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ON AN INVARIANT DESIGN OF FEEDBACKS FOR BILINEAR CONTROL SYSTEMS OF SECOND ORDER

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The problem of linear feedback design for bilinear control systems guaranteeing their conditional closed-loop stability is considered. It is shown that this problem can be reduced to investigating the conditional stability of solutions of quadratic systems of differential equations depending on parameters of the control law. Sufficient conditions for stability in the cone of a homogeneous quadratic system are obtained. For second-order systems, invariant conditions of conditional asymptotic stability are found.

Keywords: bilinear control systems, invariant design, asymptotic stability, second order systems

1. Introduction

Consider a bilinear control system whose state equation is

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \left(\boldsymbol{A}_0 + \sum\limits_{i=1}^{m} \boldsymbol{u}_i(t)\boldsymbol{A}_i\right)\boldsymbol{x}(t), \\ \boldsymbol{x}(t) \in \mathbb{R}^n, \quad \mathbf{u}(t) = \big(u_1(t), \dots, u_m(t)\big)^T \in \mathbb{R}^m, \end{cases} \tag{1}$$

and the output equation has the form

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t), \quad \boldsymbol{y}(t) \in \mathbb{R}^p, \quad \boldsymbol{y}(0) = (y_{10}, \dots, y_{p0})^T. \tag{2}$$

Here $\mathbb{R}^n$, $\mathbb{R}^m$ and $\mathbb{R}^p$ are real vector spaces of dimensions n, m and p, respectively, $\boldsymbol{x}(t)$, $\boldsymbol{u}(t)$ and $\boldsymbol{y}(t)$ are vectors of states, inputs and outputs, $\boldsymbol{y}(0)$ is a vector of initial values, $\boldsymbol{A}_i : \mathbb{R}^n \to \mathbb{R}^n$ and $\boldsymbol{C} : \mathbb{R}^n \to \mathbb{R}^p$ are real linear mappings of appropriate real spaces, $i = 0, \dots, m$.

Definition 1. System (1) is called homogeneous if $\boldsymbol{A}_0 = 0$, and non-homogeneous otherwise.

Fixing bases in spaces $\mathbb{R}^n$ and $\mathbb{R}^p$, we will write the matrices of operators $\boldsymbol{A}_i$ and $\boldsymbol{C}$ in selected bases as A_i and $C = (c_1, \dots, c_n)$, respectively, where $c_1, \dots, c_n$

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are the columns of the matrix C, $i = 0, \dots, m$. For arbitrary column vectors a and b, we will denote their scalar product by (a, b). Recall now the definition of the conditional stability of solutions of differential equations (Demidovich, 1967).

Definition 2. The trivial solution $\boldsymbol{x}(t) \equiv 0$ of the system of differential equations

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{F}\big(t, \boldsymbol{x}(t)\big), \tag{3}$$

with the vector of initial values $\boldsymbol{x}(0) = (x_{10}, \dots, x_{n0})^T$, where $\boldsymbol{F}(t, \boldsymbol{x}) = (F_1(t, x_1, \dots, x_n), \dots, (F_n(t, x_1, \dots, x_n))^T \in \mathbb{R}^n$ stands for some vector-valued function, is called conditionally stable if there exists a manifold of initial values $\boldsymbol{\Theta} \subset \mathbb{R}^n$ such that for any solution $\boldsymbol{x}(t)$ satisfying the conditions

$$\boldsymbol{x}(0) \in \boldsymbol{\Theta} \quad \text{and} \quad \|\boldsymbol{x}(0)\| < \delta(\epsilon),$$

the following inequality is fulfilled:

$$\|\boldsymbol{x}(t)\| < \epsilon \ \text{ for all } \ t > 0.$$

Moreover, if

$$\lim_{t \to \infty} \|\boldsymbol{x}(t)\| = 0,$$

then the solution $x(t) \equiv 0$ is called conditionally asymptotically stable. (Here ϵ is a given positive number and $\delta = \delta(\epsilon)$ is a function of ϵ.)

For the system (1)–(2) formulate now the following problem:

Problem of the synthesis of a static feedback law. Construct a matrix $K = (k_1^T, \dots, k_m^T)^T \in \mathbb{R}^{m \times p}$, where $k_1, \dots, k_m$ are row vectors, of a linear control law $\boldsymbol{u}(t) = K\boldsymbol{y}(t)$ such that the trivial solution of the closed-loop system

$$\dot{\boldsymbol{x}}(t) = \left(A_0 + \sum_{i=1}^{n} \sum_{j=1}^{m} x_i(t)(k_j, c_i)A_j \right) \boldsymbol{x}(t), \tag{4}$$

with a vector of initial values $\boldsymbol{x}_0 = \{x_{10}, \dots, x_{n0}\} \in \boldsymbol{\Theta}$, is conditionally asymptotically stable.

The system (4) can be rewritten as follows:

$$\left\{ \begin{aligned} \dot{x}_1(t) &= \sum_{j=1}^{n} d_{1j}x_j(t) + \boldsymbol{x}^T(t)B_1\boldsymbol{x}(t), \\ &\vdots \\ \dot{x}_n(t) &= \sum_{j=1}^{n} d_{nj}x_j(t) + \boldsymbol{x}^T(t)B_n\boldsymbol{x}(t). \end{aligned} \right. \tag{5}$$

Here $D = (d_{ij})$, $B_1, \dots, B_n \in \mathbb{R}^{n \times n}$ are some real matrices, and additionally, $B_1, \dots, B_n$ are symmetric.

Definition 3. The system of equations (5) is called quadratic. If $D = 0$, (5) is called homogeneous quadratic.

Definition 4. A quadratic (or homogeneous quadratic) system (5) is termed regular if the matrices $B_1, \dots, B_n$ are linear independent and there exist numbers $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ such that rank $(\alpha_1 B_1 + \dots + \alpha_n B_n) = n$.

Theorem 1. *For the regular system (5), let the following conditions be fulfilled:*

1. *all initial values* $x_{i0} \geq 0$,
2. *the forms* $(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) B_i (x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)^T$ *are all positive semi-definite,*
3. $d_{ij} \geq 0$, $i \neq j$, *and*
4. *there exist some positive numbers* r_i *such that the form* $\boldsymbol{x}^T (\sum_{i=1}^n r_i B_i) \boldsymbol{x}$ *is non-positive,* $i, j = 1, \dots, n$.

Then any solution of the system (5) is limited for $t \to \infty$.

Proof. (a) We will firstly show that all solutions of the system (5) are non-negative, $\forall t \geq 0$. Denote by Ω the discrete set of all points $t^* \in [0, \infty)$ such that $x_i(t^*_{ij}) = 0$, and $t^*_{i1} < t^*_{i2} < \dots < t^*_{ij} < \dots$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots$. Let $t^*_{k1} = \min_{t^*} \Omega$ (the least zero of the function $x_k(t)$). From Conditions 1–3 and the k-th equation of the system (5), we obtain that the derivative $\dot{x}_k(t^*_{k1}) \geq 0$ and therefore, the function values $x_k(t)$ are non-decreasing at the point t^*_{k1}. Thus, $\dot{x}_k(t^*_{k1}) = 0$ and t^*_{k1} is the minimum point of the function $x_k(t)$, or in the interval $[0, t^*_{k1})$ there exists a point t_1 such that $x_k(t_1) < 0$. But the latter contradicts Condition 1 and the minimality of t^*_{k1}. This implies $x_k(t) \geq 0$ $\forall t \in [0, \infty)$, and the set Ω can contain only points $t^*_{k1}, t^*_{k2}, \dots$ which are local minima of the function $x_k(t)$.

Similarly, let $t^*_{p1} = \min_{t^*} (\Omega - \{t^*_{k1}, t^*_{k2}, \dots\})$ (the least zero of the function $x_p(t)$). Then it is easy to check that $x_p(t) \geq 0$ $\forall\, t \in [0, \infty)$, and the set $\Omega - \{t^*_{k1}, t^*_{k2}, \dots\}$ contains only points $t^*_{p1}, t^*_{p2}, \dots$ that are local minima of functions $x_p(t)$, and so on.

As a result, Ω is a set that includes only points of local minima of functions $x_i(t)$ (or an empty set), $i = 1, \dots, n$. Therefore, by virtue of Condition 1, all solutions of the system (5) are non-negative.

(b) Let $r_1, \dots, r_n$ be numbers satisfying Condition 4. Then, after integration of the equations of (5), we obtain the following equality:

$$\begin{aligned} r_1 x_1(t) + \dots + r_n x_n(t) &= r_1 x_{10} + \dots + r_n x_{n0} \\ &\quad + \int_0^t \big(h_1 x_1(\xi) + \dots + h_n x_n(\xi)\big) \mathrm{d}\xi \\ &\quad + \int_0^t \boldsymbol{x}^T(\xi) \left(\sum_{i=1}^n r_i B_i \right) \boldsymbol{x}(\xi)\, \mathrm{d}\xi, \end{aligned} \tag{6}$$

where $h_1, \dots, h_n$ are some real constants.

It is easy to check that (6) can be transformed to

$$\begin{aligned} r_1x_1(t)+\cdots+r_nx_n(t) &= r_1x_{10}+\cdots+r_nx_{n0} \\ &\quad + \int\limits_0^t \left(F-\left(\sum_{i=1}^n \alpha_{1i}x_i(\xi)+\beta_1\right)^2-\cdots\right. \\ &\quad \left. -\left(\sum_{i=1}^n \alpha_{ni}x_i(\xi)+\beta_n\right)^2\right)\mathrm{d}\xi, \end{aligned} \tag{7}$$

where $F=\sum\limits_{i=1}^n \beta_i^2 \geq 0$ and $\alpha_{ij},\beta_i \in \mathbb{R}$ are some constants, $i,j=1,\dots,n$.

Let us assume now that for some (or for all) $k\in\{1,\dots,n\}$ $\lim_{t\to\infty} x_k(t)=\infty$. By the regularity of (5), the matrix $A=(\alpha_{ij})=r_1B_1+\cdots+r_nB_n\in\mathbb{R}^{n\times n}$ has an inverse. Then

$$\lim_{\xi\to\infty}\|A\boldsymbol{x}+\boldsymbol{b}\| = \lim_{\xi\to\infty}\left(\sum_{i=1}^n \alpha_{1i}x_i(\xi)+\beta_1\right)^2+\cdots+\left(\sum_{i=1}^n \alpha_{ni}x_i(\xi)+\beta_n\right)^2=\infty.$$

(Here $\boldsymbol{b}=(b_1,\dots,b_n)^T\in\mathbb{R}^n$.) But in this case, the integral on the right-hand side of (7) tends to $-\infty$ as $t\to\infty$. Therefore, from the same formula (7) it follows that for some (or for all) $k\in\{1,\dots,n\}$ we must have $\lim\limits_{t\to\infty} x_k(t)<0$, which is impossible. Thus $\|\boldsymbol{x}(t)\|<\infty$, which completes the proof. ■

Theorem 2. *For the system (5) let all conditions of Theorem 1 be fulfilled. If $d_{ij}=0$ when $i\neq j$ and $d_{ii}\leq 0$, $i,j=1,\dots,n$, then the trivial solution $\boldsymbol{x}(t)\equiv 0$ of this system is conditionally asymptotically stable. (Here $\boldsymbol{\Theta}=\{x_1\geq 0,\dots,x_n\geq 0\}$ is the first orthant.)*

Proof. Note that in the equality (6) all the constants $h_i\leq 0$, $i=1,\dots,n$. Therefore, $r_1\dot{x}_1(t)+\cdots+r_n\dot{x}_n(t)\leq 0$, and the function $r_1x_1(t)+\cdots+r_nx_n(t)$ is monotonously decreasing. Then, recalling that for $x_i(t)\geq 0$ both the integrals on the right-hand side of (6) are non-positive decreasing functions, we have $\lim_{t\to\infty} x_k(t)=0$. The last conclusion completes the proof. ■

The results of Theorem 2 can be naturally generalized to more general classes of systems of ordinary differential equations.

Definition 5. An autonomous system of differential equations (3) is called regular if the functions $F_1(x),\dots,F_n(x)$ are independent. (In other words, if $\det\partial\boldsymbol{F}/\partial\boldsymbol{x}\not\equiv 0$.)

Theorem 3. *Suppose that for the regular autonomous system (3) the following conditions are fulfilled:*

1. *initial values $x_{i0}\geq 0$,*
2. *for all $x_j\geq 0$ $(j\neq i)$ the functions $F_i(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n)$ are non-negative, and*

3. *there exist some positive numbers* r_i *such that the function* $r_1 F_1(\boldsymbol{x}) + \cdots + r_n F_n(\boldsymbol{x})$ *is non-positive for all* $x_i \geq 0,\ i = 1, \ldots, n$.

Then the trivial solution of the system (3) is conditionally asymptotically stable.

2. Conditional Stability of Homogeneous Quadratic Systems

Let the system (5) with the vector of initial values $\boldsymbol{x}^T(0) = (x_{10}, \ldots, x_{n0})$ be homogeneous:

$$\begin{cases} \dot{x}_1(t) &= \boldsymbol{x}^T(t) B_1 \boldsymbol{x}(t), \\ &\vdots \\ \dot{x}_n(t) &= \boldsymbol{x}^T(t) B_n \boldsymbol{x}(t). \end{cases} \tag{8}$$

The foregoing analysis shows that the quadratic part of the system (5) strongly influences the stability of the solutions. In this connection, we will investigate in detail the system (8).

Consider a matrix $\rho_1 B_1 + \cdots + \rho_n B_n \in \mathbb{R}^{n \times n}$, where $\rho_1, \ldots, \rho_n$ are arbitrary real parameters. Introduce for this matrix basic symmetrical functions (Gantmacher, 1990): $\sigma_1(\rho_1, \ldots, \rho_n) = \operatorname{tr}(\rho_1 B_1 + \cdots + \rho_n B_n) = \{$it is the sum of all principal minors of the first order$\}$, $\sigma_2(\rho_1, \ldots, \rho_n) = \{$it is the sum of all principal minors of the second order$\}, \ldots, \sigma_n(\rho_1, \ldots, \rho_n) = \det(\rho_1 B_1 + \cdots + \rho_n B_n)$.

Consider the set of equations

$$\sigma_1(\rho_1, \ldots, \rho_n) = r,\ \sigma_2(\rho_1, \ldots, \rho_n) = 0,\ \ldots,\ \sigma_n(\rho_1, \ldots, \rho_n) = 0, \tag{9}$$

with respect to the unknowns $\rho_1, \ldots, \rho_n$ and a known non-zero constant $r \in \mathbb{R}$.

It is easy to show (Gantmacher, 1990) that for common matrixes $B_1, \ldots, B_n$ the system (9) has n linearly independent solutions

$$\boldsymbol{f}_1 = (\rho_{11}, \rho_{12}, \ldots, \rho_{1n}),\ \ldots,\ \boldsymbol{f}_n = (\rho_{n1}, \rho_{n2}, \ldots, \rho_{nn}),$$

(in general, they are complex).

Form a non-singular matrix $F^{-1} = (\boldsymbol{f}_1{}^T \ldots, \boldsymbol{f}_n{}^T)^T \in \mathbb{C}^{n \times n}$ and introduce into the system (8) a new variable $\boldsymbol{z}(t) = (z_1(t), \ldots, z_n(t))^T \in \mathbb{C}^n$ using the formula $\boldsymbol{x}(t) = F \boldsymbol{z}(t)$.

Theorem 4. *If the homogeneous system (8) is regular, there exists a non-singular matrix* $P \in \mathbb{C}^{n \times n}$ *such that, after the change of variables* $\boldsymbol{x} = P \boldsymbol{z}(t)$*, (8) will have the form*

$$\begin{cases} \dot{z}_1(t) &= -\beta_{11} z_1^2(t) + \sum\limits_{2 \leq i \leq n} \beta_{1i} z_i^2(t), \\ &\vdots \\ \dot{z}_k(t) &= -\beta_{22} z_k^2(t) + \sum\limits_{1 \leq i \leq n, i \neq k} \beta_{ki} z_i^2(t), \\ &\vdots \\ \dot{z}_n(t) &= -\beta_{nn} z_n^2(t) + \sum\limits_{1 \leq i \leq n-1} \beta_{ni} z_i^2(t), \end{cases}$$

or

$$\begin{pmatrix} \dot z_1(t) \\ \dot z_2(t) \\ \vdots \\ \dot z_n(t) \end{pmatrix} = - \begin{pmatrix} \beta_{11} & -\beta_{12} & \dots & -\beta_{1n} \\ -\beta_{21} & \beta_{22} & \dots & -\beta_{1n} \\ \dots & \dots & \dots & \dots \\ -\beta_{n1} & -\beta_{n2} & \dots & \beta_{nn} \end{pmatrix} \begin{pmatrix} z_1^2(t) \\ z_2^2(t) \\ \vdots \\ z_n^2(t) \end{pmatrix}, \tag{10}$$

where β_{ij} *are some complex numbers,* $\boldsymbol{z}(0) = P^{-1}\boldsymbol{x}(0) = (z_{10}, \dots, z_{n0})^T$.

Proof. Denote by $q_1, \dots, q_n \in \mathbb{C}^n$ the columns of the matrix F, and define through $(z_1^2, z_1z_2, \dots, z_1z_n, z_2^2, \dots, z_2z_n, \dots, z_n^2)^T$ a vector $\boldsymbol{z}^2$ of dimension $n(n+1)/2$. Then, in the new variables $z_1, \dots, z_n$, the system (8) will have the form $\dot{\boldsymbol{z}}(t) = W\boldsymbol{z}^2(t)$, where

$$W = F^{-1} \begin{pmatrix} q_1^TB_1q_1, 2q_1^TB_1q_2, \dots, 2q_i^TB_1q_j, \dots, q_n^TB_1q_n, \\ q_1^TB_2q_1, 2q_1^TB_2q_2, \dots, 2q_i^TB_2q_j, \dots, q_n^TB_2q_n, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ q_1^TB_nq_1, 2q_1^TB_nq_2, \dots, 2q_i^TB_nq_j, \dots, q_n^TB_nq_n \end{pmatrix},$$

is an $\left(n \times \frac{n(n+1)}{2}\right)$-matrix, $i \le j$.

As any row of a matrix F^{-1} is a solution of the system (9), it is clear that, by virtue of the inertia law (Gantmacher, 1990), any quadratic form on the right-hand side of the system $\dot{\boldsymbol{z}}(t) = W\boldsymbol{z}^2(t)$ has rank 1. Therefore, the last system can be rewritten (probably after one more change $z_i \to \delta_i z_i$, where $\delta_i = 1$ or -1, $i = 1, \dots, n$) as

$$\begin{cases} \dot z_1(t) &= -(\gamma_{11}z_1 + \dots + \gamma_{1n}z_n)^2, \\ &\vdots \\ \dot z_n(t) &= -(\gamma_{n1}z_1 + \dots + \gamma_{nn}z_n)^2, \end{cases} \tag{11}$$

where $\gamma_{ij} \in \mathbb{C}$ are some constants.

Now, if we introduce the change of variables in the system (11),

$$\begin{cases} z_1 &\to \gamma_{11}z_1 + \dots + \gamma_{1n}z_n, \\ &\vdots \\ z_n &\to \gamma_{n1}z_1 + \dots + \gamma_{nn}z_n, \end{cases} \qquad (\boldsymbol{z} \Rightarrow F_1\boldsymbol{z}),$$

then, after appropriate renamings, we obtain the system (10). (It is easy to see that actually $\gamma_{ij} = \beta_{ij}$ and $P = F_1F$.) The proof is thus completed. ■

Lemma 1. *Let*

$$D = \begin{pmatrix} b_1 & a_{12} & \dots & -a_{1n} \\ -a_{21} & b_2 & \dots & -a_{2n} \\ \dots & \dots & \dots & \dots \\ -a_{n1} & -a_{n2} & \dots & b_n \end{pmatrix} \tag{12}$$

be a real $(n \times n)$-matrix possessing the following properties:

1. $b_i \geq 0, \ i = 1, \dots, n,$
2. $a_{ij} \geq 0$ *for all* $i \neq j, \ i, j = 1, \dots, n,$
3. $\det D > 0$ *and for at least one* $i \in \{1, \dots, n\}$*, the cofactors* A_{ij} *of the elements of the* i*-th row of the matrix* D *are non-negative,* $j = 1, \dots, n.$
   Then there exist some positive numbers $\zeta_1, \dots, \zeta_n$ *such that all non-zero elements of the matrix* $D^{-1}(\zeta_1, \dots, \zeta_n)$ *being the inverse of the matrix*

$$D(\zeta_1, \dots, \zeta_n) = \begin{pmatrix} b_1\zeta_1 & -a_{12}\zeta_2 & \dots & -a_{1n}\zeta_n \\ -a_{21}\zeta_1 & b_2\zeta_2 & \dots & -a_{2n}\zeta_n \\ \dots & \dots & \dots & \dots \\ -a_{n1}\zeta_1 & -a_{n2}\zeta_2 & \dots & b_n\zeta_n \end{pmatrix},$$

   are positive.

Proof. Assume that in Condition 3 we have $i = 1$ and $A_{1j} \geq 0, j = 1, \dots, n$. Consider the set of equations

$$\begin{pmatrix} b_1 & -a_{12} & \dots & -a_{1n} \\ -a_{21} & b_2 & \dots & -a_{2n} \\ \dots & \dots & \dots & \dots \\ -a_{n1} & -a_{n2} & \dots & b_n \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_n \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{13}$$

Then we get $\zeta_1 = A_{11}/\det D, \dots, \zeta_n = A_{1n}/\det D$, and the numbers $\zeta_1, \dots, \zeta_n$ are non-negative by virtue of Condition 3. (It is clear that these numbers are the elements of the first column of D^{-1}.)

Replace the vector on the right-hand side of the system (13) with the vector $(1 + \eta_1, \dots, \eta_n)^T$, where $\eta_i > 0$ and the magnitudes η_i are sufficiently small, $i = 1, \dots, n$. Then, taking into account the continuous dependence of the solution of (13) on sufficiently small changes on the right-hand side, it is possible to indicate numbers $\eta_i > 0$ such that the solutions of the perturbed system (13) will conserve the same sign as the non-perturbed one. Therefore, there exist positive ζ_i's such that $b_i\zeta_i \geq \sum_{j=1, j\neq i}^{n} a_{ij}\zeta_j$. (The system (13) was just introduced to prove the last inequality, and the single vector on the right-hand side of this system can be replaced with any vector with non-negative elements.) The proof of the following lemma can be obtained from Lederman's result (Bellman, 1976). ■

Condition 4 of Theorem 1 is hard to check. Below we present a simple sufficient criterion for the conditional stability of the system (8).

Theorem 5. *Let us assume that the regular system (8) is such that the matrix* P *from Theorem 4, reducing (8) to the form of (10), is real. If:*

1. *the matrix* $G = (\beta_{ij}) \in \mathbb{R}^{n \times n}$ *possesses the properties of the matrix* D *from Lemma 1,*
2. *initial values* $z_{i0} \geq 0, \ i = 1, \dots, n,$

then the trivial solution of the system (10) is conditionally asymptotically stable.

Proof. Introducing the changes of variables $z_i \to \zeta_i z_i$, we obtain that the elements of the matrix G satisfy Lederman's conditions (Bellman, 1976). Then, according to Lemma 1, all the elements of the matrix G^{-1} are non-negative. Multiply both the sides of the system (10) by G^{-1}. Then we have

$$G^{-1}\begin{pmatrix} \dot{z}_1(t) \\ \vdots \\ \dot{z}_n(t) \end{pmatrix} = -\begin{pmatrix} z_1^2(t) \\ \vdots \\ z_n^2(t) \end{pmatrix}. \tag{14}$$

Now multiply both the sides of (14) by a row vector $(r_1, \dots, r_n)$ whose all coordinates are strictly positive. Then we obtain

$$s_1\dot{z}_1(t) + \cdots + s_n\dot{z}_n(t) = -\big(r_1 z_1^2(t) + \cdots + r_n z_n^2(t)\big),$$

where $(s_1, \dots, s_n) = (r_1, \dots, r_n)G^{-1}$. According to Lemma 1, all the elements of the matrix G^{-1} are non-negative. Therefore there always exist positive numbers $r_1, \dots, r_n$ such that $s_1 > 0, \dots, s_n > 0$. The reminder of the proof proceeds on the same lines as the proof of part (b) of Theorem 1. ■

3. Example

Set $n = 2$. Rewrite the system (8), with initial values $x_{10} > 0$, $x_{20} > 0$, in the following form:

$$\begin{cases} \dot{x}_1(t) = b_{111}x_1^2(t) + b_{112}x_1(t)x_2(t) + b_{121}x_2(t)x_1(t) + b_{122}x_2^2(t), \\ \dot{x}_2(t) = b_{211}x_1^2(t) + b_{212}x_1(t)x_2(t) + b_{221}x_2(t)x_1(t) + b_{222}x_2^2(t), \end{cases} \tag{15}$$

where $b_{112} = b_{121}$ and $b_{212} = b_{221}$. In this case Theorem 2 gives the following results: $b_{122} \geq 0$, $b_{211} \geq 0$, $r_1 b_{111} + r_2 b_{211} < 0$,

$$\det\begin{pmatrix} r_1 b_{111} + r_2 b_{211} & r_1 b_{112} + r_2 b_{212} \\ r_1 b_{112} + r_2 b_{212} & r_1 b_{122} + r_2 b_{222} \end{pmatrix} > 0.$$

Write

$$r_1 b_{111} + r_2 b_{211} = x < 0, \quad r_1 b_{122} + r_2 b_{222} = y < 0,$$

$$G = \begin{pmatrix} b_{111} & b_{122} \\ b_{211} & b_{222} \end{pmatrix} \in \mathbb{R}^{2\times 2}, \quad H = \begin{pmatrix} b_{111} & b_{112} \\ b_{211} & b_{212} \end{pmatrix} \in \mathbb{R}^{2\times 2},$$

$$Q = \begin{pmatrix} b_{121} & b_{122} \\ b_{221} & b_{222} \end{pmatrix} \in \mathbb{R}^{2\times 2}.$$

Then from the last relations we have

$$r_1 = \frac{b_{222}x - b_{211}y}{\det G}, \quad r_2 = \frac{-b_{122}x + b_{111}y}{\det G},$$

$$(\det G)^2 xy - (x \det H + y \det Q)^2 > 0. \tag{16}$$

As r_1 and r_2 should be positive for $x < 0$ and $y < 0$, we must have $b_{222} < 0$, $b_{111} < 0$ and $\det G > 0$. The analysis of inequalities (16) gives the condition $\det(G^2) - 4\det(HQ) > 0$.

Theorem 6. *Let $n = 2$. Then the conditions*

1. $x_{10} \geq 0, x_{20} \geq 0$,
2. $b_{111} < 0, b_{222} < 0, b_{122} \geq 0, b_{211} \geq 0$,
3. $\det G > 0$, *and*
4. $\det(G^2) - 4\det(HQ) > 0$

are sufficient for the conditional asymptotic stability of the system (15).

There is one peculiarity here. Let us consider the system (9) for $n = 2$. As r is a non-zero number, this system is reduced to the equation $\det(\rho_1 B_1 + \rho_2 B_2) = 0$. The discriminant of this equation is equal to $\det(G^2) - 4\det(HQ)$. Therefore, Condition 4 of Theorem 6 means that the system (8) is reduced to a real system (10). Thus, if the system (8) is not reducible to a real system, its trivial solution is unstable (for all $x_{10} \geq 0,\ \ x_{20} \geq 0$), even conditionally.

The problem of the synthesis of a state feedback (in (2) we have $\det C \neq 0$) for $n = 2$ and a homogeneous system (4) has an obvious solution: conditions of Theorem 6 become now dependant on the feedback matrix K, and all is reduced to investigation of a system of four linear inequalities, one square inequality, and one inequality of the fourth degree with respect to the unknown elements k_{ij} of this matrix.

4. Invariant Analysis

It is possible to show that the restrictions given by Theorems 1–3 or 6 can be essentially weakened. Their principal disadvantage consists in the fact that these restrictions have uninvariant character. In fact, in the autonomous system (3) introduce a new variable $\boldsymbol{z}$ defined by $\boldsymbol{x} = S\boldsymbol{z}$. Then the system (3) becomes $\dot{\boldsymbol{z}}(t) = S^{-1}\boldsymbol{F}(S\boldsymbol{z})(t)$ with initial values $\boldsymbol{z}(0) = S^{-1}\boldsymbol{x}(0)$. Let us assume now that for the system (3) the conditions of Theorem 6 are not fulfilled. However, one can see that they will be satisfied for the transformed system. So it is clear that it is necessary to add to the conditions of Theorem 6 an invariant character. If the functions $F_1(\boldsymbol{x}), \ldots, F_n(\boldsymbol{x})$ are polynomials, it is possible to do so as follows.

For simplicity, assume that the above-mentioned polynomials are homogeneous of degree k. It is well-known (Sibirsky, 1982) that any polynomial system of homogeneous differential equations can be given by an appropriate mixed tensor $\boldsymbol{T}^i_{j_1,\ldots,j_k}$ once contravalent and k times covalent, $i, j_1, \ldots, j_k = 1, \ldots, n$. (The tensor contains n^{k+1} coordinates.) The transformed system is then determined by the tensor $\boldsymbol{T}^i_{j_1,\ldots,j_k}(S)$. Let $S \in GL(n, \boldsymbol{C})$, where $GL(n, \boldsymbol{C})$ is a complete linear group. Recall that a polynomial $g(\boldsymbol{T}^i_{j_1,\ldots,j_k})$ is named an invariant of the group $GL(n, \boldsymbol{C})$ if $\forall S \in GL(n, \boldsymbol{C})\ \ g(\boldsymbol{T}^i_{j_1,\ldots,j_k}(S)) = (\det S)^l \times g(\boldsymbol{T}^i_{j_1,\ldots,j_k})$, where l is some integer. It is required to find a basis of polynomial invariants with respect to an actuation of the group $GL(n, \boldsymbol{C})$ generated by a change of variables S on the vector space of tensors $\boldsymbol{C}^d = \{\boldsymbol{T}^i_{j_1,\ldots,j_k}\}$ of an appropriate dimension d. (For example, in case $N = k = 2$ we have $d = 8$; in case $n = 3, k = 2$ we have $d = 27$.) Furthermore, all conditions of Theorem 6 might be rewritten in terms of polynomials $g(\boldsymbol{T}^i_{j_1,\ldots,j_k})$. This problem is one the of most difficult ones in the theory of invariants. We hope to obtain its partial solution in the next publications.

Here we briefly describe a method of constructing a base of invariants for the system (15). In Theorem 6, only Condition 4 has invariant character. Here, the invariant polynomial will be a polynomial $g(\boldsymbol{T}^i_{j_1j_2}) = \det(G^2) - 4\det(HQ)$.

The tensor describing the system (15) has the form

$$\boldsymbol{T}^i_{j_1j_2} = \begin{pmatrix} b_{111} & b_{112} & b_{121} & b_{122} \\ b_{211} & b_{212} & b_{221} & b_{222} \end{pmatrix} \in \mathbb{C}^8.$$

Group $GL(2, \mathbb{C})$ operates on $\mathbb{C}^8$ based on the rule

$$\forall S \in GL(2,\mathbb{C}) \quad \boldsymbol{T}^i_{j_1j_2}(S) = S^{-1} \begin{pmatrix} b_{111} & b_{112} & b_{121} & b_{122} \\ b_{211} & b_{212} & b_{221} & b_{222} \end{pmatrix} \left(S \bigotimes S\right).$$

On the conjugate space $\mathbb{C}^{*\,8}$, the system (15) is described by the conjugate tensor

$$(\boldsymbol{T}^*)^i_{j_1j_2} = \begin{pmatrix} b_{111} & b_{112} \\ b_{211} & b_{212} \\ b_{112} & b_{122} \\ b_{212} & b_{222} \end{pmatrix}.$$

Group $GL(2, \mathbb{C})$ operates on $\mathbb{C}^{*8}$ as follows:

$$\forall S \in GL(2,\mathbb{C}) \quad (\boldsymbol{T}^*)^i_{j_1j_2}(S) = \left(S^{-1} \bigotimes S^T\right) \begin{pmatrix} b_{111} & b_{112} \\ b_{211} & b_{212} \\ b_{112} & b_{122} \\ b_{212} & b_{222} \end{pmatrix} S.$$

Based on tensors $\boldsymbol{T}^i_{j_1j_2}$ and $(\boldsymbol{T}^*)^i_{j_1j_2}$, construct the matrices

$$A = G, \quad A_1 = H, \quad A_2 = Q,$$

$$A_3 = \begin{pmatrix} b_{111} & b_{112} \\ b_{112} & b_{122} \end{pmatrix}, \quad A_4 = \begin{pmatrix} b_{211} & b_{212} \\ b_{212} & b_{222} \end{pmatrix}, \quad A_5 = \begin{pmatrix} b_{211} & b_{212} \\ b_{112} & b_{122} \end{pmatrix}.$$

Using the symbolical method (Sibirsky, 1982), we will find a basis of the ring of invariants distinct from that offered by Sibirsky (1982):

$$I_1 = \det(A^2) - 4\det(A_1A_2), \quad I_2 = \det(A_1A_2 - A_2A_1),$$

$$\begin{aligned} I_3 \ =\ & (\operatorname{tr} A_2)^2(\det A_1 + \det A_4) + (\operatorname{tr} A_1)^2(\det A_2 + \det A_3) \\ & + 2(\operatorname{tr} A_1)\operatorname{tr}(A_2)\det A_5. \end{aligned}$$

(It is easy to verify that $I_k(\boldsymbol{T}^i_{j_1j_2}(S)) = (\det S)^2 I_k(\boldsymbol{T}^i_{j_1j_2})$, $k = 1,\ldots,3$. However, the procedure of construction of this basis is rather non-trivial and is completely omitted in the present work.) Then Theorem 6 can be represented in the following form.

Theorem 7. *Let $I_1 > 0$, $I_2 > 0$, $I_3 < 0$. Then in $\mathbb{R}^2$ there is a closed cone $\boldsymbol{K}$ with forming equations $\alpha_1x_1 + \beta_1x_2 = 0$ and $\alpha_2x_1 + \beta_2x_2 = 0$ such that for any initial vector $\boldsymbol{x}_0 \in \boldsymbol{K}$ the trivial solution of (15) will be conditionally asymptotically stable.*

Proof. As was noted above, the condition $I_1 > 0$ guarantees the existence of a real matrix $S \in GL(2,\mathbb{R})$ reducing the system (15) to (10) with real β_{ij}. Therefore it is possible to think that the system (15) is given already in this form.

In this case the invariants I_1, I_2 and I_3 are as follows:

$$I_1 = (\beta_{11}\beta_{22} - \beta_{12}\beta_{21})^2, \quad I_2 = \beta_{12}\beta_{21}(\beta_{11}\beta_{22} - \beta_{12}\beta_{21}),$$

$$I_3 = \beta_{12}(\beta_{11})^3 + 2(\beta_{12}\beta_{21})^2 + \beta_{21}(\beta_{22})^3.$$

It is clear that the condition $I_1 > 0$ is always fulfilled. Therefore we need to study the condition $I_2 > 0$. Assume that $\beta_{11}\beta_{22} \neq 0$. Then with the help of new changes of variables we can achieve that in the system (10) $|\beta_{11}| = |\beta_{22}| = 1$. Two situations are possible here: (a) $\beta_{11}\beta_{22} > 0$ and (b) $\beta_{11}\beta_{22} < 0$.

The first situation can be split up into two cases:
(i) $\beta_{12}\beta_{21} > 0$, $1 - \beta_{12}\beta_{21} > 0$ and (ii) $\beta_{12}\beta_{21} < 0$, $1 - \beta_{12}\beta_{21} < 0$.
It is clear that the second case is impossible. Therefore, in the situation (a), there should be $\beta_{12}\beta_{21} > 0$.

The second situation implies the inequality $I_2 = \beta_{12}\beta_{21}(-1 - \beta_{12}\beta_{21}) > 0$. Here, the case $\beta_{12}\beta_{21} > 0$ is impossible. In the case of $\beta_{12}\beta_{21} < 0$ we obtain $(-1 - \beta_{12}\beta_{21}) < 0$, which implies the same result.

Finally, we obtain $\beta_{11}\beta_{22} > 0$ and $\beta_{12}\beta_{21} > 0$ and we have again two situations: (a) $\beta_{11} = \beta_{22} = 1$ and (b) $\beta_{11} = \beta_{22} = -1$. In the following we will take advantage of the condition $I_3 < 0$.

The first situation implies the inequality $\beta_{12} + 2(\beta_{12}\beta_{21})^2 + \beta_{21} < 0$ (which is impossible if $\beta_{12} > 0$, $\beta_{21} > 0$); or, in case $\beta_{12} < 0$, $\beta_{21} < 0$, the inequality $-|\beta_{12}| + 2(\beta_{12}\beta_{21})^2 - |\beta_{21}| < 0$. But the last inequality is always satisfied if $1 - \beta_{12}\beta_{21} > 0$, which is identified by the condition $I_2 > 0$. (This is true as $|\beta_{21}| < 1/|\beta_{12}|$, $2(\beta_{12}\beta_{21})^2 < |\beta_{12}| + |\beta_{21}| < |\beta_{12}| + 1/|\beta_{12}|$. But the discriminant $(\beta_{12}\beta_{21})^4 - 1$ of the last inequality is always negative and therefore, it has a solution for any positive number $\beta_{12}\beta_{21}$ satisfying the condition $1 - \beta_{12}\beta_{21} > 0$.) Thus there should be $\beta_{12} < 0$, $\beta_{21} < 0$.

The second situation is opposite to the first one and implies the inequalities $\beta_{12} > 0$ and $\beta_{21} > 0$.

It is easy to check that both of the last situations are reduced to each other with the help of an appropriate diagonal transformation $S_1 \in GL(2, \mathbb{R})$. Therefore, the inequalities $I_1 > 0$, $I_2 > 0$, $I_3 < 0$ really define a non-empty subset of conditionally stable systems.

As to the form of a cone $\boldsymbol{K}$ on which the conditional asimptotical stability is reached, it is determined by the matrix F from Theorem 4. The search for this matrix can be simplified if we take advantage of the matrix W from the proof of Theorem 4. This is possible if we define the columns $q_1, q_2 \in \mathbb{R}^2$ from the equation $q_1^T[B_1q_2, B_2q_2] = 0$, which is equivalent to the square-law equation $\det[B_1q, B_2q] = 0$. Then $F = [q_1, q_2]^{-1}$ and generators of the cone are set by the vector equation $F^{-1}\boldsymbol{x}(0) = (\alpha_1x_{10} + \beta_1x_{20}, \alpha_2x_{10} + \beta_2x_{20})^T = 0$. The interior of a cone is one of four sectors into which straight lines $\alpha_1x_{10} + \beta_1x_{20} = 0$, $\alpha_2x_{10} + \beta_2x_{20} = 0$ divide the plane $\mathbb{R}^2$. (Thus $F \in GL(2, \mathbb{R})$, and here $P = F$ is the resulting transformation, reducing the system (15) to (10)). The proof is thus completed. ■

5. Conclusion

The problems of the stability of quadratic systems were considered in the works of Isidori (1995) and Khalil (1995). We remark that our results intersect to some extent with those in the book by Borisenko *et al.* (1988). However, they reduced all the proofs of the existence theorems to the so-called Vashevsky's systems. (If $F_i(\boldsymbol{x})$ are polynomials, Borisenko *et al.* require that these polynomials look like $(a_1x_1^k + \cdots + a_nx_n^k)^l$, where k, l are odd numbers which are different, generally speaking, for different i's.) Here there are no such restrictions. For completeness, we also remark that Theorems 1–6 considerably generalize the results of Belozyorov and Poddubnaya (2000). The monograph of Zubov (1974) is also devoted to investigating the stability of systems (8). However, the basic tools there are Lyapunov's functions, whose determination constitutes a very difficult task. Again in (Zubov, 1974) the conditions of stability have uninvariant character. In the present work the invariant approach to investigate stability is offered. In particular, for $n = 2$, the explicit form of invariants responsible for conditional asymptotic stability in the from of a trivial solution is obtained.

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