The nodal equation (6) for the $k$-th branch has the form

$$\boldsymbol{a}_n^T \boldsymbol{x}_n = I_n^{(k)}, \tag{9}$$

where $U_n^{(k)}$ and $I_n^{(k)}$ are the $k$-th coordinates of the inner vectors $\boldsymbol{U}_n$ and $\boldsymbol{I}_n$, respectively. The solution to the problem (8)–(9) is then given by

$$\boldsymbol{x}_n = \frac{1}{\mu_n}\left(\boldsymbol{x}_{n-1} + \lambda_n \boldsymbol{a}_n\right), \tag{10}$$

where

$$\mu_n = \sqrt{\frac{\left(\boldsymbol{x}_{n-1}^T \boldsymbol{x}_{n-1}\right)\left(\boldsymbol{a}_n^T \boldsymbol{a}_n\right) - \left(\boldsymbol{a}_n^T \boldsymbol{x}_{n-1}\right)^2}{q_n\left(\boldsymbol{a}_n^T \boldsymbol{a}_n\right) - \left(I_n\right)^2}}, \quad \lambda_n = \frac{I_n \mu_n - \boldsymbol{a}_n^T \boldsymbol{x}_{n-1}}{\boldsymbol{a}_n^T \boldsymbol{a}_n}.$$

The existence condition for the solution is thus

$$q_n > \frac{I_n^2}{(U_n)^2 + (\Delta U_n)^2}.$$

The recurrent equation (10) is non-linear, but it becomes linear when we omit the condition (8) (Siwczyński, 1998):

$$\begin{bmatrix} g_n \\ c_n \end{bmatrix} = \frac{1}{(U_n)^2 + (\Delta U_n)^2} \left( \begin{bmatrix} (\Delta U_n) & -U_n \Delta U_n \\ -U_n \Delta U & (U_n)^2 \end{bmatrix} \begin{bmatrix} g_{n-1} \\ c_{n-1} \end{bmatrix} + \begin{bmatrix} U_n I_n \\ \Delta U_n I_n \end{bmatrix} \right).$$

The periodical solution for periodical changes of parameters can be found by means of the so-called Poincaré projection (Kudrewicz, 1996). As has been demonstrated by numerical experiments, the periodical solution generally exists.

The synthesis of variable parameters in an elementary branch ot the $(RL)_n$-class is performed analogously.

## 5. Numerical Experiment

Figure 6 shows the graph of a chain network (it can be infinite). The matrices $\boldsymbol{P}$ and $\boldsymbol{C}$ have the following forms:

$\boldsymbol{P} =$

| | 0 | 3 | 5 | 7 | 9 | 11 |
|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| 2 | −1 | 1 | 0 | 0 | 0 | 0 |
| 3 | 0 | 1 | 0 | 0 | 0 | 0 |
| 4 | 0 | −1 | 1 | 0 | 0 | 0 |
| 5 | 0 | 0 | 1 | 0 | 0 | 0 |
| 6 | 0 | 0 | −1 | 1 | 0 | 0 |
| 7 | 0 | 0 | 0 | 1 | 0 | 0 |
| 8 | 0 | 0 | 0 | −1 | 1 | 0 |
| 9 | 0 | 0 | 0 | 0 | 1 | 0 |
| 10 | 0 | 0 | 0 | 0 | −1 | 0 |

$\boldsymbol{C} =$

| | 0 | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|
| 1 | −1 | 1 | 0 | 0 | 0 | 0 |
| 2 | 0 | 1 | 0 | 0 | 0 | 0 |
| 3 | 0 | −1 | 1 | 0 | 0 | 0 |
| 4 | 0 | 0 | 1 | 0 | 0 | 0 |
| 5 | 0 | 0 | −1 | 1 | 0 | 0 |
| 6 | 0 | 0 | 0 | 1 | 0 | 0 |
| 7 | 0 | 0 | 0 | −1 | 1 | 0 |
| 8 | 0 | 0 | 0 | 0 | 1 | 0 |
| 9 | 0 | 0 | 0 | 0 | −1 | 1 |
| 10 | 0 | 0 | 0 | 0 | 0 | 1 |

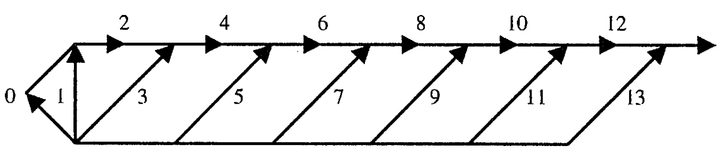


Fig. 6. Circuit graph of a chain structure.

Taking $\boldsymbol{v} = [u^0]$ and $\boldsymbol{j} = [i^0]$, we obtain the following matrix products:

$$\boldsymbol{P}^T\boldsymbol{P} = \begin{array}{c|c:ccccc} & 0 & 3 & 5 & 7 & 9 & 11 \\ \hline 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ \hdashline 3 & -1 & 3 & -1 & 0 & 0 & 0 \\ 5 & 0 & -1 & 3 & -1 & 0 & 0 \\ 7 & 0 & 0 & -1 & 3 & -1 & 0 \\ 9 & 0 & 0 & 0 & -1 & 3 & -1 \\ 11 & 0 & 0 & 0 & 0 & -1 & 3 \\ & \boldsymbol{B}_{21} & & & \boldsymbol{B}_{22} & & \end{array} \qquad \boldsymbol{C}^T\boldsymbol{C} = \begin{array}{c|c:ccccc} & 0 & 2 & 4 & 6 & 8 & 10 \\ \hline 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ \hdashline 2 & -1 & 3 & -1 & 0 & 0 & 0 \\ 4 & 0 & -1 & 3 & -1 & 0 & 0 \\ 6 & 0 & 0 & -1 & 3 & -1 & 0 \\ 8 & 0 & 0 & 0 & -1 & 3 & -1 \\ 10 & 0 & 0 & 0 & 0 & -1 & 3 \\ & \boldsymbol{B}_{21} & & & \boldsymbol{B}_{21} & & \end{array}$$

The systems of equations for the corresponding minimization problems have the following forms:

$\boldsymbol{U}^T\boldsymbol{U} \to \min \Rightarrow \boldsymbol{B}_{22}\boldsymbol{u} = -\boldsymbol{B}_{21}\boldsymbol{v}$ :

$$\begin{bmatrix} 3 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 3 & -1 \\ & & & \vdots & & & \end{bmatrix} \begin{bmatrix} u^3 \\ u^5 \\ u^7 \\ u^9 \\ u^{11} \\ u^{13} \\ \vdots \end{bmatrix} = \begin{bmatrix} u^0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

$\boldsymbol{I}^T\boldsymbol{I} \to \min \Rightarrow \boldsymbol{B}_{22}\boldsymbol{i} = -\boldsymbol{B}_{21}\boldsymbol{j}$ :

$$\begin{bmatrix} 3 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 3 & -1 \\ & & & \vdots & & & \end{bmatrix} \begin{bmatrix} i^2 \\ i^4 \\ i^6 \\ i^8 \\ i^{10} \\ i^{12} \\ \vdots \end{bmatrix} = \begin{bmatrix} i^0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

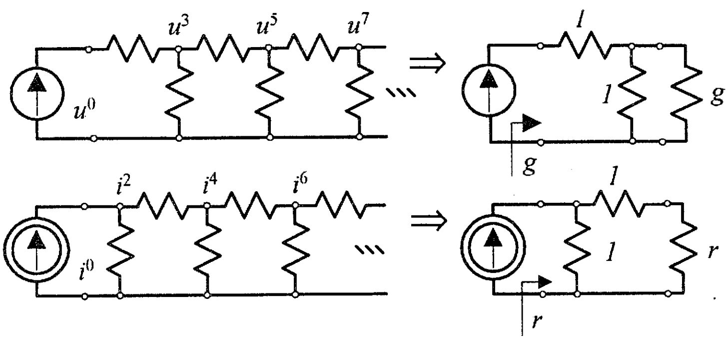


Fig. 7. Diagrams of equivalent infinite chains.

These equations describe infinite and homogeneous resistance chains with unit resistances (Fig. 7). In order to solve them, it is convenient to calculate a conductance $g$ and a resistance $r$ (input) of a chain. By means of the diagrams presented in Fig. 7, we get

$$\frac{1}{g} = 1 + \frac{1}{1+g}, \qquad \frac{1}{r} = \frac{1}{1+r}$$

or

$$g^2 + g - 1 = 0.$$

Thus

$$g = \frac{\sqrt{5}-1}{2}.$$

Accordingly, it is easy to calculate the distribution of the voltages and currents:

$$u^{2n+3} = qu^{2n+1}, \quad i^{2n+2} = qi^{2n}, \quad n = 0, 1, 2, \ldots,$$

where

$$q = \frac{3-\sqrt{5}}{2}.$$

The diagrams of Fig. 7 give the initial distribution of the voltages or currents in our network. The diagrams presented in Fig. 8(a) enable us to calculate the individual iterative distributions.

A numerical experiment with the iterative procedure presented in Fig. 4 and the system of differential equations (1) has been carried out. The alternative iterative

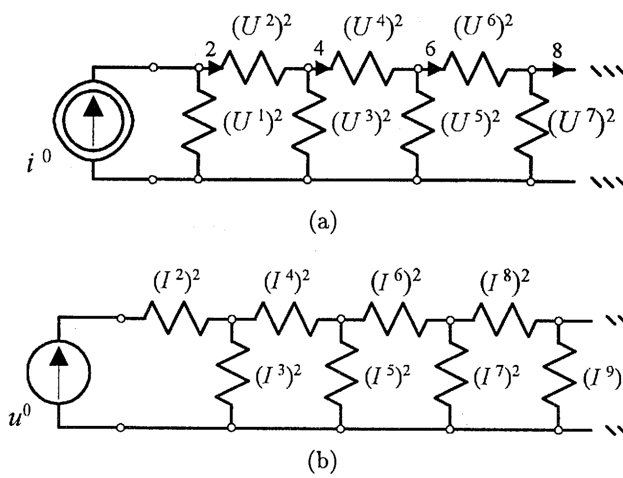


Fig. 8. The chain of resistances $\boldsymbol{C}^T(\operatorname{diag}\boldsymbol{U})^2\boldsymbol{C}$ (a) and the chain of conductances $\boldsymbol{P}^T(\operatorname{diag}\boldsymbol{I})^2\boldsymbol{P}$ (b).

procedure has proved substantially faster. In Fig. 9 the following graphs are presented:

- voltage signals of the chain obtained by means of the differential equation (1) and the procedure of Fig. 4,
- resistances ($u/i$ ratio) for individual elements, and
- power distribution for individual elements.

All the results obtained correspond to the signal values at a given moment.

## References

Kudrewicz J. (1996): *Nonlinear Electrical Networks.* — Warsaw: WNT (in Polish).

Siwczyński M. (1998): *Design of correction networks—Application of artifical neural networks.* — Jakość i Użytkowanie Energii Elektrycznej, Vol.2, pp.69–75 (in Polish).

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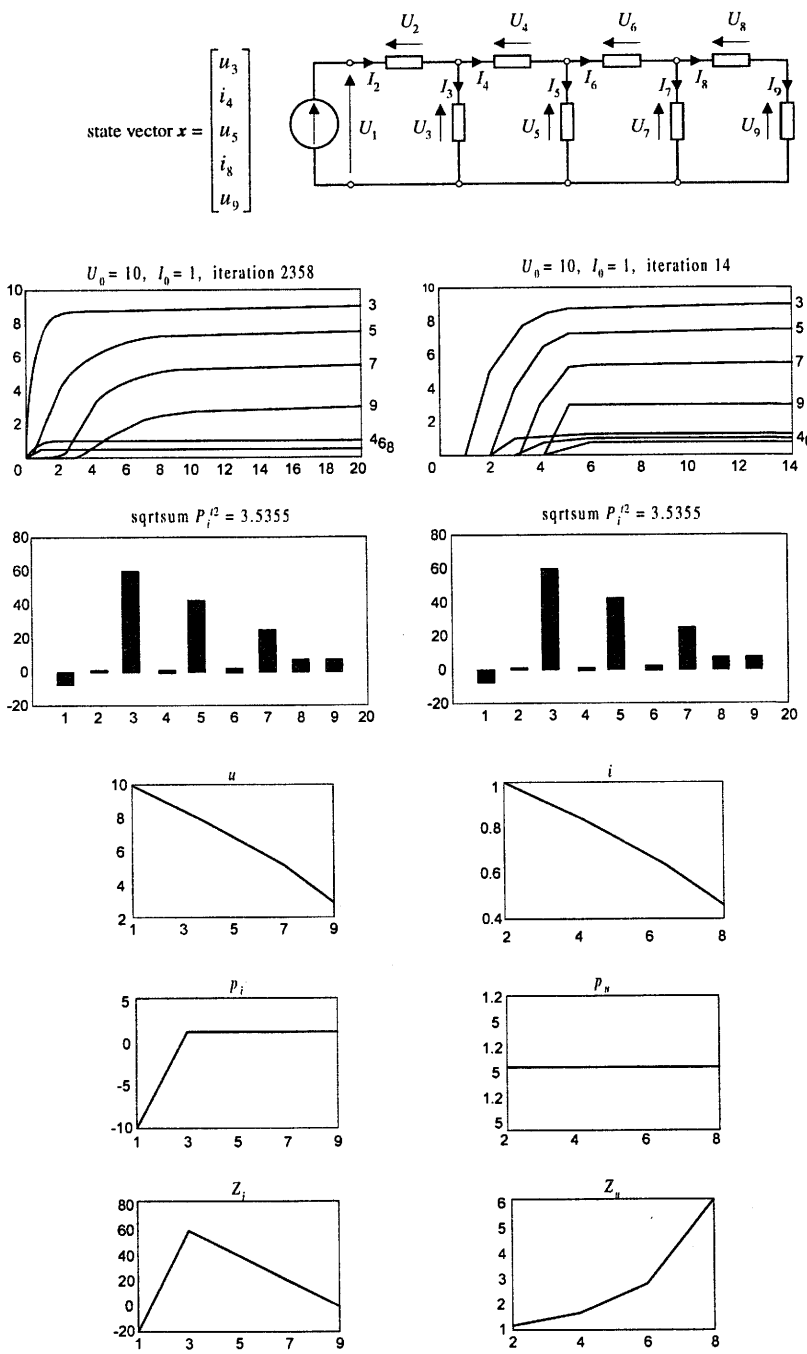


Fig. 9. Results of numerical experiments.