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Brief note

VIBRATIONAL ANALYSIS OF DAMPED ORTHOTROPIC RECTANGULAR PLATES WITH SIMPLE SUPPORT ON ELASTIC WINKLER FOUNDATION SUBJECTED TO MOVING LOADS

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The model of a damped orthotropic rectangular plate resting on a Winkler foundation with a simple support has the fourth order differential equation governing it, which is reduced to a second order coupled differential equation by separating the variables. The coupled differential equation was solved using numerical schemes .The classical condition used as an illustrative example is the simple support condition. It is observed that damping plays a very significant role in the vibration of solid structures, as it has been shown that the deflection profile depends greatly on the damping ratio. The deflection profile also proves to be more stable in the presence of foundation coupled with viscous damping. The results obtained were discussed and graphically presented.

Key words: Winkler foundation, orthotropic rectangular plates, viscous damping, and dynamic moving loads.

1. Background

Vibration of rectangular plates is an interesting subject because of its wide applications in structural engineering and transport engineering. Structures such as railway bridges, highway bridges, cranes, road pavements etc., can actually be modeled as rectangular plates. Much research has been done on rectangular plates. In Zimmermann, *et al*. [1] the differential equation relating to the breaking of railway bridges was developed and well discussed. Some other research works focused more on vibration of solids and structures under moving loads, such work includes Dobyns [2] who analyzed a simply supported orthotropic plate subjected to static and dynamic loads, which included the numerical solution of a plate subjected to blast loads that were modeled as a triangular function, an exponential function and a stepped triangular function. Fryba [3], found that the theoretical considerations are applicable in calculations relating to dynamic stresses in railway and highway bridges, suspension bridges, rails, sleepers, cranes etc. In mechanics moving loads are defined as loads that vary in both time and space. In Gbadeyan and Dada [4] the dynamic response of plates on an elastic foundation to distributed moving loads was investigated and it was reported that the natural frequency of rectangular plates traversed by moving concentrated forces is greater than that of plates subjected to moving concentrated masses and that the presence of foundation modulus reduces the deflection of the plate.

In most of the works, the type of plates considered are such which found applications in the modeling of the dynamic response of rigid concrete pavements. In Alisjarbana and Wangsadinata [5], an orthotropic rectangular plate was used to model the dynamics of a rigid roadway pavement under a dynamic load, the method used was the modified Bolotin Method. The dynamic moving traffic load is expressed as a concentrated load of harmonically varying magnitude, moving straight along the plate with a constant

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velocity, it was found that this dynamic load approach may lead to more economic solutions as compared to those obtained from the conventional static load approach.

Viscous damping is the dissipation of energy and the consequence reduction or decay of motion. To understand the control and mechanical response of vibrating structures, viscous damping should be properly definedd. Most of the early works neglected damping, but recently, interesting studies and results have emerged on the effects of viscous damping on the vibration of rectangular plates on an elastic foundation. Some of such works include Gbadeyan, *et al.* [6]. It was found that the deflection profile of the plate depends on the magnitude of the damping coefficient. Alisjahbana and Wangsadinata [7] studied the behavior of orthotropic damped plates with different stiffener configurations subjected to a stepped triangular blast loading. In Idowu, *et al.* [8], the effect of viscous damping on the dynamic response of isotropic rectangular plates on Pasternak foundations are studied and it was shown that viscous damping in the presence of a Pasternak foundation actually reduces the build-up of amplitude, thereby reducing the possibility of resonance. In this work, we pay attention to the vibration of the dynamic behavior of a damped orthotropic rectangular plate resting on a Winkler foundation subjected to dynamic loads.

2. Problem formulation

The governing equation of the problem is given as

$$\left[\alpha_1 \frac{\partial^4 w}{\partial x^4} + 2\alpha_2 \frac{\partial^4 w}{\partial x^2 y^2} + \alpha_3 \frac{\partial^4 w}{\partial y^4}\right] + M\frac{\partial^2 w}{\partial t^2} + 2M\gamma\frac{\partial w}{\partial t} + Kw = p(x,y,t) \tag{2.1}$$

where $$\alpha_1, \alpha_2, \alpha_3 = \frac{Eh^3}{12(1-v)},$$

$w = w(x, y, t)$ is the deflection of the plate.
t = time in seconds.
γ =viscous damping coefficient.
α_1 = flexural rigidity in the x direction.
α_2 = effective torsional rigidity.
α_3 = flexural rigidity in the y direction.
v = Poisson's ratio
E= Young's modulus
M= mass density per unit area
h= thickness of the plate
$P(x, y, t)$= the applied load, which is

$$P(x,y,t) = \frac{1}{r}\left(mg - m\frac{d^2W}{dt^2}\right)\left[H\left(x - vt + \frac{r}{2}\right) - H\left(x - vt - \frac{r}{2}\right)\right]\partial(y - y_1) \tag{2.2}$$

r= length of the load.
H(x)= Heaviside step function
$\partial(x)$ = Dirac delta function
g= acceleration due to gravity.
v= velocity.
K= foundation stiffness.

The above governing partial differential Eq.(2.1) was developed under the following assumptions:

- the small strain in the system is still governed by Hook's law,
-the plate is resting on an elastic foundation.
-the load is taken to be a distributed time load,
-there is no deformation in the middle of the plate, i.e the plate remains the same before and after bending.

3. Method of solution

The governing equation of the problem is solved using separation of variable in series form. We assume the following;
Let

$$W(x, y, t)= \sum_{n=1}^{N}\sum_{m=1}^{M} A_{mn}(t) W_n(x) W_m(y) \tag{3.1}$$

where $n=1, 2, 3,\ldots,N$ and $m=1, 2, 3,\ldots M$,
M and N are fixed positive integers and $A_{mn}(t)$ is a function of time.

If we substitute Eq.(3.1) into Eq.(2.1), we get;

$$\begin{aligned}&\left[\propto_1 \sum_{n=1}^{N}\sum_{m=1}^{M} A_{mn}(t) w_n^{iv}(x) w_m(y) + 2\propto_2 \sum_{n=1}^{N}\sum_{m=1}^{M} A_{mn}(t) w_n^{ii}(x) w_m^{ii}(y) + \right.\\ &\left. +\propto_3 \sum_{n=1}^{N}\sum_{m=1}^{M} A_{mn}(t) w_n(x) w_m^{iv}(y)\right] + M\sum_{n=1}^{N}\sum_{m=1}^{M} \ddot{A}_{mn}(t) w_n(x) w_m(y) + \\ &+2M\gamma\sum_{n=1}^{N}\sum_{m=1}^{M} \dot{A}_{mn}(t) w_n(x) w_m(y) + K\sum_{n=1}^{N}\sum_{m=1}^{M} A_{mn}(t) w_n(x) w_m(y) = p(x,y,t).\end{aligned} \tag{3.2}$$

When we substitute $p(x, y, t)$, from Eq.(2.2).

$$\begin{aligned}&\sum_{n=1}^{N}\sum_{m=1}^{M}\Big[A_{mn}(t)\Big\{\propto_1 w_n^{iv}(x) w_m(y) + 2\propto_2 w_n^{ii}(x) w_m^{ii}(y) + \propto_3 w_n(x) w_m^{iv}(y)\Big\} + \\ &+M\ddot{A}_{mn}(t) w_n(x) w_m(y) + 2M\gamma \dot{A}_{mn}(t) w_n(x) w_m(y) + KA_{mn}(t) w_n(x) w_m(y)\Big] = \\ &= \frac{1}{r}\left(mg - m\frac{d^2W}{dt^2}\right)\left[H\left(x - vt + \frac{r}{2}\right) - H\left(x - vt - \frac{r}{2}\right)\right]\partial(y - y_1).\end{aligned} \tag{3.3}$$

The equation governing the undamped free vibration of an orthotropic rectangular plate is as follows (Idowu and Aguda [9])

$$\left[\propto_1 \frac{\partial^4 \mathrm{w}}{\partial \mathrm{x}^4} + 2\propto_2 \frac{\partial^4 \mathrm{w}}{\partial \mathrm{x}^2 \mathrm{y}^2} + \propto_3 \frac{\partial^4 \mathrm{w}}{\partial \mathrm{y}^4}\right] + Kw + \omega^2 Mw = 0 . \tag{3.4}$$

By substituting Eq.(3.1) into Eq.(3.4), and taking $\mu_{mn} = -\omega^2 M$, we have that

$$\begin{gathered}\propto_1 w_n^{iv}(x) w_m(y)+2 \propto_2 w_n^{ii}(x) w_m^{ii}(y)+\propto_3 w_n(x) w_m^{iv}(y)+ \\ +\mathrm{K} w_n(x) w_m(y)=\mu_{mn} w_n(x) w_m(y).\end{gathered} \tag{3.5}$$

On substituting Eq.(3.5) into Eq.(3.3), and putting Eq.(3.1) in the RHS of Eq.(3.3), we have the simplified equation governing the vibration problem of a damped orthotropic rectangular plate resting on a Winkler foundation subjected to dynamic loading.

$$\begin{gathered}\sum_{n=1}^{N}\sum_{m=1}^{M}\left[\mu_{mn} A_{mn}(t) M w_n(x) w_m(y)+M\ddot{A}_{mn}(t) w_n(x) w_m(y)+2M\gamma\dot{A}_{mn}(t) w_n(x) w_m(y)\right]= \\ \frac{1}{r}\left(mg-m\sum_{n=1}^{N}\sum_{m=1}^{M}\left\{\ddot{A}_{mn}(t) w_n(x) w_m(y)+2v\dot{A}_{mn}(t) w_n^{i}(x) w_m(y)+v^2 A_{mn}(t) w_n^{ii}(x) w_m(y)\right\}\right)\times \\ \times\left[H\left(x-vt+\frac{r}{2}\right)-H\left(x-vt-\frac{r}{2}\right)\right]\partial(y-y_1).\end{gathered} \tag{3.6}$$

Multiply both sides of Eq.(3.6) by $w_i(x)w_j(y)$ and integrate along the edges of the rectangular plate of dimension (*a x b*). And further apply the orthogonality of $w_n(x)$ and $w_m(y)$ with the following relation between the Dirac delta function and the Heaviside unit function.

$$\int_{x_1}^{x_2}\partial(x-x_o)dx=H(x-x_o)\,.$$

We obtain

$$\begin{gathered}\ddot{A}_{mn}(t)+2\gamma\dot{A}_{mn}(t)+\mu_{mn}A_{mn}(t)=\frac{1}{\varphi r M}\left\{mg w_j(y_1)\int_{vt-\frac{r}{2}}^{vt+\frac{r}{2}} w_i(x)dx+\right. \\ -m\sum_{n=1}^{N}\sum_{m=1}^{M}\left[\ddot{A}_{mn}(t) w_m(y_1) w_n(y_1)\int_{vt-\frac{r}{2}}^{vt+\frac{r}{2}} w_n(x) w_i(x)dx+\right. \\ \left.\left.+2v\dot{A}_{mn}(t) w_m(y_1) w_j(y_1)\int_{vt-\frac{r}{2}}^{vt+\frac{r}{2}} w_n^{i}(x) w_i(x)dx+v^2 A_{mn}(t) w_m(y_1) w_j(y_1)\int_{vt-\frac{r}{2}}^{vt+\frac{r}{2}} w_n^{ii} w_i(x)dx\right]\right\}\end{gathered} \tag{3.7}$$

where φ is an arbitrary constant.

Equation (3.7) is the generalized ordinary coupled differential equation to be solved for some specific boundary conditions.

4. Simply supported rectangular plate (as an illustrative example)

We have different kinds of classical edge supports, but for the sake of this study we limit our consideration to simply supported edges alone.

The boundary condition for simply supported rectangular plates is given as;

$$w(0,y,t)=w(a,y,t)=w_{xx}(0,y,t)=w_{xx}(a,y,t)=0\,,$$
$$w(x,0,t)=w(x,b,t)=w_{yy}(x,0,t)=w_{yy}(x,,t)=0\,,$$

with the initial condition

$$w(x,y,0)=w_t(x,y,0)=0\,. \tag{4.1}$$

The normalized deflection curve for the simply supported boundary condition for a rectangular plate has been obtained in Gbadeyan and Dada [4]

$$w_n(x)w_m(y)=\frac{2}{\sqrt{ab}}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b} \tag{4.2}$$

where $n = 1, 2, 3,\ldots$, and $m = 1, 2, 3,\ldots$.

To obtain the eigen values, we substitute (4.2) into (3.5) to have

$$\mu_{mn}=\left[\propto_1\frac{n^4\pi^4}{a^4}+2\propto_2\frac{n^2m^2\pi^4}{a^2b^2}+\propto_3\frac{m^4\pi^4}{b^4}\right]+K\,. \tag{4.3}$$

The exact governing equation for a simply supported damped orthotropic rectangular plate resting on a Winkler foundation can be obtained by putting Eq.(4.2) into Eq.(3.7).

$$\begin{aligned}
&\ddot{A}_{mn}(t)+2\gamma\dot{A}_{mn}(t)+\mu_{mn}A_{mn}(t)=\frac{1}{\varphi rM}\left[\frac{4mga}{\pi i\sqrt{ab}}\sin\frac{i\pi vt}{a}\sin\frac{i\pi y_1}{b}+\right.\\
&-m\sum_{n=1}^{N}\sum_{m=1}^{M}\left\{\frac{4}{ab}\sin\frac{m\pi y_1}{b}\sin\frac{i\pi y_1}{b}\ddot{A}_{mn}(t)\left(\frac{a}{\pi d_1}\cos\frac{d_1\pi rt}{a}\sin\frac{d_1\pi r}{2a}-\frac{a}{\pi p}\cos\frac{p\pi vt}{a}\sin\frac{p\pi r}{2a}\right)+\right.\\
&+\frac{8n\pi v}{a^2b}\dot{A}_{mn}(t)\sin\frac{m\pi y_1}{b}\sin\frac{\pi y_1}{b}\Big(\frac{a}{p\pi}\sin\frac{p\pi vt}{a}\sin\frac{p\pi r}{2a}-\frac{a}{d_1\pi}\sin\frac{d_1\pi vt}{a}\sin\frac{d_1\pi r}{2a}+\\
&\left.\left.-\frac{4\pi^2n^2}{a^3b}v^2A_{mn}(t)\sin\frac{m\pi y_1}{b}\sin\frac{j\pi y_1}{b}\left(\frac{a}{\pi d_1}\cos\frac{d_1\pi vt}{a}\sin\frac{d_1\pi r}{2a}-\frac{a}{\pi p}\cos\frac{p\pi vt}{a}\sin\frac{p\pi r}{2a}\right)\right]\right.
\end{aligned} \tag{4.4}$$

where $i\neq n$

$$\begin{aligned}
&\ddot{A}_{mn}(t)+2\gamma\dot{A}_{mn}(t)+\mu_{mn}A_{mn}(t)=\frac{1}{\varphi rM}\left[\frac{4mga}{\pi n\sqrt{ab}}\sin\frac{i\pi y_1}{b}\sin\frac{ir\pi}{2a}\sin\frac{i\pi vt}{a}+\right.\\
&-m\sum_{n=1}^{N}\sum_{m=1}^{M}\left\{\frac{4}{ab}\sin\frac{m\pi y_1}{b}\sin\frac{i\pi y_1}{b}\ddot{A}_{mn}(t)\left(\frac{r}{2}-\frac{a}{2n\pi}\sin\frac{n\pi r}{a}\cos\frac{2nvt\pi}{a}\right)+\right.\\
&+\frac{8n\pi v}{a^2b}\dot{A}_{mn}(t)\sin\frac{m\pi y_1}{b}\sin\frac{j\pi y_1}{b}\left(\frac{a}{2n\pi}\sin\frac{2nvt\pi}{a}\sin\frac{n\pi r}{a}\right)+\\
&\left.-\frac{4\pi^2n^2v^2}{a^3b}A_{mn}(t)\sin\frac{m\pi y_1}{b}\sin\frac{j\pi y_1}{b}\left(\frac{r}{2}-\frac{a}{2n\pi}\sin\frac{n\pi r}{a}\cos\frac{2nvt\pi}{a}\right)\right\}.
\end{aligned} \tag{4.5}$$

5. Results and discussion

Equations (4.4) and (4.5) are second order coupled differential equations. Equation (4.4) is for $i \neq n$, and Eq.(4.5) is for $n = i$. The coupled differential equations are solved using the finite difference method, i.e., the central difference method. The resulting tridiagonal matrix is of the form;

$$\left[2 - hP\left(t_i\right)\right]T_{i-1} + \left[-4 + 2h^2 Q\left(t_i\right)\right]T_i + \left[2 + hP\left(t_i\right)\right]T_{i+1} = 2h^2 R\left(t_i\right)$$

where h is chosen appropriately, and $i = 1, 2, 3, \ldots n$.

The resulting tridiagonal matrices were solved using MATLAB. The following values are assumed for the corresponding variables: $a = 10, b = 5$, $v = 12m/s$, $24m/s$ and $36m/s$, $K=0$, 20, 80. The values used for the flexural rigidity in the x-direction (α_1), the effective torsonal rigidity (α_2) and the flexural rigidity in the y-direction (α_3), is that of Veneer, given as 0.297, 0.21 and 0.69, respectively. The values of the damping ratios (γ) are assumed to be 0, 100, 150, respectively.

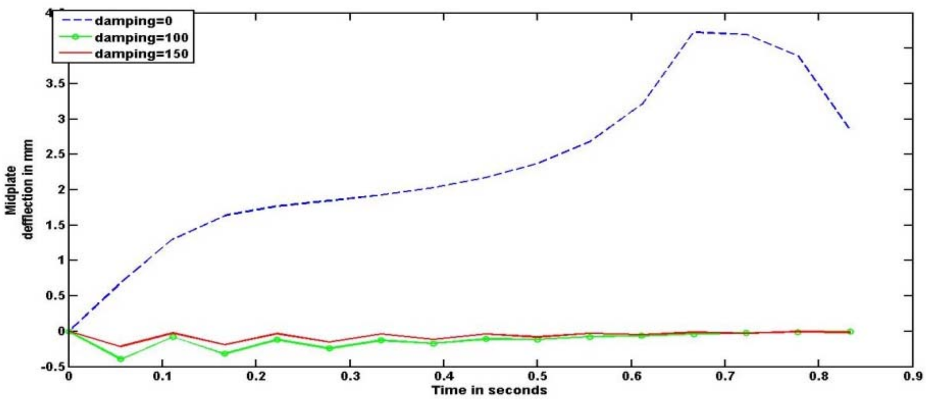

Fig.1. Deflection versus time at various damping ratios.

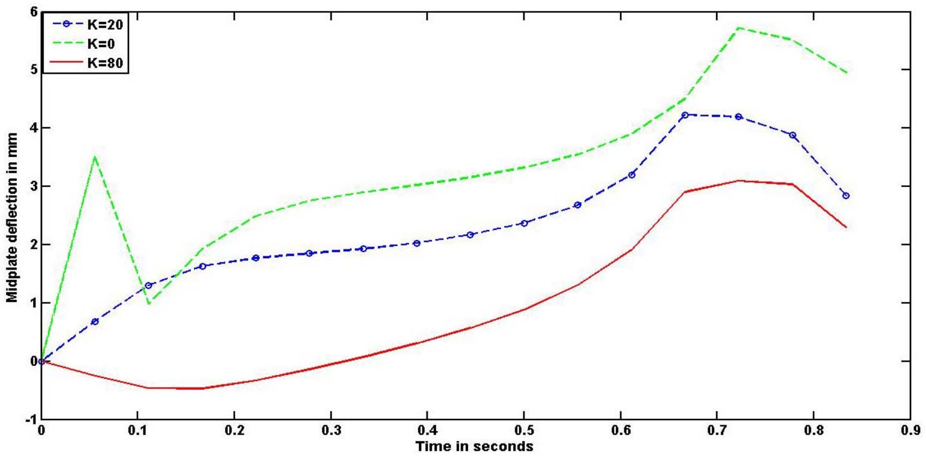

Fig.2. Mid-plate deflection versus time at various values of K.

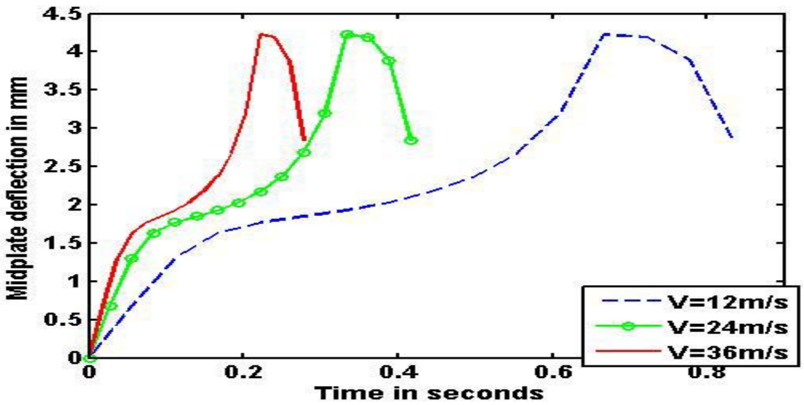

Fig.3. Deflection versus time at various velocities.

In Fig.1, we see that the maximum deflection is much higher when the damping ratio is (γ)=*0*, and as the damping ratio increases, the deflection reduces and the vibration also stabilizes with time.

In Fig.2, we can see that when the foundation modulus is reduced to zero, the mid-plate deflection increased and when the foundation modulus K, is increased, the maximum deflection is reduced.

In Fig.3, various velocities are presented. We see that at a high velocity, the maximum deflection is attained at a shorter time.

6. Conclusion

Damping plays a very significant role in the vibration of solid structures. It has been shown that the deflection profile depends greatly on the damping ratio. The deflection profile also proves to be more stable in the presence of a foundation coupled with viscous damping. These results further show that very high speed can be detrimental to solid structures, especially highway bridges and other structures subjected to dynamic loads.

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Nomenclature

E – Young's modulus
g – acceleration due to gravity
H(x) – Heaviside step function
h – thickness of plate
K – foundation stiffness
M – mass density per unit area
$P(x, y, t)$ – the applied load
r – length of the load
v – velocity
α_1 – flexural rigidity in the x direction
α_2 – effective torsional rigidity
α_3 – flexural rigidity in the y direction
γ – viscous damping coefficient

v – Poisson's ratio
$\partial(x)$ – Dirac delta function

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