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STABILITY ANALYSIS AND INTERNAL HEATING EFFECT ON OSCILLATORY CONVECTION IN A VISCOELASTIC FLUID SATURATED POROUS MEDIUM UNDER GRAVITY MODULATION

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In this paper, we investigate the combined effect of internal heating and time periodic gravity modulation in a viscoelastic fluid saturated porous medium by reducing the problem into a complex non-autonomous Ginzgburg-Landau equation. Weak nonlinear stability analysis has been performed by using power series expansion in terms of the amplitude of gravity modulation, which is assumed to be small. The Nusselt number is obtained in terms of the amplitude for oscillatory mode of convection. The influence of viscoelastic parameters on heat transfer has been discussed. Gravity modulation is found to have a destabilizing effect at low frequencies and a stabilizing effect at high frequencies. Finally, it is found that overstability advances the onset of convection, more with internal heating. The conditions for which the complex Ginzgburg-Landau equation undergoes Hopf bifurcation and the amplitude equation undergoes supercritical pitchfork bifurcation are studied.

Key words: Non-linear stability analysis, complex Ginzburg-Landau equation, gravity modulation, internal heating, bifurcation.

1. Introduction

The studies of thermal instability in porous media play a very significant role in many areas such as in petroleum industry, chemical engineering and geophysics, etc. A detailed account of the work on thermal instability in porous media has been given in the most excellent books due to Ingham and Pop [1], Nield and Bejan [2] and Vafai [3].

External regulation of convection is important either for enhancing or diminishing heat transfer in a physical system. This type of regulation (e.g., thermal, gravity, rotation and magnetic field modulation) is used effectively to control convective phenomenon. This paper is concerned with the thermal instability under gravity modulation, first studied by Gresho and Sani [4]. In this model they observed that the gravity modulation enables the system to get control on its instability either by suitably adjusting the values of frequency or the amplitude of modulation. The related studies of gravity modulation are given in [5]-[16]. The nature of non-Newtonian fluids is quite different from Newtonian fluids due to their properties such as like shear stress and shear strain. The basic idea for considering these fluids is due to their oscillatory nature. For example, industrial fluids are basically non-Newtonian. In particular, viscoelastic fluids have been important. A proper understanding of convective motion and its behaviour is necessary for controlling many

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processes such as geothermal reservoirs, filtration, and enhanced oil recovery. Green [17] was the first to study the oscillatory convection in a viscoelastic fluid layer, Vest and Arpaci [18] reported the occurrence of overstability for typical Rayleigh-Benard convection of a horizontal homogeneous Maxwellian fluid layer heated from below.

Bhatia and Steiner [19] studied thermal instability in a rotating viscoelastic fluid layer. Some studies on thermal convection of viscoelastic fluids in porous media are available in the literature. Kim *et al*. [20] investigated the critical conditions of stationary and oscillatory instabilities in a horizontal porous layer saturated with a viscoelastic fluid based on the modified Darcy-Oldroyd model. Malashetty and Kulkarni [21] used a two-field model in the energy equation to investigate the thermal non-equilibrium problem of a Maxwell fluid in a porous medium, Wang and Tan [22] performed linear and nonlinear analyses on the double-diffusive convection with the Soret effect in a Maxwell fluid-saturated porous medium.

Kumar and Bhadauria [23]-[25] studied thermal instability in a rotating porous layer saturated with a viscoelastic fluid and with double diffusive gradients. An oscillatory mode of convection is investigated by Bhadauria and Kiran [26]-[29]. These authors derived complex Ginzburg-Landau equations for obtaining finite amplitude convection under modulation, and the conditions for which the oscillatory convective flow is confirmed.

There are situations of great practical importance where the porous material offers its own source of heat. This gives a different way in which a convective flow can be set up through the local heat generation within the porous media. Such a situation can occur through radioactive decay or through, in the present perspective, a relatively weak exothermic reaction which can take place within the porous material.

To the best of our knowledge, there is no work related to the oscillatory convection under modulation with an internal heat source in the literature. It is of great practical importance, when the porous material offers its own source of heat and the internal heating can modify the convective flow through local heat generation within the porous media. To be specific, internal heat is one of the main sources of energy for celestial bodies caused by nuclear fusion and decaying of radioactive materials which keep the celestial objects warm and active. It is due to the internal heating of the earth that there exists a thermal gradient between the interior and exterior of the earth crust, saturated by multi-components fluids, which helps convective flow, thereby transferring the thermal energy towards the surface of the earth. So the role of internal heat generation becomes very important in several applications that include geophysics, reactor safety analyses, fire and combustion studies, and in storage of radioactive materials. Some of the studies on thermal instability under internal heat generation are due to [30-40]. The present study aims at finding the combined effect of gravity modulation and internal heating on oscillatory convection in a porous material saturated with a viscoelastic fluid.

2. Mathematical formulation

An infinitely extended horizontal porous medium saturated with a viscoelastic fluid heated from below and cooled from above is considered. The Darcy model has been employed in the momentum equation. Further, an internal heat source term has been included in the energy equation. The physical configuration of the model is reported in Fig.1. The corresponding mathematical model of the problem under the Boussinesq approximation [26] is as follows

$$\begin{gathered}
\nabla \cdot \boldsymbol{q} = 0, \\
\left(\bar{\lambda}_1 \frac{\partial}{\partial t} + 1\right)\left(-\nabla p + \rho \boldsymbol{g}\right) - \frac{\mu}{K}\left(\bar{\lambda}_2 \frac{\partial}{\partial t} + 1\right)\boldsymbol{q} = 0, \\
\frac{\partial T}{\partial t} + \left(\boldsymbol{q} \cdot \nabla\right)\mathrm{T} = \kappa_T \nabla^2 T + Q\left(T - T_0\right), \\
\rho = \rho_0\left\{1 - \alpha_T\left(T - T_0\right)\right\}
\end{gathered} \tag{2.1}$$

where the physical variables have their usual meanings as given in the nomenclature. The externally imposed gravitational field and the thermal boundary conditions are given by

$$\boldsymbol{g} = g_0\left\{1+\chi^2\delta\cos(\Omega t)\right\}\hat{k}, \tag{2.2}$$

$$T = \begin{cases} T_0+\Delta T, & \text{at} \quad z=0, \\ \\ T_0, & \text{at} \quad z=d \end{cases} \tag{2.3}$$

where g_0 is the mean gravity and $\hat{k}$ is the unit vector along the positive z axis.

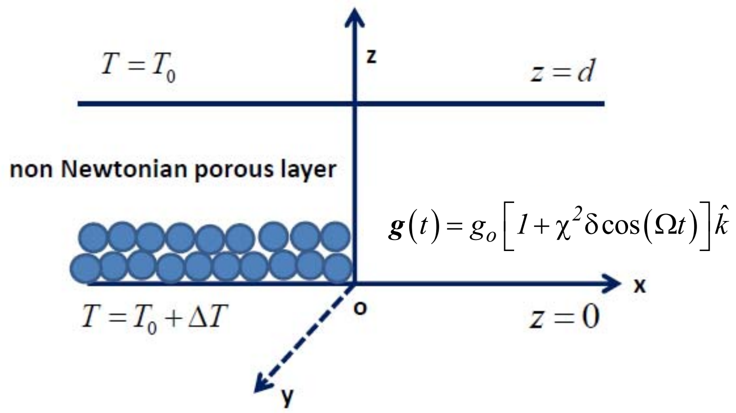

Fig.1. Physical configuration of the problem.

3. Basic state

At this state the velocity, pressure, temperature and density profiles are given by

$$\boldsymbol{q}_b = 0, \qquad p = p_b(z), \qquad T = T_b(z), \qquad \rho=\rho_b(z). \tag{3.1}$$

The following relations follow is direct from Eq.(3.1) and Eq. (2.1)

$$\frac{\partial p_b}{\partial z} = -\rho_b \boldsymbol{g}\ , \tag{3.2}$$

$$\kappa_T\frac{d^2(T_b-T_0)}{dz^2}+Q(T_b-T_0)=0, \tag{3.3}$$

$$\rho_b = \rho_0\left\{1-\alpha_T(T_b-T_0)\right\}. \tag{3.4}$$

The solution of Eq.(3.3), subject to the boundary conditions (2.3), is given by

$$T_b = T_0 + \Delta T \frac{\sin\left(\left(\sqrt{\frac{Q}{\kappa_T}}\right)\left(1-\frac{z}{d}\right)\right)}{\sin\left(\sqrt{\frac{Q}{\kappa_T}}\right)}. \tag{3.5}$$

Superimpose the finite amplitude perturbations on the basic state in the form

$$\boldsymbol{q} = q_b + q', \quad T = T_b + T', \quad p = p_b + p', \quad \rho = \rho_b + \rho' \tag{3.6}$$

where the primes represent the perturbation quantities. The dimensionless governing system as in [26] is given by

$$\left(\lambda_1 \frac{\partial}{\partial t} + 1\right) g_m R_{aD} \frac{\partial T}{\partial x} + \left(\lambda_2 \frac{\partial}{\partial t} + 1\right)\nabla^2 \psi = 0, \tag{3.7}$$

$$-\frac{\partial \psi}{\partial x}\frac{\partial T_b}{\partial z} + \left(\frac{\partial}{\partial t} - \nabla^2 - R_i\right) T = \frac{\partial(\psi, T)}{\partial(x,z)} \tag{3.8}$$

where $g_m = \left(1 + \chi^2 \delta \cos(\Omega t)\right)$, $R_{aD} = \frac{\alpha_T g_0 \Delta T K d}{\nu \kappa_T}$ is the thermal Rayleigh number, $R_i = \frac{Qd^2}{\kappa_T}$ is the internal Rayleigh number, $\nu = \frac{\mu}{\rho_0}$ is the kinematic viscosity. The above system will be solved by considering stress free and isothermal boundary conditions as given below

$$\psi = \frac{\partial^2 \psi}{\partial z^2} = T = 0 \quad \text{on} \quad z = 0, \quad z = 1. \tag{3.9}$$

The dimensionless basic temperature field $T_b(z)$ from Eq.(3.8) is given by

$$\frac{dT_b}{dz} = -\frac{\sqrt{R_i}\cos\left(\sqrt{R_i}(1-z)\right)}{\sin\left(\sqrt{R_i}\right)}. \tag{3.10}$$

Introducing a small perturbation parameter χ, a deviation from the critical state of onset of convection, the variables for a weak non-linear state is expanded in the power series in χ as [44, 45]

$$R_{aD} = R_0 + \chi^2 R_2 + \chi^4 R_4 + \cdots, \tag{3.11}$$

$$\psi = \chi\psi_1 + \chi^2\psi_2 + \chi^3\psi_3 + \ldots , \tag{3.12}$$

$$T = \chi T_1 + \chi^2 T_2 + \chi^3 T_3 + \ldots \tag{3.13}$$

where R_0 is the critical value of the Darcy-Rayleigh number at which the onset of convection takes place in the absence of gravity modulation.

4. Analysis of the periodic solutions

In order to study time periodic convective phenomenon according to [20, 26], the slow and fast time scale will be introduced by $\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + \chi^2 \frac{\partial}{\partial s}$. The above system (3.7) and (3.8) will be solved at every order of χ.

At the first order, we obtain the matrix operator (the reader may note that this is similar to the linear case)

$$\begin{bmatrix} \left(\lambda_2 \frac{\partial}{\partial \tau} + 1\right)\nabla^2 & R_0\left(\lambda_1 \frac{\partial}{\partial \tau} + 1\right)\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial x}\frac{\partial T_b}{\partial z} & \left(\frac{\partial}{\partial \tau} - \nabla^2 - R_i\right) \end{bmatrix} \begin{bmatrix} \psi_1 \\ T_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4.1}$$

The solution of the first order system with the boundary conditions (3.9) is assumed as:

$$\psi_1 = \left(A(s)e^{i\omega\tau} + \overline{A}(s)e^{-i\omega\tau}\right)\sin ax \ \sin \pi z, \tag{4.2}$$

$$T_1 = \left(B(s)e^{i\omega\tau} + \overline{B}(s)e^{-i\omega\tau}\right)\cos ax \ \sin \pi z. \tag{4.3}$$

The undetermined amplitudes are functions of slow time scale, and are related by the following expression

$$B(s) = -\frac{4\pi^2 a}{\left(c + i\omega - R_i\right)\left(4\pi^2 - R_i\right)} A(s) \tag{4.4}$$

where $c = a^2 + \pi^2$.

The values of the critical Darcy-Rayleigh number and the corresponding wave number of the system for a stationary mode of convection are as given below

$$R_0^{st} = \frac{c\left(c - R_i\right)\left(4\pi^2 - R_i\right)}{4\pi^2 a^2}, \tag{4.5}$$

$$a_c^2 = \sqrt{\pi^4 - \pi^2 R_i}. \tag{4.6}$$

For the system without internal-heating, that is, $R_i = 0$, we get

$$R_0 = \frac{c^2}{a^2}, \tag{4.7}$$

$$a_c = \pi, \tag{4.8}$$

which are same as the classical results obtained by Chandrasekhar [41]. Moreover, the critical Darcy-Rayleigh number and associated wave number of the system for the oscillatory mode of convection are obtained as

$$R_0^{osc} = \left(\frac{c\left(c - R_i\right) - \lambda_2 \omega^2}{a^2} \right) \frac{4\pi^2 - R_i}{4\pi^2}, \tag{4.9}$$

$$a_c^2 = \sqrt{\pi^4 - \pi^2 R_i + \frac{\pi^2}{\lambda_2}} \tag{4.10}$$

where ω is the oscillatory frequency, given as

$$\omega^2 = \frac{\left(\lambda_1 - \lambda_2\right)c + R_i\left(\lambda_2 - \lambda_1\right) - 1}{\lambda_1 \lambda_2}. \tag{4.11}$$

Since ω is real, and so, from relation (4.11), the necessary condition for oscillatory convection is obtained as

$$\lambda_1 > \lambda_2 + \frac{1 - R_i\left(\lambda_2 - \lambda_1\right)}{c}. \tag{4.12}$$

Now, at the second order, we have

$$\begin{bmatrix} \left(\lambda_2 \frac{\partial}{\partial \tau} + 1\right)\nabla^2 & R_0\left(\lambda_1 \frac{\partial}{\partial \tau} + 1\right)\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial x}\frac{\partial T_b}{\partial z} & \left(\frac{\partial}{\partial \tau} - \nabla^2 - R_i\right) \end{bmatrix} \begin{bmatrix} \psi_2 \\ T_2 \end{bmatrix} = \begin{bmatrix} R_{21} \\ R_{22} \end{bmatrix} \tag{4.13}$$

where

$$R_{21} = 0, \tag{4.14}$$

$$R_{22} = \frac{\partial \psi_1}{\partial x}\frac{\partial T_1}{\partial z} - \frac{\partial \psi_1}{\partial z}\frac{\partial T_1}{\partial x}. \tag{4.15}$$

The second order solution, subject to the boundary condition (3.9), is given by

$$\psi_2 = 0, \tag{4.16}$$

$$\left(\frac{\partial}{\partial \tau} - \nabla^2 - R_i\right) T_2 = R_{22}. \tag{4.17}$$

Keeping [20, 26] in mind, the second order temperature term written as

$$T_2 = \left\{ T_{20} + T_{22}\,\mathrm{e}^{2i\omega\tau} + \overline{T}_{22}\,\mathrm{e}^{-2i\omega\tau} \right\} \sin\left(2\pi z\right) \tag{4.18}$$

where T_{22} and T_{20} are temperature fields having the terms with the frequency 2ω and independent of fast time scale, respectively. The solutions of the second order problems are

$$T_{20} = \frac{\pi a}{8\pi^2 - 2R_i}\{A(s)\overline{B}(s) + \overline{A}(s)B(s)\}, \tag{4.19}$$

$$T_{22} = \frac{\pi a}{8\pi^2 + 4i\omega - 2R_i} A(s)B(s). \tag{4.20}$$

The horizontally averaged Nusselt number, $\mathrm{Nu}(s)$, for the oscillatory mode of convection is given by

$$\mathrm{Nu}(s) = 1 + \left[\chi^2 \left(\frac{\partial T_2}{\partial z} \right)_{z=0} \Bigg/ \left(\frac{\partial T_b}{\partial z} \right)_{z=0} \right]. \tag{4.21}$$

By using Eqs (3.10), (4.18)-(4.20), we can simplify Eq.(4.21) as

$$\begin{aligned} \mathrm{Nu}(s) = 1 + \Bigg[& \frac{4\left(c - R_i\right)\pi^2 a^2}{\left(\left(c-R_i\right)^2 + \omega^2\right)\left(8\pi^2 - 2R_i\right)} + \\ & \frac{2\pi^2 a^2}{\sqrt{\left(c-R_i\right)^2 + \omega^2}\sqrt{\left(8\pi^2 - 2R_i\right)^2 + 16\omega^2}} \Bigg] \left(\frac{4\pi^2}{4\pi^2 - R_i} \right) \frac{\tan\sqrt{R_i}}{\sqrt{R_i}} |A(s)|^2 . \end{aligned} \tag{4.22}$$

It is clear that gravity modulation is effective at third order and affects $\mathrm{Nu}(s)$ through $A(s)$ which is evaluated at third order.

At the third order, we have

$$\begin{bmatrix} \left(\lambda_2 \dfrac{\partial}{\partial\tau} + 1\right)\nabla^2 & \mathrm{R}_0 \left(\lambda_1 \dfrac{\partial}{\partial\tau} + 1\right)\dfrac{\partial}{\partial x} \\ -\dfrac{\partial}{\partial x}\dfrac{\partial T_b}{\partial z} & \left(\dfrac{\partial}{\partial\tau} - \nabla^2 - R_i\right) \end{bmatrix} \begin{bmatrix} \psi_3 \\ T_3 \end{bmatrix} = \begin{bmatrix} R_{31} \\ R_{32} \end{bmatrix}, \tag{4.23}$$

where

$$R_{31} = -\lambda_2 \frac{\partial}{\partial s}\left(\nabla^2\psi_1\right) - R_0\lambda_1 \frac{\partial}{\partial s}\frac{\partial T_1}{\partial x} - \left(R_2 + R_0\delta\cos\left(\Omega s\right)\right)\left(\lambda_1 \frac{\partial}{\partial\tau} + 1\right)\frac{\partial T_1}{\partial x}, \tag{4.24}$$

$$R_{32} = \frac{\partial\psi_1}{\partial x}\frac{\partial T_2}{\partial z} - \frac{\partial T_1}{\partial s}. \tag{4.25}$$

Using first and second order solutions, the expressions of R_{31} and R_{32} will be determined. Now, using the solvability condition for the existence of third order solution, we derive the complex Gingburg-Landau equation for finite amplitude convection.

$$\frac{dA(s)}{ds} - \gamma^{-1} F(s) A(s) + \gamma^{-1} k \left| A(s) \right|^2 A(s) = 0 \tag{4.26}$$

where

$$\gamma = \lambda_2 c - \frac{4R_2 \lambda_1 a^2 \pi^2}{\left(c + i\omega - R_i\right)\left(4\pi^2 - R_i\right)} + \frac{4R_2 a^2 \pi^2 \left(1 + i\omega\lambda_1\right)}{\left(c + i\omega - R_i\right)^2 \left(4\pi^2 - R_i\right)},$$

$$F(s) = \frac{4R_2 a^2 \pi^2 \left(1 + i\omega\lambda_1\right)\left(1 + \delta\cos(\Omega s)\right)}{\left(c + i\omega - R_i\right)\left(4\pi^2 - R_i\right)},$$

and

$$k = -\left(\frac{4\pi^2 a^4 R_2 \left(1 + \lambda_1 i\omega\right)\left(c - R_i\right)}{\left(c + i\omega - R_i\right)\left(\left(c - R_i\right)^2 + \omega^2\right)\left(8\pi^2 - 2R_i\right)} + \right.$$
$$\left. + \frac{4\pi^2 a^4 R_2 \left(\left(c - R_i\right)^2 + \omega^2 \lambda_1\right)}{\left(c + i\omega - R_i\right)\left(\left(c - R_i\right)^2 + \omega^2\right)\left(8\pi^2 - 2R_i + 4i\omega\right)} \right) \frac{4\pi^2}{4\pi^2 - R_i}.$$

Writing $A(s)$ in the phase-amplitude form, we get

$$A(s) = \left|A(s)\right| e^{i\varphi}. \tag{4.27}$$

Now, substituting Eq.(4.27) in Eq.(4.26), we get the following expression for the amplitude $\left|A(s)\right|$ as

$$\frac{d\left|A(s)\right|^2}{ds} - 2p_r \left|A(s)\right|^2 + 2l_r \left|A(s)\right|^4 = 0, \tag{4.28}$$

$$\frac{d\left(\mathrm{ph}\left(A(s)\right)\right)}{ds} = p_i - l_i \left|A(s)\right|^2 \tag{4.29}$$

where $\gamma^{-1} F(s) = p_r + ip_i$, $\gamma^{-1} k = l_r + il_i$ and ph(.) represents the phase shift. Equation (4.28) solved numerically using the function NDSolve of Mathematica, subject to the suitable initial condition $A(0) = a_0$, where a_0 is the chosen initial amplitude of convection. In our computation, we assume $R_2 = R_0$ to keep the parameters to a minimum.

5. Bifurcation analysis

A qualitative study of a dynamical system is very important, thus it has attracted attention of researchers during the last few decades. Bifurcations are important scientifically, since they which provide models of transitions and instabilities when as some control parameter is varied [42]. In the present paper, two types of bifurcation (1) Pitchfork bifurcation (this bifurcation is common in physical problems that have a symmetry); (2) Hopf bifurcation (bifurcation corresponding to the presence of $\lambda_{1,2} = \pm i\omega_0, \omega_0 > 0$) are discussed, for details cf. [43].

5.1. Hopf bifurcation

The complex Ginzburg-Landau Eq.(4.26) can be written as

$$\frac{dA(s)}{ds} = (\alpha + i\beta)A(s) - (\xi + i\varsigma)|A(s)|^2 A(s) = 0 \tag{5.1}$$

where

$$\alpha = \mathrm{Re}\left[\frac{F(s)}{\gamma}\right], \quad \beta = \mathrm{I}_m\left[\frac{F(s)}{\gamma}\right], \quad \xi = \mathrm{Re}\left[\frac{k}{\gamma}\right], \quad \text{and} \quad \varsigma = I_m\left[\frac{k}{\gamma}\right].$$

Putting $A(s) = \rho_1 e^{i\phi}$ in Eq.(5.1) and equating the real and imaginary parts, we get

$$\dot{\rho}_1 = \rho_1\left(\alpha - \xi\rho_1^{\,2}\right), \tag{5.2}$$

$$\dot{\phi} = \beta - \varsigma\rho_1^{\,2}. \tag{5.3}$$

Clearly, Eq.(5.2) has an equilibrium point $\rho_1 = 0$, for all values of α. Further, Eq.(5.3) describes the rotation. Hence, Eq.(5.1) has an equilibrium at the point $A(s) = 0$, i.e., origin. This equilibrium is a stable focus for $\alpha < 0$ and an unstable focus for $\alpha > 0$ and at $\alpha = 0$ (critical value) the equilibrium is non-linearly stable and topologically equivalent to the focus. This equilibrium is surrounded, for $\alpha > 0$, by an isolated limit cycle that is unique and stable. All orbits starting outside or inside the cycle except at the origin tend to the cycle as $s \to +\infty$, see Fig.2. This is an Andronov-Hopf bifurcation. Here, we assume $\lambda_{\cdot 1}$ as the bifurcation parameter, the critical value of the Hopf-bifurcation is $\lambda_{\cdot 1} = 0.14336936$. The phase portrait diagram for the critical value shows that the system (5.1) undergoes Hopf-bifurcation around the origin (see Fig.3). This bifurcation can also be represented in the (x, y, α) coordinate system see Fig.4. Figure 5 shows that the α family of limit cycles forms a paraboloid surface.

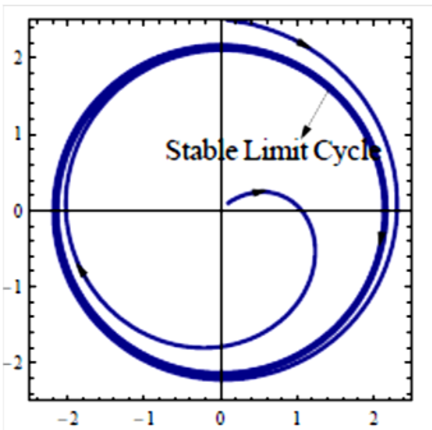

Fig.2. Stable limit cycle for λ_1 greater than critical value. $\lambda_1 = 0.5, \lambda_2 = 0.1, \Omega = 1, \delta = 0.02, R_i = 0.4$.

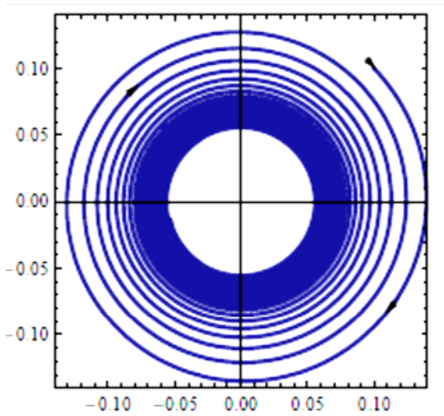

Fig.3. Hopf bifurcation w.r.t. bifurcation parameter λ_1. $\lambda_1 = 0.14336936, \lambda_2 = 0.1,\ \Omega = 1, \delta = 0.02, R_i = 0.4$.

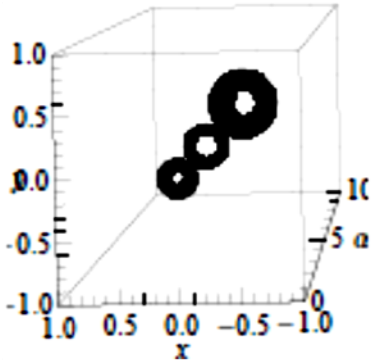

Fig.4. Supercritical Hopf bifurcation in the phase parameter (x, y, α) space.

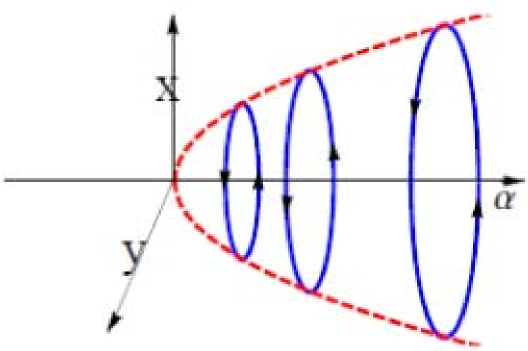

Fig.5. Paraboloid surface formed by α family of limit cycles shown in Fig.4.

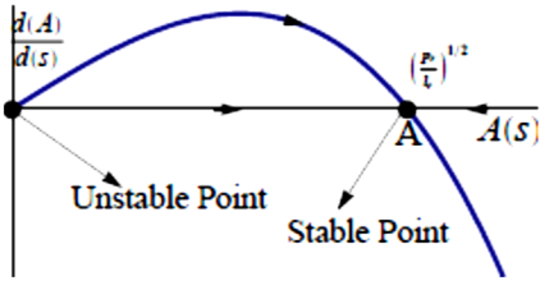

Fig.6. The point A is stable while the origin is unstable. $\lambda_1 = 0.5, \lambda_2 = 0.1, \Omega = 2, \delta = 0.02, R_i = 1$.

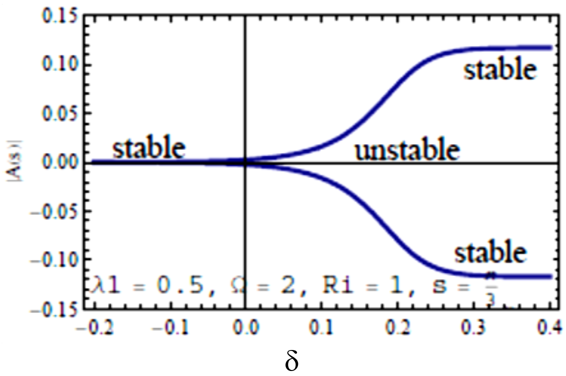

Fig.7. Supercritical Pichfork Bifurcation. $\lambda_1 = 0.5, \lambda_2 = 0.1, \Omega = 2, \delta = 0.02, R_i = 1, s = \frac{\pi}{3}$.

5.2. Pitchfork bifurcation

Eq.(4.28) can be written as

$$\frac{d|A(s)|}{ds} - p_r|A(s)| + l_r|A(s)|^3 = 0. \tag{5.4}$$

The steady state solution (equilibrium point) of Eq.(5.4) is $|A(s)| = 0$ for all values of p_r, l_r and $|A(s)| = \pm\sqrt{\frac{p_r}{l_r}}$ for $p_r > 0,\ l_r > 0$.

The unsteady state solution of Eq.(5.4) is given by

$$|A(s)|^2 = \frac{A_0^2}{\frac{l_r}{p_r}A_0^2 + \left(1 - \frac{l_r}{p_r}A_0^2\right)e^{-2p_r s}}, \qquad p_r > 0, \qquad l_r > 0 \tag{5.5}$$

where A_0 is the initial value of the amplitude. From Eq.(5.5), it is clear that as $s \to -\infty$, $|A(s)| \to 0$ and if $s \to \infty$, then $|A(s)|$ grows towards $\sqrt{\frac{p_r}{l_r}}$ whenever $0 < A_0 < \sqrt{\frac{p_r}{l_r}}$ and decreases towards $\sqrt{\frac{p_r}{l_r}}$ whenever

$A_0 > \sqrt{\frac{p_r}{l_r}}$. Thus, the equilibrium point $A(s) = 0$ is the only equilibrium point whenever $p_r < 0$ and it is stable. If $\Pr = 0$, then the origin of the only equilibrium point is still stable. When $\Pr > 0$ and $l_r > 0$ then $|A(s)| = 0$ is still an equilibrium point, but becomes unstable and two new stable equilibrium points appear on either side of $|A(s)| = 0$, symmetrically located at $|A(s)| = \pm\sqrt{\frac{p_r}{l_r}}$, Fig.14. This is known as supercritical pitchfork bifurcation, depicted in Fig.7.

6. Results and discussion

This paper investigates the combined effect of internal heating and gravity modulation on oscillatory convection in a porous medium saturated with a viscoelastic fluid. A weakly nonlinear stability analysis is performed to investigate the effect of gravity modulation on heat transport in the presence of internal heat generation. This analysis helps us to derive an amplitude equation in terms of a complex coefficient which gives a finite amplitude of convection. The present model is valid for oscillatory mode of convection whenever the relation (28) holds for the viscoelastic parameters. The new contribution to the recent study by Bhadauria and Kiran [26] is the effect of heat generation and bifurcation analysis. The results of the corresponding problem are depicted in Figs 8-13, where the graphs are drawn for Nu versus slow time s. Relation Eq.(4.22) confirms that the value of Nu starts with 1, thus showing the conduction state initially. The value of Nu increases as time varies, thus showing the convection state; becomes oscillatory, thus showing the modulation effect. The numerical values of Nu have been obtained from the expression (4.22) while solving the amplitude Eq.(4.28).

The effect of the internal Rayleigh number R_i on heat transport is to increase the heat transport in the system as depicted in Fig.8. It is to be noted that when the value of R_i increases, the internal energy of the system increases. The moderate values of R_i are considered to avoid domination in the system. This confirms the results obtained recently in the studies [34, 35]. The effects of relaxation parameter λ_1 (see, Fig.9) and retardation parameter λ_2 (see, Fig.10) are found to destabilize or stabilize the system, respectively, and consequently increase or decrease the heat transfer in the system. The effects of the amplitude δ and frequency Ω of modulation on heat transport are given in Fig.11 and Fig.12, respectively. Figure 11 confirms that an increment in the amplitude of modulation increases the magnitude of Nu, and so, enhances the heat transfer and advances the onset of convection. An opposite effect is obtained in the case of frequency of modulation, see Fig.12. Moreover, the effect of gravity modulation decreases as the frequency of modulation increases. The effect disappears altogether when the frequency of modulation becomes very large. The proposed results of internal heating have been compared with the results of non-internal heating system, reported in Fig.13. It is observed that in the presence of an internal heat source, the magnitude of Nu is larger than that in the absence of internal-heating, i.e., the heat transport in the system is due to internal-heating. Thus, internal-heating advances the onset of convection, confirming the results obtained by Bhadauria *et al*. [36]-[38].

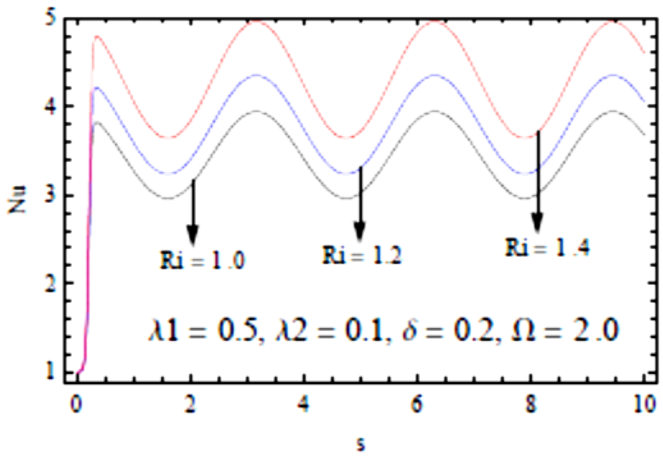

Fig.8. Effect of R_i on Nu for fixed values of the other parameters.

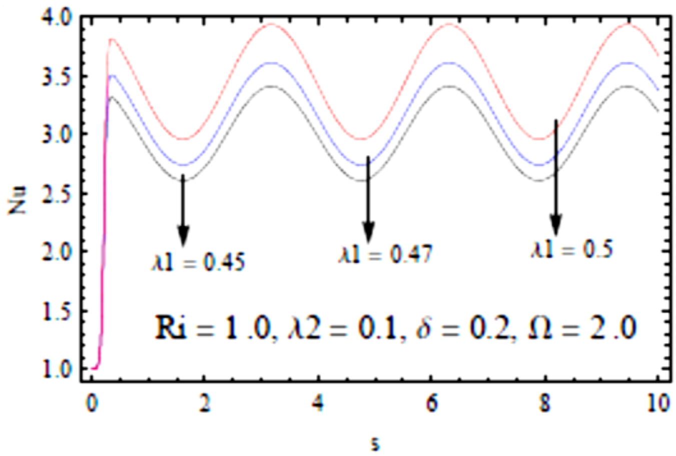

Fig.9. Effect of λ_1 on Nu for fixed values of the other parameters.

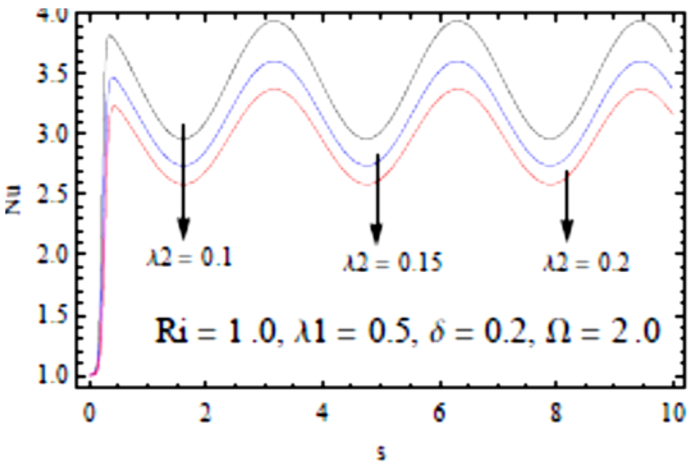

Fig.10. Effect of λ_2 on Nu for fixed values of the other parameters.

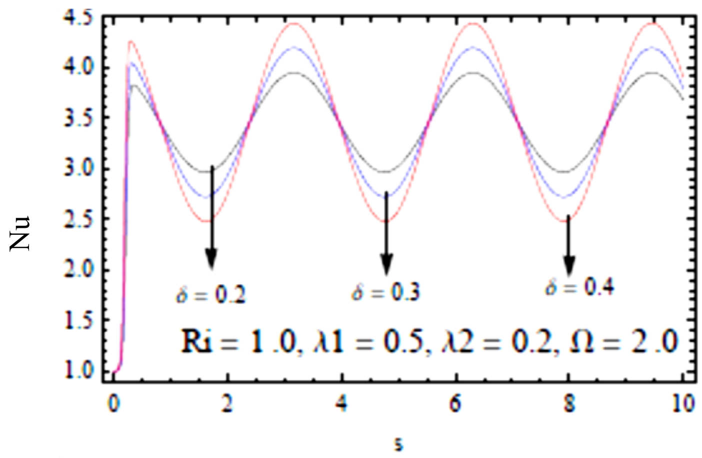

Fig.11. Effect of δ on Nu for fixed values of the other parameters.

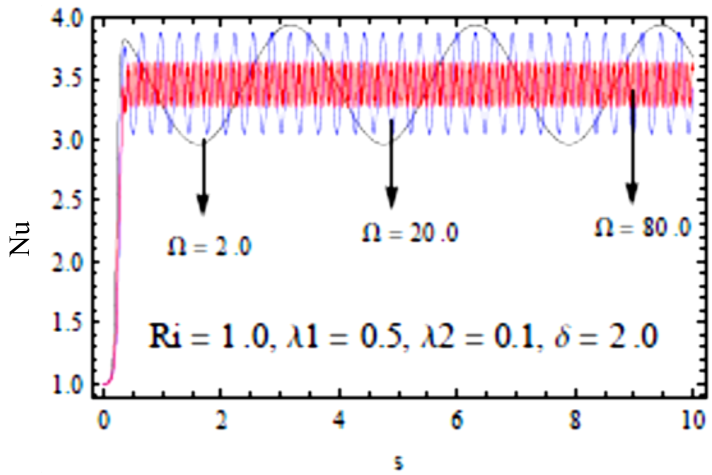

Fig.12. Effect of Ω on Nu for fixed values of the other parameters.

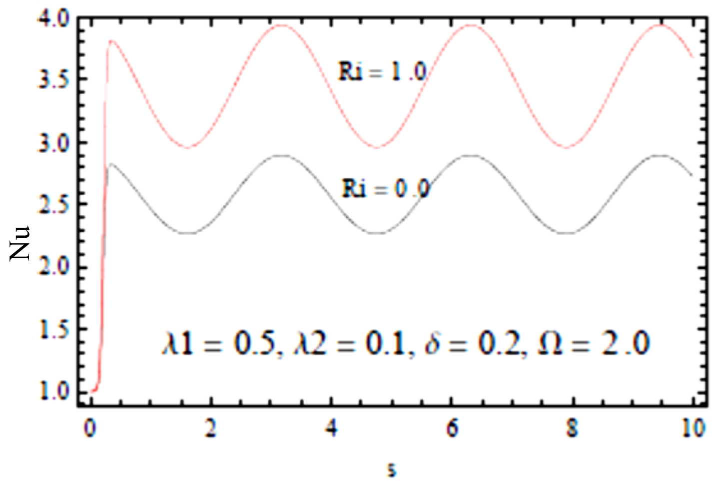

Fig.13. Comparison between internal and non internal-heating system.

In Figs 14-15, the streamlines and corresponding isotherms are depicted, respectively, at $s = 0.0,\ 0.13,\ 0.16,\ 0.2,\ 0.3,\ 0.43$ for $\lambda_1 = 0.5$, $\lambda_2 = 0.1$, $\delta = 0.2$, $\Omega = 2$, $\chi = 0.5$, and $R_i = 0.1$. From these figures, it is observed that initially when time is small, the magnitude of streamlines is also small [Figs 14a, b] and isotherms are straight [Figs 15a, b], that is the system is in conduction state. However, as time increases, the magnitude of streamlines increases [Figs 14c, d] and the isotherms lose their evenness [Figs 15c, d]. This shows that convection is taking place in the system. The system achieves the steady state beyond $s = 0.2$ as there is no change in the streamlines and isotherms, [Figs 14e, f and 15e, f].

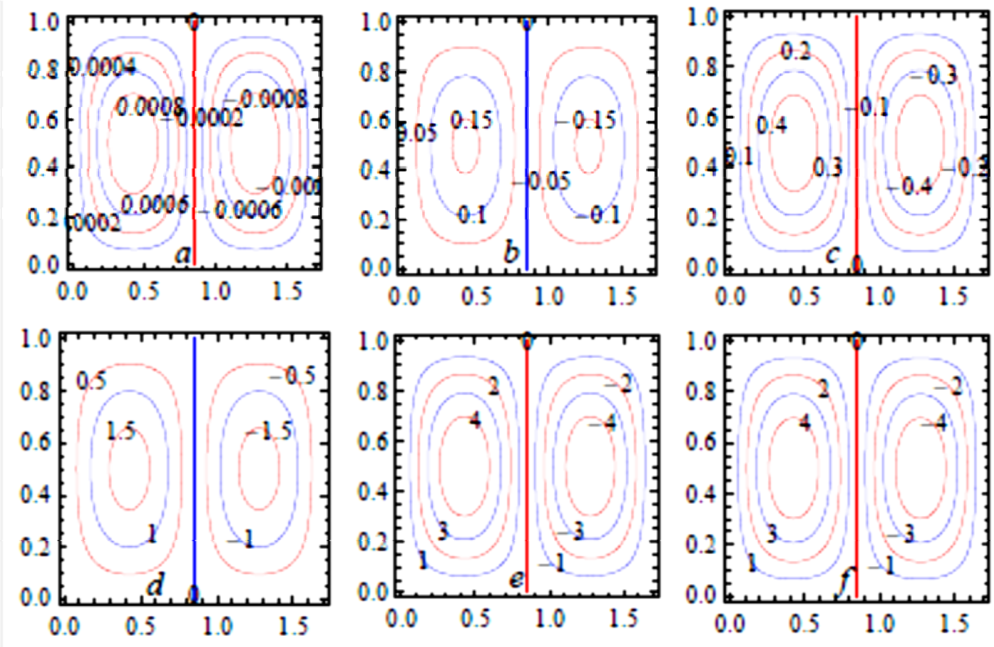

Fig.14. Streamlines at (a) $s = 0.0$, (b) $s = 0.13$, (c) s = 0.16, (d) $s = 0.2$, (e) $s = 0.3$, (f) $s = 0.43$.

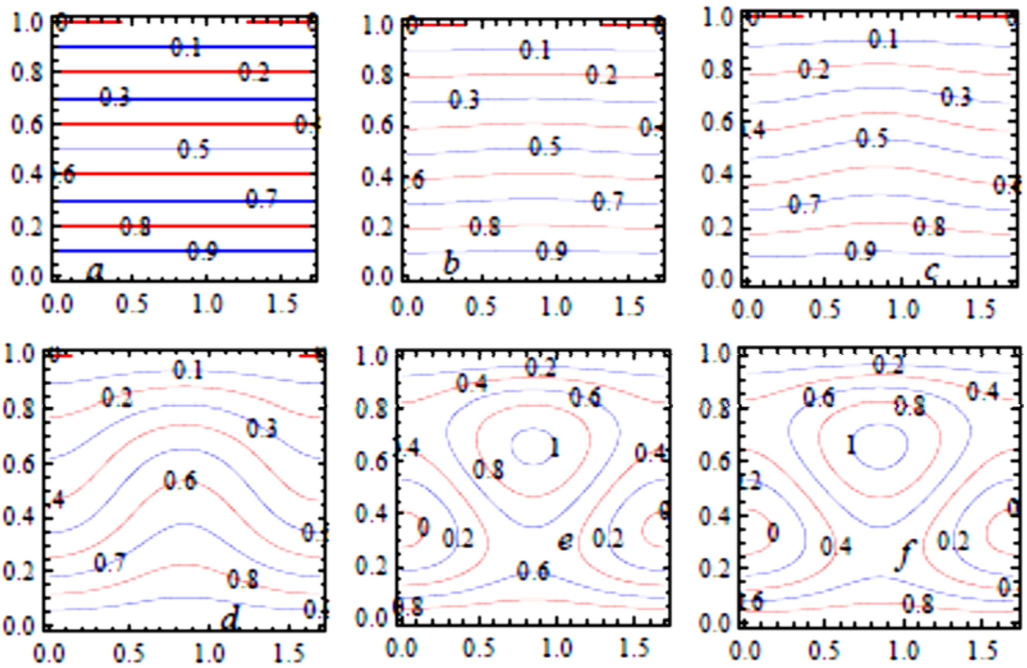

Fig.15. Isotherms at (a) $s = 0.0$, (b) $s = 0.13$, (c) $s = 0.16$, (d) $s = 0.2$, (e) $s = 0.3$, (f) $s = 0.43$.

Also, by taking λ_1 as the bifurcation parameter, we have shown that the system represented by Landau Eq.(4.26) undergoes Hopf bifurcation. Further, the limit cycles are found to be stable circles. A sketch of this bifurcation in the (x, y, α) coordinate plane, results in a paraboloid surface. Apart from the Hopf bifurcation, we have also shown that the system represented by the amplitude Eq.(4.28) undergoes the supercritical pichfork bifurcation. Figure 16 shows that as s increases the amplitude $|A(s)|$ becomes stable and remains stable for all future of time.

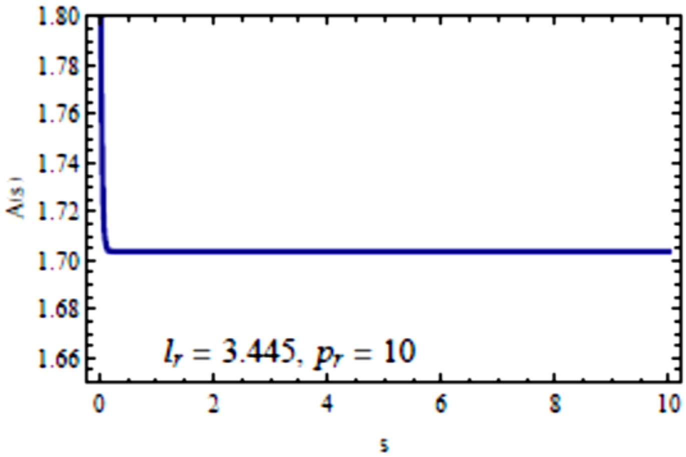

Fig.16. Phase portrait diagram for $A(s)$ and s, shows that $A(s)$ is stable when s increases.

7. Conclusions

This paper deals with a combined effect of internal heating and gravity modulation on oscillatory convection. By performing a weak nonlinear stability analysis using the Gingburg-Landau equation, following conclusions are drawn:

a) The nature and the effect of viscoelastic fluid is in agreement with followed the studies of Kim *et al.* [20], Bhadauria and Kiran [26]-[29].
b) The effects of amplitude and frequency of modulation are, respectively to increase and decreases the heat transfer.
c) The effect of internal Rayleigh number is to increase the heat transport in the system.
d) The equilibrium point of the Ginzburg-Landau equation losses its stability as λ_1 increases.
e) If $p_r > 0$ and $l_r > 0$, then the amplitude equation gives two more equilibrium points, which are stable while the origin losses its stability.

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Nomenclature

$A(s)$ – amplitude of convection

a – wave number

d – depth of fluid layer
$\boldsymbol{g}$ – acceleration due to gravity
K – permeability
Nu – Nusselt number
p – reduced pressure
$R_{a\,D}$ – thermal Rayleigh number
R_i – internal Rayleigh number
s – slow time scale
T – temperature
t – time
Q – internal heat source
$\boldsymbol{q}$ – fluid velocity (u,v,w)
(x,z) – horizontal and vertical co-ordinates
α_T – coefficient of thermal expansion
ΔT – temperature difference
δ – amplitude of gravity modulation
γ – heat capacity ratio
κ_T – effective thermal diffusivity
$\bar{\lambda}_1$ – stress relaxation time
$\bar{\lambda}_2$ – strain retardation time
ν – kinematic viscosity
ρ – fluid density
τ – fast time scale
ϕ – porosity
χ – perturbation parameter
ψ – stream function
Ω – frequency of modulation
ω – oscillatory frequency
0 – reference value

Superscripts

, – perturbed quantity
* – dimensionless quantity

Subscripts

b basic state
c critical

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

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