

EFFECTS OF PRESSURE GRADIENT ON CONVECTIVE HEAT TRANSFER IN A BOUNDARY LAYER FLOW OF A MAXWELL FLUID PAST A STRETCHING SHEET

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The pressure gradient term plays a vital role in convective heat transfer in the boundary layer flow of a Maxwell fluid over a stretching sheet. The importance of the effects of the term can be monitored by developing Maxwell's equation of momentum and energy with the pressure gradient term. To achieve this goal, an approximation technique, i.e. Homotopy Perturbation Method (HPM) is employed with an application of algorithms of Adams Method (AM) and Gear Method (GM). With this approximation method we can study the effects of the pressure gradient (m), Deborah number (β), the ratio of the free stream velocity parameter to the stretching sheet parameter (ϵ) and Prandtl number (Pr) on both the momentum and thermal boundary layer thicknesses. The results have been compared in the absence and presence of the pressure gradient term m . It has an impact of thinning of the momentum and boundary layer thickness for non-zero values of the pressure gradient. The convergence of the system has been taken into account for the stretching sheet parameter ϵ . The result of the system indicates the significant thinning of the momentum and thermal boundary layer thickness in velocity and temperature profiles. On the other hand, some results show negative values of $f'(\eta)$ and $\theta(\eta)$, which indicates the case of fluid cooling.

Key words: Homotopy Perturbation Method (HPM), pressure gradient parameter, convective heat transfer, Maxwell fluid, stretching sheet.

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1. Introduction

The convective heat transfer phenomena are the subject of research which has attracted several researchers due to numerous industrial and biological engineering applications such as compulsive convection in conduits, production of papers, fibre glass, different types of paints, and electronic chips, material processing, crystal growth, nuclear reactor cooling, food processing and movement of biological fluids, drilling muds, shampoo, ketchup, cement, sludge, grease, granular suspension, aqueous foams, slurries, plastics, etc. [1, 2]. Although a lot of investigation has been done on the convective heat transfer in the boundary layer flow, there will always be room for deeper and wider understanding of these phenomena. Fathizadeh and Rashidi [3], Aziz [4], Ishak [5] and Shagaiya and Daniel [6] studied the convective heat transfer in the boundary layer flow of Newtonian fluids. In recent years, Bhattacharyya *et al.* [2], Surati and Timol [7], Patel and Timol [8] and Hayat *et al.* [9] used non-Newtonian fluids to analyse the convective heat transfer in the boundary layer flows.

Hayat *et al.* [1], Malik *et al.* [10], Singh and Agarwal [11], Javed *et al.* [12], Shateyi [13], Mukhopadhyay [14], Bhattacharyya [15], Ibrahim and Suneetha [16], Rehman and Nadeem [17] and Saleh *et al.* [18] theoretically studied the behaviour of the fluid flow and boundary layer thickness along with the convective heat transfer in the flow of Newtonian and non-Newtonian fluids over a stretching sheet with various boundary conditions and parameters, but they did not discuss the effects of the pressure gradient on various rheological quantities.

The Homotopy Perturbation Method (HPM) is an effective and powerful analytical method which was first developed by Ji-Huan [19] for solving nonlinear differential equations. It has been well established that the analytical solutions obtained through this method converge rapidly [20]. As an approximation technique, several researchers [20-23] have applied it to solve various kinds of nonlinear differential equations and also to solve problems from several branches of engineering and biological sciences. Later on, this method was significantly improved by Ji-Huan [20] and other researchers [24-26].

Fan *et al.* [27] and Ishak in [28] propounded that their analytical solution to temperature profile converges to θ for a wide range of values of the free convective parameter γ and the unsteadiness parameter, respectively A . Bhattacharyya *et al.* in [29, 30] analysed the effects of slip velocity at the boundary on the heat transfer in the stagnation point flow over a stretching sheet. Motsa and Sibanda in [31] obtained semi-analytical solution to the MHD flow over a nonlinear stretching sheet. Rahman *et al.* [32] investigated heat transfer characteristics in a micropolar fluid flow along a non-linear stretching sheet with a temperature-dependent viscosity and variable surface temperature. Hayat *et al.* in [33] analysed the mixed convection stagnation point flow of a Casson fluid with convective boundary conditions. Kazem *et al.* in [34] derived the analytical solutions to a stagnation-point flow past a porous stretching sheet with heat generation and used several parameters such as the porosity parameter M , the stretching parameter C , the Prandtl number Pr and the dimensionless heat generation/absorption coefficient B to study the convergence of the obtained analytical solution. To the knowledge of the authors the effects of the pressure gradient on the convective heat transfer in a boundary layer flow of a Maxwell fluid past a stretching sheet have not been studied yet, and thus it is proposed to study the effects of the pressure gradient on the convective heat transfer of a non-Newtonian Maxwell fluid.

The present study employs He's HPM to obtain the approximate analytical solution to the convective heat transfer in the boundary layer flow of a Maxwell fluid over a stretching sheet in the presence and absence of the pressure gradient. To obtain the solution of this problem, two main algorithms such as Adams Method (AM) [35] and Gear Method (GM) [36] have been used [37] with an application of HPM. The effect of the pressure gradient m on the velocity and temperature profiles is analysed for different values of the Deborah number β , the ratio of the stretching velocity parameter ε and Prandtl numbers Pr . To the knowledge of the authors, the proposed research problem has not been studied so far.

2. Basics of HPM

The solution methodology of the HPM is explained hereunder: Consider the nonlinear differential equation

$$A(u) - f(r) = 0, \quad r \in \Omega, \quad (2.1)$$

along with the boundary conditions

$$B(u, \partial u / \partial n) = 0, \quad r \in \Gamma \quad (2.2)$$

where $A(u)$ is a differential operator, B is the boundary condition, $f(r)$ is an analytic function, Ω is the boundary of the domain Γ . The differential operator $A(u)$ in Eq.(2.1) can be split into two parts, such as the linear differential operator $L(u)$ and nonlinear differential operator $N(u)$, therefore, Eq.(2.1) reduces to the form given below:

$$L(u) + N(u) - f(r) = 0. \quad (2.3)$$

By the HPM [10, 37], a homotopy $v(r, p) : \Omega \times [0, 1] \rightarrow R$ is constructed, which satisfies

$$H(v, p) = (1 - p)[L(v) - L(u_o)] + p[A(v) - f(r)] = 0, \quad p \in [0, 1], \quad r \in \Omega \quad (2.4)$$

or

$$H(v, p) = L(v) - L(u_o) + pL(u_o) + p[N(v) - f(r)] = 0 \quad (2.5)$$

where $p \in [0, 1]$ is a parameter which is embedded, u_o is the initial approximated solution of Eq.(2.1), where the boundary conditions are fulfilled. Clearly, from Eq.(2.4) or Eq.(2.5), H takes the following forms:

$$H(v, 0) = L(v) - L(u_o) = 0, \quad (2.6)$$

$$H(v, 1) = A(v) - f(r) = 0. \quad (2.7)$$

The transformation of p from 0 to 1 is referred to as $v(r, p)$ from $u_o(r)$ to $u(r)$. Topologically, this is known as a deformation, besides $L(v) - L(u_o)$, $A(v) - f(r)$ are termed as homotopic. In this study, the embedding parameter p is a small parameter. Assume that the solution of Eq.(2.4) or Eq.(2.5) can be written as a power series in p :

$$v = v_0 + pv_1 + p^2v_2 + \dots \quad (2.8)$$

Setting $p = 1$ results in the approximate solution of Eq.(2.1), we get

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots \quad (2.9)$$

The coupling of the perturbation method and the homotopy method is called the homotopy perturbation method, which has eliminated limitations of the traditional perturbation methods. On the other hand, the proposed technique can take full advantage of the traditional perturbation techniques.

3. Mathematical formulation

Consider the convective heat transfer in the boundary layer flow of a Maxwell fluid over a stretching sheet in the presence of the pressure gradient which is governed by the continuity and the momentum equations. The governing equations of continuity, momentum and energy may be written in usual notation as [2, 38]:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (3.1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y} + \lambda \left[u^2 \frac{\partial^2 u}{\partial x^2} + v^2 \frac{\partial^2 v}{\partial y^2} + 2uv \frac{\partial^2 u}{\partial x \partial y} \right] = -\frac{1}{\rho} \frac{dP}{dx} + \nu \frac{\partial^2 u}{\partial x^2}, \quad (3.2)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{\kappa}{\rho c_p} \frac{\partial^2 T}{\partial y^2} \quad (3.3)$$

where u and v are the components of velocity in the x and y directions, respectively, ν is the kinematic fluid viscosity, ρ is the fluid density, μ is the coefficient of fluid viscosity, λ is the relaxation time, T is the temperature, κ is the fluid thermal conductivity and c_p is the specific heat. Subject to the boundary conditions:

$$u = U_w(x) = ax^m, \quad v = 0, \quad T = T_w \quad \text{at } y = 0, \quad \eta = 0, \quad (3.4)$$

$$u = U_\infty(x) = bx^m, \quad v = 0, \quad T \rightarrow T_\infty \quad \text{at } y \rightarrow \infty, \quad \eta = \eta_\infty.$$

Now, the stream function $\psi(x, y)$ is introduced as:

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \quad (3.5)$$

For an external flow $-\frac{1}{\rho} \frac{dP}{dx}$ can be replaced by $U_\infty \frac{dU_\infty}{dx}$, and taking into consideration Eq.(3.5), Eq.(3.1) is identically satisfied and Eqs (3.2) and (3.3) can be reduced to the following forms:

$$\begin{aligned} & \frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} + \lambda \left[\left(\frac{\partial \psi}{\partial y} \right)^2 \frac{\partial^2 \psi}{\partial x \partial y} + \left(\frac{\partial \psi}{\partial x} \right)^2 \frac{\partial^3 \psi}{\partial y^3} + \right. \\ & \left. - 2 \frac{\partial \psi}{\partial y} \frac{\partial \psi}{\partial x} \frac{\partial^3 \psi}{\partial x \partial y^2} \right] = U_\infty \frac{dU_\infty}{dx} + \nu \frac{\partial^3 \psi}{\partial y^3} \end{aligned} \quad (3.6)$$

and

$$\frac{\partial \psi}{\partial y} \frac{\partial T}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial T}{\partial y} = \frac{\kappa}{\rho c_p} \frac{\partial^2 T}{\partial y^2}. \quad (3.7)$$

Here, we have introduced the dimensionless variables as below:

$$\eta = y \sqrt{\frac{U_w(x)}{\nu x}}, \quad \psi = f(\eta) \sqrt{\nu x U_w(x)},$$

$$\theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty} \quad \text{and} \quad \left\{ U_\infty(x) = bx^m, m = \frac{x}{U_\infty(x)} \frac{dU_\infty(x)}{dx} \right\}. \quad (3.8)$$

Based on Eq.(3.8), we have used similarity transformation to reduce the governing differential equations; Eq.(3.6) and Eq.(3.7) to ordinary non-linear differential equations, Eq.(3.9) and Eq.(3.10), respectively.

$$f''' + m(1 - f'^2) + \left(\frac{m+1}{2}\right) f f'' - \frac{\beta}{2} [(m-1)(3-m)] \eta f'^2 f'' + 4m(m+1) f'^3 + (m+1)^2 f^2 f''' - 2(m+1)(3m-1) f f' f'' = 0, \quad (3.9)$$

$$\theta'' + \frac{\text{Pr}(m+1)}{2} f \theta' = 0 \quad (3.10)$$

where $\beta = \lambda U_\infty(x) / 2x$ is the Deborah number [39] and $\text{Pr} = \mu c_p / \kappa$ is the Prandtl number [40]. The boundary conditions are obtained from the similarity variables as given below:

$$f(0) = 0, \quad \eta = f'(0) = 1, \quad f'(\eta_\infty) = \varepsilon, \quad \theta(0) = 1, \quad \theta(\eta_\infty) = 0 \quad (3.11)$$

where $\varepsilon = \frac{b}{a}$, b is the free stream velocity parameter, a is the stretching sheet parameter, ε is the ratio of the free stream velocity parameter to the stretching sheet parameter.

4. HPM Solution

According to the HPM method from Eqs (3.9) and (3.10) we get:

$$(1-p)(f''' - f'''_0) + p f''' + \left(\frac{m+1}{2}\right) f f'' + m(1 - f'^2) - \frac{\beta}{2} [(m-1)(3-m)] \eta f'^2 f'' + 4m(m+1) f'^3 + (m+1)^2 f^2 f''' - 2(m+1)(3m-1) f f' f'' = 0. \quad (4.1)$$

$$(1-p)(\theta'' - \theta''_0) + p \left(\theta'' + \frac{\text{Pr}(m+1)}{2} f \theta' \right) = 0, \quad (4.2)$$

$$f = f_0 + p f_1 + p^2 f_2 + \dots, \quad (4.3)$$

$$\theta = \theta_0 + p \theta_1 + p^2 \theta_2 + \dots \quad (4.4)$$

Assuming $f'''=0$, $\theta''=0$, and using Eq.(3.11), Eq.(4.3) and Eq.(4.4) in Eqs (4.1)-(4.2) and then simplifying the resulting equations and thereafter equating the terms involving the like powers of p -terms, we have the following equations:

$$p^0 : f_0'''=0, \quad f_0(0)=0, \quad f_0'(0)=1, \quad f_0'(\eta_\infty)=\varepsilon, \tag{4.5}$$

$$\theta_0''=0, \quad \theta_0(0)=1, \quad \theta_0(\eta_\infty)=0,$$

$$p^1 : f_1'''=m(f_0'^2-1)-\left(\frac{m+1}{2}\right)f_0f_0''+2m(m+1)\beta f_0'^2+(1-2m-3m^2)\beta f_0f_0'f_0''+ \\ +\frac{1}{2}(4m-3\eta-m^2\eta)\beta f_0'^2f_0'', \quad f_1(0)=0, \quad f_1'(0)=1, \quad f_1'(\eta_\infty)=\varepsilon, \tag{4.6}$$

$$\theta_1''=-\frac{\text{Pr}(m+1)}{2}f_0\theta_0', \quad \theta_1(0)=0, \quad \theta_1(\eta_\infty)=0,$$

$$p^2 : f_2'''=2mf_0'f_1'-\left(\frac{m+1}{2}\right)(f_1f_0''+f_0f_1'')\delta\beta m(m+1)f_0'^2f_1'+ \\ +(1-2m-3m^2)\beta\times(f_0'f_1f_0''+f_0f_0'f_1''+f_0f_1'f_0'')+ \eta\beta(-3+4m-m^2)+ \\ +\left(f_0f_1'f_0''+\frac{1}{2}f_0'^2f_1''\right)+\beta\frac{(m+1)^2}{2}f_0'^2f_1'', \tag{4.7}$$

$$f_2(0)=0, \quad f_2'(0)=1, \quad f_2'(\eta_\infty)=\varepsilon, \quad \theta_2''=-\frac{\text{Pr}(m+1)}{2}(f_0\theta_1'+f_1\theta_0'),$$

$$\theta_2(0)=0, \quad \theta_2(\eta_\infty)=0,$$

$$p^3 : f_3'''=mf_1'^2+2mf_0'f_2'-\left(\frac{m+1}{2}\right)(f_2f_0''+f_1f_1''+f_0f_2'')+ \delta\beta m(m+1)\times \\ \times\left(f_1'^2f_0'+f_0'^2f_2'\right)(1-2m-3m^2)\beta\times(f_0'f_2f_0''+f_1f_1'f_0''+f_0f_2'f_0''+f_0'f_1f_1''+ \\ +f_1'f_0f_1''+f_0'f_0f_2'')+ \eta\beta(-3+4m-m^2)\left(f_0'f_2'f_0''+\frac{1}{2}f_1'^2f_0''+f_0'f_1'f_1''+\frac{1}{2}f_0'^2f_2''\right)+ \\ +\beta(1+2m+m^2)\left(f_0f_1f_1''+\frac{1}{2}f_0'^2f_2''\right), \tag{4.8}$$

$$f_3(0)=0, \quad f_3'(0)=1, \quad f_3'(\eta_\infty)=\varepsilon,$$

$$\theta_3''=-\frac{\text{Pr}(m+1)}{2}(f_0\theta_2'+f_1\theta_1'+f_2\theta_0'), \quad \theta_3(0)=0, \quad \theta_3(\eta_\infty)=0. \tag{4.9}$$

Solving Eqs (4.5)-(4.9), we get

$$f_0 = \frac{I}{2\eta_\infty}(-\eta^2 + 2\eta\eta_\infty + \eta^2\varepsilon), \quad (4.10)$$

$$f_1 = \frac{I}{480\eta_\infty^2} \left(4\beta\eta^6 - 2\beta\eta^5\eta_\infty + 10\beta\eta^4\eta_\infty^2 - 2\beta\eta^2\eta_\infty^4 - 4\beta\eta^6\varepsilon^3 + \right. \\ \left. -12\beta\eta^2\varepsilon^3\eta_\infty^4 + 12\beta\eta^6\varepsilon^2 - 12\beta\eta^5\varepsilon^2\eta_\infty - 6\beta\eta^2\varepsilon^2\eta_\infty^4 - \dots \right), \quad (4.11)$$

$$f_2 = \frac{I}{2419200\eta_\infty^5} \left(2772m^3\beta^2\varepsilon^5\eta^{10} - 1596m^2\beta^2\varepsilon^5\eta^{10} + 2828\beta^2\varepsilon^5\eta^{10} - 868\beta^2\varepsilon^5\eta^{10} + \right. \\ \left. -13860m^3\beta^2\varepsilon^4\eta^{10} + 7980m^2\beta^2\varepsilon^4\eta^{10} - 14140m\beta^2\varepsilon^4\eta^{10} + 4340\beta^2\varepsilon^4\eta^{10} + \dots \right), \quad (4.12)$$

$$f_3 = \eta + \frac{\eta^2}{83026944000\eta_\infty} \left(-92085840m^5\beta^3\varepsilon^7\eta_\infty^6 + 664356000m^4\beta^2\varepsilon^4\eta^{10} + \right. \\ \left. + 2319791936m^3\beta^3\varepsilon^7\eta_\infty^6 + \dots \right) - \frac{I}{4838400\eta_\infty^7} \left(-\frac{2848\beta^3\eta^{14}}{13} + \frac{3353}{3}\beta^2\eta^4\varepsilon^5\eta_\infty^{10} + \right. \\ \left. + \frac{19936}{13}\beta^3\eta^{14}\varepsilon + \frac{9966}{13}\beta^3\eta^{14}m^5 + 5775\beta^3\eta^4m^5\varepsilon^5\eta_\infty^{10} + 3465\beta^2\eta^4m^4\varepsilon^5\eta_\infty^{10} + \right. \\ \left. + \frac{38824}{39}\beta^3\eta^{14}m^3 - \frac{186634}{3}\beta^3\eta^4m^3\varepsilon^5\eta_\infty^{10} + \frac{30542}{39}\beta^3\eta^{14}m - \frac{57379}{3}\beta^3\eta^4m\varepsilon^5\eta_\infty^{10} + \right. \\ \left. - \frac{231794}{39}\beta^3\eta^{14}m\varepsilon + \dots \right), \quad (4.13)$$

$$\theta_0 = \frac{I}{\eta_\infty}(\eta_\infty - \eta), \quad (4.14)$$

$$\theta_1 = \frac{I}{48\eta_\infty^2} \left(-\eta^4 m \text{Pr} + 4\eta^3 m \text{Pr} \eta_\infty - 3\eta m \text{Pr} \eta_\infty^3 + \eta^4 m \text{Pr} \varepsilon + \right. \\ \left. -\eta m \text{Pr} \varepsilon \eta_\infty^3 - \eta^4 \text{Pr} + 4\eta^3 \text{Pr} \eta_\infty - 3\eta \text{Pr} \eta_\infty^3 + \eta^4 \text{Pr} \varepsilon - \eta \text{Pr} \varepsilon \eta_\infty^3 \right), \quad (4.15)$$

$$\theta_2 = \frac{I}{80640\eta_\infty^4} \left(18m^2 \text{Pr} \beta \varepsilon^3 \eta^8 + 12m \text{Pr} \beta \varepsilon^3 \eta^8 - 6 \text{Pr} \beta \varepsilon^3 \eta^8 - 54m^2 \text{Pr} \beta \varepsilon^2 \eta^8 + \right. \\ \left. -36m \text{Pr} \beta \varepsilon^2 \eta^8 + 18 \text{Pr} \beta \varepsilon^2 \eta^8 - 18m^2 \text{Pr} \beta \eta^8 - 12m \text{Pr} \beta \eta^8 + \dots \right), \quad (4.16)$$

$$\theta_3 = \frac{I}{638668800\eta_\infty^6} \left(23184m\eta \text{Pr} \beta^2 \varepsilon^3 \eta_\infty^{11} + 2772m^4 \text{Pr} \beta^2 \eta^{12} + \right. \\ \left. + 149688 \text{Pr} \beta^2 \eta \varepsilon^5 \eta_\infty^{11} - 1176m^3 \text{Pr} \beta^2 \eta^{12} - 770472 \text{Pr} m^3 \beta^2 \eta \varepsilon^5 \eta_\infty^{11} + \right. \\ \left. -1232m^2 \text{Pr} \beta^2 \eta^{12} - 211464m^2 \eta \varepsilon^5 \text{Pr} \beta^2 \eta_\infty^{11} + 6160\beta^2 \eta^{12} m^2 \text{Pr} \varepsilon + \right. \\ \left. -1960\beta^2 \eta^{12} m \text{Pr} + 569160 \text{Pr} m \beta^2 \eta \varepsilon^5 \eta_\infty^{11} + 9700\beta^2 \eta^{12} m \text{Pr} \varepsilon + \dots \right). \quad (4.17)$$

5. Results and discussion

The primary objective of this study is to discuss the effects of various flow parameters such as the pressure gradient (m), dimensionless variable (η_∞), the ratio of the free stream velocity parameter to the stretching sheet parameter (ε), Deborah number (β) and Prandtl number (Pr) on the important rheological measurements such as velocity and temperature profiles. The analytic solutions obtained in Eqs (4.10)-(4.16) are evaluated for different values of the parameters to obtain data for plotting appropriate graphs and then the results are analysed.

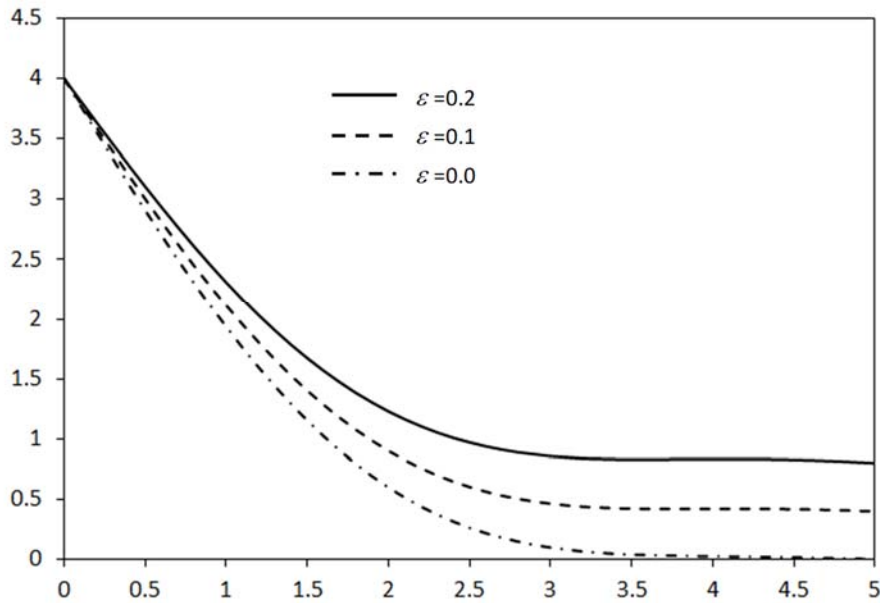


Fig.1. Velocity profile for $f'(\eta)$ for different values of ε when $\beta = -0.1$, $m = 0$ and $\eta_\infty = 5$.

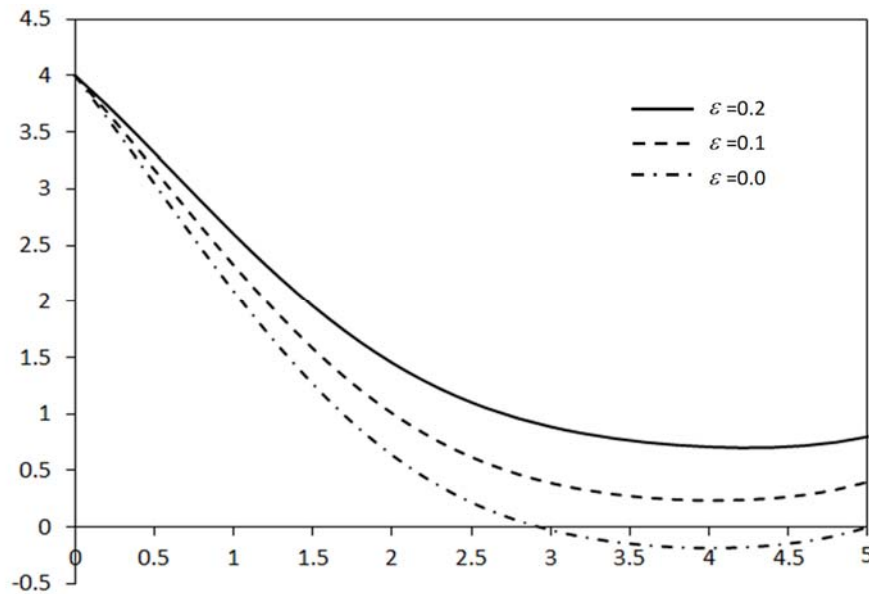


Fig.2. Velocity profile for $f'(\eta)$ for different values of ε when $\beta = -0.1$, $m = -0.2$ and $\eta_\infty = 5$.

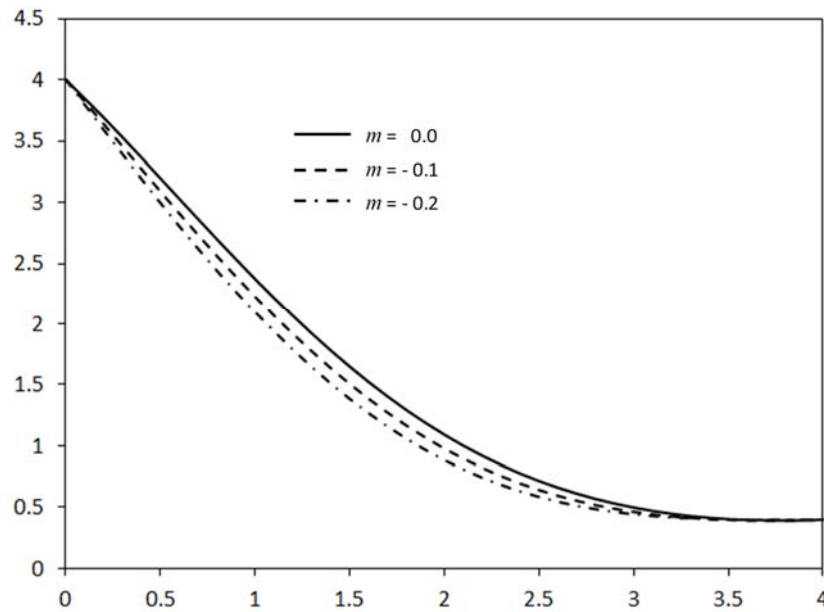


Fig.3. Velocity profile for $f'(\eta)$ for different values of m when $\beta = -0.1$, $\varepsilon = 0.1$ and $\eta_\infty = 4$.

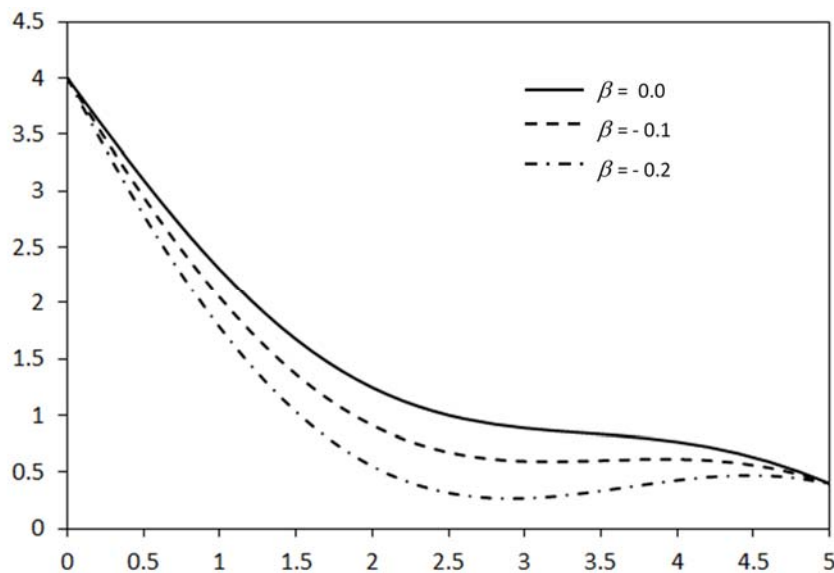


Fig.4. Velocity profile $f'(\eta)$ for different values of β when $m = 0.1$, $\varepsilon = 0.1$ and $\eta_\infty = 5$.

The solution system has become convergent for the considered values of ε as discussed in Fig.1. for the velocity profile. As ε is a ratio, it converges to the ratio value 0.4 instead of converging to 0.1 , 0.2 or 0.8 , since in the present study, boundary layer thickness ranges from 4 for the value of $f'(\eta)$, whereas as reported earlier, it starts from 1 (as mentioned in Section 1). For the temperature profile, it is shown to be convergent to “ 0 ” for the considered values of ε as shown in Fig.5. These results are similar to the cases discussed in Section 1. Reported results are in the presence and absence of the pressure gradient m for the sake of comparison. In Figs 1 and 2, we have plotted velocity profiles for $m = 0$ and $m = -0.2$ respectively. In Fig.1., it can be observed that the momentum boundary layer thickness decreases with the decreasing value of ε . The boundary layer thickness becomes significantly thinner. Note that in Fig.2., a curve for

$\varepsilon = 0$ shows some negative values for $f'(\eta)$. This is the case of fluid cooling. It is possible when the buoyancy force opposes the stretching motion of the surface, thus the fluid flow caused by the upward motion of the surface is opposed by the free convection currents which were induced by the buoyancy force and hence $f'(\eta)$ decreases [27].

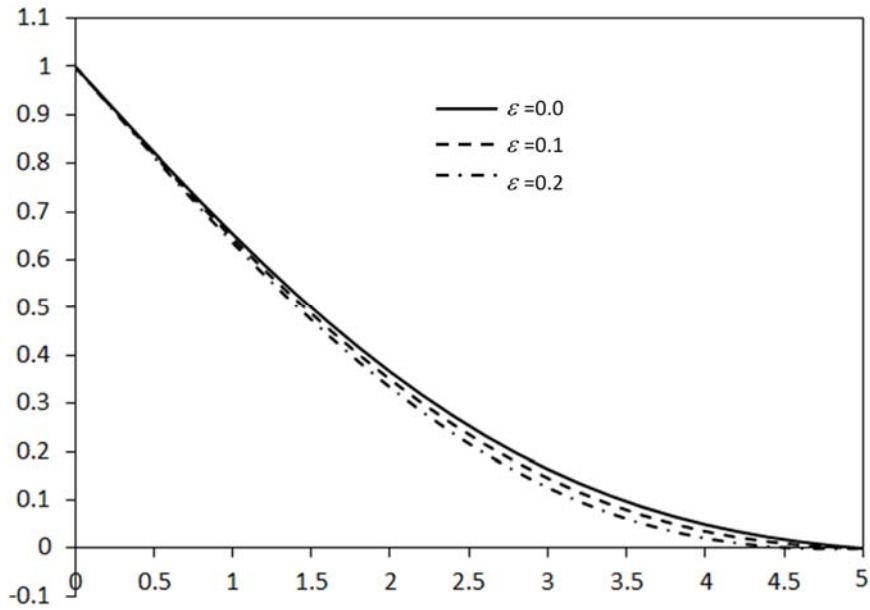


Fig.5. Energy profile for $\theta(\eta)$ for different values of ε at $\eta_\infty = 5$, $\beta = -0.1$, $m = 0$ and $Pr = 0.3$.

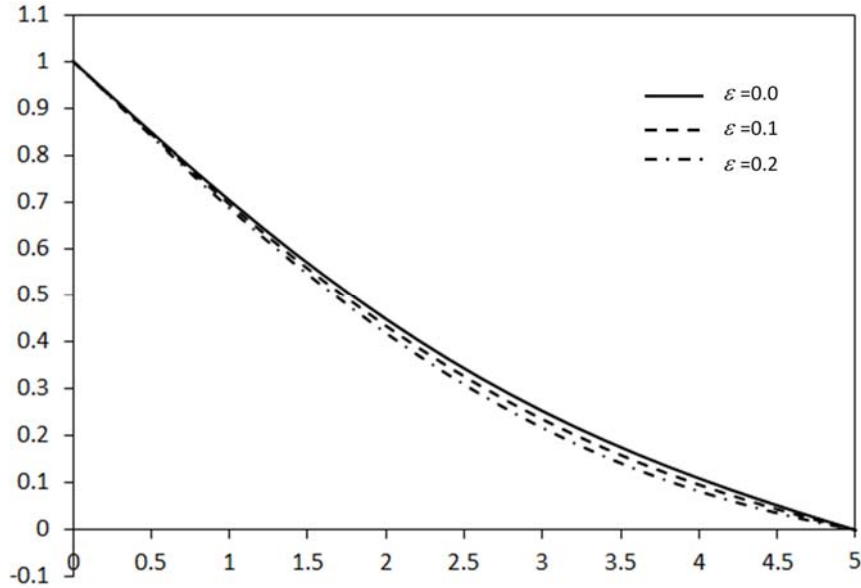


Fig.6. Energy profile for $\theta(\eta)$ for different values of ε at $\eta_\infty = 5$, $\beta = -0.1$, $m = -0.2$ and $Pr = 0.3$.

Figure 3 shows the increasing behaviour of the fluid flow for the decreasing values of m and thus the thickness of the boundary layer increases. Figure 4 illustrates the velocity profile for different values of β at $\eta_\infty = 5$ when $m = 0$ and $m = 0.1$, respectively. Both the figures show the increasing behaviour of the

fluid flow for the decreasing values of β so that the boundary layer thickness increases. Figures 5-6 depict the energy profiles for ε in the absence and presence of the pressure gradient m . From these figures, it is observed that the thermal boundary layer thickness decreases with the increasing values of ε . On comparing these three figures, one can note that the thermal boundary layer thickness decreases when the pressure gradient parameter m increases from -0.2 to 0.2 and hence the thermal boundary layer thickness decreases.

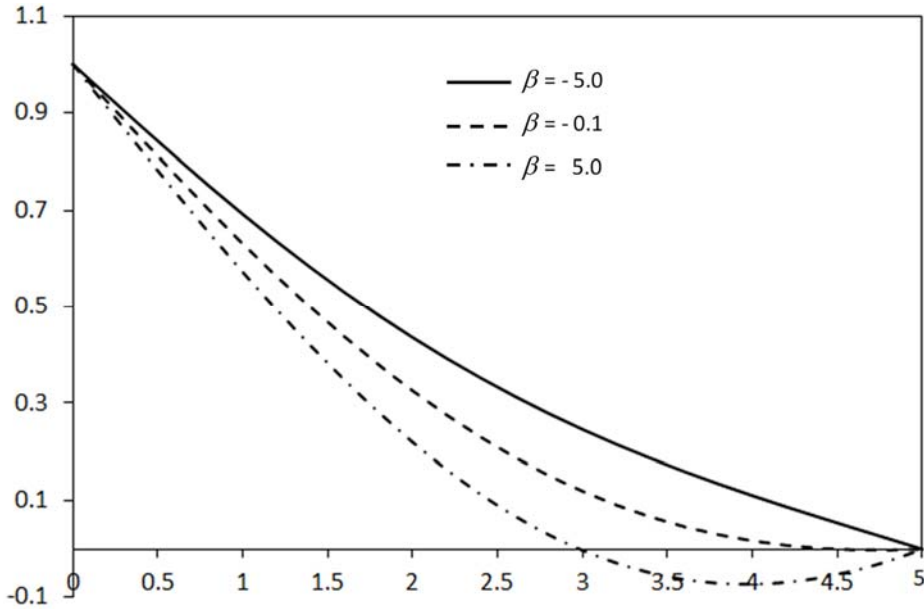


Fig.7. Energy profile for $\theta(\eta)$ for the different values of β at $m = 0.05$, $\varepsilon = 0.1$, $Pr = 0.3$ and $\eta_\infty = 5$.

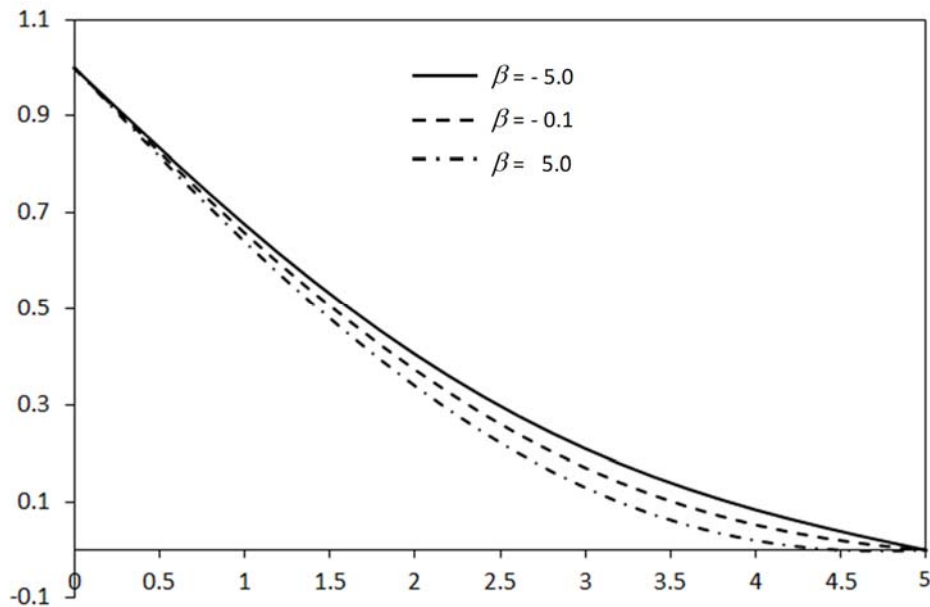


Fig.8. Energy profile for $\theta(\eta)$ for different values of β at $m = -0.05$, $\varepsilon = 0.1$, $Pr = 0.3$ and $\eta_\infty = 5$.

Figures 7-8 show the energy profile for β in the absence and presence of the pressure gradient m , respectively. Figure 7 shows that the thermal boundary layer thickness increases with the increasing values

of β . On the other hand, in Fig.8., the behaviour of the thermal boundary layer thickness changed to a decreasing trend with the increasing values of β which is the case in the presence of m and hence the thermal boundary layer thickness decreases. In all two figures the values of β range from $-5 \leq \beta \leq 5$. From Fig.7., it is noticed that the negative values of $\theta(\eta)$ for the value of $\beta = 0.05$ lead to the fluid cooling case. Physically, this is possible when the fluid is cooled. The values $\beta > -0.1$ indicate the fluid becomes cooled. Figure 9 shows the energy profiles for different values of the Prandtl number Pr in the absence and presence of the pressure gradient m . In the specified circumstances, the thermal boundary layer thickness decreases. On the other hand, all of these three figures show the variation of energy profiles when the pressure gradient m ranges from -0.2 to 0.2 and it is observed that the thermal boundary layer thickness decreases.

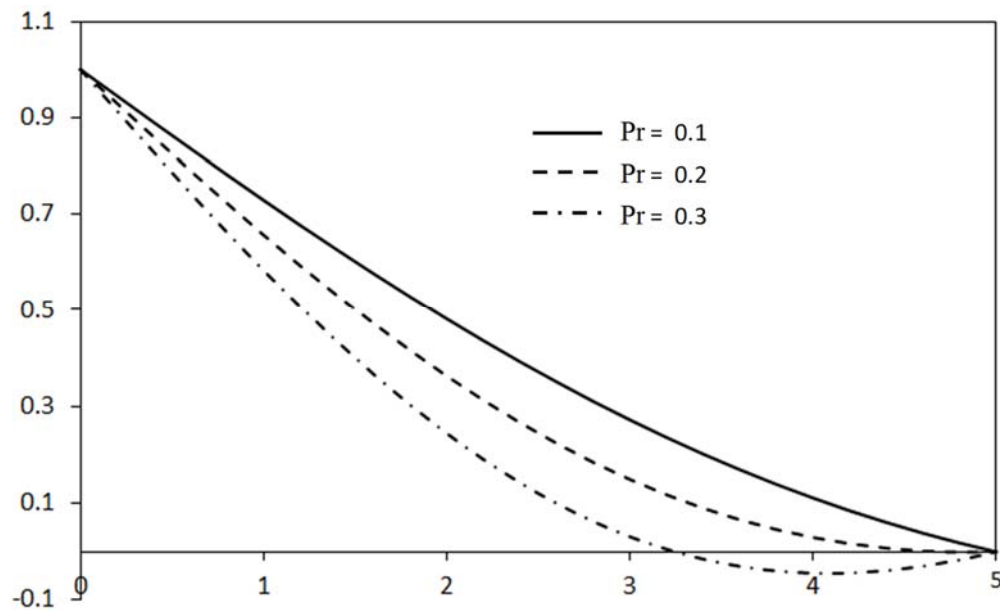


Fig.9. Energy profile for $\theta(\eta)$ for different values of Pr at $\eta_{\infty} = 5$, $\varepsilon = 0.1$, $m = 0.2$ and $\beta = -0.1$.

It is clear from Fig.9. that the negative values of $\theta(\eta)$ for the value of $Pr = 0.3$ lead to the fluid cooling case. Physically, this is possible when the fluid is cooled. When the Prandtl number $Pr > 0.2$, the fluid cools down.

6. Conclusion

In this research problem, partial differential equations have been established for the convective heat transfer in the boundary layer flow of a Maxwell fluid over a stretching sheet in the presence and absence of the pressure gradient. To get a solution of such a problem, initially similarity transformation technique has been used to transform these equations into an ODE (Ordinary Differential Equation) and then an approximation technique has been implemented through an application of Adam and Gear Methods.

The results have been reported in the absence and presence of the pressure gradient for the sake of comparison. From the plots of velocity profiles $f'(\eta)$, it is observed that the boundary layer thickness increases, except in the two cases where the boundary layer thickness decreases for the case of ε in Figs 1 and 2. The thermal boundary layer thickness has decreased for all results, except in Fig.7. which shows that the thermal boundary layer thickness increased. As a whole, results indicate a significant thinning in the momentum and thermal boundary layer thickness in both velocity and temperature profiles. On the other

hand, some figures exhibit negative values for $f'(\eta)$ and $\theta(\eta)$ which indicate the case of fluid cooling. It is also noted that the momentum and boundary layer thickness decrease for non-zero values of the pressure gradient m .

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Nomenclature

- a – stretching sheet parameter
- A – general differential operator
- AM – Adams Method
- b – free stream velocity parameter
- B – boundary operator
- c_p – specific heat
- f – dimensionless velocity function
- GM – Gear Method
- H – convective heat transfer coefficient
- HPM – homotopy perturbation method
- L – linear operator
- m – pressure gradient parameter
- n – number of approximations
- N – non-linear operator
- NM – numerical method
- p – embedding parameter of homotopy
- P – pressure
- Pr – Prandtl Number
- $q(r)$ – analytic function
- T – temperature
- T_w – wall temperature
- T_∞ – local ambient temperature
- u, v – fluctuating velocity components in x and y directions
- x, y – coordinates along and perpendicular to the plate
- β – Deborah Number
- Γ – boundary of the domain Ω
- δ – boundary layer thickness
- ε – ratio of free stream velocity parameter to stretching sheet parameter
- η – dimensionless variable

- θ – dimensionless temperature
 κ – fluid thermal conductivity
 λ – relaxation time or Maxwell parameter
 μ – coefficient of fluid viscosity
 P – fluid density
 ν – kinematics fluid viscosity
 ψ – streamline function

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