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MEMORY DEPENDENT TRIPLE-PHASE-LAG THERMO-ELASTICITY IN THERMO-DIFFUSIVE MEDIUM

Anand Kumar YADAV*
Mathematics, Shishu Niketan Model Sr Sec School, INDIA
E-mail: yadavanand977@gmail.com

Aarti SINGH
Department of Mathematics, Maharishi Markandeshwar (Deemed to be University), Mullana, Ambala, INDIA

Paweł JURCZAK
Department of Mechanical Engineering, Department of Mechanical Engineering, Institute of Mechanical Engineering, University of Zielona Góra, POLAND

The objective of the paper is to look at the propagation and reflection of plane waves in a thermo-diffusion isotropic medium. The reflection of plane waves in a thermo-diffusion medium was investigated in this study with reference to triple phase lag thermo-elasticity. The memory dependent derivative (MDD) is applied for this investigation. The fundamental equations are framed and solved for a particular plane. The four plane waves that are propagating across the medium are, shown namely: longitudinal displacement, *P-wave*, thermal diffusion *T-wave*, mass diffusion *MD-wave* and shear vertical SV-wave. These four plane wave velocities are listed for a specific medium, illustrating the impact of the diffusion coefficient and are graphically represented. Expressions for the reflection coefficient for the incidence plane wave are produced from research on the reflection of plane waves from the stress-free surface. It should be noted that these ratios are graphically represented and shown when diffusion and memory dependent derivative (MDD) factors are in play. The new model is relevant to many different fields, including semiconductors, earth- engineering, and electronics, among others, where thermo-diffusion elasticity is significant. Diffusion is a technique that can be applied to the production of integrated circuits, MOS transistors, doped polysilicon gates for the base and emitter in transistors, as well as for efficient oil extraction from oil reserves. Wave propagation in a thermos-diffusion elastic media provides crucial information about the presence of fresh and enhanced waves in a variety of technical and geophysical contexts. For experimental seismologists, developers of new materials, and researchers, this model might be useful in revising earthquake estimates.

Keywords: diffusion, triple phase lag thermo-elasticity, reflection coefficients, Memory Dependent Derivative (MDD).

1. Introduction

Engineering structural problems involve calculating deformations, deflections, and internal forces within structures. Thermo-elasticity, an integral offshoot of elasticity, is crucial in stress analysis and mechanical behaviors in materials like steel, wood, concrete, coal, polymers, metals, composites, and rocks and concrete. Biot [1] developed the classical theory of thermo-elasticity (CTT), which governed heat transfer within structures. Lord and Shulman [2] modified Fourier's law by adding a new parameter called one relaxation time to remove the paradox of Biot [1]. Later, researchers [3-6] also developed different generalized heat models. Tzou [7] introduced the dual time delay model (DPL) in heat law, which was later extended by Roy Choudhuri [8] to the three-phase lags heat transfer system (TPL). This model describes transient heat

* To whom correspondence should be addressed

conduction in materials with thermal relaxation phenomenon, considering temperature gradient, heat-flux, and heat storage as constitutive elements and three separate time lags.

Thermo-diffusion is a phenomenon in mixtures of substances, particularly fluids and solid materials, where components move due to temperature gradients, i.e. different particles within the mixture tend to move in response to this temperature difference. It is crucial in fields like chemical engineering, geophysics, astrophysics, and material science, as it optimizes processes and designs more efficient systems involving mixtures of different substances. Nowacki, and some eminent researchers [9-13] extensively investigated dynamical issues of diffusion and developed the coupled theory of diffusion of heat and mass. Othman and Eraki [14] analyzed generalized thermoelastic diffusion in a magnetized medium under the effect of initial stress utilizing TPL concept. Yadav [15] studied the effect of diffusion in an orthotropic medium. Mondal and Kanoria [16] enhanced the Fourier law in the context of memory-dependent derivative for TPL model. Yadav [17] examined the effect of TPL and diffusion in semiconductors.

Yadav [18,19] studied thermo-diffusion in detail. Many investigators have studied the reflection problem in a thermoelastic medium, isotropic micropolar and diffusive medium, eminent of them are Marin and his co-workers [20], Othman *et al.* [21], Yadav [22], Abbas *et al.* [23], Singh *et al.* [24], Sheoran *et al.* [25], Saeed *et al.* [26].

Yadav [27] explored the effect of a magnetic field and multi-phase-lag on photothermal-plasma wave in a two-temperature diffusive-semiconductor. Yadav *et al.* [28] examined the results of nonlocal impedance in a micropolar porous thermo-diffusive medium. Yadav *et al.* [29] investigated piezo-electric waves in an orthotropic hygro-thermo-elastic medium. Regarding the hypothesis of elastic substances with a dipolar structure, Marin *et al.* [30] presented a generalization of the Saint- Venant principle. By using the Eigen value method, Alzahrani *et al.* [31] investigated a two-dimensional porous substance with different conductivities. Abouelregal and Marin [32] analyzed thermoelastic vibrations in a nonlocal nanobeam. Singh [33] investigated how generalized thermoelastic diffusion affects SV wave behaviour in an elastic solid at free surface.

The new memory-dependent derivatives model in thermo-elasticity theory is suitable for various fields, including semiconductors, earth-engineering, and electronics, where thermo-diffusion elasticity is significant. It addresses the historical behaviour of displacement fields and temperature fields over time, unlike traditional elasticity theory which considers instantaneous values and Fourier's law without time-dependent considerations. The wave propagation is the easiest and most cost-effective method to detect oil and mineral deposits without drilling into the earth. Since seismic wave technologies offer higher accuracy, higher resolution, and are more cost-effective than drilling, which is extensive and time-consuming, almost all oil companies rely on seismic interpretation. Diffusion is a technique used in various fields, including integrated circuits, MOS transistors, and oil production from oil reserves. Wave propagation in thermo-diffusion elastic media provides crucial information for seismic studies, material creation, and researchers. This work contributes to the understanding of thermal lagging's effects and helps create more sophisticated models capturing heat flow.

2. Memory dependence derivative origin

Fractional calculus has been employed throughout the past few decades in a variety of fields, including control engineering, electromagnetic engineering, aerospace engineering, nuclear physics, signal processing, and quantum mechanics. Unconventional building methods are also necessary for the ongoing development of novel materials. One of fractional calculus' most considerable benefits across numerous applications is the fact that it is nonlocal. A fractional-order derivative (FOD) is a generalisation of an integral-order derivative and it is an effective method for outlining memory phenomenon. Although it does not mimic any physical process, the memory function of a fractional derivative is known as the kernel function. Fractional derivatives continue to trail the integer-order calculus far behind due to the ambiguous physical meaning. It was modelled after the letter L'Hospital wrote to Leibnitz regarding the definition of the half-order derivative in 1695. Also, the use of fractional-order derivatives to describe memory processes is one approach. Differentiation could be the antithesis of a integration. Since then, a number of definitions have been created. The most widely used non-local FODs are the Riemann-Liouville (RL-FOD) and Caputo (C-FOD) types.

2.1. Definition (RL-FOD)

The Riemann-Liouville FOD is defined as

$$ {}^{0}D_t^{\alpha}\varepsilon(t)=\frac{d^{\alpha}\varepsilon(t)}{dt^{\alpha}}=\frac{d^n}{dt^n}\left(\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{\varepsilon(s)}{(t-s)^{1+\alpha-n}}ds\right), \qquad n-1\le\alpha<n, \tag{2.1}$$

and $\Gamma(.)$ is the Gamma function, n is integer of order α, which is dependent on the strain history $\varepsilon(t)$ from 0 to t.

2.2. Definition (C-FOD)

The Caputo FOD ${}^{0}D_t^{\alpha}\varepsilon(t)$ of order α with respect to time is defined as

$$ {}^{0}D_t^{\alpha}\varepsilon(t)=\frac{d^{\alpha}\varepsilon(t)}{dt^{\alpha}}=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{1}{(t-s)^{1+\alpha-n}}\frac{d^n\varepsilon(s)}{dt^n}ds, \qquad n-1\le\alpha<n. \tag{2.2}$$

A material model has been studied by Mainardi [34]; this model provides a formula for memory phenomenon in diverse fields. The simulation uses the form

$$ {}^{0}D_t^{\alpha}\varepsilon(t)=\kappa\sigma(t) \tag{2.3}$$

where ${}^{0}D_t^{\alpha}\varepsilon(t)$ is the FOD, which is reliant on the history of strains from 0 to t. For integral value of $\alpha=n$

$$ {}^{0}D_t^{\alpha}\varepsilon(t)=\frac{d^n\varepsilon(t)}{dt^n} \tag{2.4}$$

where, $\kappa>0$ is a constant, $\varepsilon(t)$ is the strain history and $\sigma(t)$ is the stress history from time 0 to t. In the below mentioned concept, memory dependence means non-locality in time. In general, a memory process consists of two stages: the first is brief, with permanent retention at the beginning, and it cannot generally be overlooked; the other is guided by the fractional model equation (2.3). Most of the time, the genesis is not the crucial transition between the initial stage and the working stage. Comparing this to the standard fractional models of one step is considerably different. The most important idea is that a fractional derivative's order acts as a memory index. The dimensionless form of the solution of Eq.(2.3) is

$$ \Im(\chi)=\chi^{\beta}-(\chi-1)^{\beta} \tag{2.5}$$

where

$$ \chi=\frac{t}{t_m}, \qquad \Im(\chi)=\frac{\varepsilon(t)}{\varepsilon_m}$$

where ε_m is a creeping strain as time passes $t=t_m$. $\Im(\chi)$ is the memory function. Equation (2.5) implies that if β increases, $\Im(\chi)$ increases. Also, Eq.(2.5) shows that the process of forgetting is slowed down as the index value β increases. In particular, at $\beta=0$, $\Im=0$, meaning that “nothing is memorized”, and $\Im=1$, for $\beta=1$ conveys the message "nothing is forgotten". Therefore, the fractional order β is essentially referred to as the memory effect index. Diethelm [35] by inculcating a kernel function revised the Caputo-type fractional-order derivative which is termed the memory function, in which a physical process does not change. Wang and Li [36] incorporated first exposure to Fourier's theory of heat flow by MDD to provide new hyperbolic-type heat equation models with measuring memory.

$$D_{\tau}^{(1)} f(t) = \frac{1}{\tau} \int_{t-\tau}^{t} k(t-\xi) f'(\xi) d\xi. \tag{2.6}$$

If $k(t-\xi)=1$,

$$D_{\tau}^{(1)} f(t) = \frac{1}{\tau} \int_{t-\tau}^{t} f'(\xi) d\xi = \frac{f(x,t) - f(x,t-\tau)}{\tau}$$

where $\tau(>0)$ is the delay time, which is also freely selectable. They introduced fractional derivative (MDD) in terms of an integral form of a common derivative with arbitrary "phase time lag" and kernel function $k(t-\xi)$ (as memory effect), can be taken freely in line with the necessity of applications over a slipping interval $[(t-\tau), t]$. The magnitude of MDD should be always smaller than that of the normal partial derivative.

Later, Ezzat *et al.* [37,38] proposed the first order MDD, instead of fractional calculus, in the rate of heat flux in LS theory [2] to denote memory-dependence as:

$$\left(1+\tau_0 D_{\tau_0}\right) q_i = -KT_{,i} \Rightarrow q_i + \tau_0 D_{\tau_0} q_i = -KT_{,i}, \tag{2.7}$$

$$D_{\tau_0} q_i = D_{\tau_0}^{(1)} q_i = \frac{1}{\tau_0} \int_{t-\tau_0}^{t} k(t,\xi) q_i^{(1)}(\xi) d\xi \tag{2.8}$$

where, $k(t,\xi)$ is the kernel function, D_{τ_0} is MDD term of timing, τ_0 is the time delay parameter, for present time as *t*, and $[t-\tau_0, t)$ is the past-time interval. According to Ezzat *et al.* [39] memory kernel, is taken as follows:

$$k(t-\xi) = 1 - \frac{2B}{\tau_0}(t-\xi) + \frac{A^2}{\tau_0^2}(t-\xi)^2 \tag{2.9}$$

here A, B are parametric coefficients, the values of which are freely selectable and a selection of four kernels is made as

$$k(t-\xi) = 1 - \frac{2B}{\tau_0}(t-\xi) + \frac{A^2}{\tau_0^2}(t-\xi)^2 = \begin{cases} 1, & \text{if} \quad A=0, B=0, \\ 1-\dfrac{(t-\xi)}{\tau_0}, & \text{if} \quad A=0, B=\dfrac{1}{2}, \\ 1-(t-\xi), & \text{if} \quad A=0, B=\dfrac{\tau_0}{2}, \\ \left(1-\dfrac{(t-\xi)}{\tau_0}\right)^2, & \text{if} \quad A=1, B=1. \end{cases} \tag{2.9}$$

Later, Karamany and Ezzat [40] proposed the first order MDD, rather than fractional calculus, within the rate of heat-flux in the LS theory [2] to denote memory-dependence as

$$\left(1+\tau_d D_{\tau_d}\right)\eta_i = -DP_{,i} \tag{2.10}$$

where, $D_{\tau d}$, is MDD with respect to time, τ_d is a time lag parameter due to diffusion, for the present time as t, we could say $[t-\tau_0, t)$ is the past time interval. Studies in recent years have extensively examined numerous issues in thermo-elasticity based on MDD and fractional-order derivative [41-52]. These research works studied thermal waves with micro concentration in thermo-diffusion rotator. Yadav *et al.* [53] investigated the reflection of hydrothermal waves was examined in a nonlocal theory of linked thermo-elasticity.

Many researchers have examined wave propagation in an elastic medium under different parameters. Othman *et al.* [54] studied the effect of fractional and magnetic field parameters on plane waves of generalized - thermoelastic diffusion with reference to a temperature dependent elastic medium. Othman and Eraki [55] investigated the effect of gravity on generalized thermoelastic diffusion due to laser pulse heating medium utilizing DPL concept. Othman, and Mondal [56,57] investigated the effect of memory dependent-derivative in a thermoelastic and micropolar medium. Othman and his co-worker [58,59] investigated the effect of diffusion and rotation and initial stress on the plane wave propagation in micropolar as well as thermoelastic medium. Lotfy K. and Hassan [60] investigated a thermal shock problem under normal mode method for two-temperature theory. Lotfy and his co-workers [61-63] studied the effect of variable thermal conductivity, magnetic field, functionally gradation properties on the plasma waves in a semiconductor. Mahdy *et al.* [64] investigated the effect of laser pulses heating, variable thermal conductivity and hyperbolic two-temperature theory using a magneto-photothermal theory of semiconductor. Yasein *et al.* [65] studied elasto-photo-thermal waves in a semiconductor medium under the influence of variable thermal conductivity during photothermal excitation subjected to thermal ramp type heating. Mahdy *et al.* [66] studied numerical methods to examine rubella ailment disease.

3. Conceptualization of problem

According to Lord and Shulman [2], the basic equations in triple phase lag theory of a thermo-elastic medium with mass diffusion with MDD heat flow developed by Ezzat *et al.* [37-39] as follows

– entropy equation

$$-q_{i,i} = \rho T_0 \dot{S}, \tag{3.1}$$

– mass conservation equation

$$-\dot{C} = \eta_{i,i}, \tag{3.2}$$

– constitutive relations

$$\left.\begin{aligned} &\tau_{ij} = \lambda\delta_{ij}e_{kk} + 2\mu e_{ij} - \beta_T T\delta_{ij} - \beta_C C\delta_{ij}, \beta_T = (3\lambda+2\mu+\kappa)\alpha_t, 2e_{ij} = (u_{i,j}+u_{j,i}) \\ &\eta_i = -DP_{,i}, P = bC - \beta_C e_{kk} - aT, P_{,ii} = bC_{,ii} - \beta_C e_{kk,ii} - aT_{,ii}, \beta_C = (3\lambda+2\mu+\kappa)\alpha_c, \\ &\rho T_0 S = \beta^T T_0 e_{ij} + \rho C_E T + aT_0 C, \dot{\upsilon} = T, -\ddot{C} = \dot{\eta}_{i,i}, \end{aligned}\right\}, \tag{3.3}$$

– equations of motion

$$\mu\nabla^2\boldsymbol{u} + (\lambda+\mu)\nabla(\nabla.\boldsymbol{u}) - \beta_T\nabla T - \beta_C\nabla C = \rho\ddot{\boldsymbol{u}}. \tag{3.4}$$

Considering the plane strain problem in *x*-*y* plane equation (3.4) become

$$(\lambda+2\mu)\frac{\partial^2 u}{\partial x^2}+(\lambda+\mu)\frac{\partial^2 v}{\partial x\partial y}+\mu\frac{\partial^2 u}{\partial y^2}-\beta_T\frac{\partial T}{\partial x}-\beta_C\frac{\partial C}{\partial x}=\rho\frac{\partial^2 u}{\partial t^2}, \tag{3.5}$$

$$(\lambda+2\mu)\frac{\partial^2 v}{\partial y^2}+(\lambda+\mu)\frac{\partial^2 u}{\partial x\partial y}+\mu\frac{\partial^2 v}{\partial x^2}-\beta_T\frac{\partial T}{\partial y}-\beta_C\frac{\partial C}{\partial y}=\rho\frac{\partial^2 v}{\partial t^2}. \tag{3.6}$$

According to Roy Choudhuri [8] the heat equation in TPL model is stated as

$$-K\left(1+\tau_T\frac{\partial}{\partial t}\right)T_{,i}-K^*\left(1+\tau_V\frac{\partial}{\partial t}\right)\upsilon_{,i}=\left(1+\tau_q\frac{\partial}{\partial t}+\frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right)q_i, \tag{3.7}$$

with the condition

$$\frac{\partial\upsilon}{\partial t}=T. \tag{3.8}$$

Using Eqs (3.1), (3.3) and (3.8) in Eq.(3.7) we get

$$\begin{aligned}&K\frac{\partial}{\partial t}\left(1+\tau_T\frac{\partial}{\partial t}\right)\nabla^2 T+K^*\left(1+\tau_V\frac{\partial}{\partial t}\right)\nabla^2 T=\\&=\left(1+\tau_q\frac{\partial}{\partial t}+\frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right)\left(\beta_T T_0\frac{\partial^2 e_{ij}}{\partial t^2}+\rho C_E\frac{\partial^2 T}{\partial t^2}+aT_0\frac{\partial^2 C}{\partial t^2}\right),\end{aligned} \tag{3.9}$$

Following Ezzat *et al.* [37, 38] and Wang and Li [36] utilising the definition of MDD in TPL theory, the heat flow in TPL with MMD effect having freely taken kernel $k(t-\xi)$ over a slipping interval $[(t-\tau_i), t]$ is

$$D_{\tau_i}f(t)=\frac{1}{\tau_i}\int_{t-\tau_i}^{t}k(t-\xi)f^{/}(\xi)d\xi \tag{3.10}$$

here $k(t-\xi)$ is differentiable regarding variables t and ξ, that demonstrate the memory effect on the delay interval $[(t-\tau_i), t]$, and $0 \le K(t-\xi) < 1$ *for* $\xi \in [(t-\tau_i), t]$. Here τ_i takes the values τ_1,τ_2,τ_3, for TPL heat flow and it takes the values τ_4,τ_5,τ_6, for the TPL diffusion equation.
Equation (3.9) takes the form

$$\begin{aligned}&K(1+\tau_T D_{\tau_2})\nabla\dot{T}+K^*(1+\tau_V D_{\tau_3})\nabla T=\\&=\left(1+\tau_q D_{\tau_1}+\frac{\tau_q^2}{2}D_{\tau_1}^2\right)\left\{\beta_T T_0\left(\frac{\partial\ddot{u}}{\partial x}+\frac{\partial\ddot{v}}{\partial y}\right)+\rho C_E\ddot{T}+aT_0\ddot{C}\right\}\end{aligned} \tag{3.11}$$

where τ_q, τ_T, τ_V are the phase lag of heat flux due to thermal inertia, phase lag of temperature gradient, phase lag of thermal displacement gradient, and τ_1, τ_2, τ_3 are the TPL time lags for MDD and $D_{\tau_1}, D_{\tau_2}, D_{\tau_3}$, are memory dependent derivatives, respectively.

Heat equation (3.9) in TPL theory with MDD effect in the *x*-*y* plane can be written as

$$\begin{gathered}K(1+\tau_T D_{\tau_2})\frac{\partial}{\partial t}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+K^*(1+\tau_V D_{\tau_3})\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)= \\ =\left(1+\tau_q D_{\tau_1}+\frac{\tau_q^2}{2}D_{\tau_1}^2\right)\left\{\beta_T T_0\left(\frac{\partial \ddot{u}}{\partial x}+\frac{\partial \ddot{v}}{\partial y}\right)+\rho C_E\frac{\partial^2 T}{\partial t^2}+aT_0\frac{\partial^2 C}{\partial t^2}\right\}\end{gathered} \tag{3.12}$$

where τ_q, τ_T, τ_V are the phase lag of heat flux due to thermal inertia, phase lag of temperature gradient, phase lag of thermal displacement gradient, and τ_1, τ_2, τ_3 are the delay times due to TPL model for MDD, $D_{\tau_1}, D_{\tau_2}, D_{\tau_3}$, are memory dependent derivatives, respectively. Using Eq.(3.10), the Eq.(3.12) becomes

$$\begin{gathered}K\left(\frac{\partial^2 \dot{T}}{\partial x^2}+\frac{\partial^2 \dot{T}}{\partial y^2}\right)+K\frac{\tau_T}{\tau_2}\int_{t-\tau_2}^{t}k(t-\xi)\left(\frac{\partial^4 T}{\partial \xi^2\partial x^2}+\frac{\partial^4 T}{\partial \xi^2\partial y^2}\right)d\xi+K^*\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right) \\ +K^*\frac{\tau_V}{\tau_3}\int_{t-\tau_3}^{t}k(t-\xi)\left(\frac{\partial^3 T}{\partial \xi\partial x^2}+\frac{\partial^3 T}{\partial \xi\partial y^2}\right)d\xi=\left\{\beta_T T_0\left(\frac{\partial \ddot{u}}{\partial x}+\frac{\partial \ddot{v}}{\partial y}\right)+\rho C_E\frac{\partial^2 T}{\partial t^2}+aT_0\frac{\partial^2 C}{\partial t^2}\right\}+ \\ +\frac{\tau_q}{\tau_1}\int_{t-\tau_1}^{t}k(t-\xi)\left\{\beta_T T_0\left(\frac{\partial^4 u}{\partial \xi^3\partial x}+\frac{\partial^4 v}{\partial \xi^3\partial y}\right)+\rho C_E\frac{\partial^3 T}{\partial \xi^3}+aT_0\frac{\partial^3 C}{\partial \xi^3}\right\}d\xi+ \\ \frac{\tau_q^2}{2\tau_1}\int_{t-\tau_1}^{t}k(t-\xi)\left\{\beta_T T_0\left(\frac{\partial^5 u}{\partial \xi^4\partial x}+\frac{\partial^5 v}{\partial \xi^4\partial y}\right)+\rho C_E\frac{\partial^4 T}{\partial \xi^4}+aT_0\frac{\partial^4 C}{\partial \xi^4}\right\}d\xi,\end{gathered} \tag{3.13}$$

$$\begin{gathered}K\left(1+\frac{\tau_T}{\tau_2}G_2(t)\right)\left(\frac{\partial^2 \dot{T}}{\partial x^2}+\frac{\partial^2 \dot{T}}{\partial y^2}\right)+K^*\left(1+\frac{\tau_V}{\tau_3}G_3(t)\right)\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)= \\ =\left(1+\frac{\tau_q}{\tau_1}G_1(t)+\frac{\tau_q^2}{2\tau_1}G_{12}(t)\right)\left\{\beta_T T_0\left(\frac{\partial \ddot{u}}{\partial x}+\frac{\partial \ddot{v}}{\partial y}\right)+\rho C_E\frac{\partial^2 T}{\partial t^2}+aT_0\frac{\partial^2 C}{\partial t^2}\right\}\end{gathered} \tag{3.14}$$

where

$$G_j(t)=\int_{t-\tau_j}^{t}k(t-\xi)f^{/}(\xi)d\xi, \qquad j=1,2,3, \qquad G_{12}(t)=D_\tau^2 f(t)=\int_{t-\tau_1}^{t}k(t-\xi)f^{//}(\xi)d\xi \ .$$

Using Eq.(2.9), we get

$$G_j(\tau_j,\omega)=\frac{1}{\tau_j^2\omega^2}\Big\{e^{i\omega\tau_j}\left(2A^2-\tau_j^2\omega^2-A^2\tau_j^2\omega^2+2B\tau_j^2\omega^2-2iA^2\tau_j\omega+2iB\tau_j\omega\right)6+$$
$$+\left(\tau_j^2\omega^2-2Bi\tau_j\omega-2A^2\right)\Big\},\qquad j=1,2,3,$$

$$G_{12}(\tau_1,\omega)=\frac{1}{4i\omega\tau_1^2}\Big\{e^{2i\omega\tau_4}\left(-2\omega^2\tau_1^2+4B\omega^2\tau_1^2+2Bi\omega\tau_1-2A^2\omega^2\tau_1^2-2iA^2\omega\tau_1+A^2\right)+$$
$$+\left(-A^2-2Bi\omega\tau_1+2\omega^2\tau_1^2\right)\Big\}.$$

Diffusion equation in TPL theory in context of the LS theory is

$$-\left[D\frac{\partial}{\partial t}(1+\tau_n\frac{\partial}{\partial t})+D^*(1+\tau_p\frac{\partial}{\partial t})\right]P_{,i}=\left(1+\tau_k\frac{\partial}{\partial t}+\frac{\tau_k^2}{2}\frac{\partial^2}{\partial t^2}\right)\dot{\eta}_i\Rightarrow$$
$$\Rightarrow\ -\left[D\frac{\partial}{\partial t}(1+\tau_n\frac{\partial}{\partial t})+D^*(1+\tau_p\frac{\partial}{\partial t})\right]P_{,ii}=\left(1+\tau_k\frac{\partial}{\partial t}+\frac{\tau_k^2}{2}\frac{\partial^2}{\partial t^2}\right)\dot{\eta}_{i,i},\tag{3.15}$$

Using constitutive relations (3.2), (3.3) and (2.10) in Eq.(3.15) we get

$$\left(D\frac{\partial}{\partial t}(1+\tau_n\frac{\partial}{\partial t})+D^*(1+\tau_p\frac{\partial}{\partial t})\right)\left(\beta_C\nabla^2\left(\nabla.\vec{u}\right)-b\nabla^2C+a\nabla^2T\right)=$$
$$=-\left(1+\tau_k\frac{\partial}{\partial t}+\frac{\tau_k^2}{2}\frac{\partial^2}{\partial t^2}\right)\ddot{C},\tag{3.16}$$

which can be written as

$$D\beta_C\frac{\partial}{\partial t}\left(1+\tau_n\frac{\partial}{\partial t}\right)\nabla^2\left(\nabla.\vec{u}\right)+D^*\beta_C\left(1+\tau_p\frac{\partial}{\partial t}\right)\nabla^2\left(\nabla.\vec{u}\right)+$$
$$+Da\frac{\partial}{\partial t}\left(1+\tau_n\frac{\partial}{\partial t}\right)\nabla^2T+D^*a\left(1+\tau_p\frac{\partial}{\partial t}\right)\nabla^2T-Db\frac{\partial}{\partial t}\left(1+\tau_n\frac{\partial}{\partial t}\right)\nabla^2C+\tag{3.17}$$
$$-D^*b\left(1+\tau_p\frac{\partial}{\partial t}\right)\nabla^2C=-\left(1+\tau_k\frac{\partial}{\partial t}+\frac{\tau_k^2}{2}\frac{\partial^2}{\partial t^2}\right)\ddot{C}.$$

Using $\left(\nabla.\vec{u}\right)=e_{ij}$ and considering the plane strain problem in the *x*-*y* plane Eq.(3.17) becomes

$$
\begin{aligned}
&D\beta_C \frac{\partial}{\partial t}\left(1+\tau_n \frac{\partial}{\partial t}\right)\left(\frac{\partial^3 u}{\partial x^3}+\frac{\partial^3 v}{\partial x^2 \partial y}\right)+D\beta_C \frac{\partial}{\partial t}\left(1+\tau_n \frac{\partial}{\partial t}\right)\left(\frac{\partial^3 u}{\partial x \partial y^2}+\frac{\partial^3 v}{\partial y^3}\right)+\\
&+D^*\beta_C\left(1+\tau_p \frac{\partial}{\partial t}\right)\left(\frac{\partial^3 u}{\partial x^3}+\frac{\partial^3 v}{\partial x^2 \partial y}\right)+D^*\beta_C\left(1+\tau_p \frac{\partial}{\partial t}\right)\left(\frac{\partial^3 u}{\partial x \partial y^2}+\frac{\partial^3 v}{\partial y^3}\right)+\\
&+Da \frac{\partial}{\partial t}\left(1+\tau_n \frac{\partial}{\partial t}\right)\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+D^* a\left(1+\tau_p \frac{\partial}{\partial t}\right)\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+\\
&-Db \frac{\partial}{\partial t}\left(1+\tau_n \frac{\partial}{\partial t}\right)\left(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}\right)-D^* b\left(1+\tau_p \frac{\partial}{\partial t}\right)\left(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}\right)=\\
&=-\left(1+\tau_k \frac{\partial}{\partial t}+\frac{\tau_k^2}{2} \frac{\partial^2}{\partial t^2}\right)\ddot{C}.
\end{aligned} \tag{3.18}
$$

Equation (3.18) can be written as

$$
\begin{aligned}
&D\beta_C \frac{\partial}{\partial t}\left(1+\tau_n \frac{\partial}{\partial t}\right)\left(\frac{\partial^3 u}{\partial x^3}+\frac{\partial^3 v}{\partial x^2 \partial y}+\frac{\partial^3 u}{\partial x \partial y^2}+\frac{\partial^3 v}{\partial y^3}\right)+\\
&+D^*\beta_C \frac{\partial}{\partial t}\left(1+\tau_p \frac{\partial}{\partial t}\right)\left(\frac{\partial^3 u}{\partial x^3}+\frac{\partial^3 v}{\partial x^2 \partial y}+\frac{\partial^3 u}{\partial x \partial y^2}+\frac{\partial^3 v}{\partial y^3}\right)+\\
&+Da \frac{\partial}{\partial t}\left(1+\tau_n \frac{\partial}{\partial t}\right)\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+D^* a\left(1+\tau_p \frac{\partial}{\partial t}\right)\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+\\
&-Db \frac{\partial}{\partial t}\left(1+\tau_n \frac{\partial}{\partial t}\right)\left(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}\right)-D^* b\left(1+\tau_p \frac{\partial}{\partial t}\right)\left(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}\right)=\\
&=-\left(1+\tau_k \frac{\partial}{\partial t}+\frac{\tau_k^2}{2} \frac{\partial^2}{\partial t^2}\right)\ddot{C}.
\end{aligned} \tag{3.19}
$$

Diffusion equation in TPL theory with MDDs following Ezzat *et al.* [37, 38] and Karamany and Ezzat [40] Eq.(3.19) becomes

$$
\begin{aligned}
&D\beta_C \frac{\partial}{\partial t}(1+\tau_n D_{\tau_5})\left(\frac{\partial^3 u}{\partial x^3}+\frac{\partial^3 v}{\partial x^2 \partial y}+\frac{\partial^3 u}{\partial x \partial y^2}+\frac{\partial^3 v}{\partial y^3}\right)++D^*\beta_C\left(1+\tau_p D_{\tau_6}\right)\left(\frac{\partial^3 u}{\partial x^3}+\right.\\
&\left.+\frac{\partial^3 v}{\partial x^2 \partial y}+\frac{\partial^3 u}{\partial x \partial y^2}+\frac{\partial^3 v}{\partial y^3}\right)+Da \frac{\partial}{\partial t}\left(1+\tau_n D_{\tau_5}\right)\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+\\
&+D^* a\left(1+\tau_p D_{\tau_6}\right)\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+-Db \frac{\partial}{\partial t}\left(1+\tau_n D_{\tau_5}\right)\left(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}\right)+\\
&-D^* b\left(1+\tau_p D_{\tau_6}\right)\left(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}\right)=-\left(1+\tau_k D_{\tau_4}+\frac{\tau_k^2}{2} D_{\tau_4}^2\right)\ddot{C},
\end{aligned} \tag{3.20}
$$

where τ_k, τ_n, τ_p are the phase lags due to diffusion of TPL model and τ_4, τ_5, τ_6 are the TPL time lags for MDD respectively, and $D_{\tau_5}, D_{\tau_6}, D_{\tau_4}$, are memory dependent derivatives due to diffusion of TPL model. Using Eq.(3.10) in Eq.(3.20) we get

$$\begin{gathered}
D\beta_C\left(\frac{\partial^4 u}{\partial t\partial x^3}+\frac{\partial^4 v}{\partial t\partial x^2\partial y}+\frac{\partial^4 u}{\partial t\partial x\partial y^2}+\frac{\partial^4 v}{\partial t\partial y^3}\right)+D\beta_C\frac{\tau_n}{\tau_5}\int\limits_{t-\tau_5}^{t}k(t-\xi)\left(\frac{\partial^5 u}{\partial\xi^2\partial x^3}+\right.\\
\left.+\frac{\partial^5 v}{\partial\xi^2\partial x^2\partial y}+\frac{\partial^5 u}{\partial\xi^2\partial x\partial y^2}+\frac{\partial^5 v}{\partial\xi^2\partial y^3}\right)d\xi+D^*\beta_C\left(\frac{\partial^4 u}{\partial t\partial x^3}+\frac{\partial^4 v}{\partial t\partial x^2\partial y}+\frac{\partial^4 u}{\partial t\partial x\partial y^2}+\right.\\
\left.+\frac{\partial^4 v}{\partial t\partial y^3}\right)+D^*\beta_C\frac{\tau_p}{\tau_6}\int\limits_{t-\tau_6}^{t}k(t-\xi)\left(\frac{\partial^5 u}{\partial\xi^2\partial x^3}+\frac{\partial^5 v}{\partial\xi^2\partial x^2\partial y}+\frac{\partial^5 u}{\partial\xi^2\partial x\partial y^2}+\frac{\partial^5 v}{\partial\xi^2\partial y^3}\right)d\xi+\\
+Da\left(\frac{\partial^2\dot{T}}{\partial x^2}+\frac{\partial^2\dot{T}}{\partial y^2}\right)++Da\frac{\tau_n}{\tau_5}\int\limits_{t-\tau_5}^{t}k(t-\xi)\left(\frac{\partial^4 T}{\partial\xi^2\partial x^2}+\frac{\partial^4 T}{\partial\xi^2\partial y^2}\right)d\xi+\\
+D^*a\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+D^*a\frac{\tau_p}{\tau_6}\int\limits_{t-\tau_6}^{t}k(t-\xi)\left(\frac{\partial^3 T}{\partial\xi\partial x^2}+\frac{\partial^3 T}{\partial\xi\partial y^2}\right)d\xi+\\
-Db\left(\frac{\partial^3 C}{\partial t\partial x^2}+\frac{\partial^3 C}{\partial t\partial y^2}\right)-Db\frac{\tau_n}{\tau_5}\int\limits_{t-\tau_5}^{t}k(t-\xi)\left(\frac{\partial^4 C}{\partial\xi^2\partial x^2}+\frac{\partial^4 C}{\partial\xi^2\partial y^2}\right)d\xi+\\
-D^*b\left(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}\right)-D^*b\frac{\tau_p}{\tau_6}\int\limits_{t-\tau_6}^{t}k(t-\xi)\left(\frac{\partial^3 C}{\partial\xi\partial x^2}+\frac{\partial^3 C}{\partial\xi\partial y^2}\right)d\xi+\\
+\ddot{C}+\frac{\tau_k}{\tau_4}\int\limits_{t-\tau_4}^{t}k(t-\xi)\frac{\partial^3 C}{\partial\xi^3}d\xi+\frac{\tau_k^2}{2\tau_4}\int\limits_{t-\tau_4}^{t}k(t-\xi)\frac{\partial^4 C}{\partial\xi^4}d\xi=0.
\end{gathered}\tag{3.21}$$

Equation (3.21) can be written as

$$\begin{gathered}
D\beta_C\frac{\partial}{\partial t}\left(1+\frac{\tau_n}{\tau_5}G_5(t)\right)\left(\frac{\partial^3 u}{\partial x^3}+\frac{\partial^3 v}{\partial x^2\partial y}+\frac{\partial^3 u}{\partial x\partial y^2}+\frac{\partial^3 v}{\partial y^3}\right)+D^*\beta_C\left(1+\frac{\tau_p}{\tau_6}G_6(t)\right)\left(\frac{\partial^3 u}{\partial x^3}+\right.\\
\left.+\frac{\partial^3 v}{\partial x^2\partial y}+\frac{\partial^3 u}{\partial x\partial y^2}+\frac{\partial^3 v}{\partial y^3}\right)+Da\frac{\partial}{\partial t}\left(1+\frac{\tau_n}{\tau_5}G_5(t)\right)\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+\\
+D^*a\left(1+\frac{\tau_p}{\tau_6}G_5(t)\right)\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)-Db\frac{\partial}{\partial t}\left(1+\frac{\tau_n}{\tau_5}G_5(t)\right)\left(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}\right)+\\
-D^*b\left(1+\frac{\tau_p}{\tau_6}G_6(t)\right)\left(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}\right)=-\left(1+\frac{\tau_k}{\tau_4}G_4(t)+\frac{\tau_k^2}{2\tau_4}G_{42}(t)\right)\ddot{C}
\end{gathered}\tag{3.22}$$

where

$$G_j(t)=\int\limits_{t-\tau_j}^{t}k(t-\xi)f^{/}(\xi)d\xi, j=4,5,6,\ \ G_{42}(t)=D_\tau^2 f(t)=\int\limits_{t-\tau_4}^{t}k(t-\xi)f^{//}(\xi)d\xi.$$

Using Eq.(2.9) we get

$$G_j(\tau_j,\omega)=\frac{1}{\tau_j^2\omega^2}\left\{e^{i\omega\tau_j}\left(2A^2-\tau_j^2\omega^2-A^2\tau_j^2\omega^2+2B\tau_j^2\omega^2-2iA^2\tau_j\omega+2iB\tau_j\omega\right)+\right.$$
$$\left.+\left(\tau_j^2\omega^2-2Bi\tau_j\omega-2A^2\right)\right\},\qquad j=4,5,6,$$

$$G_{42}(\tau_4,\omega)=\frac{1}{4i\omega\tau_4^2}\left\{e^{2i\omega\tau_4}\left(-2\omega^2\tau_4^2+4B\omega^2\tau_4^2+2Bi\omega\tau_4-2A^2\omega^2\tau_4^2-2iA^2\omega\tau_4+A^2\right)+\right.$$
$$\left.+\left(-A^2-2Bi\omega\tau_4+2\omega^2\tau_4^2\right)\right\}.$$

For the plane strain problem in x-y plane and using the Helmholtz's representation, u and v are read in terms of potentials Φ and ψ

$$u=\frac{\partial\Phi}{\partial x}-\frac{\partial\psi}{\partial y},\qquad v=\frac{\partial\Phi}{\partial y}+\frac{\partial\psi}{\partial x}. \tag{3.23}$$

Using Eqs (3.23), Eqs (3.5)-(3.6), (3.14) and (3.22) become

$$\frac{\mu}{\rho}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)-\ddot{\psi}=0, \tag{3.24}$$

$$\left(\frac{\lambda+2\mu}{\rho}\right)\left(\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}\right)-\overline{\beta}_T T-\overline{\beta}_C C-\ddot{q}=0, \tag{3.25}$$

$$K\left(1+\frac{\tau_T}{\tau_2}G_2(t)\right)\left(\frac{\partial^2\dot{T}}{\partial x^2}+\frac{\partial^2\dot{T}}{\partial y^2}\right)+K^*\left(1+\frac{\tau_V}{\tau_3}G_3(t)\right)\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)=$$
$$=\left(1+\frac{\tau_q}{\tau_1}G_1(t)+\frac{\tau_q^2}{2\tau_1}G_{12}(t)\right)\left\{\beta_T T_0\left(\frac{\partial\ddot{\Phi}}{\partial x^2}+\frac{\partial\ddot{\Phi}}{\partial y^2}\right)+\rho C_E\frac{\partial^2 T}{\partial t^2}+aT_0\frac{\partial^2 C}{\partial t^2}\right\}, \tag{3.26}$$

$$D\beta_C\left(1+\frac{\tau_n}{\tau_5}G_5(t)\right)\frac{\partial}{\partial t}\left(\frac{\partial^4\Phi}{\partial x^4}+2\frac{\partial^4\Phi}{\partial x^2\partial y^2}+\frac{\partial^4\Phi}{\partial y^4}\right)+D^*\beta_C\left(1+\frac{\tau_p}{\tau_6}G_6(t)\right)\left(\frac{\partial^4\Phi}{\partial x^4}+\right.$$
$$\left.+2\frac{\partial^4\Phi}{\partial x^2\partial y^2}+\frac{\partial^4\Phi}{\partial y^4}\right)+Da\left(1+\frac{\tau_n}{\tau_5}G_5(t)\right)\frac{\partial}{\partial t}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+$$
$$+D^*a\left(1+\frac{\tau_p}{\tau_6}G_6(t)\right)\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)-Db\left(1+\frac{\tau_n}{\tau_5}G_5(t)\right)\frac{\partial}{\partial t}\left(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}\right)+$$
$$-D^*b\left(1+\frac{\tau_p}{\tau_6}G_6(t)\right)\left(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}\right)+\left(1+\frac{\tau_k}{\tau_4}G_4(t)+\frac{\tau_k^2}{2\tau_4}G_{42}(t)\right)\ddot{C}=0. \tag{3.27}$$

4. Solution to the problem

There are different methods to calculate the velocities of the waves. The wave solution method is applied in this study since it is very accurate and simple to apply. In this method suitable wave potentials are defined and using these potentials in the governing partial differential equations, we get simple homogenous algebraic equations. Solving these equations, we get the phase velocity equation. This phase velocity defines a complex slowness vector, which is used to calculate the motion of material particles.
The solution to Eqs (3.24) to (3.27) are as outlined below

$$\left(\Phi,\psi,T,C\right)=\left(\bar{\Phi},\bar{\psi},\bar{T},\bar{C}\right)e^{ik(x\sin\theta+y\cos\theta-V^{*}t)} \tag{4.1}$$

where $\bar{\Phi},\bar{\psi},\bar{T}$ and $\bar{C}$ are constants, V^{*} is the phase velocity and k is the wave number. Using, Eq.(4.1) into Eqs (3.24) to (3.27), we get

$$\left(\frac{\mu}{\rho}-V^{*2}\right)\bar{\psi}=0, \tag{4.2}$$

$$\left(V^{*2}-\frac{\lambda+2\mu}{\rho}\right)\omega^{2}\bar{\Phi}-\bar{\beta}_{T}V^{*2}\bar{T}-\bar{\beta}_{C}V^{*2}\bar{C}=0, \tag{4.3}$$

$$\omega^{2}\beta_{T}T_{0}\bar{\Phi}+\left\{\frac{K^{*}\left(1+\dfrac{\tau_{V}}{\tau_{3}}G_{3}(t)\right)-i\omega K\left(1+\dfrac{\tau_{T}}{\tau_{2}}G_{2}(t)\right)}{\left(1+\dfrac{\tau_{q}}{\tau_{1}}G_{1}(t)+\dfrac{\tau_{q}^{2}}{2\tau_{1}}G_{12}(t)\right)}-\rho C_{E}V^{*2}\right\}\bar{T}-aT_{0}V^{*2}\bar{C}=0, \tag{4.4}$$

$$\begin{gathered}\omega^{2}\beta_{C}\left\{\frac{D^{*}\left(1+\dfrac{\tau_{p}}{\tau_{6}}G_{6}(t)\right)-i\omega D\left(1+\dfrac{\tau_{n}}{\tau_{5}}G_{5}(t)\right)}{\left(1+\dfrac{\tau_{k}}{\tau_{4}}G_{4}(t)+\dfrac{\tau_{k}^{2}}{2\tau_{4}}G_{42}(t)\right)}\right\}\bar{\Phi}+\\
-aV^{*2}\left\{\frac{D^{*}\left(1+\dfrac{\tau_{p}}{\tau_{6}}G_{6}(t)\right)-i\omega D\left(1+\dfrac{\tau_{n}}{\tau_{5}}G_{5}(t)\right)}{\left(1+\dfrac{\tau_{k}}{\tau_{4}}G_{4}(t)+\dfrac{\tau_{k}^{2}}{2\tau_{4}}G_{42}(t)\right)}\right\}\bar{T}\\
+V^{*2}\left\{b\frac{D^{*}\left(1+\dfrac{\tau_{p}}{\tau_{6}}G_{6}(t)\right)-i\omega D\left(1+\dfrac{\tau_{n}}{\tau_{5}}G_{5}(t)\right)}{\left(1+\dfrac{\tau_{k}}{\tau_{4}}G_{4}(t)+\dfrac{\tau_{k}^{2}}{2\tau_{4}}G_{42}(t)\right)}-V^{*2}\right\}\bar{C}=0,\end{gathered} \tag{4.5}$$

Equations (4.2) to (4.5) become

$$\left(E_{11}-V^{*2}\right)\bar{\psi}=0, \tag{4.6}$$

$$\left(V^{*2}-E_{22}\right)\bar{\Phi}-\bar{\beta}_T V^{*2}\bar{T}-\bar{\beta}_C\bar{C}=0, \tag{4.7}$$

$$\beta_T T_0\bar{\Phi}+\left\{E_{33}-\rho C_E V^{*2}\right\}\bar{T}-aT_0\bar{C}=0, \tag{4.8}$$

$$\beta_C E_{44}\bar{\Phi}+aV^{*2}E_{44}\bar{T}+\left(bE_{44}-V^{*2}\right)\bar{C}=0 \tag{4.9}$$

where, for the possibility that Eqs (4.7) to (4.9) with non-trivial solution require

$$P_0V^6+P_1V^4+P_2V^2+P_3=0, \tag{4.10}$$

where,

$$P_0=1, P_1=\bar{E}_{33}+E_{44}(\varepsilon^*-b)-E_{22}-\varepsilon,$$

$$P_2=\bar{E}_{33}(bE_{44}+E_{22})-E_{22}E_{44}(\varepsilon^*-b)+E_{44}\left(\varepsilon b-\frac{\beta_C^2}{\rho}\right), \qquad P_3=\bar{E}_{33}E_{44}\left(\frac{\beta_C^2}{\rho}-bE_{22}\right),$$

$$E_{11}=\frac{\mu}{\rho}, \qquad E_{22}=\frac{\lambda+2\mu}{\rho}, \qquad \varepsilon=\frac{\beta_T^2 T_0}{\rho^2 C_E}, \qquad \varepsilon^*=\frac{a^2T_0}{\rho C_E}, \qquad \bar{E}_{33}=\frac{E_{33}}{\rho C_E}, \qquad \bar{E}_{44}=\frac{E_{44}}{\rho},$$

$$E_{33}=\frac{K^*\left(1+\frac{\tau_V}{\tau_3}G_3(t)\right)-i\omega K\left(1+\frac{\tau_T}{\tau_2}G_2(t)\right)}{\left(1+\frac{\tau_q}{\tau_1}G_1(t)+\frac{\tau_q^2}{2\tau_1}G_{12}(t)\right)},$$

$$E_{44}=\frac{D^*\left(1+\frac{\tau_p}{\tau_6}G_6(t)\right)-i\omega D\left(1+\frac{\tau_n}{\tau_5}G_5(t)\right)}{\left(1+\frac{\tau_k}{\tau_4}G_4(t)+\frac{\tau_k^2}{2\tau_4}G_{42}(t)\right)}$$

where $\omega=kV^*$ is the wave's circular frequency. If $V_i^{*-1}=c_i^{*-1}-i\,\omega^{-1}q_i, i=1,2,3,4$, then the real parts V_1^*, V_2^*, V_3^* of the three zeros of Eq.(4.10) are three coupled plane wave's velocities namely longitudinal (P), thermal (T), mass diffusion (MD) and the solution to Eq.(4.6) that is $V_4^*=\sqrt{\frac{\mu}{\rho}}$, is the velocity of SV wave. In general, the velocity of a longitudinal wave is given by $V_L=\sqrt{\frac{\lambda+2\mu}{\rho}}$ and velocity of a transverse wave is given by $V_T = \sqrt{\frac{\mu}{\rho}}$ but due to the diffusion, MDD parameters and TPL these velocities get affected. The variations due to triple phase lag, memory dependent derivative and diffusion are extensively studied in this research. In this research, the modified phase velocities, are calculated using Eq.(4.10).

5. Reflection from a free surface

A thermoelastic solid with diffusion ($y \le 0$) in generalized triple phase lag with memory-dependent thermo-elasticity is investigated at thermal insulation/ isothermal stress-free condition at surface $y = 0$. The incident of coupled P wave at incident angle (θ_0) with the normal, will result in four reflected waves: longitudinal (P), thermal (T), mass diffusion (MD) and vertically shear (SV) waves in the half-space. θ_0, θ_1 ($\theta_0 = \theta_1$) are the incident and reflected angle of P wave, $\theta_2, \theta_3, \theta_4$ are the reflected angle the of reflected T, MD, SV waves , respectively, c_0^*, c_1^* ($c_0^* = c_1^*$) are the velocities of the reflected and incident P wave, c_2^*, c_3^*, c_4^*, are the velocities of reflected T, MD, SV waves and k_j, $(j = 1,2,.,4)$ are complex wavenumbers, respectively. The complete geometry is depicted in Figure 1.
The appropriate potentials of the reflected and incident waves in the half-space are taken as

$$\Phi = A_0 \exp\left\{ik_1\left(x sin\theta_0 + y\cos\theta_0 - c_1^* t\right)\right\} + \sum_{j=1}^{3} A_j \exp\left\{ik_j\left(x sin\theta_j - y\cos\theta_j - c_j^* t\right)\right\}, \tag{5.1}$$

$$\begin{gathered} T = d_1 A_0 \exp\left\{ik_1\left(x\sin\theta_0 + y\cos\theta_0 - c_1^* t\right)\right\} + \\ + \sum_{j=1}^{3} d_j A_j \exp\left\{ik_j\left(x sin\theta_j - y\cos\theta_j - c_j^* t\right)\right\}, \end{gathered} \tag{5.2}$$

$$\begin{gathered} C = \varsigma_1 A_0 \exp\left\{ik_1\left(x\sin\theta_0 + y\cos\theta_0 - c_1^* t\right)\right\} + \\ + \sum_{j=1}^{3} \varsigma_j A_j \exp\left\{ik_j\left(x sin\theta_j - y\cos\theta_j - c_j^* t\right)\right\}, \end{gathered} \tag{5.3}$$

$$\psi = X \exp\left\{ik_4\left(x\sin\theta_4 - y\cos\theta_4 - c_4^* t\right)\right\}, \tag{5.4}$$

and the coupling coefficient $\frac{d_i}{k_i^2}$, $\frac{\varsigma_i}{k_i^2}$, for $(s = 1,2,3)$, are as

$$\begin{gathered} \frac{d_i}{k_i^2} = \frac{\bar{\beta}_C \beta_T T_0 c_s^{*2} - a T_0 c_s^{*2}\left[c_s^{*2} - E_{33}\right]}{\bar{\beta}_C\left(\rho C_E c_s^{*2} - E_{33}\right) - \bar{\beta}_T a T_0 c_s^{*2}}, \\ \frac{\varsigma_i}{k_i^2} = \frac{\left(\rho C_E c_s^{*2} - E_{33}\right)\left[c_s^{*2} - E_{22}\right] - \bar{\beta}_T \beta_T T_0 c_s^{*2}}{\bar{\beta}_C\left(\rho C_E c_s^{*2} - E_{33}\right) - \bar{\beta}_T a T_0 c_s^{*2}}. \end{gathered} \tag{5.5}$$

The required conditions on the boundary at the surface $y = 0$, are follow as

$$\sigma_{yy} = 0, \qquad \sigma_{yx} = 0, \qquad \frac{\partial P}{\partial y} + h_1 P = 0, \qquad \frac{\partial T}{\partial y} + l_1^* T = 0 \tag{5.6}$$

where

$$\tau_{yy} = \lambda \frac{\partial^2 \Phi}{\partial x^2} + (\lambda + 2\mu)\frac{\partial^2 \Phi}{\partial y^2} + 2\mu \frac{\partial^2 \psi}{\partial x \partial y} - \beta_T T - \beta_C C, \tag{5.7}$$

$$\tau_{yx} = 2\mu \frac{\partial^2 \Phi}{\partial x \partial y} + \mu \frac{\partial^2 \psi}{\partial x^2} - \mu \frac{\partial^2 \psi}{\partial y^2}, \tag{5.8}$$

$$\tau_{xx} = \lambda \left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} \right) + 2\mu \frac{\partial^2 \Phi}{\partial x^2} - 2\mu \frac{\partial^2 \psi}{\partial x \partial y} - \beta_T T - \beta_C C, \tag{5.9}$$

$$P = bC - \beta_C \frac{\partial^2 \Phi}{\partial x^2} - \beta_C \frac{\partial^2 \Phi}{\partial y^2} - aT \tag{5.10}$$

here l_1^* is the heat transfer coefficient at the surface where $l_1^* \to 0$ for thermal insulation and $l_1^* \to \infty$ for the isothermal surface condition. h_1 is the concentration diffusion coefficient at the surface, where $h_1 \to 0$ corresponds to impermeable surfaces and $h_1 \to \infty$ refers to iso-concentrated surfaces. The potentials stated in equations (5.1) to (5.4) must satisfy the boundary conditions (5.6) and follow Snell's law

$$\frac{\sin\theta_0}{c_1^*} = \frac{\sin\theta_s}{c_s^*}, \qquad k_s c_s^* = \omega(say), \qquad (s = 1, 2, ..., 4). \tag{5.11}$$

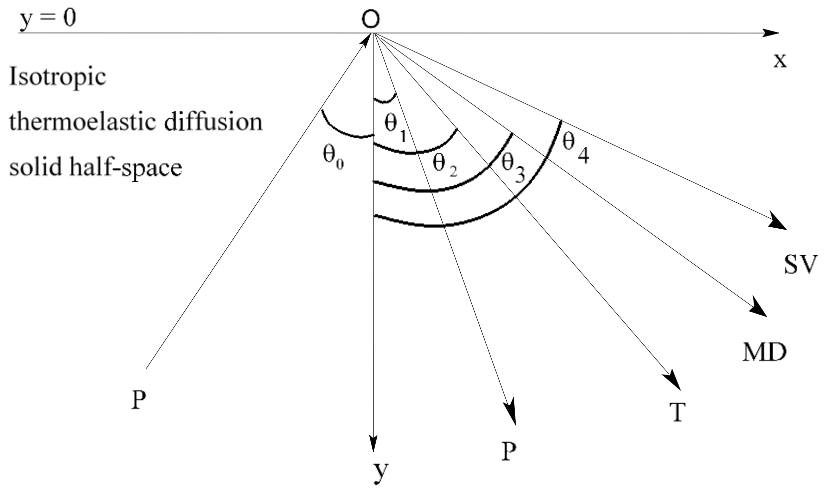

Fig.1. The geometry of the problem.

Using the potentials given by Eqs (5.1) to (5.4), the boundary conditions (5.6) and the system of equations (5.11) of four non-homogeneous equations is obtained as

$$\sum_{j=1}^{4} D_{ij} Z_j = B_i \,.,\left(i = 1,2,3,4\right) \tag{5.12}$$

where

$$D_{1j} = \left[\lambda + 2\mu\left\{1 - \left(\frac{c_j^*}{c_1^*}\right)^2 \sin^2\theta_0\right\} + \frac{\beta_T d_j}{k_j^2} + \frac{\beta_C \varsigma_j}{k_j^2}\right]\left(\frac{c_1^*}{c_j^*}\right)^2,$$

$$D_{14} = -2\mu \sin\theta_0 \frac{c_1^*}{c_4^*}\sqrt{1 - \left(\frac{c_4^*}{c_1^*}\right)^2 \sin^2\theta_0}\,,$$

$$B_1 = -[\lambda + 2\mu\cos^2\theta_0 + \frac{\beta_T d_1}{k_1^2} + \frac{\beta_C \varsigma_1}{k_1^2}],$$

$$D_{2j} = 2\mu \sin\theta_0 \frac{c_1^*}{c_j^*}\sqrt{1 - \left(\frac{c_j^*}{c_1^*}\right)^2 \sin^2\theta_0}\,,$$

$$D_{24} = -\mu \sin^2\theta_0 + \mu\left(\frac{c_1^*}{c_4^*}\right)^2\left\{1 - \left(\frac{c_4^*}{c_1^*}\right)^2 \sin^2\theta_0\right\},$$

$$B_2 = 2\mu \sin\theta_0 \cos\theta_0,$$

(a) for impermeable surface

$$D_{3j} = \left[\frac{b\varsigma_j}{k_j^2} + \beta_C - a\frac{d_j}{k_j^2}\right]\left(\frac{c_1^*}{c_j^*}\right)^3\sqrt{1 - \left(\frac{c_j^*}{c_1^*}\right)^2 \sin^2\theta_0}\,,$$

$$D_{34} = 0, B_3 = \left(\frac{b\varsigma_1}{k_1^2} + \beta_C - a\frac{d_1}{k_1^2}\right)\cos\theta_0,$$

(b) for iso-concentrated surface

$$D_{3j} = \left[\frac{b\varsigma_j}{k_j^2} + \beta_C - a\frac{d_j}{k_j^2}\right]\left(\frac{c_1^*}{c_j^*}\right)^2, \qquad D_{34} = 0, B_3 = -\left(\frac{b\varsigma_1}{k_1^2} + \beta_C - a\frac{d_1}{k_1^2}\right),$$

(c) for thermally insulated

$$D_{4j} = \frac{d_j}{k_j^2}\left(\frac{c_1^*}{c_j^*}\right)^3 \sqrt{1-\left(\frac{c_j^*}{c_1^*}\right)^2 \sin^2\theta_0}, \qquad D_{44}=0, \qquad B_4 = \frac{d_1}{k_1^2}\cos\theta_0,$$

(d) for isothermal

$$D_{4j} = \frac{d_j}{k_j^2}\left(\frac{c_1^*}{c_j^*}\right)^2, \qquad D_{44}=0, \qquad B_4 = -\frac{d_1}{k_1^2} \qquad \text{for} \quad (j=1,2,3),$$

$$Z_1 = \frac{A_1}{A_0}, \qquad Z_2 = \frac{A_2}{A_0}, \qquad Z_3 = \frac{A_3}{A_0}, \qquad Z_4 = \frac{X}{A_0}$$

where $|Z_s|$, $(s=1,2,.,4)$ are reflection coefficients of reflected $P, T, MD,$ and SV waves, respectively.

6. Numerical findings and discussion

In this study, velocities of waves and reflection coefficients are calculated for a thermo-diffusive medium. Fortran software is used for a numerical experiment to investigate the impact of frequency, material constants and memory-dependent derivative (MDD) parameters, for aluminium chosen from Sharma [67] at $T_0 = 293K$, $\lambda = 7.89\times 1010\,Nm^{-2}$, $\mu = 1.98\times 1010\,Nm^{-2}$, $\rho = 2.7\times 103 kgm^{-3}$, $C_E = 2.361\times 103\,J.Kg^{-1}K^{-1}$, $K = 4.92\times 103 NK^{-1}s^{-1}$, $K^* = 4.92\times 103\,NK^{-1}s^{-1}$, $\alpha_t = 1.78\times 10^{-5}K^{-1}$, $D=0.4, D^*=0.4$, $a=0.6, b=0.7,$ $\omega = 20Hz,$ $\tau_T = 0.03,$ $\tau_V = 0.01,$ $\tau_q = 0.02,$ $\tau_n = 0.02,$ $\tau_p = 0.03,$ $\tau_k = 0.01,$ $\tau_1 = 0.05,$ $\tau_2 = 0.04,$ $\tau_3 = 0.03,$ $\tau_4 = 0.05,$ $\tau_5 = 0.04,$ $\tau_6 = 0.03.$

For the above numerical data, the velocities of P, T, MD waves are calculated from Eq.(4.10) and these of SV waves from (4.6) using a Fortran program. The reflection amplitudes $|Z_1|,|Z_2|,|Z_3|$ and $|Z_4|$ from the system of Eq. (5.12) of reflected $P, T, MD,$ and SV waves vs. incident angle of P wave $0^0 < \theta_0 < 90^0$ are computed numerically by using a Fortran program for different values of $A=1, B=1,$ $a=0.6,$ $b=0.7,$ $D=0.4, D^*=0.4,$ $\tau_T = 0.03,$ $\tau_V = 0.01,$ $\tau_q = 0.02,$ $\tau_n = 0.02,$ $\tau_p = 0.03,$ $\tau_k = 0.01,$ $\omega = 20Hz.$ The fluctuation of the reflection coefficients for distinct three values of frequency ω, diffusion parameter a, and memory dependent derivative parameters $A, B,$ are depicted in Figures 4(a)-4(d) to 6(a)-6(d) and denoted by a blue curve using solid circles, purple curve using solid triangle, red curve using single tick, respectively, for three different values.

6.1. Impact of frequency

Variation of velocities of $P, T, MD,$ and SV wave against frequency is investigated as frequency ω, varies from $10 < \omega < 25$ at the fixed value of MMD parameter; $A=1, B=1,$ diffusion parameters a, b, D, D^*; $a=0.6,$ $b=0.7,$ $D=0.4,$ $D^*=0.4,$ and time delay parameters; $\tau_T = 0.03,$ $\tau_V = 0.01,$ $\tau_q = 0.02,$ $\tau_n = 0.02,$ $\tau_p = 0.03,$ $\tau_k = 0.01.$ The variations are depicted in Fig.2. These fluctuations in velocities of waves are denoted by a blue curve using solid circles, purple curve using solid triangle, red curve using single tick, green curve with double tick respectively. The velocity of $P, T, MD,$ wave increases as frequency ω, increases from $10 < \omega < 25 Hz$ while that of SV waves almost remains same. The variation of velocity is shown in Fig.2 on the scale from $0 - 3.6 \times 10^4 m/s$. The fluctuation of reflection coefficient $|Z_1|, |Z_2|, |Z_3|$ and $|Z_4|$ of reflected P, T, MD and SV waves in contrast to the incident angle of P wave $0^0 < \theta_0 < 90^0$ at various values of frequency $\omega = 15, 20, 25 Hz$ is shown in Figs 4(a)-4(d). It is noticed that the reflection coefficient of reflected P, MD and SV waves increases as frequency ω increases as $\omega = 15, 20, 25 Hz$.

6.2. Impact of diffusion parameters

Variations of velocities against the diffusion parameter b, $0 < b < 1$ at the fixed value of MMD parameter; $A = 1, B = 1$, diffusion parameters; a, $D, D^*; a = 0.6, D = 0.4, D^* = 0.4$; time delay parameters; $\tau_T = 0.03$, $\tau_V = 0.01$, $\tau_q = 0.02$, $\tau_n = 0.02$, $\tau_p = 0.03$, $\tau_k = 0.01$, and frequency; $\omega = 25 Hz$, are depicted in Fig.3.

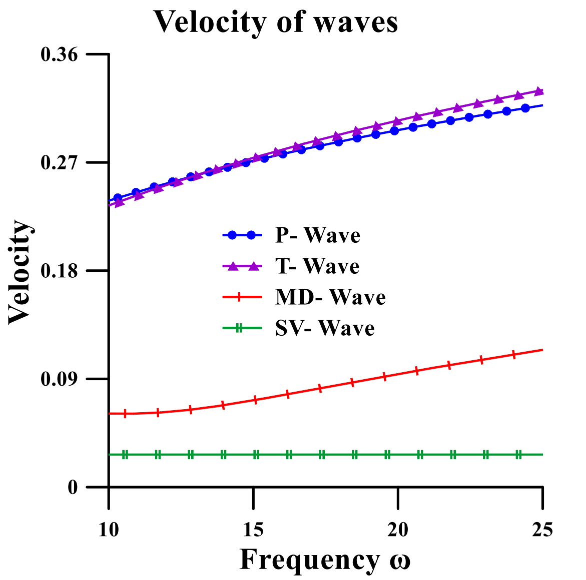

Fig.2. The dependence of the velocity of P, T, MD and SV waves against frequency $10 < \omega < 25 Hz$.

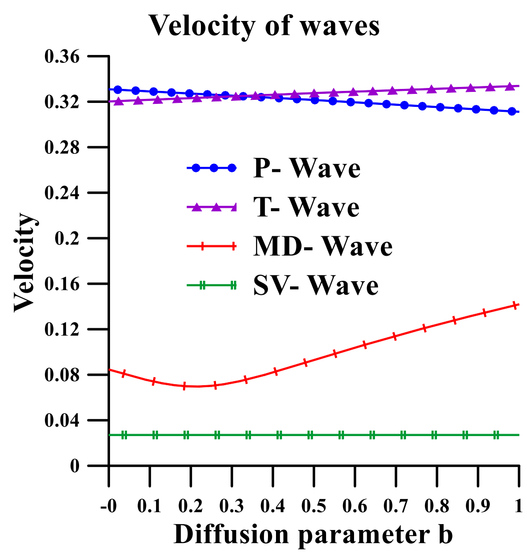

Fig.3. The fluctuation in the velocity of P, T, MD and SV waves against the diffusion parameter b.

These fluctuations in velocities of waves are denoted by a blue curve using solid circles, purple curve using solid triangle, red curve using single tick, green curve with double tick respectively. The velocity of SV wave is obtained by multiplying each value by 10. It is concluded that velocity of T, MD, increases as the diffusion parameter b increases from $0 < b < 1$. The variation of velocity is shown in Fig.3 on the scale from $0 - 3.6 \times 10^4 m/s$. The variation of reflection constants $|Z_1|, |Z_2|, |Z_3|$ and $|Z_4|$ is against the incident angle of P wave $0^0 < \theta_0 < 90^0$ for three values of diffusion constants a, $a = 0.40, 1.5, 2.5$ at $k(t - \xi) = \left(\frac{2\tau_i - (t - \xi)}{2\tau_i} \right)^2$, for $A = 0.5, B = 0.5$, is shown in Figs 5(a)-5(d). It is also noticed that reflection amplitudes of P, T, and MD, waves decrease as the incident angle increases $0^0 \le \theta_0 > 90^0$ and the diffusion parameter a, increases from $a = 0.40, 1.5, 2.5$.

6.3. Impact of the MDD parameters and kernel functions

The reflection coefficients $|Z_1|,|Z_2|,|Z_3|$ and $|Z_4|$ of reflected P, T, MD, and SV waves against the incident angle of P wave $0^0 < \theta_0 < 90^0$ are computed numerically by using a Fortran program for various values of $a = 0.6$, $b = 0.7$, $D = 0.4$, $D^* = 0.4$, $\tau_T = 0.03$, $\tau_V = 0.01$, $\tau_q = 0.02$, $\tau_n = 0.02$, $\tau_p = 0.03$, $\tau_k = 0.01$, $\omega = 20\,Hz$.

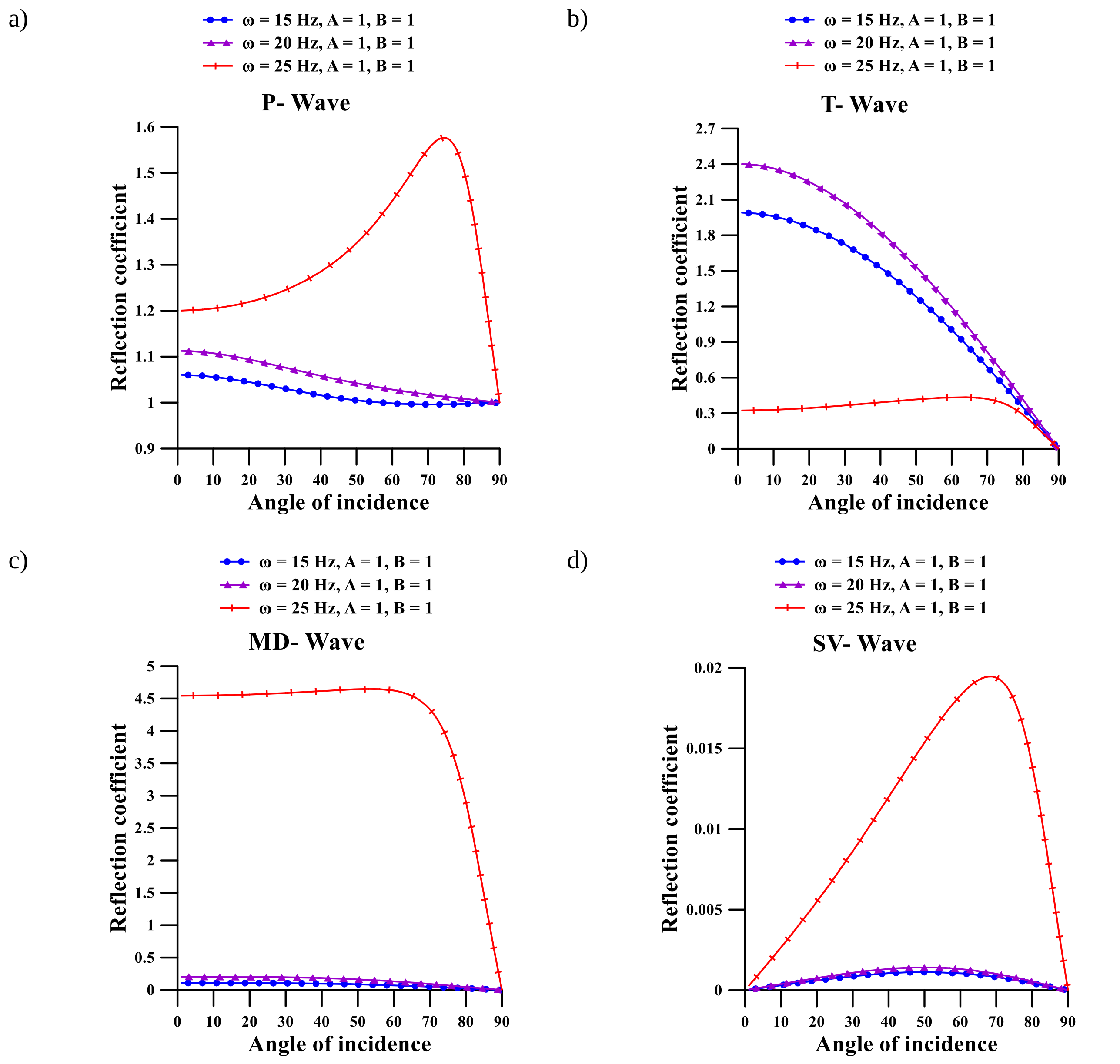

Fig.4.a-d. Variations of the reflection coefficient of reflected P, T, MD and SV waves against the angle of incidence of P wave $0^0 < \theta_0 < 90^0$ at different values of frequency $\omega = 15, 20, 25\,Hz$.

The effect of the MDD parameter $(A=0, B=0)$, $(A=0, B=1)$, $(A=1, A=1)$, is shown in Figs 6(a)-6(d) for time delay $\tau_T = 0.03$, $\tau_V = 0.01$, $\tau_q = 0.02$, $\tau_n = 0.02$, $\tau_p = 0.03$, $\tau_k = 0.01$. The fluctuation of reflection coefficient of reflected P, T, MD, and SV waves against the incident angle of P wave at different values of the MDD parameter (A, B), and different values of the kernel function $k(t-\xi)=1$, for $A=0, B=0$, $k(t-\xi)=1-\dfrac{2(t-\xi)}{\tau_i}$, for $A=0, B=1$ and $k(t-\xi)=\left(1-\dfrac{(t-\xi)}{\tau_i}\right)^2$, for $A=1, B=1$, is studied. It is seen that as the values the MDD parameters (A, B) increase the reflection coefficients of reflected P, T, waves increase while these of MD, and SV waves decrease as the angle of incidence changes from $0^0 < \theta_0 < 90^0$.

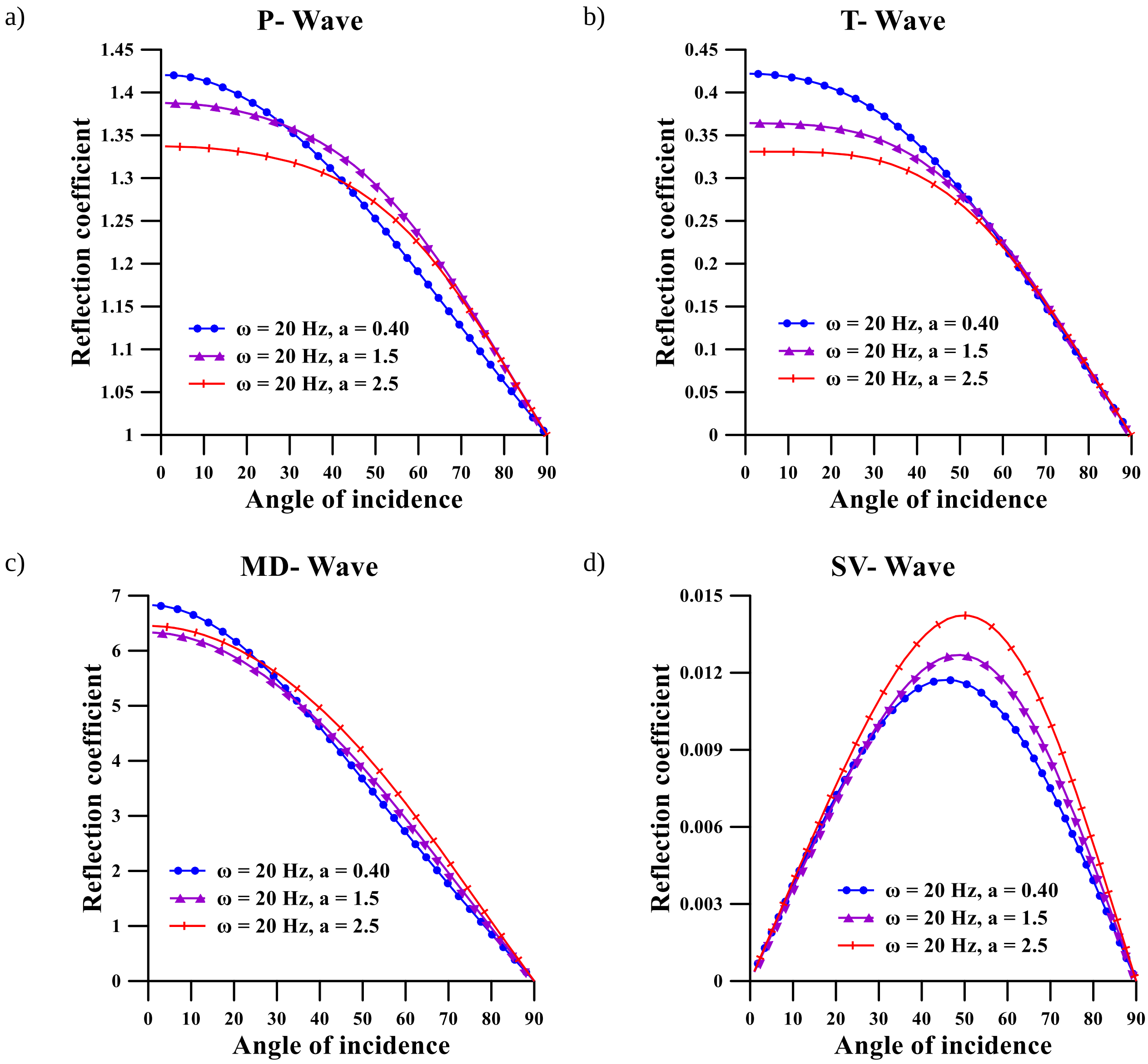

Fig.5.a-d. The aberration of the reflection coefficient of reflected P, T, MD and SV waves against incident angle of P wave $0^0 < \theta_0 < 90^0$ at distinct values of the diffusion parameter a, $a = 0.40, 1.5, 2.5$.

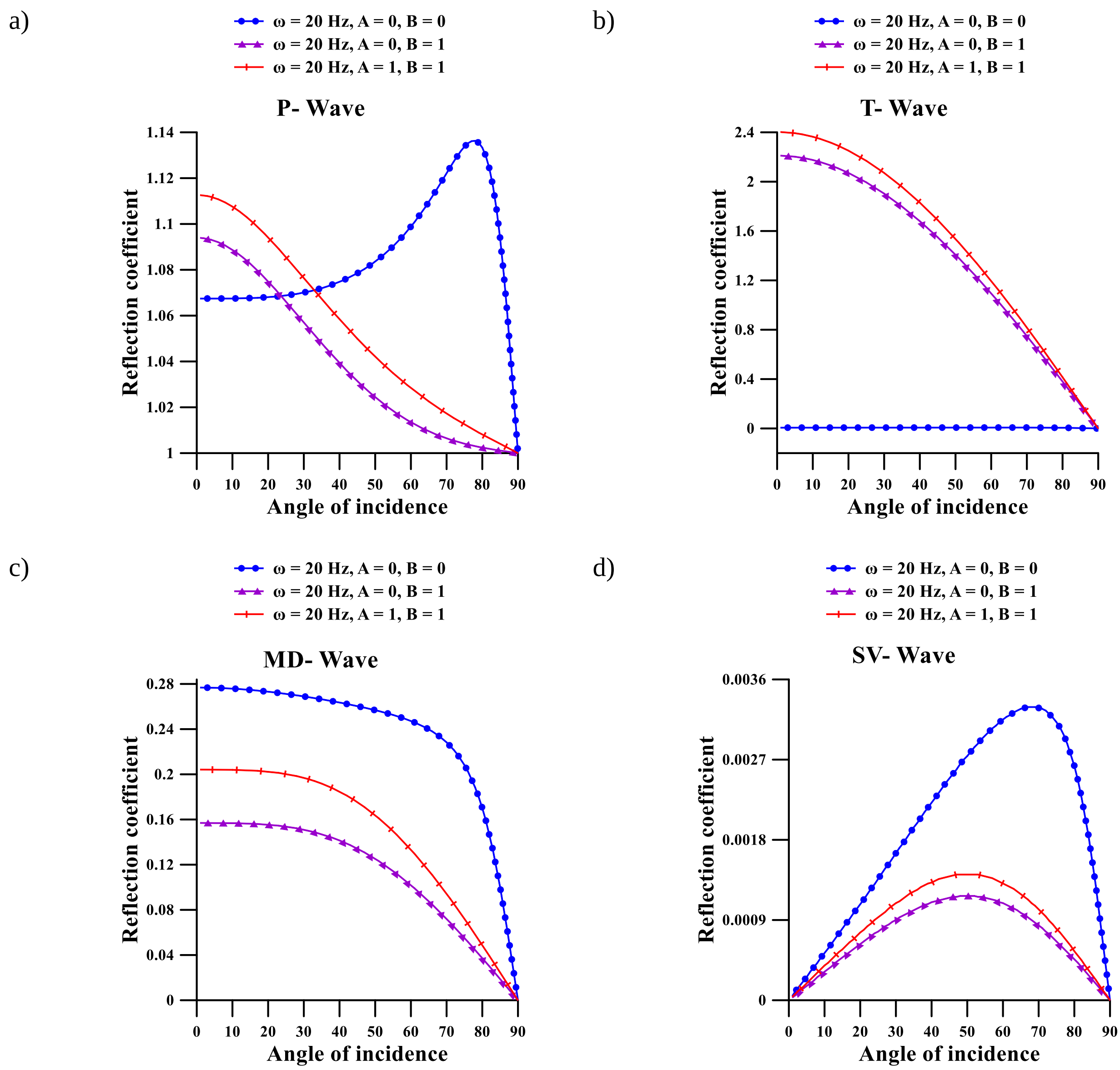

Fig.6.ad. The change in the reflection coefficient of reflected P, T, MD and SV waves against P wave's angle of incidence $0^0 < \theta_0 < 90^0$ at different values of MDD parameters $(A=0, B=0)$, $(A=0, B=1)$, $(A=1, B=1)$.

7. Conclusions

The reflection coefficients $|Z_1|, |Z_2|$ and $|Z_3|$ of reflected P, T, and SV waves are substantially affected by frequency ω, diffusion parameter a, and MDD parameters A, B, against the incident angle of P wave $0^0 < \theta_0 < 90^0$.

1) It is seen from the graph that as MDD parameters (A, B) increase from $(A=0, B=0)$, $(A=0, B=1)$, $(A=1, B=1)$, the value of reflection amplitudes $|Z_1|$, $|Z_2|$ of reflected P and T increases while, $|Z_3|, |Z_4|$ and that of MD, and SV waves decrease as the angle of incidence changes from $0^0 < \theta_0 < 90^0$.

2) It is noticed that the reflection coefficient $|Z_1|$ $|Z_3|,|Z_4|$ of reflected P, MD, and SV waves increases as frequency ω increases from $\omega = 15, 20, 25\,Hz$ and the angle of incidence shifts between $0^0 < \theta_0 < 90^0$.

These modified values of seismic signals (phase velocities from Eq.(4.10), reflection coefficients from Eq.(5.12)) may help researchers in geophysical exploration, in locating oil wells and minerals deposits and experimental seismologists in correcting earthquake estimation. It will help many industries which are preparing devices like MOS transistors, doped polysilicon gates and imaging processing devices to give more effective results. Memory dependent derivatives may be useful effectively in decision process and AI devices.

Nomenclature

a – coefficient of thermo-diffusion effect
b – constant characterizing the mass diffusion effect
C – concentration of the diffusive material
C_E – specific heat
D^*, D – coefficients of diffusion
e_{ij} – constituents of strain tensor
e_{kk} – dilatation
K, K^* – thermal conductivities
P – chemical potential per unit mass
q_i – heat flux vector, S is the entropy
T – Variation in the temperature
T_0 – uniform temperature
t – time
u_i – displacement vector
u, v – displacement vector components
V^* – wave velocity
α_c – constant of diffusion expansion
α_t – thermal expansion factor
β_T – heat transfer coefficient
β_C – diffusion coefficients
δ_{ij} – Kronecker delta
η_i – flow of the diffusing mass vector
λ, μ – Lame′s constants
ρ – density
τ_1, τ_2, τ_3 – thermal relaxation time
τ_4, τ_5, τ_6 – diffusion relaxation time
τ_{ij} – constituents of stress tensor
θ_0 – propagation angle measured from the normal to the medium surface
ω – circular frequency

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