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Feedback-Aided PD-Type Iterative Learning Control for Time-Varying Systems with Non-Uniform Trial Lengths

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Abstract

In this paper, a feedback-aided PD-type iterative learning control (ILC) design is proposed for the time-varying systems with non-uniform trial lengths to achieve asymptotic tracking of the desired trajectory. To alleviate the problem of missing information caused by non-uniform trial lengths, signals from most recently available iterations can be used for system learning by introducing an indicator function to construct recursively generated update error and input sequences. The main results are obtained by utilizing the combination of λ-norm technique and inductive analysis approach, and the design is extended to nonlinear time-varying systems. At last, the effectiveness of the proposed feedback-aided ILC design for linear and nonlinear time-varying systems is demonstrated by a numerical simulation and a single-joint robot model.

Keywords

Iterative learning control, Non-uniform trial lengths, Feedback-aided, Time-varying system

Introduction

As a branch of intelligent control, iterative learning control (ILC) has been widely studied since its first introduction and application to a robotic system by Arimoto et al. in the 1980s (Arimoto et al. 1984). The core idea of ILC is to iteratively correct the control signal of a system with a repetitive motion by using the priori information generated after the previous operation of the system, aiming to achieve accurate tracking of the desired trajectory in a finite time interval. Due to its effectiveness and ease of application, this control method has made significant progress and is broadly applied in the fields of robot control (Bouakrif and Zasadzinski 2018; Zeng et al. 2019), chemical processes (Lin et al. 2019; Tao et al. 2017), and spatially interconnected systems (Tao et al. 2021), etc.

In many conventional ILC designs, a key requirement is that as the iteration number of the control process increases, the trial length of each iteration is a fixed value, which can be called as the desired length. This ensures that the system has sufficient learning times at each time instant within the desired length, which facilitates the learning process. However, in some practical application scenarios, some iterations of the control system may end or be terminated prematurely for several reasons. For example, in Yu et al. (2018), when automatic control of high-speed trains is performed, it is difficult to maintain consistent train arrival times due to the complex environment and various random events, thus the non-uniform trial lengths problem arises when ILC is applied to this system. In addition, the trial lengths of ILC control system for ventricular assist devices described in Ketelhut et al. (2019) are not uniform in clinical practice due to variations in hemodynamic conditions caused by heart rate fluctuations. Another typical example is seen in the functional electrical stimulation for upper limb and gait assistance where some iterations are terminated early due to safety concerns (Seel et al. 2016). Difficulties are encountered when applying conventional ILC strategies to such practical systems. The application of ILC to non-uniform trial lengths systems has attracted the attention of researchers (Shen and Li 2019). The two

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most important aspects of solving this problem are the methods to handle different trial lengths uniformly and the approaches to deducing system analysis.

In a system with non-uniform trial lengths, iterations with shorter trial lengths could suffer from some information loss compared to longer iterations. To address this, Li et al. used the information of all past iterations for the design of the ILC control law by utilizing an iterative averaging operator in Li et al. (2014) and Li and Xu (2015), aiming to eliminate the effect of missing information to some extent. Two novel control law design schemes using moving average operators were proposed subsequently in Li and Shen (2017) to pay more attention to recent iterations, and the corresponding system convergence conditions were given. With a similar idea, a D-type ILC control law was designed in Wei and Li (2017) to solve the problem of non-uniform trial lengths and initial state shift. Moreover, the averaging operator was modified to improve the learning efficiency in Shi et al. (2020), where a Gaussian distribution was used to describe the distribution of different trial lengths, and bounded convergence of errors was achieved for systems with disturbances. These schemes construct the historical information as a compensating signal for the missing parts, while these higher-order strategies may dilute the influence of recent or larger errors. By using zero to fill in the missing part of error information, a full-length modified tracking error was constructed, thus the conventional one-order P-type ILC law was successfully applied in Shen et al. (2016a). Benefit from this, a control scheme based on noisy output was proposed in Shen and Saab (2021) recently, where the non-uniform trial lengths was modeled by a Markelov chain. Similar compensation methods to this one were widely used in in subsequent works (Meng and Zhang 2017; Shen et al. 2016b; Zhuang et al. 2022). Furthermore, an indicator function was used in Jin (2020), thus the highest learning priority was put on the most recent iterations that can sustain for each time instant. These designs used a variety of compensation methods to supplement the absent portion of the tracking information. This was because for upcoming iterations with unknown actual trial lengths, a sufficiently long error signal is required to update the system input.

On the other hand, multiple approaches have been used to analyze the ILC systems with non-uniform trial lengths problem. The contraction mapping technique combined with λ-norm was widely adopted in convergence analysis in many works (Li et al. 2014; Li and Shen 2017; Shi et al. 2020; Wei and Li 2017). To obtain stronger convergence properties, a modified λ-norm was introduced in Shen et al. (2016b) and a switching system approach was adopted in Shen et al. (2016a). Besides, analysis method based on successive projection framework was also studied in Zhuang et al. (2022) under an optimal ILC design. However, most of the above mentioned papers use stochastic models to describe the non-uniform trial lengths of the system, while another way to describe the problem is through deterministic models, whose key point is that the stochastic properties of the system do not need to be predicted or assumed in advance. In Jin (2020), analysis was completed under a deterministic model by using a modified composite energy function to deal with non-uniform trial lengths. By defining a maximum-pass-length error, Seel et al. (2017) obtained the convergence properties of the system through a direct analysis of adjacent iterations without using any specific probabilistic information .

Apart from these, Meng and Zhang (2017) innovatively applied inductive analysis for convergence analysis and also verified the deterministic convergence of the P-type ILC control law for systems with non-uniform trial lengths under a persistent full-learning property, while its processing of the missing signal is still similar to that in Shen et al. (2016a), and it also tends to update the full-length input signal right after the end of previous iteration. This makes such strategies do not take advantage of the latest information generated by the current iteration in time, therefore the aforementioned articles mostly adopt the typical open-loop learning approach.

By combining the conventional PD-type ILC with the tracking error of the current iteration, a feedback-aided learning algorithm can be formed (Sun et al. 2013). This scheme utilizes the derivation of the tracking error from the previous and the current iterations as the rectification of the input signal to improve the tracking performance. It has been used in the quantized system or against the initial state error problem (Bi et al. 2018), thus the system output can be corrected in a more timely manner to achieve better control results (Sun et al. 2015). It was further applied to systems with non-uniform trial lengths, but it was analyzed for time-invariant systems under a stochastic model (Wang et al. 2021).

In this paper, we aim to realize the control of time-varying systems under non-uniform trial length problem. A recursively updated error sequence is constructed to form the full-length error signal, and an update input sequence is constructed to store historical valid inputs. Using both sequences a feedback-aided PD-type ILC control law for time-varying systems with non-uniform trial lengths is designed. Furthermore, under an iteration recurrence interval, the non-uniform trail lengths is described by a deterministic model. The convergence of the ILC design is analyzed using both contraction mapping technique and inductive analysis approach, and the design is further extended to nonlinear systems. The effectiveness of the ILC design is showed through two simulation examples by being compared with the iterative averaging operator design in Li et al. (2014).

The main contributions of this paper can be summarized as follows:

- The full-length recursively updated error and input sequences are designed so that the learning process can make greater use of the the latest running information from most recent iterations, benefit from which the storage burden of the system is reduced.
- Through the utilization of the update sequences and tracking errors in the current iteration, a feedback-aided PD-type ILC design is applied to systems with non-uniform trial lengths to acquire better tracking performance.
- The application of the feedback-aided PD-type ILC design has also been extended to nonlinear systems, and its convergence property under uncertain initial state conditions is analyzed.

The remainder of the article is organized as follows: Section II carries out the problem formulation of the ILC problems with non-uniform trial lengths. In Section III, a feedback-aided PD-type ILC design is proposed, and its convergence analysis is presented. Moreover, the previous design is extended to nonlinear time-varying systems in Section IV. Two simulation examples are performed in Section V, while Section VI draws the conclusion.

The main notations in the article are as listed below. $\mathbb{Z}^+$ represents the set of positive integers. $\mathbb{R}^n$ denotes the n-dimensional space. $\|\cdot\|$ is the Euclidean norm, with which $\|\mathbf{f}(t)\|_\lambda = \sup_{t\in\{0,1,2,...,T\}} \alpha^{-\lambda t}\|\mathbf{f}(t)\|$ indicates the λ-norm of a vector function $\mathbf{f}(t)$ where $\lambda > 0$, $\alpha > 1$. Further notations will be introduced as needed in the following sections.

Problem formulation

We consider a discrete-time time-varying linear system as follows:

$$\begin{cases} x_k(t+1) = A_t x_k(t) + B_t u_k(t), \\ \quad\; y_k(t) = C_t x_k(t), \end{cases} \tag{1}$$

where k and t are the trial and time indexes, $k \in \mathbb{Z}^+$, $t \in \{0, 1, ..., N_k\}$ and N_k is the trial length of kth iteration. Note that $x_k(t) \in \mathbb{R}^n$, $u_k(t) \in \mathbb{R}^p$ and $y_k(t) \in \mathbb{R}^q$ respectively represent the state, input and output. A_t, B_t and C_t are system matrices with appropriate dimensions, and $C_{t+1}B_t$ is full-rank. Assume that the system has a unique ideal control input $u_d(t) \in \mathbb{R}^p$, under which the system can achieve

$$\begin{cases} x_d(t+1) = A_t x_d(t) + B_t u_d(t), \\ \quad\; y_d(t) = C_t x_d(t), \end{cases} \tag{2}$$

in which $x_d(t)$ and $y_d(t)$ are desired state and output respectively.

Due to the existence of the non-uniform trial length problem, the trial length of each iteration may be not identical when the system is put into operation. Let the actual trial length of kth iteration be N_k, it is reasonable to assume that there is a minimum value N_L and a maximum value N_H for N_k. Since we are only focusing on the system tracking performance over the desired time period, we can simply take $N_H = N_d$, in which N_d is the desired trial length. With this, we can obtain $N_k \in [N_L, N_d]$ for all $k \in \mathbb{Z}^+$, which means $t \in \{0, 1, 2, ..., N_L, ..., N_k, ..., N_d\}$.

Remark 1. *The trial lengths of practical system may be less than or exceed the desired one, the former being the non-uniform trial length problem described earlier. While when the actual trial length is larger than the desired length, the redundant part of the information is immediately discarded because it is not in our consideration, thus this will transform into a full-length case. Therefore, only the case where the actual length does not exceed the desired length is discussed in this paper.*

Based on the features of iterative learning control and the setup above, we should use the historical information from previous iterations to update the control input $u_k(t)$ for the upcoming one, so that the system can meet the control target (2) as the number of iterations increases.

Prior to the design and convergence analysis of the ILC control law design, the following lemma and assumptions need to be introduced.

Lemma 1. *(Meng and Zhang 2017) Consider an iterative system*

$$z_{k+1} = Dz_k + d_k, \quad k \in \mathbb{Z}^+, \tag{3}$$

where z_k and $d_k \in \mathbb{R}^n$ represent the state and a bounded external input. D is the system matrix with proper dimension. Then $\lim_{k\to\infty} d_k = 0$ implies $\lim_{k\to\infty} z_k = 0$, if and only if $\rho(D) < 1$, where $\rho(D)$ is the spectral radius of the system matrix D.

Assumption 1. *For any given iteration number $k \in \mathbb{Z}^+$ and a time instant $t \in \{0, 1, ..., N_k\}$, in the past σ consecutive iterations, there is at least one iteration with a trial length greater than or equal to t.*

Remark 2. *The constant σ can be referred to as the iteration recurrence interval, and when considered for the time instant N_d, the Assumption 1 can also be known as a persistent full-learning property in Meng and Zhang (2017), which is intended to improve the learning efficiency of the system. This assumption ensures that as the number of iterations k goes to infinity, the system can have enough iterations for* *each time instant's learning. This obviously relaxes the system's requirement for trial length distribution probabilities. The specific value of σ can be obtained from a priori knowledge or prediction results in real engineering, but in fact, only Assumption 1 is required to be satisfied for convergence analysis in this paper, and the actual value of σ is not required.*

Assumption 2. *The initial state of the system can be reset precisely at the beginning of each iteration, which means*

$$x_k(0) = x_d(0), \forall k \in \mathbb{Z}^+.$$

ILC design and convergence analysis

In this section, an update error sequence and an update input sequences are constructed by using an indicator function, by utilizing which an ILC update law is developed, and its convergence proof is accomplished by dividing the trial length into two parts.

Introduce an indicator function $\mathbb{I}_k(t), t \in [0, N_d]$. $\mathbb{I}_k(t) = 1$ if and only if the system can run up to the time instant t at the kth iteration, and on the contrary, the function value will be 0 when the system process ends before time instant t, that is $\mathbb{I}_k(t) = 1, t \leq N_k$ for any $k \in \mathbb{Z}^+$. It should be noted that, under the influence of the non-uniform trial lengths, for the kth iteration with a trial length of N_k, the system's output of time instants $N_k + 1, ..., N_d$ are not available, which means tracking error during this time period cannot be calculated. In light of this situation, the tracking error for this part is considered to be zero, thus the modified tracking error of the kth iteration can be described as

$$e_k^*(t) = \begin{cases} e_k(t), & t \in [0, N_k], \\ 0, & t \in [N_k + 1, N_d], \end{cases} \quad k \in \mathbb{Z}^+, \tag{4}$$

where $e_k(t) \triangleq y_d(t) - y_k(t)$ is the error between the desired trajectory of the system and the actual output tracking trajectory. Using the previously proposed indicator function, the above equation can also be expressed simply as $e_k^*(t) \triangleq \mathbb{I}_k(t)e_k(t), t \in [0, N_d]$. For upcoming iterations, the trial length is unpredictable, thus it is necessary to calculate the full length input $u_k(t), t \in [0, N_d - 1]$. For this purpose, a sequence of update errors is constructed as follows:

$$E_k(t) = \begin{cases} [1 - \mathbb{I}_k(t)]\, E_{k-1}(t) + e_k^*(t), & k \geq 1, \\ e_k(t), & k = 0. \end{cases} \tag{5}$$

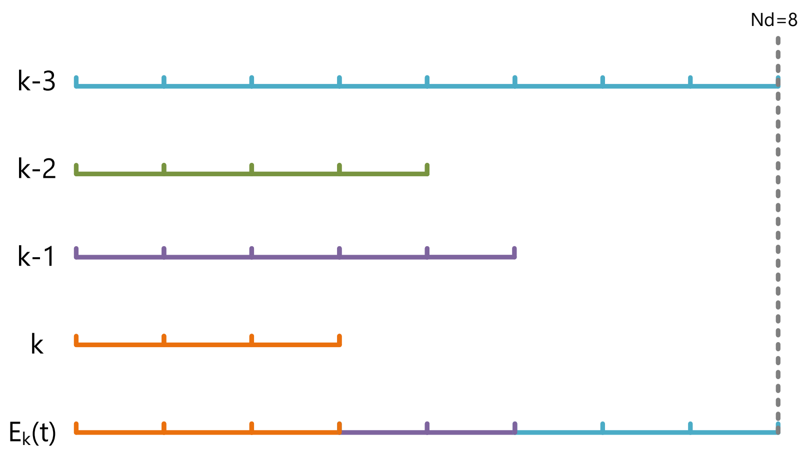

Figure 1. Composition of $E_k(t)$ in Scenario 1 with $N_d = 8$.

It can be seen from (5) that $E_k(t)$ is recursively generated, and uses the current iteration's latest information to update after every time instant's system execution.

Remark 3. *The construction of $E_k(t)$ stores each time instant's error generated by the nearest iteration that have corresponding available information. In a sense, its composition can be approximated as equivalent to a moving averaging operator in Li and Shen (2017) with a fixed window size of 1. However, one significant difference is that the search mechanism has been replaced by a recursive generation method, thus there is no need to store all the information from several previous iterations, which can reduce the storage burden on the system.*

It is easy to see that the composition of $E_k(t)$ may contain error information from multiple iterations when the trial lengths of the system are non-uniformly distributed. As depicted in Figure 1, the trial lengths from $(k-3)$th to kth iteration are all different, and only the $(k-3)$th iteration reaches the desired trial length $N_d = 9$. Hence, according to the recursion rule above, we can obtain the $E_k(t)$ for kth iteration as

$$E_k(t) = \begin{cases} e_k(t), & t \in [0,3], \\ e_{k-1}(t), & t \in [4,5], \\ e_{k-3}(t), & t \in [6,8]. \end{cases} \tag{6}$$

By utilizing the update error sequence $E_k(t)$, there is a sufficient error information can be used to calculate the input for the next iteration. However, it should be noted that for any upcoming iteration, when using $E_k(t)$ directly to update $u_{k+1}(t)$, all the $u_{k+1}(t), t \in [0, N_d - 1]$ will be updated, whether they are actually put into operation

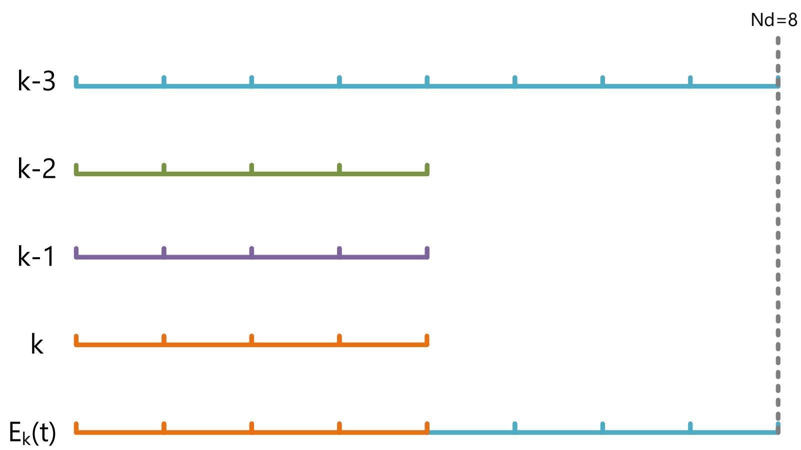

Figure 2. Composition of $E_k(t)$ in Scenario 2 with $N_d = 8$.

or not. In this case, as the situation illustrated in Figure 2 occurs, since several successive iterations do not run to the 5th and subsequent time instants, if $u_{k+1}(t)$ is updated directly using $u_k(t)$ and $E_k(t)$, the inputs at these time instants are repeatedly updated with the same error information from $(k-3)$th iteration without actually being put into operation to obtain the new tracking error, which is obviously redundant.

To avoid such a problem, it is necessary to introduce an updated input sequence $U_k(t)$ with the same idea as the construction of $E_k(t)$:

$$U_k(t) = \begin{cases} [1 - \mathbb{I}_k(t+1)]\, U_{k-1}(t) + \mathbb{I}_k(t+1) u_k(t), & k \geq 1, \\ u_k(t), & k = 0. \end{cases} \tag{7}$$

The reason why the indicator function in (7) is $\mathbb{I}_k(t+1)$ is that the input at time instant t needs to obtain the corresponding output information for the error calculation at next time instant $t+1$. The updated input sequence generated by such recursion stores the most recent updated input which has its corresponding output result at each time instant. By utilizing $E_k(t)$ and $U_k(t)$, a feedback-aided PD-type ILC control law is given as follows:

$$u_{k+1}(t) = U_k(t) + L_t E_k(t+1) + \Gamma_t E_k(t) + K_t E_{k+1}(t), \tag{8}$$

in which $t \in [0, N_d - 1]$ for all $k \in \mathbb{Z}^+$, with L_t, Γ_t and K_t are learning gains for the system to be designed. The following theorem can be obtained by applying the ILC control law (8) to the system (1).

Theorem 1. *For the discrete linear time-varying system (1), when ILC update law (8) is applied with Assumptions 1 and 2 satisfied, if the appropriate learning gain is chosen such that*

$$0 < \|I - C_{t+1} B_t L_t\| < 1, \forall t \in [0, N_d], \tag{9}$$

then as the number of iterations k approaches infinity, the tracking goal described in (2) can be achieved on the time interval $t \in [0, N_d]$, that is $\lim_{k \to \infty} e_k^*(t) = 0, \forall t \in [0, N_d]$, where I is a unit matrix with appropriate dimensions.

Proof. The proof of Theorem 1 is completed in two separate parts.

Part-I. Prove that Theorem 1 holds when $t \in [0, N_L]$. Noticing that for all the iteration, time instant N_L is reachable, namely, $N_k \geq N_L, \forall k \in \mathbb{Z}^+$. Then the errors on that time interval are available and will be updated at each iteration. It is easy to conclude that $E_k(t) = e_k^*(t) = e_k(t), \forall k \in \mathbb{Z}^+, t \in [0, N_L]$. From system (1), we have

$$x_k(t) = \left(\prod_{j=0}^{t-1} A_j \right) x_k(0) + \sum_{j=0}^{t-1} \left(\prod_{l=0}^{t-j-2} A_{t-1-l} \right) B_j u_k(j), \tag{10}$$

by defining $\Delta u_{k+1}(t) \triangleq u_{k+1}(t) - u_k(t)$ and $\Delta x_{k+1}(t) \triangleq x_{k+1}(t) - x_k(t), t \in [0, N_L]$, which further leads to

$$\begin{aligned} \Delta x_{k+1}(t) = & \left(\prod_{j=0}^{t-1} A_j \right) \Delta x_{k+1}(0) \\ & + \sum_{j=0}^{t-1} \left(\prod_{l=0}^{t-j-2} A_{t-1-l} \right) B_j \Delta u_{k+1}(j) \\ = & \sum_{j=0}^{t-1} \left(\prod_{l=0}^{t-j-2} A_{t-1-l} \right) B_j \Delta u_{k+1}(j). \end{aligned} \tag{11}$$

where $x_{k+1}(0) - x_k(0) = 0$ by noticing Assumption 2. Further, from the definition of $e_{k+1}^*(t+1)$ we can obtain

$$\begin{aligned} & e_{k+1}^*(t+1) \\ & = e_{k+1}(t+1) \\ & = e_k(t+1) - C_{t+1} \left[x_{k+1}(t+1) - x_k(t+1) \right] \\ & = e_k(t+1) - C_{t+1} \sum_{j=0}^{t} \left(\prod_{l=0}^{t-j-1} A_{t-l} \right) B_j \Delta u_{k+1}(j), \end{aligned} \tag{12}$$

in the formula, $t \in \psi$ and $\psi = [0, N_L - 1]$. It is easy to conclude that when $t \in \psi$, $U_k(t) = u_k(t)$, thus according to the designed ILC control law it implies that

$$\begin{aligned} e_{k+1}(t+1) = &(I - C_{t+1}B_tL_t)\, e_k(t+1) \\ &- C_{t+1}\sum_{j=0}^{t-1}\left(\prod_{l=0}^{t-j-1} A_{t-l}\right) B_jL_je_k(j+1) \\ &- C_{t+1}\sum_{j=0}^{t}\left(\prod_{l=0}^{t-j-1} A_{t-l}\right) B_j\Gamma_je_k(j) \\ &- C_{t+1}\sum_{j=0}^{t}\left(\prod_{l=0}^{t-j-1} A_{t-l}\right) B_jK_je_{k+1}(j). \end{aligned} \tag{13}$$

Next, taking the norm at both ends of equation (13) yields

$$\begin{aligned} \|e_{k+1}(t+1)\| \le &\|I - C_{t+1}B_tL_t\|\|e_k(t+1)\| \\ &+ l_1\sum_{j=0}^{t-1}\alpha^{t-j}\|e_k(j+1)\| \\ &+ l_2\sum_{j=0}^{t}\alpha^{t-j+1}\|e_k(j)\| \\ &+ l_3\sum_{j=0}^{t}\alpha^{t-j+1}\|e_{k+1}(j)\|, \end{aligned} \tag{14}$$

in which $\alpha \ge \sup_{t\in\psi}\|A_t\|$, $l_1 \triangleq \sup_{t\in\psi}\|C_{t+1}\|\|B_t\|\|L_t\|$, $l_2 \triangleq \sup_{t\in\psi}\|C_{t+1}\|\|B_t\|\|\Gamma_t\|\alpha^{-1}$ and $l_3 \triangleq \sup_{t\in\psi}\|C_{t+1}\|\|B_t\|\|K_t\|\alpha^{-1}$ respectively. By noticing $e_k(0) = C_0[x_d(0) - x_k(0)] = 0, \forall k \in \mathbb{Z}^+$, from the above equation it can be further derived that

$$\begin{aligned} \|e_{k+1}(t+1)\| \le &\|I - C_{t+1}B_tL_t\|\|e_k(t+1)\| \\ &+ (l_1 + l_2)\sum_{j=0}^{t-1}\alpha^{t-j}\|e_k(j+1)\| \\ &+ l_3\sum_{j=0}^{t-1}\alpha^{t-j}\|e_{k+1}(j+1)\|. \end{aligned} \tag{15}$$

Multiplying both sides of the above equation by $\alpha^{-\lambda(t+1)}$ and taking the supremum with respect to the time interval $t \in \psi$ gives rise to

$$
\begin{aligned}
&\sup_{t\in\psi}\alpha^{-\lambda(t+1)}\|e_{k+1}(t+1)\| \\
&\leq \rho\sup_{t\in\psi}\alpha^{-\lambda(t+1)}\|e_k(t+1)\| \\
&\quad + (l_1+l_2)\sup_{t\in\psi}\alpha^{-\lambda(t+1)}\sum_{j=0}^{t-1}\alpha^{t-j}\|e_k(j+1)\| \\
&\quad + l_3\sup_{t\in\psi}\alpha^{-\lambda(t+1)}\sum_{j=0}^{t-1}\alpha^{t-j}\|e_{k+1}(j+1)\|,
\end{aligned}
\tag{16}
$$

where $\rho \triangleq \sup_{t\in\psi}\|I - C_{t+1}B_tL_t\|$ is applied. According to the definition of λ-norm, we can obtain

$$
\begin{aligned}
&\sup_{t\in\psi}\alpha^{-\lambda(t+1)}\sum_{j=0}^{t-1}\alpha^{t-j}\|e_k(j+1)\| \\
&= \sup_{t\in\psi}\alpha^{-(\lambda-1)t}\sum_{j=0}^{t-1}\alpha^{-\lambda(j+1)}\|e_k(j+1)\|\alpha^{(\lambda-1)j} \\
&\leq \|e_k(t+1)||_\lambda\sup_{t\in\psi}\alpha^{-(\lambda-1)t}\sum_{j=0}^{t-1}\alpha^{(\lambda-1)j} \\
&\leq \rho_1\|e_k(t+1)||_\lambda,
\end{aligned}
\tag{17}
$$

where $\rho_1 \triangleq \frac{1-\alpha^{-(\lambda-1)(N_L-1)}}{\alpha^{\lambda-1}-1}$. By combining (16) and (17), it follows that

$$
\|e_{k+1}(t+1)||_\lambda \leq \rho_0\|e_k(t+1)||_\lambda,
\tag{18}
$$

where $\rho_0 \triangleq \frac{\rho+(l_1+l_2)\rho_1}{1-l_3\rho_1}$. By Theorem 1, $0<\rho<1$, so that when λ is chosen sufficiently large, it is feasible that $\rho_0 < 1$ with $1-l_3\rho_1 > 0$, which implies that $\lim_{k\to\infty}\|e_k(t+1)||_\lambda = 0, \forall t \in [0, N_L-1]$, and it can be further concluded that $\lim_{k\to\infty} e_k(t) = 0, \forall t \in [0, N_L]$. Meanwhile, from $E_k(t) = e_k^*(t) = e_k(t), \forall k \in \mathbb{Z}^+, t \in [0, N_L]$, the result $\lim_{k\to\infty} e_k(t) = E_k(t) = e_k^*(t) = 0, \forall k \in \mathbb{Z}^+, t \in [0, N_L]$ can be easily derived, this completes the Part-I of the proof. □

Part-II. Prove that Theorem 1 holds when $t \in [N_L+1, N_d]$. It has already been proved that Theorem 1 holds when $t \in [0, N_L]$ in Part-I. By inductive analysis method, assume that for any $T \in [N_L, N_d-1]$, $\lim_{k\to\infty} e_k^*(t) = 0, \forall t \in [0, T]$ holds, thus $\lim_{k\to\infty} E_k(t) = 0, \forall t \in [0, T]$ can be obtained, now we need to prove that $\lim_{k\to\infty} e_k^*(t) = 0$ when $t = T+1$.

Define a subset $\partial(T+1)$, which contains all the iterations whose trial length $N_k \geq T+1$. Rearrange the elements of $\partial(T+1)$ in the order of their occurrence, thus we have $\partial(T+1) = \{k_i : i \in \mathbb{Z}^+\}$. And from Assumption 1 it is clear that the distance between k_i and k_{i+1} is less than or equal to σ. With this subset, it is easy to obtain

$$\lim_{k\to\infty} e_k^*(t) = 0 \Leftrightarrow \lim_{i\to\infty} e_{k_i}^*(t) = 0, \tag{19}$$

when $t = T+1$ and

$$e_{k_i}(T+1) = e_{k_i}^*(T+1) = E_{k_i}(T+1), i \in \mathbb{Z}^+. \tag{20}$$

By using the above equation (20) and $u_{k_{i+1}}(t) - u_{k_i}(t)$ is defined as $\bar{\Delta} u_{k_{i+1}}(t), t \in [0, T]$, we can have

$$\begin{aligned} & e_{k_{i+1}}^*(T+1) \\ & = e_{k_i}^*(T+1) - C_{T+1}\left[x_{k_{i+1}}(T+1) - x_{k_i}(T+1)\right] \\ & = e_{k_i}^*(T+1) - C_{T+1}A_T\left[x_{k_{i+1}}(T) - x_{k_i}(T)\right] \\ & \quad - C_{T+1}B_T\bar{\Delta} u_{k_{i+1}}(T). \end{aligned} \tag{21}$$

Because for all $k \notin \partial(T+1)$, $\mathbb{I}_k(T+1) = 0$, this means that $U_k(T)$ and $E_k(T+1)$ are not updated at the end of these iterations, namely, $U_{k_{i+1}-1}(T) = U_{k_i}(T) = u_{k_i}(T), E_{k_{i+1}-1}(T+1) = E_{k_i}(T+1) = e_{k_i}^*(T+1), i \in \mathbb{Z}^+$. According to the ILC update law (8), one could have

$$\begin{aligned} & \bar{\Delta} u_{k_{i+1}}(T) \\ & = u_{k_{i+1}}(T) - u_{k_i}(T) \\ & = L_T E_{k_i-1}(T+1) + \Gamma_T E_{k_{i+1}-1}(T) + K_T E_{k_{i+1}}(T) \\ & = L_T E_{k_i}(T+1) + \Gamma_T E_{k_{i+1}-1}(T) + K_T E_{k_{i+1}}(T). \end{aligned} \tag{22}$$

Then substituting (22) into (21) and defining $\bar{\Delta} x_{k_{i+1}}(t) \triangleq x_{k_{i+1}}(t) - x_{k_i}(t), t \in [0, T]$ further yields

$$\begin{aligned} & e_{k_{i+1}}^*(T+1) \\ & = (I - C_{T+1}B_T L_T)E_{k_i}(T+1) - C_{T+1}A_T\bar{\Delta} x_{k_{i+1}}(T) \\ & \quad - C_{T+1}B_T[\Gamma_T E_{k_{i+1}-1}(T) + K_T E_{k_{i+1}}(T)] \\ & = (I - C_{T+1}B_T L_T)e_{k_i}^*(T+1) - C_{T+1}A_T\bar{\Delta} x_{k_{i+1}}(T) \\ & \quad - C_{T+1}B_T[\Gamma_T E_{k_{i+1}-1}(T) + K_T E_{k_{i+1}}(T)], \end{aligned} \tag{23}$$

For $\bar{\Delta}x_{k_{i+1}}(T)$, we can recursively get

$$\begin{aligned}\bar{\Delta}x_{k_{i+1}}(T) =& A_{T-1}\bar{\Delta}x_{k_{i+1}}(T-1) + B_{T-1}\bar{\Delta}u_{k_{i+1}}(T-1)\\ =&(\prod_{j=0}^{T-1} A_j)\bar{\Delta}x_{k_{i+1}}(0)\\ &+\sum_{j=0}^{T-1}(\prod_{l=0}^{t-j-2} A_{t-1-l})B_j\bar{\Delta}u_{k_{i+1}}(j)\\ =&\sum_{j=0}^{T-1}(\prod_{l=0}^{t-j-2} A_{t-1-l})B_j\bar{\Delta}u_{k_{i+1}}(j),\end{aligned} \tag{24}$$

in which $\bar{\Delta}x_{k_{i+1}}(0) = x_{k_{i+1}}(0) - x_{k_i}(0) = 0$ according to Assumption 2.

From the generation rule for the update input sequence $U_k(t)$ it follows that $U_{k_i}(t)$ will be updated only if the trial length of k_ith iteration $N_{k_i} \geq t+1$. Considering this, define a number of subsets $\bar{\partial}(t), t \in [0, T]$ between k_ith and k_{i+1}th iteration, similarly, arrange the elements in $\bar{\partial}(t)$ in the order of occurrence as $\bar{\partial}(t) = \{k_j^t : j \in \{0, 1, ..., \max\}\}$. This means that for input $u_{k_{i+1}}(t)$ of time instant $t \in [0, T]$, it will only be updated at the end of the iterations in $\bar{\partial}(t+1)$. It is easy to conclude that k_1^t and k_i are the same iteration, and also that $k_{\max}^t$ and k_{i+1} are the same iteration. Thus when $t \in [0, T]$, we can have

$$\begin{aligned}&\bar{\Delta}u_{k_{i+1}}(t)\\ &= u_{k_{i+1}}(t) - u_{k_i}(t)\\ &= u_{k_{\max}^{t+1}}(t) - u_{k_1^{t+1}}(t)\\ &= L_t\sum_{m=1}^{\max -1} E_{k_m^{t+1}-1}(t+1) + \Gamma_t\sum_{m=1}^{\max -1} E_{k_m^{t+1}-1}(t)\\ &\quad + K_t\sum_{m=1}^{\max -1} E_{k_m^{t+1}-1}(t).\end{aligned} \tag{25}$$

Again, by the definition of $\partial(T+1)$ and $\bar{\partial}(t)$, it can be seen that $k_j^t \to \infty$ as $k_i \to \infty$, and $k_i \to \infty$ as $k \to \infty, \forall t \in [0, T]$. As previously assumed, $\lim_{k\to\infty} E_k(t) = 0, \forall t \in [0, T]$. Meanwhile, from Assumption 1 it is clear that there are at most $\sigma - 1$ iterations between the k_ith and k_{i+1}th iteration, which means for each time instant $t \in [0, T]$, there are a finite number of elements in $\bar{\partial}(t)$, thus $\lim_{i\to\infty} \bar{\Delta}u_{k_{i+1}}(t) = 0$ is obtained. As a result we can ensure that $\lim_{i\to\infty} \bar{\Delta}x_{k_{i+1}}(T) = 0$.

Based on the above, applying Lemma 1 to equation (23), we can conclude that $\lim_{i\to\infty} e^*_{k_i}(T+1) = 0$ if and only if $\rho(I - C_{T+1}B_T L_T) < 1$, that is, $\lim_{k\to\infty} e^*_k(T+1) = 0$. Meanwhile, from the definition of spectral radius, it is easy to obtain that $\rho(I - C_{T+1}B_T L_T) \le \|I - C_{T+1}B_T L_T\|$. Therefore, based on inductive analysis, we can conclude that $\lim_{k\to\infty} e^*_k(t) = 0, \forall t \in [N_L+1, N_d]$, and this completes the second part of the proof. □

Combining Part-I and Part-II of the proof, we can finally have that when under Assumption 1 and 2, if the learning gain of the system is chosen such that $0 < \|I - C_{t+1}B_t L_t\| < 1, \forall t \in [0, N_d]$ is satisfied, the tracking goal (2) can be achieved, namely, $\lim_{k\to\infty} e^*_k(t) = 0, \forall t \in [0, N_d]$. Thus Theorem 1 is proved. □

Assumption 2 is also referred to as the identical initial condition, which guarantees the previously derived convergence properties of the system. The resetting of initial state is one of the most important topics in the area of ILC (Park 2005). In some practical industrial scenarios, the identical initial condition may not be maintained for all iterations. More specifically, the initial state of the system may fluctuate in a small bounded range around the desired initial state. In the coming section, the control law proposed in this paper is extended to nonlinear systems, and the convergence property is analyzed in the case that Assumption 2 is not satisfied.

Extension to nonlinear systems

Consider a class of discrete nonlinear time-varying affine systems as follows:

$$\begin{cases} x_k(t+1) = f(x_k(t), t) + B_t u_k(t), \\ \quad y_k(t) = C_t x_k(t), \end{cases} \tag{26}$$

in which $t \in [0, N_d]$, $f(x_k(t), t) \in \mathbb{R}^n$ is the time-varying nonlinear function. For this system, the following additional lemma and assumptions are given:

Lemma 2. *(Sun and Wang 2001) For every $k \in \mathbb{Z}^+$, a_k and b_k are non-negative, a_0 and b_0 are bounded. Thus when $\lim_{k\to\infty} b_k = b_\infty$, if $0 \le \rho < 1$ is satisfied, the following inequality*

$$a_{k+1} = \rho a_k + b_k, \quad k \in \mathbb{Z}^+, \tag{27}$$

implies that $\limsup_{k\to\infty} a_k \le \frac{b_\infty}{1-\rho}$.

Assumption 3. *The nonlinear function $f(x_k(t), t) \in \mathbb{R}^n$ satisfies global Lipschitz condition, which means there exist a constant $k_f > 0$, for all $t \in [0, N_d]$ and*

$x_1(t), x_2(t) \in \mathbb{R}^n$,

$$\|f(x_1(t),t) - f(x_2(t),t)\| \le k_f \|x_1(t) - x_2(t)\|. \tag{28}$$

Assumption 4. *The initial state of the system changes in a bounded neighborhood of the desired state at each iteration, satisfying*

$$||x_d(0) - x_k(0)|| \le \omega, \omega > 0, \forall k \in \mathbb{Z}^+. \tag{29}$$

These lead to the following theorem:

Theorem 2. *For the discrete nonlinear time-varying affine system (26) with ILC law (8) applied, and Assumption 1, 3 and 4 hold, if the appropriate learning gain is chosen such that*

$$0 < \|I - C_{t+1} B_t L_t\| < 1, \forall t \in [0, N_d], \tag{30}$$

then as the number of iteration k approaches infinity, the tracking error of the system converges to a bounded area proportional to ω, i.e.

$$\lim_{k\to\infty} ||e_k^*(t)|| \le \vartheta\omega, \forall t \in [0, N_d], \tag{31}$$

where ϑ is a suitable constant greater than 0.

Proof. The proof of Theorem 2 can be proved using similar steps as Theorem 1, only noticing that the global Lipschitz condition is applied and Assumption 3 is combined at the appropriate time to construct the inequality, recursively we have

$$\begin{aligned}
&\|x_{k+1}(t+1) - x_k(t+1)\| \\
&\le \|f(x_{k+1}(t),t) - f(x_k(t),t)\| \\
&\quad + \|B_t\|\|u_{k+1}(t) - u_k(t)\| \\
&\le (k_f)^{t+1}\|x_{k+1}(0) - x_k(0)\| \\
&\quad + \sum_{i=0}^{t} {k_f}^{t-i}\|B_i\|\|\Delta u_{k+1}(i)\| \\
&\le 2(k_f)^{t+1}\omega + k_b \sum_{i=0}^{t} {k_f}^{t-i}\|\Delta u_{k+1}(i)\|,
\end{aligned} \tag{32}$$

where $k_b \triangleq \sup_{t\in\psi}\|B_t\|$. With this, for the Part-I of the proof, when $t \in \psi$ the equation (12) in the proof of Theorem 1 can be rewritten into

$$\begin{aligned}
&e^*_{k+1}(t+1)\\
&= e_{k+1}(t+1)\\
&= e_k(t+1) - C_{t+1}\left[x_{k+1}(t+1) - x_k(t+1)\right]\\
&= e_k(t+1) - C_{t+1}\left[f(x_{k+1}(t),t) - f(x_k(t),t)\right]\\
&\quad - C_{t+1}B_t\Delta u_{k+1}(t)\\
&= \left(I - C_{t+1}B_tL_t\right)e_k(t+1)\\
&\quad - C_{t+1}\left[f(x_{k+1}(t),t) - f(x_k(t),t)\right]\\
&\quad - C_{t+1}B_t\Gamma_t e_k(t) - C_{t+1}B_tK_te_{k+1}(t).
\end{aligned} \tag{33}$$

Then, taking the norm on both ends of the equation (33) and substituting (32) subsequently gives rise to

$$\begin{aligned}
&\|e_{k+1}(t+1)\|\\
&\leq \|(I - C_{t+1}B_tL_t)\|\,\|e_k(t+1)\|\\
&\quad + k_f\,\|C_{t+1}\|\,\|x_{k+1}(t) - x_k(t)\|\\
&\quad + k_b\,\|C_{t+1}\|\,\|\Gamma_t\|\,\|e_k(t)\| + k_b\,\|C_{t+1}\|\,\|K_t\|\,\|e_{k+1}(t)\|\\
&\leq \|(I - C_{t+1}B_tL_t)\|\,\|e_k(t+1)\|\\
&\quad + k_fk_b\,\|C_{t+1}\|\sum_{i=0}^{t-1}{k_f}^{t-i-1}\,\|\Delta u_{k+1}(i)\| + 2{k_f}^t\omega\\
&\quad + k_b\,\|C_{t+1}\|\,\|\Gamma_t\|\,\|e_k(t)\| + k_b\,\|C_{t+1}\|\,\|K_t\|\,\|e_{k+1}(t)\|,
\end{aligned} \tag{34}$$

by replacing $k_c \triangleq \sup_{t\in\psi}\|C_{t+1}\|$ which further leads to

$$\begin{aligned}
\|e_{k+1}(t+1)\| \leq{}& \|(I - C_{t+1}B_tL_t)\|\,\|e_k(t+1)\|\\
&+ k_1\sum_{i=0}^{t-1}\alpha^{t-i}\|e_k(i+1)\|\\
&+ k_2\sum_{i=0}^{t}\alpha^{t-i+1}\|e_k(i)\|\\
&+ k_3\sum_{i=0}^{t}\alpha^{t-i+1}\|e_{k+1}(i)\| + 2{k_f}^t\omega,
\end{aligned} \tag{35}$$

where the parameters $\alpha \geq k_f$, and denote $k_1 \triangleq \sup\limits_{t\in\psi} k_c k_b \|L_t\|$, $k_2 \triangleq \sup\limits_{t\in\psi} k_c k_b \|\Gamma_t\| \alpha^{-1}$ and $k_3 \triangleq \sup\limits_{t\in\psi} k_c k_b \|K_t\| \alpha^{-1}$ for simplicity.

Then by noticing Assumption 3, the equation 15 in Theorem 1 turns into

$$\begin{aligned} \|e_{k+1}(t+1)\| \leq \|I - C_{t+1}B_t L_t\| \|e_k(t+1)\| \\ + (k_1 + k_2)\sum_{i=0}^{t-1} \alpha^{t-i} \|e_k(i+1)\| \\ + k_3 \sum_{i=0}^{t-1} \alpha^{t-i} \|e_{k+1}(i+1)\| + \phi\omega, \end{aligned} \tag{36}$$

with $\phi \triangleq \left[k_c(k_2 + k_3)\alpha^{t+1} + 2{k_f}^t\right]$. Next, by applying the λ-norm technique and a derivation procedure similar to that of Theorem 1 further yields

$$\|e_{k+1}(t+1)||_\lambda \leq \bar{\rho}_0 \|e_k(t+1)||_\lambda + \frac{\alpha^{-(\lambda-1)t}}{1 - l_3\bar{\rho}_1}\phi\omega. \tag{37}$$

Because we only focus on system tracking performance for a finite period of time, there can always be a constant $\bar{\phi}$ such that $\sup\limits_{t\in\psi} \frac{\alpha^{-(\lambda-1)t}}{1-l_3\bar{\rho}_1}\phi < \bar{\phi}$, thus one could have

$$\|e_{k+1}(t+1)||_\lambda \leq \bar{\rho}_0 \|e_k(t+1)||_\lambda + \bar{\phi}\omega. \tag{38}$$

According to Lemma 2, (38) further implies

$$\limsup_{k\to\infty} \|e_{k+1}(t+1)\|_\lambda \leq \frac{\bar{\phi}}{1 - \bar{\rho}_0}\omega, \tag{39}$$

and this leads to

$$\limsup_{k\to\infty} \|e_{k+1}(t+1)\| \leq \frac{\alpha^{\lambda(t+1)}\bar{\phi}}{1 - \bar{\rho}_0}\omega, t \in \psi, \tag{40}$$

from which it is easy to obtain that the tracking error $e_k(t)$ is bounded to a region proportional to ω while the iteration number k rises when $t \in [0, N_L]$. Furthermore, the second part of the proof can be done using the same idea as Theorem 1. Through the utilization of Assumptions 3 and 4, the inequality that is similar to (23) and satisfies Lemma 2 can be formed, which leads to the bounded convergence of $\|e_k^*(t)\|$ for $t \in [N_L + 1, N_d]$ part. Combining the above obtained results, we can always find a suitable constant ϑ such that $\limsup_{k\to\infty} \|e_k^*(t)\| \leq \vartheta\omega$, thus the proof of Theorem 2 is completed. □

Simulation Results

To verify the effectiveness of the proposed ILC control law, two separate simulation examples were performed. Firstly, Example 1 applies the proposed ILC control law to a numerical simulation model of a linear time-varying system. The second example is a nonlinear time-varying model of a single-joint robot system.

Example 1

Consider a discrete linear time-varying system (Shen et al. 2016b)

$$\begin{cases} x_k(t+1) &= \begin{pmatrix} 0.2e^{(-t/100)} & -0.6 & 0 \\ 0 & 0.5 & \sin(t) \\ 0 & 0 & 0.7 \end{pmatrix} x_k(t) \\ & \quad + \begin{pmatrix} 0 \\ 0.3\sin(t) \\ 1 \end{pmatrix} u_k(t), \\ y_k(t) &= \begin{pmatrix} 1 & 0.1 & 1+0.1\cos(t) \end{pmatrix} x_k(t), \end{cases} \tag{41}$$

whose initial state is set as $x_k(0) = x_d(0) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$. The system's trial length varies between 45 and 55, which means $N_L = 45$, $N_d = 55$ and $N_k \in [45, 55]$. The desired output trajectory of the system is $y_k(t) = \sin(2\pi t/50) + \sin(2\pi t/5)$, $t \in [0, 55]$.

Further, apply ILC control law (8) with learning gains set as $L_t = 0.5$, $\Gamma_t = 0.1$, $K_t = 0.2$, $\forall t \in [0, N_d]$. This will satisfy the requirement in Theorem 1 that $0 < \|I - C_{t+1}B_tL_t\| < 1, t \in [0, N_d]$. It should be noted that this control law design does not require the distribution probabilities of the actual trial lengths of the system iterations, and only needs to satisfy the iteration recurrence interval σ in Assumption 1. In this example, σ is set as 20, this means that even the time instant $t = 55$ with a minimum number of arrivals will be reached at least once in 20 consecutive iterations. This requirement ensures the effectiveness of the ILC control law for the trial lengths varying part of the system.

Setting the total iteration number of the system to 50, it can be seen from Figure 3 that the output of the system is very close to the desired trajectory by the third iteration, and the output of 50th iteration almost coincides with the desired one.

As illustrated in Figure 4 that the tracking error of the system decreases as the number of iterations increases and converges to zero asymptotically. In detail, the trial length of the 3rd iteration is 45, and the 15th iteration's trial length is 52, its error

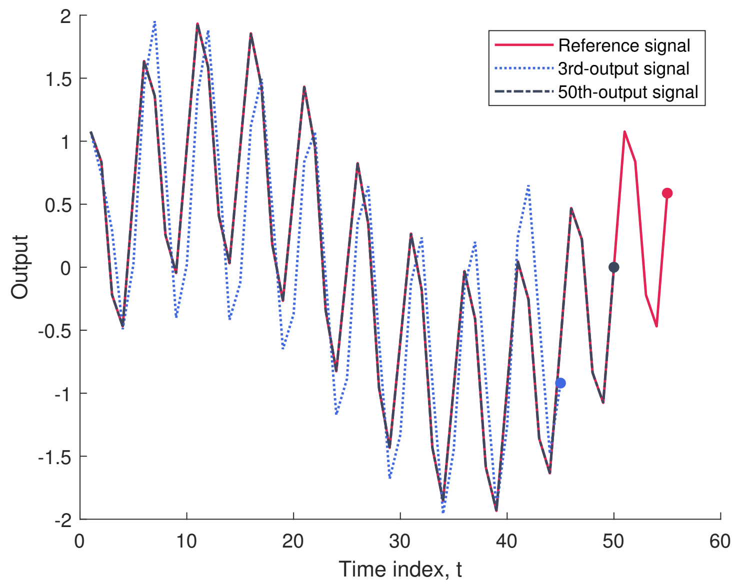

Figure 3. The reference signal and the output profiles of 3rd and 80th iterations.

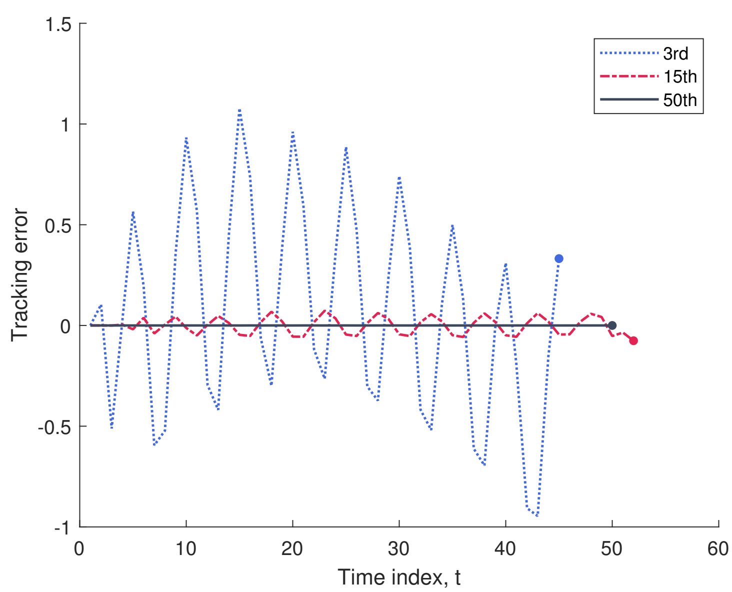

Figure 4. The tracking errors of 2nd, 15th and 50th iterations.

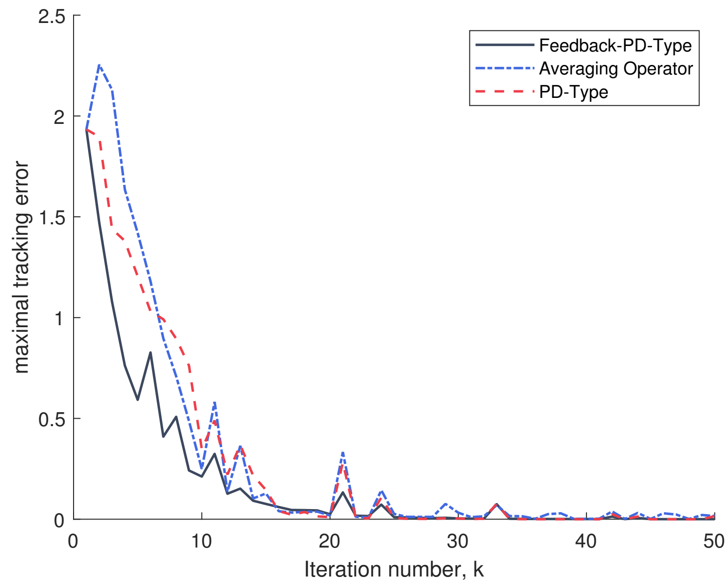

Figure 5. The maximal tracking error along all iterations.

decreases significantly compared to the second iteration. N_{50} = 50, and the tracking error within its trial length has almost converged to zero.

For comparison, the control output results using the iterative averaging operator design and the conventional PD-type design are also depicted in Figure 5, where the maximum tracking error is defined as $\max_{t\in[0,N_k]}\|e_k(t)\|$. One could see that the control law proposed in this paper can better track the desired trajectory with a faster convergence speed.

As mentioned before, the analysis of convergence requires only the presence of σ but not its specific value. However, it should be pointed out that a smaller value of σ accelerates the convergence of the system since it implies more learning times for any given time instant. A special case is that when σ equals 1, the non-uniform trial lengths problem vanishes and the control of given system can be transformed into a traditional ILC problem. To demonstrate this, the maximal tracking error along all iterations under different values of σ is illustrated in Figure 6, from which it can be concluded that the maximal tracking error decreases faster under a smaller value of σ.

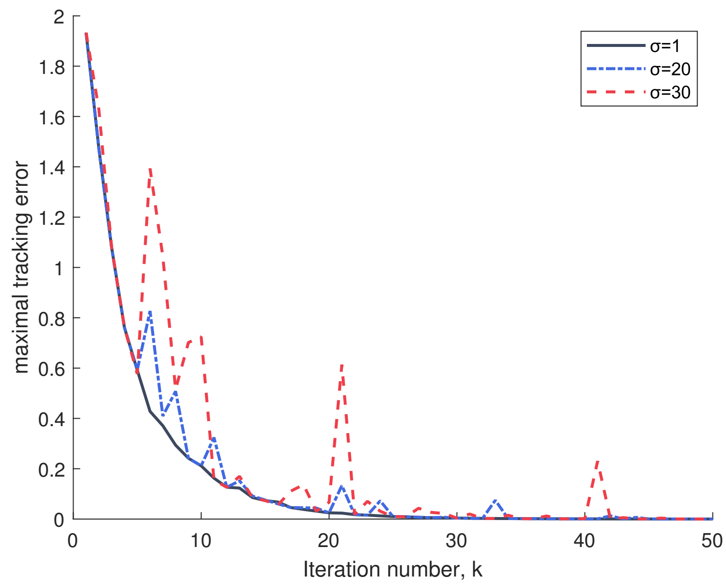

Figure 6. The maximal tracking error profiles under different σ.

Example 2

Consider a single-joint robot system (Wu and Chen 2010), whose dynamic model can be represented as

$$J_M\ddot{\theta}(t) + Mgl\sin(\theta(t)) = \tau(t), \tag{42}$$

where $J_M = 1.33Ml^2$ is the momentum of inertia, $\theta(t)$ is the rotation angle of the force arm of the single-joint robot, $M = 10kg$ is for mass, $g = 9.8m/s^2$ is the acceleration of gravity, $l = 2.5m$ is the rotation distance of the center of mass to the center of the connecting rod, $\tau(t)$ is for acting torque. Let the sampling period of the system be 0.1, with $\theta(t) = x^{(1)}(t)$, $\theta(t+1) = x^{(2)}(t)$ and $u(t) = \tau(t)$, then the system can be further described as

$$\begin{cases} x^{(1)}(t+1) = x^{(2)}(t), \\ x^{(2)}(t+1) = -{J_M}^{-1}Mgl\sin(x^{(1)}(t)) + -{J_M}^{-1}u(t). \end{cases} \tag{43}$$

The output signal is $y(t) = (0.4 + 0.2\sin t)x^{(2)}(t)$, which means the system matrix $B_t = \begin{bmatrix} 0 & 0.012 \end{bmatrix}^T$ and $C_t = \begin{bmatrix} 0 & 0.4 + 0.2\sin t \end{bmatrix}$ in (26). The desired tracking trajectory of the system $y_d(t) = 20 + 1.5t - t^2$, and the desired trial length is 5.5 with a total of 55 sampling points. The actual trial length of the system per iteration varies

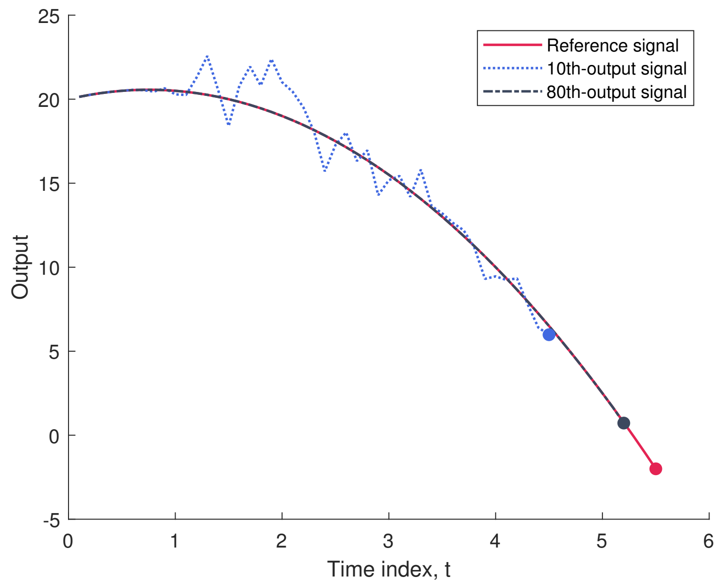

Figure 7. The reference signal and the output profiles of 3rd and 80th iterations.

between 4.5 and 5.5, that is, $N_L = 4.5$, $N_d = 5.5$ and $N_k \in [4.5, 5.5]$. σ is set to 20 as in the previous example.

According to Theorem 2, the ILC control law gains are set to $L_t = 120$, $\Gamma_t = 25$, $K_t = 20$, $\forall t \in [0, N_d]$, the initial state is set to $x_k(0) = \begin{bmatrix} 0 & 50 \end{bmatrix}^T$ with 80 total iterations.

Similar to Example 1, in the Figure 7 it can be seen that the trial length of the 10th iteration is 4.5, its output is close to the desired tracking trajectory, but there is still some error. The final iteration trial length is 5.2, its output can track the desired trajectory well, and the two almost coincide.

The tracking error given in Figure 8 shows that the output error of the system decreases significantly as the number of iterations k increases. Like Example 1, as can be seen in Figure 9, the proposed ILC control law is compared with the designs using the iterative averaging operator and conventional PD-type ILC for the maximal tracking error in $[0, N_k]$, we can find that the proposed control law can well realize the control of nonlinear time-varying systems and can achieve faster tracking error convergence. To verify the effectiveness of the proposed control law under Assumption 4, the initial states of the system are set to fluctuate within 0.05% and 0.005% around the desired state respectively. These tracking performances are worse than the case

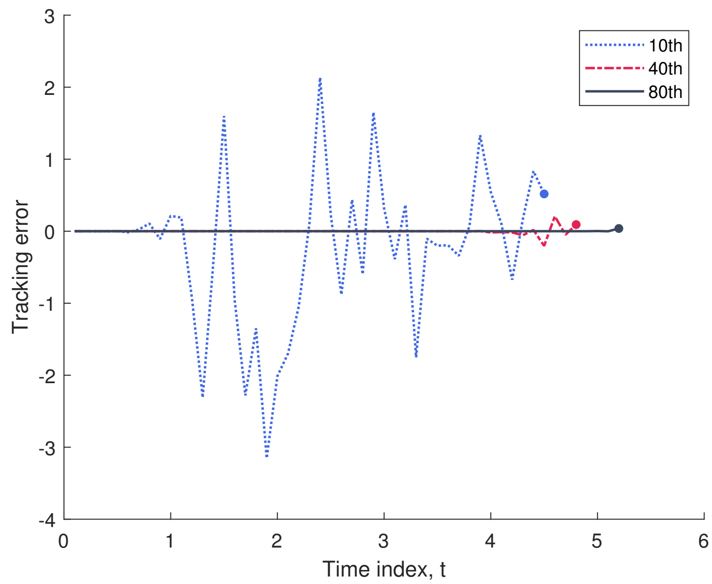

Figure 8. The tracking errors of 3rd, 20th and 80th iterations.

where the initial state is always equal to the desired state, but the control law still has robust performance. As depicted in Figure 10, fluctuations in the initial state lead to bounds on the tracking errors, and as the fluctuation increases, the error will converge to a larger range.

Conclusion

In this paper, we construct a recursively generated update error sequence and an update input sequence for the non-uniform trial length problem that arises when applying ILC to discrete time-varying systems, using which a feedback-aided PD-type ILC law is developed to achieve asymptotic tracking of the system output to the desired trajectory. A corresponding theorem is given for the control law gain selection of linear discrete time-varying systems, and it is proved through the combination of contraction mapping methodology and inductive analysis approach that the design can guarantee the asymptotic convergence of the tracking error. Furthermore, the design is extended to nonlinear systems and the convergence property under uncertain initial state conditions is demonstrated. Finally, the validity of the design in this paper is verified through a numerical simulation and a single-joint robot model by comparing with the ILC design using iterative averaging operator. For future works, the

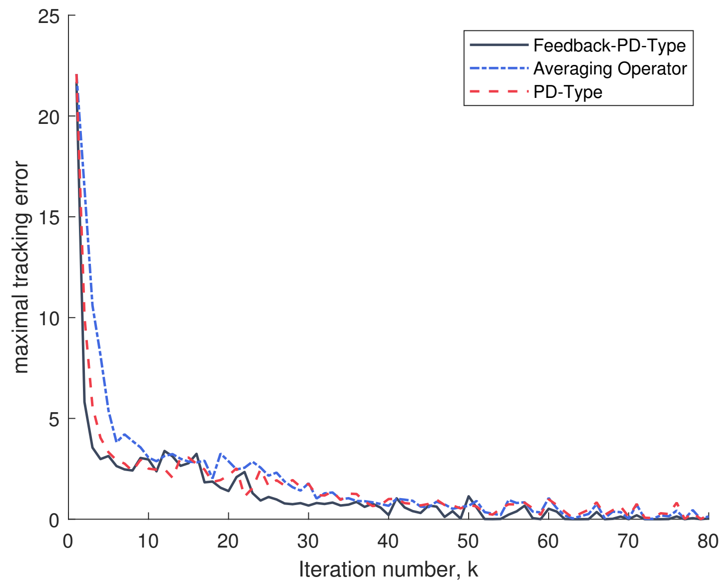

Figure 9. The maximal tracking error along all iterations.

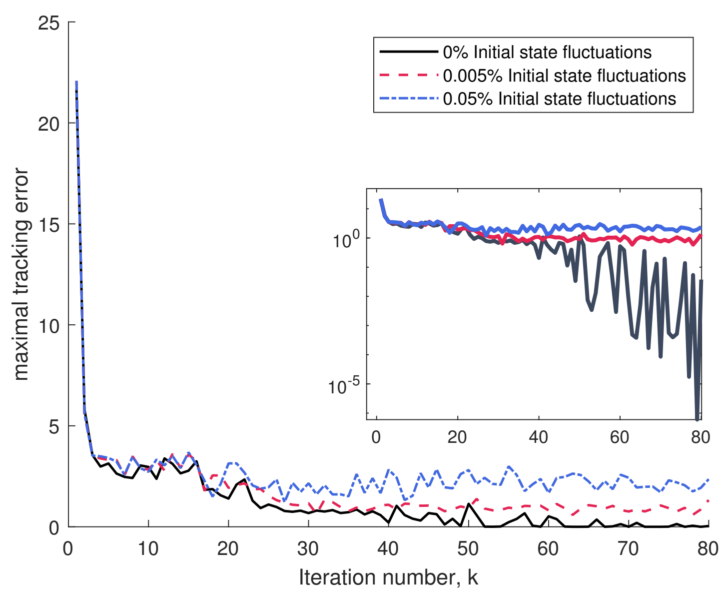

Figure 10. The maximal tracking error under uncertain initial state conditions.

construction of recursively generated higher-order error sequences will be investigated, and more kinds of systematic uncertainties will be taken into consideration.

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