

APPLICATION OF FRACTIONAL ORDER THEORY OF THERMOELASTICITY IN AN ELLIPTICAL DISK AND ASSOCIATED THERMAL STRESSES

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In this article, a time fractional-order theory of thermoelasticity is applied to an isotropic homogeneous elliptical disk. The lower and upper surfaces of the disk are maintained at zero temperature, whereas the sectional heat supply is applied on the outer curved surface. Thermal deflection and associated thermal stresses are obtained in terms of Mathieu function of the first kind of order $2n$. Numerical evaluation is carried out for the temperature distribution, Thermal deflection and thermal stresses and results of the resulting quantities are depicted graphically.

Key words: elliptical disk, fractional-order theory, Caputo derivative, integral transform, thermal stresses, thermal deflection.

1. Introduction

Fractional calculus is a very useful and interesting subject for mathematicians. It is a generalization of the concept of ordinary differentiation $D^\alpha f(t)$ to non integer values of α that goes back to the beginning of the concept of differential calculus. Many models of physical processes such as heat conduction, wave propagation, diffusion, electric theories and viscoelasticity have been modified by fractional calculus. In the second half of the 19th century the theory of fractional integrals and derivatives was established. Abel [1] was first to apply fractional derivatives during the formulation of the Tautochrone problem by using fractional calculus in the solution of an integral equation. Caputo and Mainardi [2, 3] and Caputo [4] introduced a model of dissipation based on memory. Ning [5] used the variable separation method to obtain an analytical solution for the time-fractional heat conduction equation in the spherical coordinate system.

Lenzi *et al.* [6] studied exact solutions of the fractional diffusion equation by Green function approach. Povstenko [7, 8] investigated variation of time-fractional differential operators with memory effects. Also, Povstenko [9] solved analytically fractional heat conduction in a space with a source varying harmonically in time and calculated associated thermal stresses. Abbas [10] studied the fractional order theory with thermoelastic diffusion using the Laplace and Eigen value approach for an infinite elastic medium within a spherical cavity.

Deshmukh [11] solved a two dimensional thermoelastic problem of a thin circular plate by a quasi-static approach in the context of the fractional order theory. Youssef [12] in 2010 introduced the generalized

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thermoelastic theory for heat conduction in the context of the fractional integral. Also, the uniqueness of the solutions is proved in [13]. Lamba [14] presented mathematical modeling of thermoelastic hollow cylinder subjected to convective boundary condition on the upper and lower plane surface by the application of the fractional order theory. Tripathi [15] studied a dynamic thermoelastic problem of fractional order for a thick circular plate with finite wave speeds. Kukla [16] presented an analytical solution to the problem of time-fractional heat conduction in a sphere. Lamba [17] applied the fractional order theory to determine thermal deflection and stresses of a circular disk subjected to axisymmetric and partially distributed heat supply on the upper surface while the lower surface is kept thermally insulated. Lamba [18] applied the fractional order theory to analyze the temperature distribution, displacement, thermal stresses function and deflection on the outer curved surface of a solid circular cylinder.

Due to the advantage of physical, mechanical, as well as thermal properties of elliptical structures materials, a copper (pure) elliptical disk is used extensively and much more frequently in modern engineering applications as compared to other shape objects. Both analytical and numerical techniques are the best methodology to solve such problems. But there are only few studies concerned with elliptical objects for both steady and transient state heat conduction problems. McLachlan [19, 20] solved the heat conduction problem for an elliptical cylinder in the form of an infinite Mathieu function series. Dhakate [21] determined thermally induced transverse vibration of a uniform thin elliptical disk with elastic supports at both radial boundaries by using the integral transform technique. Gupta [22] determined the boundary value problem for elliptic cylinders by introducing a finite transform involving Mathieu functions. Bhad [23] investigated the thermoelastic problems on an elliptical plate in which interior heat sources are generated within the solid. Dhakate [24] used Berger's approximate methods to determine the large deflection of a heated thin annular sector plate with clamped edges under transient temperature distribution.

From the above a literature it is clear that till now no work has dealt with the thermoelastic bending analysis in the context of the fractional-order theory of thermoelasticity. So the authors have been motivated to conduct this study. In this paper, a mathematical model of a fractional-order thermoelastic problem for a thin simply supported elliptical disk subjected to sectional heat supply on the outer curved surface is developed. The numerical computations have been carried out for a copper (pure) elliptical disk and the results for temperature distribution, thermal deflection, and stresses are discussed and illustrated graphically.

2. Formulation of the problem

By following Varghese [21], here we consider a thin elliptical disk (as shown in Fig.1) with the space variables (ξ, η, z) of thickness ℓ occupying space D defined by $\xi_i < \xi < \xi_o$ and $0 < \eta < 2\pi$, whose lower and upper surfaces are maintained at zero temperature, whereas the sectional heat supply is applied on the outer curved surface.

The relation between Cartesian and elliptical coordinates is defined as

$$x = c \cosh \xi \cos \eta, \quad y = c \sinh \xi \sin \eta, \quad z = z, \quad h^2 = 2 \left[c^2 (\cosh 2\xi - \cos 2\eta) \right].$$

By using the transformation $\xi + i\eta = \cosh^{-1}[(x + iy)/c]$ where $2c = 2\sqrt{a_o^2 - b_o^2} = 2\sqrt{a_i^2 - b_i^2}$ denotes the distance between their common foci. The semi-major axis varies from the inner diameter $2a_i$ to the outer diameter $2a_o$ and the semi-minor axis varies from the inner diameter $2b_i$ to the outer diameter $2b_o$. Whereas $\xi \in (\xi_i, \xi_o)$ and at the inner and outer elliptical boundary are given as $\xi_i = \tanh^{-1}(b_i/a_i)$ $\xi_o = \tanh^{-1}(b_o/a_o)$ respectively.

A mathematical model is prepared considering a nonlocal Caputo type time fractional heat conduction equation of order α for a thin elliptical disk.

The definition of the Caputo type fractional derivative is given by [25]

$$\frac{d^\alpha f(t)}{dt^\alpha} = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{d^n f(\tau)}{d\tau^n} d\tau, & n-1 < \alpha < n, \\ \frac{d^n f(\tau)}{dt^n}, & \alpha = n \end{cases} \quad (2.1)$$

with the following Laplace transform rule

$$L\left\{\frac{d^\alpha f(t)}{dt^\alpha}\right\} = s^\alpha L\{\bar{f}(s)\} - \sum_{k=0}^{n-1} f^{(k)}(0^+) s^{\alpha-1-k}, \quad n-1 < \alpha < n \quad (2.2)$$

in which s is the transform parameter.

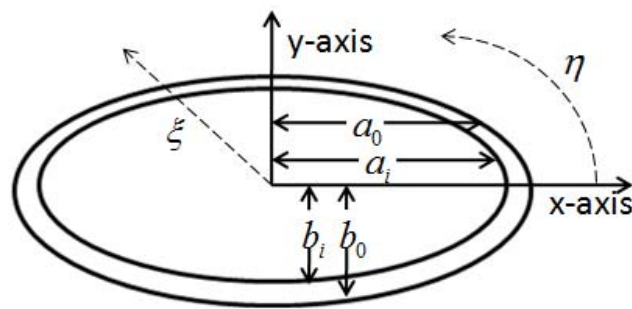


Fig.1. Geometry of a thin elliptical disk.

3. Temperature distribution

The temperature of the disk $\theta(\xi, \eta, z, t)$ satisfies the time fractional order differential equation (Following [8] and [21])

$$\left\{ h^2 \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) + \frac{\partial^2}{\partial z^2} \right\} \theta(\xi, \eta, z, t) = \frac{1}{\kappa} \frac{\partial^\alpha}{\partial t^\alpha} \theta(\xi, \eta, z, t) \quad (3.1)$$

subjected to boundary and initial conditions

$$\left. \begin{aligned} \theta(\xi, \eta, z, t) \Big|_{t=0} &= 0 & ; & & 0 < \alpha \leq 2 \\ \frac{\partial \theta}{\partial t}(\xi, \eta, z, t) \Big|_{t=0} &= 0 & ; & & 1 < \alpha \leq 2 \\ \theta(\xi, \eta, z, t) \Big|_{\xi=\xi_i} &= 0, \quad kD_{RL}^{l-\alpha} \frac{\partial \theta}{\partial \xi}(\xi, \eta, z, t) \Big|_{\xi=\xi_o} &= f_l(z) \delta(t) \\ \theta(\xi, \eta, z, t) \Big|_{z=0} &= 0, \quad \theta(\xi, \eta, z, t) \Big|_{z=l} &= 0 \end{aligned} \right\} \quad (3.2)$$

where $\kappa = \lambda / \rho C$ represents thermal diffusivity, λ is the thermal conductivity of the material, ρ is the density and C is the calorific capacity.

4. Thermal deflection and thermal stresses

The expressions of thermally induced resultant force and resultant moment are given as [21]

$$M_\theta(\xi, \eta, t) = \alpha E \int_0^\ell z \theta(\xi, \eta, z, t) dz, \quad N_\theta(\xi, \eta, t) = \alpha E \int_0^\ell \theta(\xi, \eta, z, t) dz \quad (4.1)$$

where E denotes the Young modulus and α is the coefficient of linear thermal expansion of the material of the disk, respectively.

The governing differential equation for the disk in terms of normal deflection $\omega(\xi, \eta, t)$ is

$$\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right)^2 \omega(\xi, \eta, t) = - \frac{I}{2b^2 h^4 \rho \ell (1-\nu)} \nabla^2 M_\theta(\xi, \eta, t) \quad (4.2)$$

where $b^2 = D / 2\rho\ell$,

here D denotes the flexural stiffness of the disk and is defined as $D = E\ell^3 / 12(1-\nu^2)$.

The basic relations of resultant bending moments per unit width (M_{ij}), resultant forces (N_{ij} , $i, j = \xi, \eta$) are defined [23] as

$$N_\xi = N_\eta = N_{\xi\eta} = 0, \quad (4.3)$$

$$\left. \begin{aligned} M_\xi &= -h^2 D \left\{ \left(\frac{\partial^2 \omega}{\partial \xi^2} + \nu \frac{\partial^2 \omega}{\partial \eta^2} \right) - \frac{c^2 h^2 (1-\nu)}{2} [\sinh 2\xi \frac{\partial \omega}{\partial \xi} - \sin 2\eta] \frac{\partial \omega}{\partial \eta} \right\} - \frac{M_\theta}{1-\nu}, \\ M_\eta &= -h^2 D \left\{ \left(\nu \frac{\partial^2 \omega}{\partial \xi^2} + \frac{\partial^2 \omega}{\partial \eta^2} \right) + \frac{c^2 h^2 (1-\nu)}{2} [\sinh 2\xi \frac{\partial \omega}{\partial \xi} - \sin 2\eta] \frac{\partial \omega}{\partial \eta} \right\} - \frac{M_\theta}{1-\nu}, \\ M_{\xi\eta} &= -h^2 D (1-\nu) \left\{ \frac{\partial \omega}{\partial \xi} \sin 2\eta + \frac{\partial \omega}{\partial \eta} \sinh 2\xi - \frac{2}{c^2 h^2} \frac{\partial^2 \omega}{\partial \xi \partial \eta} \right\}. \end{aligned} \right\} \quad (4.4)$$

The aforementioned Eq.(4.2) and first equation of (4.4) must satisfy the initial condition

$$\omega \Big|_{t=0} = \frac{\partial \omega}{\partial t} \Big|_{t=0} = 0 \quad \text{for} \quad \xi_i \leq \xi \leq \xi_o, \quad 0 \leq \eta \leq 2\pi, \quad (4.5)$$

$$\omega \Big|_{\xi=\xi_i} = M_\xi \Big|_{\xi=\xi_i} = 0 \quad \text{for all} \quad \eta \quad \text{in} \quad 0 \leq \eta \leq 2\pi. \quad (4.6)$$

The stress functions $\sigma_{\xi\xi}$, $\sigma_{\eta\eta}$ and $\sigma_{\xi\eta}$ in terms of resultant forces and resultant moments are given [23-24] as

$$\begin{aligned} \sigma_{\xi\xi} &= \frac{I}{\ell} N_{\xi} + \frac{12z}{\ell^3} M_{\xi} + \frac{I}{I-\nu} \left(\frac{I}{\ell} N_{\theta} + \frac{12z}{\ell^3} M_{\theta} - \alpha E \theta \right), \\ \sigma_{\eta\eta} &= \frac{I}{\ell} N_{\eta} + \frac{12z}{\ell^3} M_{\eta\eta} + \frac{I}{I-\nu} \left(\frac{I}{\ell} N_{\theta} + \frac{12z}{\ell^3} M_{\theta} - \alpha E \theta \right), \\ \sigma_{\xi\eta} &= \frac{I}{\ell} N_{\xi\eta} - \frac{12z}{\ell^3} M_{\xi\eta}. \end{aligned} \tag{4.7}$$

Equations (2.1) to (4.7) constitute the mathematical formulation of the problem under consideration.

5. Solution to the problem

5.1. Solution for temperature distribution

To obtain the expression for the temperature function $\theta(\xi, \eta, z, t)$, we use the finite Fourier cosine transformation with respect to the variable z and then introduce the finite integral transform ([22] and [21]) involving Mathieu functions as

$$\bar{f}(q_{2n,m}) = \int_{\xi_i}^{\xi_0} \int_0^{2\pi} f(\xi, \eta) (\cosh 2\xi - \cos 2\eta) Be_{2n}(\xi, q_{2n,m}) ce_{2n}(\eta, q_{2n,m}) d\xi d\eta, \tag{5.1}$$

with its inversion theorem at any point within the range as

$$\begin{aligned} f(\xi, \eta) &= \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{\bar{f}(q_{2n,m}) Be_{2n}(\xi, q_{2n,m}) ce_{2n}(\eta, q_{2n,m})}{\pi \int_{\xi_i}^{\xi_0} (\cosh 2\xi - \Theta_{2n,m}) Be_{2n}^2(\xi, q_{2n,m}) d\xi}, \\ \Theta_{2n,m} &= \frac{1}{\pi} \int_0^{2\pi} \cos 2\eta ce_{2n}^2(\eta, q_{2n,m}) d\eta = A_0^{(2n)} A_2^{(2n)} + \sum_{r=0}^{\infty} A_{2r}^{(2n)} A_{2r+2}^{(2n)} \end{aligned} \tag{5.2}$$

in these series A represents the functions of q , $q_{2n,m}$ is a root of the transcendental equation $Be_{2n}(\xi_0, q_{2n,m}) = 0$, and having the kernel as

$$Be_{2n}(\xi, q_{2n,m}) = Ce_{2n}(\xi, q_{2n,m}) Fey'_{2n}(\xi_i, q_{2n,m}) - Fey_{2n}(\xi, q_{2n,m}) Ce'_{2n}(\xi_i, q_{2n,m}),$$

in which prime dash denotes differentiation with respect to the variable ξ , $ce_{2n}(\eta, q)$ is the ordinary Mathieu function [20] of first kind of order n , $Ce_{2n}(\xi, q)$ is a modified Mathieu function [20] of first kind of order n and the recurrence relations for the Bessel functions Y_{2r} (are identical in form with those for J_{2r}) can be defined [20] as

$$Fey_{2n}(\xi, q) = \frac{ce_{2n}(0, q)}{A_0^{(2n)}} \sum_{r=0}^{\infty} A_{2r}^{(2n)} Y_{2r}(2k' \sinh \xi) \begin{pmatrix} |\sinh \xi| > 1 \\ R(\xi) > 0 \end{pmatrix}$$

with y in Fey which signifies the Y - Bessel function and $q = k'^2 = \lambda c^2 / 4$ [20].

Applying the finite integral transform [22] given in Eq.(5.1) and finite Fourier cosine transformation and their inversions to Eq.(2.3) and making use of the transformed boundary and initial conditions (2.4), one obtains the temperature distribution function expressed as

$$\begin{aligned} \theta(\xi, \eta, z, t) = & \frac{2}{\ell} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \cos(\beta_p z) \left\langle \int_0^t E_{\alpha} \{[-\kappa(\lambda_{2n,m}^2 + \beta_p^2)]u^{\alpha}\} \right. \\ & \times \left[-\frac{4\kappa}{c^2} \pi A_0^{2n} Be_{2n}(\xi_0) \hat{f}(q_{2n,m}, \beta_p, t-u) \right] du \Big\rangle \times Be_{2n}(\xi, q_{2n,m}) ce_{2n}(\eta, q_{2n,m}) / C_{2n,m} \end{aligned} \tag{5.3}$$

where

$$E_{\alpha}[-\kappa(\lambda_{2n,m}^2 + \beta_p^2)] t^{\alpha} = L^{-1} \left\{ \frac{I}{s^{\alpha} + \kappa(\lambda_{2n,m}^2 + \beta_p^2)} \right\}. \tag{5.4}$$

Here, $E_{\alpha}(z)$ represents the Mittag-Leffler-type function [25] described by the series representation

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad z \in C, \text{Re}(\alpha) > 0. \tag{5.5}$$

5.2. Solution for thermal deflection and stresses

Using Eqs (5.3) in Eq.(4.1), we get the resultant moment and resultant force as follows

$$\begin{aligned} M_{\theta} = & \frac{2\alpha E}{\ell} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} [(-I)^p - I] \left\langle \int_0^t E_{\alpha} \{[-\kappa(\lambda_{2n,m}^2 + \beta_p^2)]u^{\alpha}\} \times \right. \\ & \times \left[-\frac{4\kappa}{c^2} \pi A_0^{2n} Be_{2n}(\xi_0) \hat{f}(q_{2n,m}, \beta_p, t-u) \right] du \Big\rangle \times Be_{2n}(\xi, q_{2n,m}) ce_{2n}(\eta, q_{2n,m}) / \beta_p^2 C_{2n,m} \tag{5.6} \\ N_{\theta} = & 0 \end{aligned}$$

Using Eqs (5.6) in Eq.(4.2), we obtain the expression for ω as follows

$$\begin{aligned} \omega = & \frac{2\alpha E}{D\ell(I-\nu)} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} [(-I)^p - I] \left\langle \int_0^t E_{\alpha} \{[-\kappa(\lambda_{2n,m}^2 + \beta_p^2)]u^{\alpha}\} \times \right. \\ & \times \left[-\frac{4\kappa}{c^2} \pi A_0^{2n} Be_{2n}(\xi_0) \hat{f}(q_{2n,m}, \beta_p, t-u) \right] du \Big\rangle \times Be_{2n}(\xi, q_{2n,m}) ce_{2n}(\eta, q_{2n,m}) / \lambda_{2n,m}^2 \beta_p^2 C_{2n,m} \end{aligned} \tag{5.7}$$

Using Eqs (5.6) and (5.7) in Eq.(4.4), the resultant bending moments per unit width (M_{ij}) are as follows

$$\begin{aligned}
M_{\xi} = & -\frac{2\alpha E}{\ell(1-\nu)} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} [(-I)^p - I] \left\langle \int_0^t E_{\alpha} \{[-\kappa(\lambda_{2n,m}^2 + \beta_p^2)]u^{\alpha}\} \times \right. \\
& \times \left. -\frac{4\kappa}{c^2} \pi A_0^{2n} Be_{2n}(\xi_0) \hat{f}(q_{2n,m}, \beta_p, t-u) du \right\rangle \times \left\langle h^2 \{ [Be''_{2n}(\xi, q_{2n,m}) ce_{2n}(\eta, q_{2n,m}) + Be_{2n}(\xi, q_{2n,m}) \times \right. \\
& \times ce''_{2n}(\eta, q_{2n,m})] - \frac{c^2 h^2 (1-\nu)}{2} [\sinh 2\xi Be'_{2n}(\xi, q_{2n,m}) \times ce_{2n}(\eta, q_{2n,m}) + \\
& \left. + \sin 2\eta Be_{2n}(\xi, q_{2n,m}) ce'_{2n}(\eta, q_{2n,m})] \right\} - \lambda_{2n,m}^2 Be_{2n}(\xi, q_{2n,m}) ce_{2n}(\eta, q_{2n,m}) \Big/ \lambda_{2n,m}^2 \beta_p^2 C_{2n,m},
\end{aligned} \tag{5.8}$$

$$\begin{aligned}
M_{\eta} = & -\frac{2\alpha E}{\ell(1-\nu)} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} [(-I)^p - I] \left\langle \int_0^t E_{\alpha} \{[-\kappa(\lambda_{2n,m}^2 + \beta_p^2)]u^{\alpha}\} \times \right. \\
& \times \left. -\frac{4\kappa}{c^2} \pi A_0^{2n} Be_{2n}(\xi_0) \hat{f}(q_{2n,m}, \beta_p, t-u) du \right\rangle \times \left\langle h^2 \{ [v Be''_{2n}(\xi, q_{2n,m}) ce_{2n}(\eta, q_{2n,m}) + Be_{2n}(\xi, q_{2n,m}) \times \right. \\
& \times ce''_{2n}(\eta, q_{2n,m})] + \frac{c^2 h^2 (1-\nu)}{2} [\sinh 2\xi Be'_{2n}(\xi, q_{2n,m}) \times ce_{2n}(\eta, q_{2n,m}) + \\
& \left. - \sin 2\eta Be_{2n}(\xi, q_{2n,m}) ce'_{2n}(\eta, q_{2n,m})] \right\} - \lambda_{2n,m}^2 Be_{2n}(\xi, q_{2n,m}) ce_{2n}(\eta, q_{2n,m}) \Big/ \lambda_{2n,m}^2 \beta_p^2 C_{2n,m},
\end{aligned} \tag{5.9}$$

$$\begin{aligned}
M_{\xi\eta} = & -\frac{2h^2 \alpha E}{\ell} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} [(-I)^p - I] \left\langle \int_0^t E_{\alpha} \{[-\kappa(\lambda_{2n,m}^2 + \beta_p^2)]u^{\alpha}\} \times \right. \\
& \times \left. -\frac{4\kappa}{c^2} \pi A_0^{2n} Be_{2n}(\xi_0) \hat{f}(q_{2n,m}, \beta_p, t-u) du \right\rangle \times \\
& \times \left\langle \sin 2\eta Be_{2n}(\xi, q_{2n,m}) ce_{2n}(\eta, q_{2n,m}) + \sinh 2\xi Be'_{2n}(\xi, q_{2n,m}) \times \right. \\
& \times \left. ce'_{2n}(\eta, q_{2n,m}) - \frac{2}{c^2 h^2} Be'_{2n}(\xi, q_{2n,m}) ce'_{2n}(\eta, q_{2n,m}) \right\rangle / \lambda_{2n,m}^2 \beta_p^2 C_{2n,m}.
\end{aligned} \tag{5.10}$$

MATHEMATICA software was used for the graphic analysis of the thermal stresses which are calculated easily by using Eqs (5.6) to Eq.(5.10) in Eq.(4.7).

6. Results and discussion

Dimensions. For the sake of convenience, we choose the physical parameter as [23] $\bar{\xi}_i = 1.5m$, $\bar{\xi}_o = 0.75m$, $\bar{\ell} = 0.05 m$, $c = e = 0.5m$ and reference temperature T_0 as $150^{\circ}C$. The $q_{2n,m} = 1.331, 1.572, 2.628, 4.242, 6.235, 6.621, 8.728, 9.642, 10.564, 12.455, 13.846, 15.235, 17.731, 19.452, 20.323, 23.235, 24.553, 24.458, 25.231, 27.568$ are the positive and real roots of the $Be_{2n}(\xi_0, q_{2n,m}) = 0$.

7. Numerical results, discussion and remarks

The numerical calculation has been carried out for a copper (pure) elliptical disk with the following material properties: thermal diffusivity $\kappa = 112.34 \times 10^{-6} m^2 s^{-1}$, thermal conductivity $\lambda = 386 (W/mK)$, density

$\rho = 2954 \text{ kg/m}^3$, Poisson's ratio $\nu = 0.35$, thermal expansion coefficient $\alpha = 16.5 \times 10^{-6} / ^\circ\text{C}$ and modulus of elasticity $E = 70 \text{ GPa}$.

The graphs are plotted for the fractional-order parameter $\alpha = 0.5; 1; 1.5$ and 2 depicting different conductivity and fixed time $t = 0.5$. Figures 2-6 depict the distributions of temperature, radial stress and angular stress along the radial direction for various values of the fractional-order parameter α . The numerical calculation has been carried out in MATHEMATICA programming.

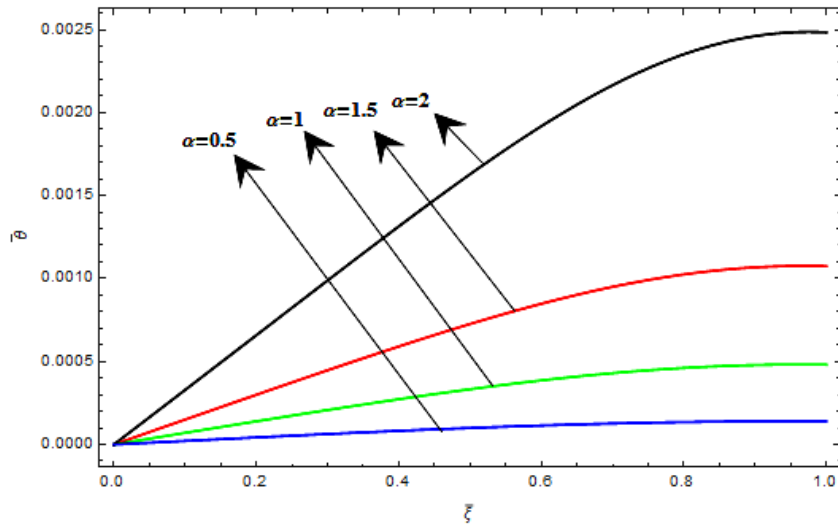


Fig.2. Temperature distribution function.

Figure 2 represents the temperature distributions along the $\bar{\xi}$ - direction for different α . For the cases $\alpha = 0.5; 1; 1.5$ and 2 , the values of the temperature show an increase with respect to $\bar{\xi}$ and it attains the highest value according to the value of α . It is clear that increasing the values of the fractional order parameter causes increasing the value of the temperature. Due to the sectional heat supply at $\bar{\xi} = 1$ of the plate, the effect of temperature at the outer edge is observed.

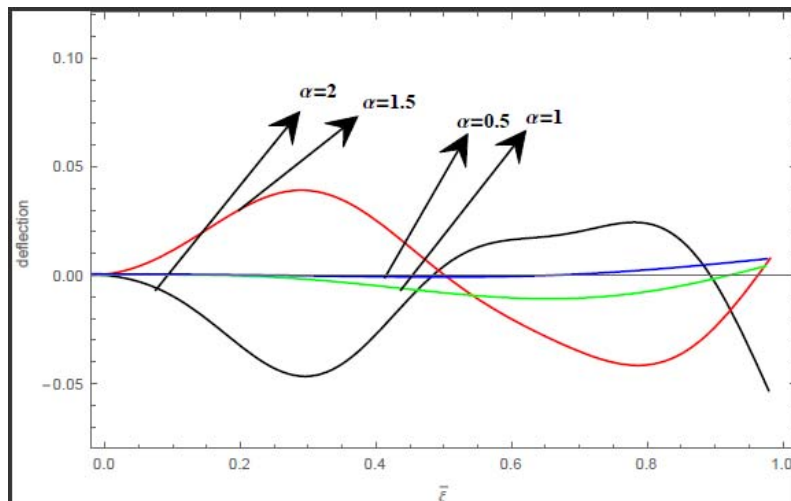


Fig.3. Thermal deflection along $\bar{\xi}$ - direction for different α .

The thermal deflection along the $\bar{\xi}$ - direction for different α is sinusoidal in nature. The figure shows that thermal deflection is zero at the inner boundary which satisfies the boundary condition $\bar{w} = 0$, at $\bar{\xi} = \bar{\xi}_i$, then decreases gradually along the radial direction for various values of α

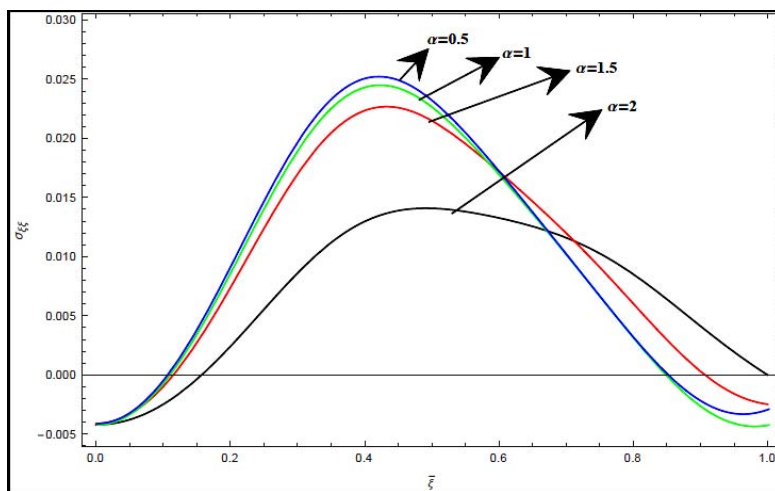


Fig.4. Radial stresses function along $\bar{\xi}$ - direction for the different α .

Figure 4 represents the radial stress distributions along along the $\bar{\xi}$ - direction for different α . The nature of the graph is sinusoidal and is highest at the middle core of the disc and represents the maximum amplitude of the wave propagation for the different values of α .

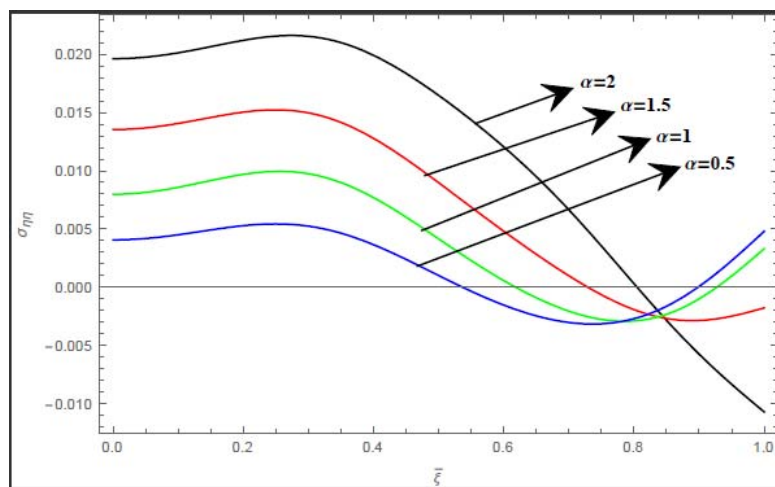


Fig.5. Tangential stresses along $\bar{\xi}$. - direction for different α .

Figure 5 illustrates tangential stresses along the $\bar{\xi}$. - direction for different α . It can be seen that the stress increases gradually with the increase in the radius and attains certain maxima and starts decreasing in outward direction.

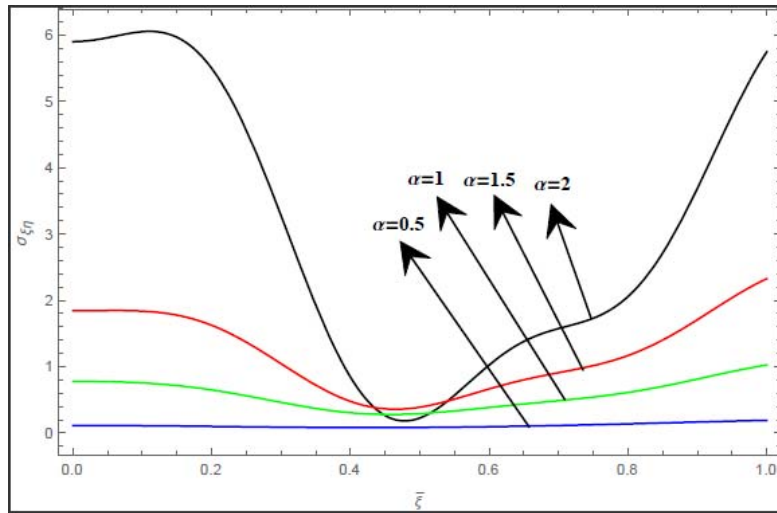


Fig.6. Shear stresses along $\bar{\xi}$. - direction for different α .

Figure 6 show shear stresses along the $\bar{\xi}$. - direction for different α . It can be seen that the stress decreases gradually with the increase in the radius and attains certain minima and starts increasing in outward direction.

Conclusion

This article is based on the uncoupled theoretical treatment of the time fractional-order heat conduction equation by neglecting the inertia term in the equation of motion. The integral transform methods are used to determine temperature distribution and the deflection in the form of ordinary and modified Mathieu functions. Also, the effect on bending stresses was investigated in the context fractional heat. It is observed that with an increase in the value of the fractional parameter corresponding speed of the waves increases, which shows that the fractional order parameter and time have a significant effect on the temperature distribution and the thermal deflection. Thus the analytical technique proposed here predicts a retarded response to physical stimuli and is widely applicable.

Nomenclature

- $2a_i$ – inner diameter of semi-major axis
- $2a_o$ – outer diameter of semi-major axis
- $2b_i$ – inner diameter of semi-minor axis
- $2b_o$ – outer diameter of semi-minor axis
- C – calorific capacity
- $Ce_{2n}(\xi, q)$ – modified Mathieu function of first kind of order n
- $ce_{2n}(\eta, q)$ – ordinary Mathieu function of first kind of order n
- $2c$ – denotes the distance between their common foci
- D – flexural stiffness of the disk
- E – Young's modulus
- $E_\alpha(z)$ – Mittag-Leffler function
- M_{ij} – resultant bending moments per unit width
- $M_\theta(\xi, \eta, t)$ – thermally induced resultant force
- N_{ij} – resultant forces

- $N_\theta(\xi, \eta, t)$ – thermally induced resultant moment
 $q_{2n,m}$ – root of the transcendental equation
 α – coefficient of linear thermal expansion
 $\theta(\xi, \eta, z, t)$ – temperature function
 (ξ, η, z) – space variables
 λ – thermal conductivity of the material
 ℓ – thickness of disk
 κ – thermal diffusivity of the material, $\kappa = \lambda / \rho C$
 ξ_i – inner elliptical boundary, $\xi_i = \tanh^{-1}(b_i / a_i)$
 ξ_o – outer elliptical boundary, $\xi_o = \tanh^{-1}(b_o / a_o)$
 ρ – density
 ν – coefficient of Poisson
 $\sigma_{\xi\xi}$ – radial stresses function
 $\sigma_{\eta\eta}$ – tangential stresses
 $\sigma_{\xi\eta}$ – shear stresses
 $\omega(\xi, \eta, t)$ – normal deflection

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