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DIFFERENTIAL KINEMATICS OF CONTEMPORARY INDUSTRIAL ROBOTS

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The paper presents a simple method of avoiding singular configurations of contemporary industrial robot manipulators of such renowned companies as ABB, Fanuc, Mitsubishi, Adept, Kawasaki, COMAU and KUKA. To determine the singular configurations of these manipulators a global form of description of the end-effector kinematics was prepared, relative to the other links. On the basis of this description , the formula for the Jacobian was defined in the end-effector coordinates. Next, a closed form of the determinant of the Jacobian was derived. From the formula, singular configurations, where the determinant's value equals zero, were determined. Additionally, geometric interpretations of these configurations were given and they were illustrated. For the exemplary manipulator, small corrections of joint variables preventing the reduction of the Jacobian order were suggested. An analysis of positional errors, caused by these corrections, was presented.

Key words: kinematics, manipulators, mechanical system, robot kinematics.

1. Introduction

The controllers of such renowned companies as ABB, Fanuc, Mitsubishi, Adept, Kawasaki, COMAU and KUKA make possible movement programming in joint space or the Cartesian space. The following commands PTP, LIN, and CIRC can be applied to programming in the Cartesian space. The mentioned commands require a start point and end point. For programming in the Cartesian space these points must be described in Cartesian coordinates, relative to the base frame, connected to the base of a manipulator. These coordinates can be obtained by a vision system. During the realization of such programmed movement the robot happens to stop before reaching the border area, and before reaching the start or the end point. The entrapment takes place when the manipulator reaches the singular configurations. It is without doubt the major problem of modern industrial robots, which makes it impossible for the robots with a vision system to cooperate with the cameras properly.

The linear system of ordinary differential equations that describes the kinematics can be applied to programming the robots in the Cartesian space. In this system a manipulator Jacobian is present. For the succesive via points interpolating trajectory programmed in the Cartesian space, joint variables of these points can be computed. To these calculations, the standard algorithm for solving the system of linear equations which is one of the elements of a computer software library, can be applied. The lack of protection in software against the reduction of the Jacobian rank in this algorithm can cause the interruption of the calculations and the robot can stop performing its operations. This rank decreases in singular configuration.

The problem of the inverse kinematics solutions in a differential form for the singular configuration is presented in Chiacchio (1996), Kozłowski (2003), Nakamura (2009), Siciliano (2010), Spong (1997), Tchoń (1997). In Chiacchio (1996), Kozłowski (2003), Nakamura (2009), Siciliano (2010) for a singular configuration it is proposed to use: singular value decomposition techniques SVD of the Jacobian, the damped least-squares inverse of the Jacobian DLS or singularity robust inverse of the Jacobian. In the work

(Tchoń, 1997) a method of avoiding singularities by using of dynamical systems of the own motion is presented. The same work presents a solution of the inverse kinematics in singular configurations, obtained using: the method a normal form, Jacobian attached, the zero and the Jacobian space. A common feature of kinematic singularity avoidance methods presented in Chiacchio (1996), Kozłowski (2003), Nakamura (2009), Siciliano (2010), Tchoń (2000) is their high computational complexity.

A very simple approach to the problem of determining the kinematic singularities based on the differential description is presented in Spong (1997), on the examples of manipulators with three degrees of freedom. Simple, because based on closed forms of the determinants of manipulators Jacobian, allowing a very simple determination of joint variables, describing singular configurations. Simple rules of linear algebra are recommended in Spong (1997) for solving the inverse kinematics in the field of speed.

Only the methods which do not require a large number of calculations can be of practical use.

This paper presents a very simple method of correction, allowing to avoidance of the singular configuration of modern industrial robots. In this method, the values of third and fifth joint variables are corrected. These values were determined from the closed form of the Jacobian determinant of modern industrial robots, (Spong, 1997). The closed form of formulas describing the third and fifth joint variables for the singular configurations of manipulators was determined. On the basis of these formulas, one is able to determine whether in the next step of the numerical calculations the freezing of the program, caused by a reduction of the Jacobian rank, will take place. These corrections prevent the reduction of the Jacobian rank and allow the use of any method of solving the inverse kinematics, including the methods described in the aforementioned works (Chiacchio, 1996; Nakamura, 2009; Siciliano, 2010; Tchoń, 2000).

In the second chapter the kinematic structure and a description of the end-effector kinematics in relation to the other links, including the base of the manipulator are presented. The formulas constituting the differential description of the kinematics are presented in the third chapter. A general description of singular configurations including their illustration and the examples of the calculations of joint variables corrections, preventing the reduction of the Jacobian rank, are presented in the fourth chapter. The fifth chapter summarizes the paper.

2. The global description of the kinematics

Figure1 illustrates the robot manipulators of the majority of modern robots, with numbered links. The link 0 is attached to the ground, other links 1-6 are movable. The link 0 will be called a base link, the last link with number 6 will be called an end-effector. The gripper, or another tool, is attached to this link. Neighboring links are connected by revolute joints. Figure 2 illustrates the manipulator kinematics schema with the co-ordinate systems (frames) associated with links according to a Denavit-Hartenberg notation (Jezierski, 2006; Kozłowski, 2003; Szkodny, 2013a; 2013b). The $x_7y_7z_7$ frame is associated with the gripper. The position and orientation of the links and tool are described by homogenous transform matrices. Matrix $\boldsymbol{A}_i$ describes the position and orientation of the i-th link frame in relation to i-1-st. $\boldsymbol{T}_6$ is a matrix that describes the position and the orientation of the end-effector frame in relation to the base link. Matrix $\boldsymbol{E}$ describes the gripper frame in relation to the end-effector frame. Matrix $\boldsymbol{X}$ describes the position and the orientation of the gripper frame in relation to the base link. Parameters Θ_i, λ_i, l_i, α_i are used in the Denavit-Hartenberg notation. For further description of the kinematics, joint variables Θ_i will be used. Variables $\Theta'_i = \Theta_i$ for i=1, 3, 4, 5, 6 and $\Theta'_2 = \Theta_2 - 90^\circ$. To simplify, the following denotations will be used: $S_i = \sin\Theta'_i$, $C_i = \cos\Theta'_i$, $S_{ij} = \sin\Theta'_{ij}$, $C_{ij} = \cos\Theta'_{ij}$, $\Theta'_{ij} = \Theta'_i + \Theta'_j$. Matrices $\boldsymbol{A}_i$ are described by Eq.(2.1a).

$$\boldsymbol{A}_1 = \begin{bmatrix} C_1 & 0 & S_1 & l_1C_1 \\ S_1 & 0 & -C_1 & l_1S_1 \\ 0 & 1 & 0 & \lambda_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \boldsymbol{A}_2 = \begin{bmatrix} -S_2 & -C_2 & 0 & -l_2S_2 \\ C_2 & -S_2 & 0 & l_2C_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} C_3 & 0 & S_3 & l_3 C_3 \\ S_3 & 0 & -C_3 & l_3 S_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad A_4 = \begin{bmatrix} C_4 & 0 & -S_4 & 0 \\ S_4 & 0 & C_4 & 0 \\ 0 & -1 & 0 & \lambda_4 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

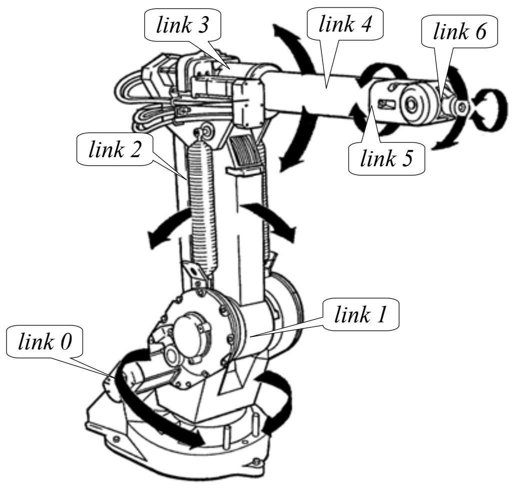

Fig.1. The manipulator.

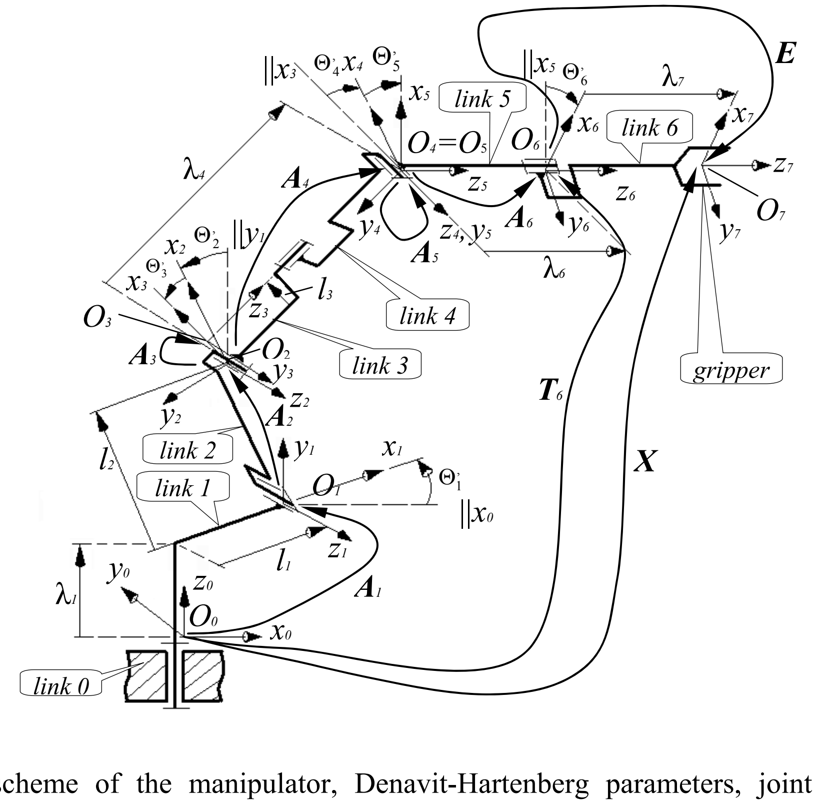

Fig.2. Kinematic scheme of the manipulator, Denavit-Hartenberg parameters, joint variables and joint frames.

$$A_5=\begin{bmatrix} C_5 & 0 & S_5 & 0 \\ S_5 & 0 & -C_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad A_6=\begin{bmatrix} C_6 & -S_6 & 0 & 0 \\ S_6 & C_6 & 0 & 0 \\ 0 & 0 & 1 & \lambda_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.1a}$$

Matrix $\boldsymbol{E}$ is described by Eq.(2.1b).

$$\boldsymbol{E}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \lambda_7 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.1b}$$

Matrix ${}^{i-1}\boldsymbol{T}_6$ describes the end-effector frame kinematics in relation to i-1-st frame. Equation (2.2a) (Szkodny, 2013a) describes these matrices.

$${}^{i-1}\boldsymbol{T}_6=\prod_{j=i}^{6}\boldsymbol{A}_j, \quad 1\le i\le 6 . \tag{2.2a}$$

From Eqs (2.1a) and (2.2a) one obtains

$${}^{5}\boldsymbol{T}_6=\begin{bmatrix} C_6 & -S_6 & 0 & 0 \\ S_6 & C_6 & 0 & 0 \\ 0 & 0 & 1 & \lambda_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad {}^{4}\boldsymbol{T}_6=\begin{bmatrix} C_5C_6 & -C_5S_6 & S_5 & \lambda_6S_5 \\ S_5C_6 & -S_5S_6 & -C_5 & -\lambda_6C_5 \\ S_6 & C_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$${}^{3}\boldsymbol{T}_6=\begin{bmatrix} C_4C_5C_6-S_4S_6 & -C_4C_5S_6-S_4C_6 & C_4S_5 & \lambda_6C_4S_5 \\ S_4C_5C_6+C_4S_6 & -S_4C_5S_6+C_4C_6 & S_4S_5 & \lambda_6S_4S_5 \\ -S_5C_6 & S_5S_6 & C_5 & \lambda_4+\lambda_6C_5 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$${}^{2}\boldsymbol{T}_6=\begin{bmatrix} C_3\left(C_4C_5C_6-S_4S_6\right)-S_3S_5C_6 & C_3\left(-C_4C_5S_6-S_4C_6\right)+S_3S_5S_6 \\ S_3\left(C_4C_5C_6-S_4S_6\right)+C_3S_5C_6 & S_3\left(-C_4C_5S_6-S_4C_6\right)-C_3S_5S_6 \\ S_4C_5C_6+C_4S_6 & -S_4C_5S_6+C_4C_6 \\ 0 & 0 \end{bmatrix.}$$

$$\begin{matrix} C_3C_4S_5+S_3C_5 & l_3C_3+\left(\lambda_4+\lambda_6C_5\right)S_3+\lambda_6C_3C_4S_5 \\ S_3C_4S_5-C_3C_5 & l_3S_3-\left(\lambda_4+\lambda_6C_5\right)C_3+\lambda_6S_3C_4S_5 \\ S_4S_5 & \lambda_6S_4S_5 \\ 0 & 1 \end{matrix}\Bigg],$$

$$
{}^{1}\boldsymbol{T}_{6}=\begin{bmatrix}
-S_{23}\left(C_4C_5C_6-S_4S_6\right)-C_{23}S_5C_6 & -S_{23}\left(-C_4C_5S_6-S_4C_6\right)+C_{23}S_5S_6 & -S_{23}C_4S_5+C_{23}C_5 & -l_2S_2-l_3S_{23}+\left(\lambda_4+\lambda_6C_5\right)C_{23}-\lambda_6S_{23}C_4S_5\\
C_{23}\left(C_4C_5C_6-S_4S_6\right)-S_{23}S_5C_6 & C_{23}\left(-C_4C_5S_6-S_4C_6\right)+S_{23}S_5S_6 & C_{23}C_4S_5+S_{23}C_5 & l_2C_2+l_3C_{23}+\left(\lambda_4+\lambda_6C_5\right)S_{23}+\lambda_6C_{23}C_4S_5\\
S_4C_5C_6+C_4S_6 & -S_4C_5S_6+C_4C_6 & S_4S_5 & \lambda_6S_4S_5\\
0 & 0 & 0 & 1
\end{bmatrix},
$$

$$
\boldsymbol{T}_{6}=\begin{bmatrix}
C_1\left(-S_{23}\left(C_4C_5C_6-S_4S_6\right)-C_{23}S_5C_6\right)+S_1\left(S_4C_5C_6+C_4S_6\right) & C_1\left(-S_{23}\left(-C_4C_5S_6-S_4C_6\right)+C_{23}S_5S_6\right)+S_1\left(-S_4C_5S_6+C_4C_6\right) & C_1\left(-S_{23}C_4S_5+C_{23}C_5\right)+S_1S_4S_5 & l_1C_1+\lambda_6S_1S_4S_5+\left(-l_2S_2-l_3S_{23}+\left(\lambda_4+\lambda_6C_5\right)C_{23}-\lambda_6S_{23}C_4S_5\right)C_1\\
S_1\left(-S_{23}\left(C_4C_5C_6-S_4S_6\right)-C_{23}S_5C_6\right)-C_1\left(S_4C_5C_6+C_4S_6\right) & S_1\left(-S_{23}\left(-C_4C_5S_6-S_4C_6\right)+C_{23}S_5S_6\right)-C_1\left(-S_4C_5S_6+C_4C_6\right) & S_1\left(-S_{23}C_4S_5+C_{23}C_5\right)-C_1S_4S_5 & l_1S_1-\lambda_6C_1S_4S_5+\left(-l_2S_2-l_3S_{23}+\left(\lambda_4+\lambda_6C_5\right)C_{23}-\lambda_6S_{23}C_4S_5\right)S_1\\
C_{23}\left(C_4C_5C_6-S_4S_6\right)-S_{23}S_5C_6 & C_{23}\left(-C_4C_5S_6-S_4C_6\right)+S_{23}S_5S_6 & C_{23}C_4S_5+S_{23}C_5 & \lambda_1+l_2C_2+\left(C_4S_5\lambda_6+l_3\right)C_{23}+\left(\lambda_4+\lambda_6C_5\right)S_{23}\\
0 & 0 & 0 & 1
\end{bmatrix}. \quad (2.2b)
$$

These matrices are necessary to the differential description of the end-effector kinematics, in relation to the base frame. The description is presented in the next chapter.

3. The differential description of the kinematics

To make the description of the manipulator kinematics independent of the shape of the gripper, the required position and orientation of the gripper frame $x_7y_7z_7$ (described by the matrix $\boldsymbol{X}_{req}$) are converted to the end-effector frame $x_6y_6z_6$. Correlation $\boldsymbol{T}_{6\,req}=\boldsymbol{X}_{req}\boldsymbol{E}^{-1}$ makes the conversion possible. Therefore, in further considerations, we will focus on the end-effector kinematics.

The movement of the $x_7y_7z_7$ frame in relation to the $x_0y_0z_0$ frame will be described by using displacement differentials ${}^{7}dx_7$, ${}^{7}dy_7$, ${}^{7}dz_7$ of origin O_7, and *x-y-z* current angle differentials ${}^{7}d\phi_{7x}$, ${}^{7}d\phi_{7y}$, ${}^{7}d\phi_{7z}$ (Craig, 1993; Szkodny, 2012). These differentials are described in the $x_7y_7z_7$ frame.

Similarly, the movement of the $x_6y_6z_6$ frame in relation to the $x_0y_0z_0$ frame will be described by using the displacement differentials ${}^6dx_6$, ${}^6dy_6$, ${}^6dz_6$ of the origin O_6, and *x-y-z* current angle differentials ${}^6d\phi_{6x}$, ${}^6d\phi_{6y}$, ${}^6d\phi_{6z}$. These differentials are described in the $x_6y_6z_6$ frame. Further, we will focus on the movement description of the end-effector and therefore one it is necessary to convert the gripper differentials to end-effector differentials. The differential equation $\boldsymbol{T}_6\,{}^6\Delta_6\boldsymbol{E} = \boldsymbol{X}\,{}^7\Delta_7$, which connects end-effector and gripper differentials, results from the work (Szkodny, 2013a). In this equation, ${}^6\Delta_6$ and ${}^7\Delta_7$ are differential transformation matrices, respectively, of the end-effector and gripper frame. These matrices have the following forms

$$
{}^6\Delta_6 = \begin{bmatrix} 0 & -{}^6d\phi_{6z} & {}^6d\phi_{6y} & {}^6dx_6 \\ {}^6d\phi_{6z} & 0 & -{}^6d\phi_{6x} & {}^6dy_6 \\ -{}^6d\phi_{6y} & {}^6d\phi_{6x} & 0 & {}^6dz_6 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad {}^7\Delta_7 = \begin{bmatrix} 0 & -{}^7d\phi_{7z} & {}^7d\phi_{7y} & {}^7dx_7 \\ {}^7d\phi_{7z} & 0 & -{}^7d\phi_{7x} & {}^7dy_7 \\ -{}^7d\phi_{7y} & {}^7d\phi_{7x} & 0 & {}^7dz_7 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
$$

From the above Eq.(3.1) results

$$
{}^6\Delta_6 = \boldsymbol{T}_6^{-1}\boldsymbol{X}\,{}^7\Delta_7\boldsymbol{E}^{-1} = \boldsymbol{E}\,{}^7\Delta_7\boldsymbol{E}^{-1}. \tag{3.1}
$$

From Eqs (3.1) and (2.1b) results Eq.(3.2), which makes it possible to compute the end-effector differentials from gripper differentials.

$$
\begin{gathered}
{}^6d\phi_{6x} = {}^7d\phi_{7x}, \qquad {}^6d\phi_y = {}^7d\phi_{7y}, \qquad {}^6d\phi_{6z} = {}^7d\phi_{7z}, \\
{}^6dx_6 = {}^7dx_7 - \lambda_7\,{}^7d\phi_{7y}, \qquad {}^6dy_6 = {}^7dy_7 + \lambda_7\,{}^7d\phi_{7x}, \qquad {}^6dz_6 = {}^7dx_7.
\end{gathered} \tag{3.2}
$$

To describe the end-effector kinematics one will apply the Cartesian differential matrix ${}^6\boldsymbol{D}_6 = \left[{}^6dx_6\ {}^6dy_6\ {}^6dz_6\ {}^6d\phi_{6x}\ {}^6d\phi_{6y}\ {}^6d\phi_{6z}\right]^T$ and the joint differential matrix $d\boldsymbol{q} = \left[d\Theta'_1\ d\Theta'_2\ d\Theta'_3\ d\Theta'_4\ d\Theta'_5\ d\Theta'_6\right]^T$ (Szkodny, 2012). Equation (3.3) connecting these matrices is a differential description of the end-effector kinematics.

$$
{}^6\boldsymbol{D}_6 = {}^6\boldsymbol{J}_6 d\boldsymbol{q}. \tag{3.3}
$$

In this equation, an end-effector Jacobian ${}^6\boldsymbol{J}_6$ is present, described in the $x_6y_6z_6$ frame, which has the form of Eq.(3.4).

$$
{}^{6}\boldsymbol{J}_{6} = \begin{bmatrix}
\frac{{}^{6}\partial x_{6}}{\partial \Theta'_{1}} & \frac{{}^{6}\partial x_{6}}{\partial \Theta'_{2}} & \frac{{}^{6}\partial x_{6}}{\partial \Theta'_{3}} & \frac{{}^{6}\partial x_{6}}{\partial \Theta'_{4}} & \frac{{}^{6}\partial x_{6}}{\partial \Theta'_{5}} & \frac{{}^{6}\partial x_{6}}{\partial \Theta'_{6}} \\
\frac{{}^{6}\partial y_{6}}{\partial \Theta'_{1}} & \frac{{}^{6}\partial y_{6}}{\partial \Theta'_{2}} & \frac{{}^{6}\partial y_{6}}{\partial \Theta'_{3}} & \frac{{}^{6}\partial y_{6}}{\partial \Theta'_{4}} & \frac{{}^{6}\partial y_{6}}{\partial \Theta'_{5}} & \frac{{}^{6}\partial y_{6}}{\partial \Theta'_{6}} \\
\frac{{}^{6}\partial z_{6}}{\partial \Theta'_{1}} & \frac{{}^{6}\partial z_{6}}{\partial \Theta'_{2}} & \frac{{}^{6}\partial z_{6}}{\partial \Theta'_{3}} & \frac{{}^{6}\partial z_{6}}{\partial \Theta'_{4}} & \frac{{}^{6}\partial z_{6}}{\partial \Theta'_{5}} & \frac{{}^{6}\partial z_{6}}{\partial \Theta'_{6}} \\
\frac{{}^{6}\partial \phi_{6x}}{\partial \Theta'_{1}} & \frac{{}^{6}\partial \phi_{6x}}{\partial \Theta'_{2}} & \frac{{}^{6}\partial \phi_{6x}}{\partial \Theta'_{3}} & \frac{{}^{6}\partial \phi_{6x}}{\partial \Theta'_{4}} & \frac{{}^{6}\partial \phi_{6x}}{\partial \Theta'_{5}} & \frac{{}^{6}\partial \phi_{6x}}{\partial \Theta'_{6}} \\
\frac{{}^{6}\partial \phi_{6y}}{\partial \Theta'_{1}} & \frac{{}^{6}\partial \phi_{6y}}{\partial \Theta'_{2}} & \frac{{}^{6}\partial \phi_{6y}}{\partial \Theta'_{3}} & \frac{{}^{6}\partial \phi_{6y}}{\partial \Theta'_{4}} & \frac{{}^{6}\partial \phi_{6y}}{\partial \Theta'_{5}} & \frac{{}^{6}\partial \phi_{6y}}{\partial \Theta'_{6}} \\
\frac{{}^{6}\partial \phi_{6z}}{\partial \Theta'_{1}} & \frac{{}^{6}\partial \phi_{6z}}{\partial \Theta'_{2}} & \frac{{}^{6}\partial \phi_{6z}}{\partial \Theta'_{3}} & \frac{{}^{6}\partial \phi_{6z}}{\partial \Theta'_{4}} & \frac{{}^{6}\partial \phi_{6z}}{\partial \Theta'_{5}} & \frac{{}^{6}\partial \phi_{6z}}{\partial \Theta'_{6}}
\end{bmatrix}. \tag{3.4}
$$

Figure 3 illustrates the differential displacement and the rotation of the end-effector, caused by the differential displacement and the rotation of the $x_i y_i z_i$ frame. The vectors ${}^{i-1}d\boldsymbol{r}_i$ and ${}^{i-1}d\phi_i$, respectively, describe the displacement and rotation of the $x'y'z'$ frame in relation to the $x_{i-1}y_{i-1}z_{i-1}$ frame, caused by a differential increase of the joint variable $d\Theta'_i$. The $x'y'z'$ frame is connected with the i-th link, and coincides with $x_{i-1}y_{i-1}z_{i-1}$ frame for $d\Theta'_i = 0$. The displacement of ${}^{i-1}d\boldsymbol{r}_{6,i}$ of $x_6y_6z_6$ frame is caused by the displacement of ${}^{i-1}d\boldsymbol{r}_i$ and rotation of vector ${}^{i-1}\boldsymbol{d}_6$ by an angle ${}^{i-1}d\phi_i$. Therefore

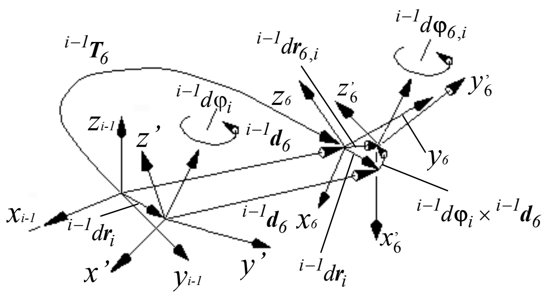

Fig.3. Differential displacements and rotations of i-th and 6-th frames.

$$
{}^{i-1}d\boldsymbol{r}_{6,i} = {}^{i-1}d\boldsymbol{r}_i + {}^{i-1}d\phi_i \times {}^{i-1}\boldsymbol{d}_6 . \tag{3.5}
$$

This movement will be recalculated in relation to the $x_6y_6z_6$ frame, using Eq.(3.6).

$$
{}^{6}dx_{6,i} = {}^{i-1}\boldsymbol{a}_6 \cdot {}^{i-1}d\boldsymbol{r}_{6,i}, \qquad {}^{6}dy_{6,i} = {}^{i-1}\boldsymbol{b}_6 \cdot {}^{i-1}d\boldsymbol{r}_{6,i}, \qquad {}^{6}dz_{6,i} = {}^{i-1}\boldsymbol{c}_6 \cdot {}^{i-1}d\boldsymbol{r}_{6,i} . \tag{3.6}
$$

The vectors ${}^{i-1}\boldsymbol{a}_6$, ${}^{i-1}\boldsymbol{b}_6$, ${}^{i-1}\boldsymbol{c}_6$ are the versors of the $x_6y_6z_6$ frame described in the $x_{i-1}y_{i-1}z_{i-1}$ frame. The rotation of the $x_6y_6z_6$ frame is the same as the rotation of $x_iy_iz_i$ frame in relation to the $x_{i-1}y_{i-1}z_{i-1}$ frame, which is a vector ${}^{i-1}d\phi_i$. This vector can be recalculated in relation to the $x_6y_6z_6$ frame by the following Eqs (3.7)

$$ {}^6d\phi_{6x,i} = {}^{i-1}\boldsymbol{a}_6 \cdot {}^{i-1}d\phi_i , \qquad {}^6d\phi_{6y,i} = {}^{i-1}\boldsymbol{b}_6 \cdot {}^{i-1}d\phi_i , \qquad {}^6d\phi_{6z,i} = {}^{i-1}\boldsymbol{c}_6 \cdot {}^{i-1}d\phi_i . \tag{3.7} $$

In Fig.2 one can see that all links are connected by revolute joints. Therefore, the angles ${}^{i-1}d\phi_i$ are vectors directed along the z_{i-1} axis, having a length equal to the differential $d\Theta_i^{`}$. Therefore

$$ {}^{i-1}d\phi_i = {}^{i-1}\boldsymbol{k}_{i-1} d\Theta_i^{`} . \tag{3.8} $$

${}^{i-1}\boldsymbol{k}_{i-1}$ is the versor of the z_{i-1} axis, described in the $x_{i-1}y_{i-1}z_{i-1}$ frame. For such joints, no displacement of the coordinate system $x'y'z'$ in relation to the $x_{i-1}y_{i-1}z_{i-1}$ frame takes place, therefore ${}^{i-1}d\boldsymbol{r}_i = 0$. Thus, from Eqs (3.5) and (3.8) one obtains the following

$$ {}^{i-1}d\boldsymbol{r}_{6,i} = {}^{i-1}d\varphi \times {}^{i-1}\boldsymbol{d}_6 = \left(-{}^{i-1}\boldsymbol{i}_{i-1}\ {}^{i-1}d_{6y} + {}^{i-1}\boldsymbol{j}_{i-1}\ {}^{i-1}d_{6x} \right) d\Theta_i^{`} . \tag{3.9} $$

The vectors ${}^{i-1}\boldsymbol{i}_{i-1}$ and ${}^{i-1}\boldsymbol{j}_{i-1}$ are versors of the x_{i-1} and y_{i-1} axes, respectively, described in $x_{i-1}y_{i-1}z_{i-1}$ frame. ${}^{i-1}d_{6x}$ and ${}^{i-1}d_{6y}$ are the x and y - coordinates of the vector ${}^{i-1}\boldsymbol{d}_6$ in the $x_{i-1}y_{i-1}z_{i-1}$ frame. After taking into account Eqs (3.9) and (3.6) one obtains the following

$$ \begin{aligned} {}^6dx_{6,i} &= \left(-{}^{i-1}a_{6x} \cdot {}^{i-1}d_{6y} + {}^{i-1}a_{6y} \cdot {}^{i-1}d_{6x} \right) d\Theta_i^{`} , \\ {}^6dy_{6,i} &= \left( -{}^{i-1}b_{6x} \cdot {}^{i-1}d_{6y} + {}^{i-1}b_{6y} \cdot {}^{i-1}d_{6x} \right) d\Theta_i^{`} , \\ {}^6dz_{6,i} &= \left(-{}^{i-1}c_{6x} \cdot {}^{i-1}d_{6y} + {}^{i-1}c_{6y} \cdot {}^{i-1}d_{6x} \right) d\Theta_i^{`} . \end{aligned} \tag{3.10} $$

${}^{i-1}a_{6x}$, ${}^{i-1}a_{6y}$ are the x and y - coordinates of the versor ${}^{i-1}\boldsymbol{a}_6$ in $x_{i-1}y_{i-1}z_{i-1}$ frame. The coordinates of the versors ${}^{i-1}\boldsymbol{b}_6$ and ${}^{i-1}\boldsymbol{c}_6$ were marked in a similar way.

From Eqs (3.7) and (3.8) one obtains the following correlations

$$ {}^6d\phi_{6x,i} = {}^{i-1}a_{6z} \cdot d\Theta_i^{`} , \qquad {}^6d\phi_{6y,i} = {}^{i-1}b_{6z} \cdot d\Theta_i^{`} , \qquad {}^6d\phi_{6z,i} = {}^{i-1}c_{6z} \cdot d\Theta_i^{`} . \tag{3.11} $$

The differential of each coordinate in the Cartesian matrix ${}^6\boldsymbol{D}_6$ is the sum of the corresponding coordinate differentials caused by differential changes in joint variables $d\Theta_i^{`}$. For example, ${}^6dx_6 = \sum_{i=1}^{6} {}^6dx_{6,i}$. In the Jacobian matrix ${}^6\boldsymbol{J}_6$ the derivatives of Cartesian coordinates of the $x_6y_6z_6$ end-effector frame with respect to the joint variables are present. The derivative $\frac{{}^6\partial x_6}{\partial\Theta_i^{`}} = \frac{\partial x_{6,i}}{\partial\Theta_i^{`}}$ only because the differential ${}^{6}dx_{6,i}$ depends on the value of $d\Theta_i^{,}$. The situation is similar with other derivatives in this matrix. Therefore, the elements of matrix ${}^{6}\boldsymbol{J}_{6}$ can be presented in the form of Eqs (3.12) and (3.13), taking into account the relations (3.10) and (3.11).

$$\frac{{}^{6}\partial x_{6}}{\partial\Theta_i^{,}}=\frac{\partial x_{6,i}}{\partial\Theta_i^{,}}=-{}^{i-1}a_{6x}\cdot{}^{i-1}d_{6y}+{}^{i-1}a_{6y}\cdot{}^{i-1}d_{6x},$$

$$\frac{{}^{6}\partial y_{6}}{\partial\Theta_i^{,}}=\frac{\partial y_{6,i}}{\partial\Theta_i^{,}}=-{}^{i-1}b_{6x}\cdot{}^{i-1}d_{6y}+{}^{i-1}b_{6y}\cdot{}^{i-1}d_{6x},\tag{3.12}$$

$$\frac{{}^{6}\partial z_{6}}{\partial\Theta_i^{,}}=\frac{\partial z_{6,i}}{\partial\Theta_i^{,}}=-{}^{i-1}c_{6x}\cdot{}^{i-1}d_{6y}+{}^{i-1}c_{6y}\cdot{}^{i-1}d_{6x},$$

$$\frac{{}^{6}\partial\phi_{6x}}{\partial\Theta_i^{,}}=\frac{\partial\phi_{6x,i}}{\partial\Theta_i^{,}}={}^{i-1}a_{6z},$$

$$\frac{{}^{6}\partial\phi_{6y}}{\partial\Theta_i^{,}}=\frac{\partial\phi_{6y,i}}{\partial\Theta_i^{,}}={}^{i-1}b_{6z},\qquad\frac{{}^{6}\partial\phi_{6z}}{\partial\Theta_i^{,}}=\frac{\partial\phi_{6z,i}}{\partial\Theta_i^{,}}={}^{i-1}c_{6z}.\tag{3.13}$$

Matrices ${}^{i-1}\boldsymbol{T}_{6}$, described by Eq.(2.2b) can be presented in the form Eq.(3.14).

$${}^{i-1}\boldsymbol{T}_{6}=\begin{bmatrix}{}^{i-1}a_{6x} & {}^{i-1}b_{6x} & {}^{i-1}c_{6x} & {}^{i-1}d_{6x}\\ {}^{i-1}a_{6y} & {}^{i-1}b_{6y} & {}^{i-1}c_{6y} & {}^{i-1}d_{6y}\\ {}^{i-1}a_{6z} & {}^{i-1}b_{6z} & {}^{i-1}c_{6z} & {}^{i-1}d_{6z}\\ 0 & 0 & 0 & 1\end{bmatrix}.\tag{3.14}$$

Quantities appearing in formulas (3.12) and (3.13) can be replaced by the corresponding elements of the matrix ${}^{i-1}\boldsymbol{T}_{6}$, resulting from the form Eq.(2.2b) and corresponding to the form Eq.(3.14). It is easy to notice that the index i in Eqs (3.12)-(3.14) is the number of column in the matrix ${}^{6}\boldsymbol{J}_{6}$. After using Eqs (2.2b), (3.12)-(3.14) and simplifications, one obtains the following elements of the end-effector Jacobian ${}^{6}\boldsymbol{J}_{6}$

$$\frac{{}^{6}\partial x_{6}}{\partial\Theta_1^{,}}=l_1(-S_4C_5C_6-C_4S_6)+l_2S_2(S_4C_5C_6+C_4S_6)+l_3S_{23}(S_4C_5C_6+C_4S_6)+$$
$$-\lambda_4C_{23}(S_4C_5C_6+C_4S_6)+\lambda_6(-C_6C_{23}S_4-C_4S_6C_{23}C_5+S_{23}S_6S_5),$$

$$\frac{{}^{6}\partial y_{6}}{\partial\Theta_1^{,}}=l_1(S_4C_5S_6-C_4C_6)-l_2S_2(S_4C_5S_6-C_4C_6)-l_3S_{23}(S_4C_5S_6-C_4C_6)+$$
$$+\lambda_4C_{23}(S_4C_5S_6-C_4C_6)+\lambda_6(C_6S_{23}S_5+S_6C_{23}S_4-C_4C_6C_{23}C_5),$$

$$\frac{{}^6\partial z_6}{\partial\Theta'_1} = -l_1S_4S_5+l_2S_4S_5S_2+l_3S_4S_5S_{23}-\lambda_4S_4S_5C_{23},$$

$$\frac{{}^6\partial\phi_{6x}}{\partial\Theta'_1} = -C_6S_{23}S_5-S_6C_{23}S_4+C_4C_6C_{23}C_5, \qquad \frac{{}^6\partial\phi_{6y}}{\partial\Theta'_1} = -C_{23}S_4C_6-C_{23}C_4S_6C_5+S_{23}S_6S_5,$$

$$\frac{{}^6\partial\phi_{6z}}{\partial\Theta'_1} = C_{23}C_4S_5+S_{23}C_5;$$

$$\frac{{}^6\partial x_6}{\partial\Theta'_2} = l_2(-C_{23}C_4C_5C_6+C_{23}S_4S_6+C_6S_{23}S_5)S_2+(S_{23}C_4C_5C_6-S_{23}S_4S_6+C_6C_{23}S_5)C_2)+$$
$$+l_3C_6S_5+\lambda_4(C_4C_5C_6-S_4S_6)-\lambda_6(S_4S_6C_5-C_6C_4),$$

$$\frac{{}^6\partial y_6}{\partial\Theta'_2} = l_2(C_{23}C_4C_5S_6+C_{23}S_4C_6-S_6S_{23}S_5)S_2-(S_{23}C_4C_5S_6+S_{23}S_4C_6+S_6C_{23}S_5)C_2)+$$
$$-l_3S_6S_5+\lambda_4(-C_4C_5S_6-S_4C_6)-\lambda_6(S_4C_5C_6+C_4S_6),$$

$$\frac{{}^6\partial z_6}{\partial\Theta'_2} = l_2((S_{23}C_4S_5-C_{23}C_5)C_2-(C_{23}C_4S_5+S_{23}C_5)S_2)-l_3C_5+\lambda_4C_4S_5,$$

$$\frac{{}^6\partial\phi_{6x}}{\partial\Theta'_2} = S_4C_5C_6+C_4S_6, \qquad \frac{{}^6\partial\phi_{6y}}{\partial\Theta'_2} = -S_4S_6C_5+C_6C_4, \qquad \frac{{}^6\partial\phi_{6z}}{\partial\Theta'_2} = S_4S_5;$$

$$\frac{{}^6\partial x_6}{\partial\Theta'_3} = l_3C_6S_5+\lambda_4(C_4C_5C_6-S_4S_6)+\lambda_6(-S_4S_6C_5+C_6C_4),$$

$$\frac{{}^6\partial y_6}{\partial\Theta'_3} = -l_3S_6S_5+\lambda_4(-C_4C_5S_6-S_4C_6)+\lambda_6(-C_4S_6-S_4C_5C_6), \qquad \frac{{}^6\partial z_6}{\partial\Theta'_3} = -l_3C_5+\lambda_4C_4S_5,$$

$$\frac{{}^6\partial\phi_{6x}}{\partial\Theta'_3} = S_4C_5C_6+C_4S_6, \qquad \frac{{}^6\partial\phi_{6y}}{\partial\Theta'_3} = -S_4S_6C_5+C_6C_4, \qquad \frac{{}^6\partial\phi_{6z}}{\partial\Theta'_3} = S_4S_5;$$

$$\frac{{}^6\partial x_6}{\partial\Theta'_4} = \lambda_6S_5S_6, \qquad \frac{{}^6\partial y_6}{\partial\Theta'_4} = \lambda_6S_5C_6, \qquad \frac{{}^6\partial z_6}{\partial\Theta'_4} = 0, \qquad \frac{{}^6\partial\phi_{6x}}{\partial\Theta'_4} = -S_5C_6, \qquad \frac{{}^6\partial\phi_{6y}}{\partial\Theta'_4} = S_5S_6,$$

$$\frac{{}^6\partial\phi_{6z}}{\partial\Theta'_4} = C_5;$$

$$\frac{{}^6\partial x_6}{\partial \Theta'_5} = \lambda_6 C_6, \qquad \frac{{}^6\partial y_6}{\partial \Theta'_5} = -\lambda_6 S_6, \qquad \frac{{}^6\partial z_6}{\partial \Theta'_5} = 0, \qquad \frac{{}^6\partial \phi_{6x}}{\partial \Theta'_5} = S_6, \qquad \frac{{}^6\partial \phi_{6y}}{\partial \Theta'_5} = C_6,$$

$$\frac{{}^6\partial \phi_{6z}}{\partial \Theta'_5} = 0; \qquad \frac{{}^6\partial x_6}{\partial \Theta'_6} = 0, \qquad \frac{{}^6\partial y_6}{\partial \Theta'_6} = 0, \qquad \frac{{}^6\partial z_6}{\partial \Theta'_6} = 0, \qquad \frac{{}^6\partial \phi_{6x}}{\partial \Theta'_6} = 0,$$

$$\frac{{}^6\partial \phi_{6y}}{\partial \Theta'_6} = 0, \qquad \frac{{}^6\partial \phi_{6z}}{\partial \Theta'_6} = 1. \tag{3.15}$$

Equation (3.3) with dependencies (3.15) is a differential description of the end-effector kinematics.

4. The singular configurations

For the singular configurations of manipulators the ${}^6\boldsymbol{J}_6$ determinant equals zero. Thus, to determine such configurations one needs to calculate the determinant. The closed form of the determinant was obtained by Symbolic Math Toolbox library of Matlab. It is described by Eq.(4.1).

$$\det {}^6\boldsymbol{J}_6 = -l_2 S_5(\lambda_4 C_3 - l_3 S_3)(-l_2 S_2 - l_3 S_{23} + \lambda_4 C_{23} + l_1). \tag{4.1}$$

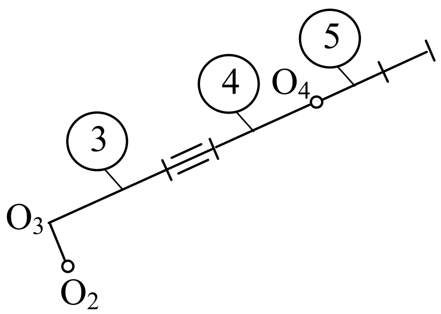

Fig.4. Configuration at which $\Theta'_5 = 0$. The z_3 and z_6 axes are collinear.

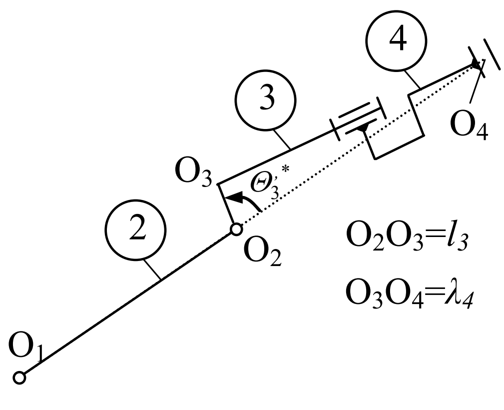

Fig.5. Configuration at which $\lambda_4 C_3 - l_3 S_3 = 0$. Points O_1, O_2 and O_4 are situated on a straight line.

The singular configurations appear in zero values of the factors on the right side of Eq.(4.1). The joint variables at which it will take place, are described by Eq.(4.2).

$$S_5 = 0 \rightarrow \Theta'^{*}_5 = 0, \qquad \lambda_4 C_3 - l_3 S_3 = 0 \rightarrow \Theta'^{*}_3 = \operatorname{arc\,tg}\left(\lambda_4 / l_3\right),$$

$$-l_2 S_2 - l_3 S_{23} + \lambda_4 C_{23} + l_1 = 0 \rightarrow \Theta'^{**}_{3(1,2)} = 2\operatorname{arc\,tg}\left(\frac{-l_3 \pm \sqrt{l_3^2 + \lambda_4^2 - \left(l_2 S_2 - l_1\right)^2}}{\lambda_4 + l_2 S_2 - l_1}\right) - \Theta'_2 \qquad (4.2)$$

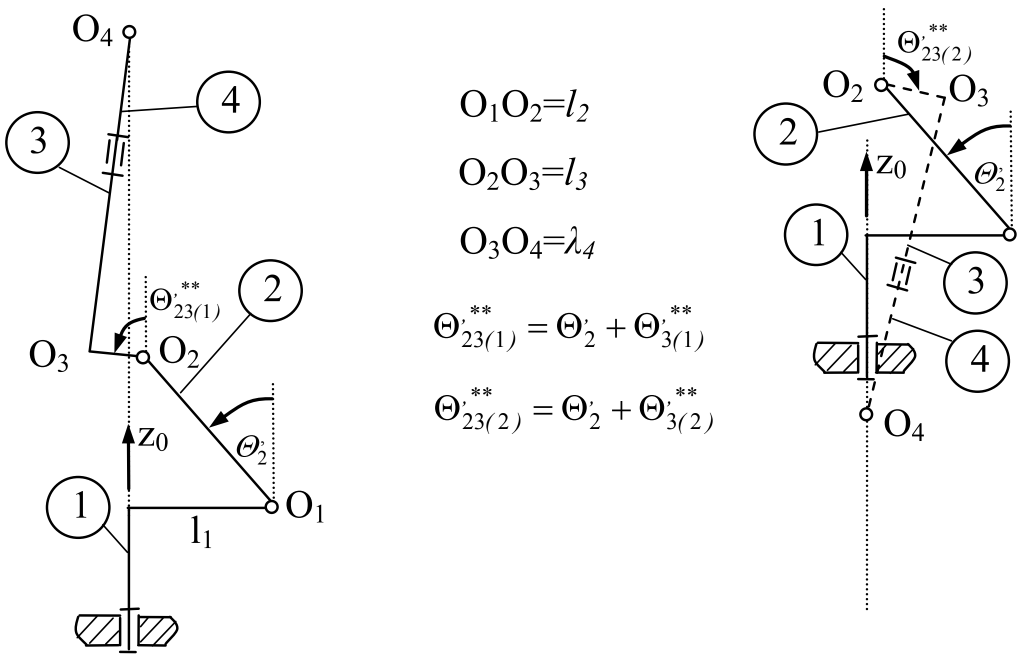

Fig.6. Configuration at which $-l_2 S_2 - l_3 S_{23} + \lambda_4 C_{23} + l_1 = 0$. Point O_4 is placed on the line passing through the z_0 axis.

Figures 4, 5 and 6 illustrate these configurations.

For example, let us analyze kinematic singularities of the manipulator IRB-1400. The manipulator has the following parameters (Szkodny, 2009) $l_1 = 150\ mm$, $l_2 = 600\ mm$, $l_3 = 120\ mm$, $\lambda_1 = 475\ mm$, $\lambda_4 = 720\ mm$, $\lambda_6 = 85\ mm$. The ranges of angle changes Θ'_3 and Θ'_5 are the following: $-70^\circ \le \Theta'_3 \le 65^\circ$, $-115^\circ \le \Theta'_5 \le 115^\circ$. The calculations below were mode in Matlab on a PC from an Intel Pentium processor with a frequency of *2* GHz.

Let us assume that $\Theta'_1 \div \Theta'_4$, $\Theta'_6 = 0^\circ$ for the configuration from Fig.4. For this configuration $\Theta'^{*}_5 = 0^\circ$. In this case the $\det{}^6\boldsymbol{J}_6 = 0$ and the $\operatorname{rank}{}^6\boldsymbol{J}_6 = 5$. The rank of the Jacobian ${}^6\boldsymbol{J}_6$ is less than 6 and therefore, a standard numerical solution algorithm of Eq.(3.3) will interrupt the calculations. One can avoid zeroing the determinant by increasing or decreasing the angle Θ'_5 by a minimum increment $\Delta\Theta'_{5\min}$, resulting from the resolution of the encoder and the gear ratio. Let us assume that a typical resolution of encoders is the following $2\pi / 4096 = 1.5 \cdot 10^{-3}$ *rad.*, and the value of the gear ratio equals 100. For such data $\Delta\Theta'_{5\min} = 1.5340 \cdot 10^{-5}$ *rad*. After the corrections $\Theta'_5 = \pm\Delta\Theta'_{5\min}$, one obtains $\det {}^6\boldsymbol{J}_6 = \mp 5.7653 \cdot 10^3$ and rank${}^6\boldsymbol{J}_6=6$. Let us estimate the gripper position error caused by this correction. The kinematic scheme in Fig.2 results in the error not greater than $\Delta\Theta'_{5\min} \cdot (\lambda_6 + \lambda_7)$. Let us assume that a typical value of the parameter gripper is the following $\lambda_7 = 150$ *mm.* For such a gripper, $\Delta\Theta'_{5\min}$ causes a change of the gripper position not greater than $3.6 \cdot 10^{-3}$ *mm.*

For the configuration from Fig.5 the joint variable $\Theta'^{*}_3 = 80.5376778^\circ$. The manipulator IRB-1400 cannot reach this variable, because it is outside the range of its changes.

Let us assume that $\Theta'_1 = 0^\circ$, $\Theta'_2 = 45^\circ$, $\Theta'_4 = 0^\circ$, $\Theta'_5 = 90^\circ$, $\Theta'_6 = 0^\circ$ for a configuration from Fig.6. For these joint variables one obtains $\Theta'^{**}_{3(1)} = 13.4676545^\circ$, $\Theta'^{**}_{3(2)} = -122.3922989^\circ$. The joint variable $\Theta'^{**}_{3(2)}$ is outside the range of its changes. For the joint variable $\Theta'^{**}_{3(1)}$ det${}^6\boldsymbol{J}_6 = 2.2918 \cdot 10^{-8}$ and rank${}^6\boldsymbol{J}_6=5$. The rank of the Jacobian ${}^6\boldsymbol{J}_6$ is less than 6 and therefore, a standard numerical solution algorithm of Eq.(3.3) will interrupt the calculations. In order to prevent a decrease in the rank of the Jacobian ${}^6\boldsymbol{J}_6$ one will increase and reduce the joint variable $\Theta'^{**}_{3(1)}$ by a minimum increment $\Delta\Theta'_{3\min}$, equal to $\Delta\Theta'_{5\min}$. After corrections $\pm\Delta\Theta'_{3\min}$ of the joint variable $\Theta'^{**}_{3(1)}$, one obtains $\det {}^6\boldsymbol{J}_6 = \pm 4.1854 \cdot 10^3$ and rank${}^6\boldsymbol{J}_6=6$. The change in $\Delta\Theta'_{3\min}$ causes an error of the gripper position, which equals max. $14.7 \cdot 10^{-3}$ *mm.*

5. Summary

A differential description of the manipulator kinematics, presented in this paper, is the basis for design of contemporary software drivers of industrial robots, independent of the software manufacturers robots.

This description can be applied in IRB manipulators series 1000, 2000, 3000, 4000, 6000; in Fanuc manipulators M6, M16, M710, M10, M900; in a KUKA manipulators KR5, KR6, KR15, KR16; in a Mitsubishi manipulators RV-1A, RV-2A, RV-3S, RV-6S, RV12S; in Adept manipulators s300, s650, s850, s1700; in Kawasaki manipulators series M, FS300N, ZHE100U, KF121.

For KUKA KRC3, Adept s300 and Mitsubishi RV-2AJ manipulators the origin of second and third link frame is at the same point $O_2=O_3$ (see Fig.2). Therefore for these manipulators one has to assume that $l_3 = 0$.

In modern industrial robot controllers actuator variables calculation is realized in two stages in master-level of control, and in a single stage in the slave-level control. In the first step, in the master-level the trajectory generator calculates the Cartesian coordinates of the trajectory via-points. The trajectory is specified by a suitable Cartesian programming command (for example, LIN). The calculated via-points in the Cartesian space are pre-assigned times. The times resulting from the speed in the Cartesian space, declared in a suitable command parameter are assigned to the points. In the second stage joint variables are determined (Craig, 1993; Szkodny, 2012) for the earlier computed via-points in the Cartesian space. To determine these variables the iterative calculation algorithm of solutions Eq.(3.3) is used. Next actuator variables are calculated for joint variables of these via-points. Such calculated actuator variables are sent from the master level into the slave level of control, with constant frequency. The actuator variables of the via-points are calculated at the time of the robot motion to the previously calculated via-points. If a via point determined by the trajectory generator in the Cartesian space is singular, calculating joint variables using the iterative algorithm, without a protection against the Jacobian rank reduction results in an undesirable interruption of the calculations and stopping the robot. The robot stops before the a singular via-point when approaching the previously calculated nonsingular via-points.

It may happen that the via-point obtained in the first stage of the calculation is near a singular via-point. In this case the iterative calculation algorithm of joint variables may be convergent, but the resulting actuator variables will be very different from the corresponding variables of the previous via-point. For large increments of joint variables and already pre-declared times of the previous and current via-points, it may happen that the average rate of change of actuator variables is much greater than the maximum actuator speed. Furthermore, in this case, the actuator variables accelerations may occur above the maximum acceleration that can be reached by the actuators. The problem of these exceedances is solved by the trajectory generator in the slave-layer of control. Command motion parameters (e.g., LIN) are sent from the master level into the slave level control. These are, among others, the declared translational movement speed and acceleration. In general, the acceleration is declared in percent of the maximum permissible values. These parameters are the basis for the generation of actuator variables in the slave level of control. Waveforms of actuator variables are generated at a constant speed sections connected with fixed segments accelerations. If the movement command, e.g., LIN, declares translational movement speed and acceleration at the level of *10*% of their maximum values, the result is that the in slave level of control the waveforms with speed and acceleration rates of not more than *10*% of the maximum actuator speed and acceleration will be generated.

In the controllers of contemporary industrial robots one can write his/her own applications. These applications can be written using Eq.(3.3) derived from this work. In this equation, differentials $d\Theta_i^{,}$ should be replaced by the increments $\Delta\Theta_i^{,}$, and differentials ${}^6dx_6$, ${}^6dy_6$, ${}^6dz_6$, ${}^6d\phi_{6x}$, ${}^6d\phi_{6y}$ and ${}^6d\phi_{6z}$ by corresponding increments ${}^6\Delta x_6$, ${}^6\Delta y_6$, ${}^6\Delta z_6$, ${}^6\Delta\phi_{6x}$, ${}^6\Delta\phi_{6y}$ and ${}^6\Delta\phi_{6z}$. Such discretized Eq.(3.3) is the basis for the second stage of the calculation at master level of control. During this stage, the joint variables of the via-pints are iteratively calculated from the discretized Eq.(3.3). Recall that at this level of control in the first stage the Cartesian coordinates of the via-points are calculated.

Assume that the joint variables of the previous via-point of trajectory were calculated iteratively and denote them by $\Theta_i^{,(-)}$. For the calculation of the joint variables of the current via-point, we will also apply the iterative calculation algorithm. In the next for example the *k*-th iteration of calculations, the initial values of the link variables $\Theta_{i\,start(k)}^{,(-)}$ are equal to the end values $\Theta_{i\,end(k-1)}^{,(-)}$ from the previous iteration. We calculate the Jacobian ${}^6\boldsymbol{J}_6\left(\Theta_{i\,start(k)}^{,(-)}\right)$ and increases $\Delta\Theta_{i(k)}^{,(-)}$ from the discretized Eq.(3.3). The end values of the joint variables in the *k*-th iteration $\Theta_{i\,end(k)}^{,(-)}$ are the sum $\Theta_{i\,start(k)}^{,(-)}+\Delta\Theta_{i(k)}^{,(-)}$. Subsequent iterative calculations cause that the end values $\Theta_{i\,end(k)}^{,(-)}$ of iteration approach values $\Theta_i^{,}$ representing the solution of the inverse kinematics of the current via-point. If the current via-point is singular, the approaching it gives rise to the risk of a reduction of the Jacobian ${}^6\boldsymbol{J}_6\left(\Theta_{i\,start(k)}^{,(-)}\right)$ rank. These protection measures are presented in the fourth chapter by using the example of the IRB-1400 manipulator. They consist in the correction of the angles $\Theta_3^{,}$ and $\Theta_5^{,}$ by minimum increments $\Delta\Theta_{3\min}^{,}$ and $\Delta\Theta_{5\min}^{,}$, seen by the servos.

If during the iterative calculations $\Theta_{3start(k)}^{,(-)}$ satisfies the inequality $\left|\Theta_{3\,start(k)}^{,(-)}-\Theta_3^{,*}\right|<\Delta\Theta_{3\min}^{,}$, we assume: $\Theta_{3end(k)}^{,(-)}=\Theta_3^{,*}-\mathrm{sign}\left(\Delta\Theta_{3(k-1)}^{,(-)}\right)\cdot\Delta\Theta_{3\min}^{,}$. If $\left|\Theta_{3\,start(k)}^{,(-)}-\Theta_{3(1,2)}^{,**}\right|<\Delta\Theta_{3\min}^{,}$, we substitute $\Theta_{3end(k)}^{,(-)}=\Theta_{3(1,2)}^{,**}-\mathrm{sign}\left(\Delta\Theta_{3(k-1)}^{,(-)}\right)\cdot\Delta\Theta_{3\min}^{,}$. Also, for the angle $\Theta_{5\,start(k)}^{,(-)}$ satisfying the inequality

$\left|\Theta_{5\,start(k)}^{,(-)} - \Theta_5^{,*}\right| = \left|\Theta_{5\,start(k)}^{,(-)}\right| < \Delta\Theta_{5\min}^{,}$, we assume: $\Theta_{5end(k)}^{,(-)} = -\mathrm{sign}\left(\Delta\Theta_{5(k-1)}^{,(-)}\right) \cdot \Delta\Theta_{5\min}^{,}$. For the other variables $\Theta_{i\,end(k)}^{,(-)} = \Theta_{i\,start(k)}^{,(-)}$.

Using such protection measures one should check the range of the changes in the gripper position caused by above angles corrections. For the sample IRB-1400 manipulator suggested corrections can at most result in changes of the position which is the sum of errors, i.e., $3.6 \cdot 10^{-3}\ mm. + 14.7 \cdot 10^{-3}\ mm. = 18.3 \cdot 10^{-3}\ mm.$ The typical positioning accuracy of industrial robots are of the order $10^{-1}\ mm.$ Thus, the errors caused by such corrections are acceptable.

If the gripper position errors caused by such corrections were equal to or greater than the required positioning accuracy, inverse kinematics equations in a global form, presented in Szkodny (2010) would need to be used. For these equations, we can apply simple criteria for selecting solutions at the joint variables. Such criteria are much simpler and faster to implement than the criteria proposed in the Jacobian methods (Chiacchio, 1996; Nakamura, 2009; Siciliano, 2010; Tchoń, 2000).

The differential description of the kinematics, including corrections to prevent the loss of the Jacobian rank, presented in this paper, constitutes the basis for the creation of our own software, free from the basic defect, namely, undesired stopping in singular positions.

Nomenclature

- $\boldsymbol{A}_i$ – homogeneous matrix describing the *i*-th link frame in relation to the *i-1*-st link
- ${}^{i-1}\boldsymbol{a}_6$ – versor of x_6 axis described in *i-1*-st frame
- ${}^{i-1}a_{6x}, {}^{i-1}a_{6y}, {}^{i-1}a_{6z}$ – coordinates of versor ${}^{i-1}\boldsymbol{a}_6$ in *i-1*-st frame
- ${}^{i-1}\boldsymbol{b}_6$ – versor of y_6 axis described in *i-1*-st frame
- ${}^{i-1}b_{6x}, {}^{i-1}b_{6y}, {}^{i-1}b_{6z}$ – coordinates of versor ${}^{i-1}\boldsymbol{b}_6$ in *i-1*-st frame
- ${}^{i-1}\boldsymbol{c}_6$ – versor of z_6 axis described in *i-1*-st frame
- ${}^{i-1}c_{6x}, {}^{i-1}c_{6y}, {}^{i-1}c_{6z}$ – coordinates of versor ${}^{i-1}\boldsymbol{c}_6$ in *i-1*-st frame
- ${}^{6}\boldsymbol{D}_6$ – Cartesian differential matrix
- ${}^{i-1}\boldsymbol{d}_6$ – vector describes the position of the *6*-th link frame origin in relation to *i-1*-st frame, described in *i-1*-st frame
- ${}^{i-1}d_{6x}, {}^{i-1}d_{6y}, {}^{i-1}d_{6z}$ – the coordinates of the vectors ${}^{i-1}\boldsymbol{d}_6$ in *i-1*-st frame
- ${}^{i-1}d\boldsymbol{r}_i$ – differential displacement of the *x'y'z'* frame origin in relation to the $x_{i-1}y_{i-1}z_{i-1}$ frame, caused by a differential increase of the joint variable $d\Theta_i^{,}$
- ${}^{i-1}d\boldsymbol{r}_6$ – differential displacement of the *6*-th link frame origin in relation to the $x_{i-1}y_{i-1}z_{i-1}$ frame, caused by a differential increase of the joint variable $d\Theta_i^{,}$
- ${}^{i-1}d\boldsymbol{\phi}_i$ – differential rotation of the *x'y'z'* frame in relation to $x_{i-1}y_{i-1}z_{i-1}$ frame, caused by a differential increase of the joint variable $d\Theta_i^{,}$
- ${}^{i-1}d\boldsymbol{\phi}_{6,i}$ – differential rotation of the *6*-th link frame in relation to $x_{i-1}y_{i-1}z_{i-1}$ frame, caused by a differential increase of the joint variable $d\Theta_i^{,}$
- $d\boldsymbol{q}$ – joint differential matrix
- $\boldsymbol{E}$ – homogeneous matrix describing the gripper frame in relation to the end-effector frame

$^{i-1}\boldsymbol{i}_{i-1}$ – versor of x_{i-1} axis described in i-1-st frame

$^{6}\boldsymbol{J}_{6}$ – end-effector Jacobian $^{6}\boldsymbol{J}_{6}$, described in the 6-th link frame

$^{i-1}\boldsymbol{j}_{i-1}$ – versor of y_{i-1} axis described in i-1-st frame

$^{i-1}\boldsymbol{k}_{i-1}$ – versor of z_{i-1} axis described in i-1-st frame

l_i – displacement along x_i axis

$\boldsymbol{T}_6$ – homogeneous matrix describing the end-effector frame in relation to base frame

$\boldsymbol{T}_{6req}$ – required matrix $\boldsymbol{T}_6$

$^{i-1}\boldsymbol{T}_6$ – homogeneous matrix describing the end-effector frame in relation to i-1-st frame

$\boldsymbol{X}$ – homogeneous matrix describing the gripper frame in relation to base frame

$\boldsymbol{X}_{req}$ – required matrix $\boldsymbol{X}$

$x_0y_0z_0$ – base frame

$x_iy_iz_i$ – i-th link frame

$x'y'z'$ – frame connected with i-th link, coinciding with i-1-st frame for $d\Theta_i^{'}=0$

α_i – rotation angle of the i-th link frame in relation to i-1-st frame about the x_{i-1} axis

$^{6}\boldsymbol{\Delta}_6$ – differential transformation matrices of the end-effector described in the 6-th frame

$^{7}\boldsymbol{\Delta}_7$ – differential transformation matrices of the gripper described in the7-th frame

Θ_i – rotation angle of the i-th link frame in relation to the i-1-st frame about z_{i-1} axis

$\Theta_i^{'}$ – the i-th link variable

λ_i – displacement along z_i axis

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