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ON PARAMETER ESTIMATION IN THE BASS MODEL BY NONLINEAR LEAST SQUARES FITTING THE ADOPTION CURVE

DARIJA MARKOVIĆ, DRAGAN JUKIĆ

Department of Mathematics
University of Osijek, Trg Ljudevita Gaja 6, HR-31 000 Osijek, Croatia
e-mail: {darija,jukicd}@mathos.hr

The Bass model is one of the most well-known and widely used first-purchase diffusion models in marketing research. Estimation of its parameters has been approached in the literature by various techniques. In this paper, we consider the parameter estimation approach for the Bass model based on nonlinear weighted least squares fitting of its derivative known as the adoption curve. We show that it is possible that the least squares estimate does not exist. As a main result, two theorems on the existence of the least squares estimate are obtained, as well as their generalization in the l_s norm ($1 \leq s < \infty$). One of them gives necessary and sufficient conditions which guarantee the existence of the least squares estimate. Several illustrative numerical examples are given to support the theoretical work.

Keywords: Bass model, least squares estimate, existence problem, data fitting.

1. Introduction

The most popular first-purchase (adoption) diffusion model in marketing research is the Bass model. It is similar in some respect to models of infectious diseases or contagion models which describe the spread of a disease through the population due to contact with infected persons (see Bailey 1975; 1957). The Bass model is distinguished from other growth models by explicitly incorporating some key behavioural assumptions from Rogers' theory of diffusion of innovation (see Rogers, 1962). Namely, Bass divided adopters (first-time buyers) into innovators and imitators. Imitators, unlike innovators, are buyers who are influenced in their adoption by the number of previous buyers. The Bass model has three parameters: the coefficient of innovation or external influence ($p > 0$), the coefficient of imitation or internal influence ($q \geq 0$), and the total market potential ($m > 0$). To capture the growth of a new durable product (innovation) due to the diffusion effect, Bass (1969) used the following Riccati differential equation with constant coefficients:

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = p\Big[m - N(t)\Big] + \frac{q}{m}N(t)\Big[m - N(t)\Big], \qquad N(0) = 0, \quad t \geq 0, \tag{1}$$

where $N(t)$ and $n(t) := \mathrm{d}N(t)/\mathrm{d}t$ are respectively the cumulative and the noncumulative number of adopters of a new product at time t. The adoption rate $n(t)$ is determined by two additive terms: the first term, $p[m - N(t)]$, represents adoptions due to innovators, whereas the second term, $(q/m)N(t)[m-N(t)]$, represents adoptions due to imitators.

To stress the fact that functions $N(t)$ and $n(t)$ depend on parameters m, p and q, we shall write $N(t; m, p, q)$ and $n(t; m, p, q)$.

The solution of (1) and the corresponding adoption rate function are given by

$$N(t; m, p, q) = m\frac{1 - \mathrm{e}^{-(p+q)t}}{1 + \frac{q}{p}\,\mathrm{e}^{-(p+q)t}}, \quad t \geq 0, \tag{2}$$

and

$$n(t; m, p, q) = m\frac{(p+q)^2}{p}\frac{\mathrm{e}^{-(p+q)t}}{(1 + \frac{q}{p}\,\mathrm{e}^{-(p+q)t})^2}, \quad t \geq 0. \tag{3}$$

The graph of the function N is known as the Bass cumulative adoption curve, and the graph of the function n is known as the Bass (noncumulative) adoption curve.

The graph of the cumulative adoption curve N is an "S-shaped" curve. If $q > p$, for this curve the point of

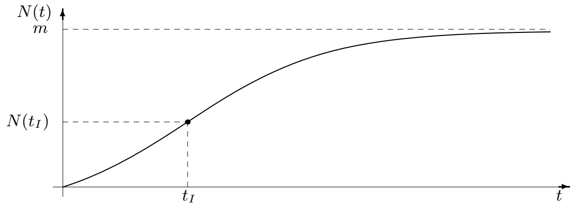

Fig. 1. Typical S-shaped Bass cumulative adoption curve.

inflection occurs at

$$t_I := \frac{1}{p+q}\ln(q/p)$$

with

$$N(t_I;m,p,q) = m\frac{(q-p)}{2q}$$

(see Fig. 1). For $q \leq p$, the graph is still S-shaped, but the point of inflection occurs at a negative value of t. Furthermore, if $q > p$, it can be easily shown that the adoption rate function n has a maximum value at t_I,

$$n(t_I;m,p,q) = m\frac{(p+q)^2}{4q},$$

and that n is symmetric about the peak t_I. In the case when $q \leq p$, the adoption rate function n is strictly decreasing on $[0,\infty)$ (see Fig. 2).

There are many applications of the Bass model in several areas like retail service, industrial technology, agricultural, educational, pharmaceutical, and consumer durable goods markets. For a review of the Bass model and its applications, see the work of Mahajan *et al.* (2000).

In practice, the unknown parameters of the Bass model are not known in advance and they must be estimated on the basis of some experimentally or empirically obtained data. This issue is known as a parameter estimation problem. There is no unique way to estimate the unknown parameters and many different methods have been proposed in the literature. Mahajan *et al.* (1986) used real diffusion data for seven products to compare the performance of four estimation procedures: Ordinary Least Squares (OLS) estimation proposed by Bass, Maximum Likelihood Estimation (MLE) proposed by Schmittlein and Mahajan (1982), Nonlinear Least Squares (NLS) estimation suggested by Srinivasan and Mason (1986), and Algebraic Estimation (AE) proposed by Mahajan and Sharma (1986). They concluded that, for the seven data sets considered in their study, the NLS procedure provides better predictions as well as more valid estimates of standard errors for the parameter estimates than the other three estimation procedures.

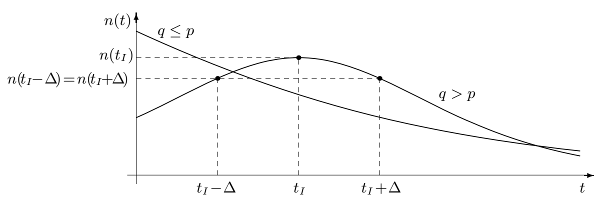

Fig. 2. Symmetry of the Bass adoption curve.

The formulation of the NLS approach is as follows: The observed number of adopters X_i in the time interval $(\tau_{i-1},\tau_i]$ is modeled as

$$X_i = N(\tau_i;m,p,q) - N(\tau_{i-1};m,p,q) + \varepsilon_i, \qquad i=1,\dots,K,$$

where ε_i is an additive error term. Here, by definition, $\tau_0 = 0$. Based on these equations, Srinivasan and Mason proposed to estimate the unknown parameters p, q and m in the sense of Least Squares (LS) by minimizing functional

$$S(m,p,q) = \sum_{i=1}^{K}\left[X_i - \big(N(\tau_i;m,p,q) - N(\tau_{i-1};m,p,q)\big)\right]^2 \quad (4)$$

on the set $\{(m,p,q) : m,p>0,\ q\geq 0\}$.

There are several other methods which can be used to estimate the unknown parameters in new product diffusion models (see, e.g., Scitovski and Meler, 2002). A very popular technique for parameter estimation is the least squares method. Numerical methods for solving the nonlinear LS problem are described by Dennis and Schnabel (1996) as well as Gill *et al.* (1981). Before starting an iterative procedure one should ask whether the LS estimate exists. For nonlinear LS problems this question is difficult to answer. The problem of nonlinear weighted LS and total least squares fitting of the Bass curve (2) is considered by Jukić (2013; 2011). Results on the existence of the LS estimate for some other special classes of functions can be found in the works of Bates and Watts (1988), Björck (1996), Demidenko (2008; 2006; 1996), Hadeler *et al.* (2007), Jukić (2013; 2009), Jukić and Marković (2010), Jukić *et al.* (2008; 2004), Marković and Jukić (2010), as well as Marković *et al.* (2009).

In this paper, we consider the parameter estimation approach for the Bass model, based on nonlinear weighted LS fitting of the Bass adoption curve (3). In Section 2, we briefly describe this approach and summarize our main results. We show that it is possible that the LS estimate for the Bass adoption curve does not exist (Proposition 1). As our main results, we present two theorems (Theorems 1 and 2) on the existence of the LS estimate, as well as their generalizations (Theorems 3 and 4) in the l_s norm ($1 \leq s < \infty$). Some numerical experiments to illustrate the efficiency of our approach are given in Section 3. To compare our approach with the NLS one proposed by Srinivasan and Mason, we used the same time series data for the seven durables. To avoid unnecessary technicalities at an early stage, all proofs are given in Section 2.3. To the best of our knowledge, there is no previous paper that has focused on the existence of the LS estimate for the Bass adoption curve.

2. Main results: LS regression existence theorems for the Bass adoption curve

In this section, we first formulate the LS fitting problem for the Bass adoption curve and then present two theorems (Theorems 1 and 2) on the existence of the least squares estimate, as well as their generalizations (Theorems 3 and 4) in the l_s norm ($1 \le s < \infty$). Their proofs are given in Section 2.3.

2.1. LS fitting problem for the Bass adoption curve. Suppose we are given the data (w_i, t_i, y_i), $i = 1, \ldots, K$, $K > 3$, where

$$0 < t_1 < t_2 < \ldots < t_K \tag{5}$$

denote the values of the independent variable,

$$y_1, \ldots, y_K > 0 \tag{6}$$

are in some way obtained respective estimates of the Bass adoption curve (3), i.e., $y_i \approx n(t_i; m, p, q)$, $i = 1, \ldots, K$, and $w_i > 0$ are the data weights which describe the assumed relative accuracy of the data. The unknown parameters m, p and q of the function (3) have to be estimated by minimizing the functional

$$\begin{aligned} F(m,p,q) &= \sum_{i=1}^{K} w_i [n(t_i; m,p,q) - y_i]^2 \\ &= \sum_{i=1}^{K} w_i \Big[\frac{m(p+q)^2}{p} \frac{\mathrm{e}^{-(p+q)t_i}}{(1+\frac{q}{p}\mathrm{e}^{-(p+q)t_i})^2} - y_i \Big]^2 \end{aligned} \tag{7}$$

on the set

$$\mathcal{P} := \{(m,p,q) : m, p > 0,\ q \ge 0\}.$$

A point $(m^\star, p^\star, q^\star) \in \mathcal{P}$ such that $F(m^\star, p^\star, q^\star) = \inf_{(m,p,q)\in\mathcal{P}} F(m,p,q)$ is called the *least squares estimate*, if it exists (see Björck, 1996; Gill *et al.*, 1981; Ross, 1990; Seber and Wild, 1989).

Data for LS estimation can be obtained in various ways. For instance, Eqn. (1) can be discretized in different ways. The most straightforward and most commonly used way is to use the finite difference method, in which case the first derivative is approximated by formulas involving only several neighboring points. To be a bit concrete, let us concentrate only on the three commonly used finite difference approximations, known as *forward*, *backward* and *centered* finite difference approximation. For this purpose, suppose that the observed cumulative number of adopters at times $0 < \tau_1 < \tau_2 < \ldots < \tau_K$ is $N_1, N_2, \ldots, N_K$, respectively. Then the observed number of adopters in the interval $(\tau_{i-1}, \tau_i]$ is given by

$$X_i = N_i - N_{i-1}, \quad i = 1, \ldots, K,$$

where $\tau_0 = 0$ and $N_0 = 0$ by definition. The forward, backward and centered finite difference discretizations of differential equation (1) and points (t_i, y_i) required for LS estimation are as follows:

- the forward finite difference discretization,

$$\frac{X_i}{\tau_i - \tau_{i-1}} = n(\tau_{i-1}; m,p,q) + \varepsilon_i, \quad i = 1, \ldots, K,$$
$$(t_i, y_i) = \Big(\tau_{i-1}, \frac{X_i}{\tau_i - \tau_{i-1}}\Big), \qquad i = 1, \ldots, K;$$

- the backward finite difference discretization

$$\frac{X_i}{\tau_i - \tau_{i-1}} = n(\tau_i; m,p,q) + \varepsilon_i, \quad i = 1, \ldots, K,$$
$$(t_i, y_i) = \Big(\tau_i, \frac{X_i}{\tau_i - \tau_{i-1}}\Big), \qquad i = 1, \ldots, K;$$

- the centered finite difference discretization

$$\frac{X_i}{\tau_i - \tau_{i-1}} = n\Big(\frac{\tau_{i-1}+\tau_i}{2}; m,p,q\Big) + \varepsilon_i, \quad i = 1, \ldots, K,$$
$$(t_i, y_i) = \Big(\frac{\tau_{i-1}+\tau_i}{2}, \frac{X_i}{\tau_i - \tau_{i-1}}\Big), \quad i = 1, \ldots, K.$$

The following proposition shows that there exist data such that the LS estimate for the Bass adoption curve (3) does not exist.

Proposition 1. *Let (w_i, t_i, y_i), $i = 1, \ldots, K$, $K > 3$, be the data such that the points (t_i, y_i), $i = 1, \ldots, K$, all lie on some exponential curve $y(t) = b\mathrm{e}^{ct}$, $b, c > 0$. Then the LS estimate for the Bass adoption curve (3) does not exist.*

Proof. Since $F(m,p,q) \ge 0$ for all $(m,p,q) \in \mathcal{P}$, and

$$\begin{aligned} \lim_{x\to\infty} F\left(\frac{bx}{c}, \frac{c}{x+1}, \frac{cx}{x+1}\right) &= \lim_{x\to\infty} \sum_{i=1}^{K} w_i \left[bx \frac{(1+x)\,\mathrm{e}^{-ct_i}}{(1+x\,\mathrm{e}^{-ct_i})^2} - y_i \right]^2 \\ &= \sum_{i=1}^{K} w_i \left(b\mathrm{e}^{ct_i} - y_i \right)^2 = 0\,, \end{aligned}$$

this means that

$$\inf_{(m,p,q)\in\mathcal{P}} F(m,p,q) = 0.$$

Furthermore, since the graph of any function of type (3) intersects the graph of exponential function $y(t) = be^{ct}$ at no more than three points, and $K > 3$, it follows that $F(m,p,q) > 0$ for all $(m,p,q) \in \mathcal{P}$, and hence the LS estimate does not exist. ∎

2.2. Existence theorems. The following theorem, whose proof is given in Section 2.3, gives a necessary and sufficient condition on the data which guarantee the existence of the LS estimate for the function (3). First, let us introduce the following notation: Let $E^\star$ be an infimum of the weighted sum of squares for the exponential function $y(t) = b\,\mathrm{e}^{ct}$ $(b, c > 0)$, i.e.,

$$E^\star := \inf_{b,c>0} \sum_{i=1}^{K} w_i (b\,\mathrm{e}^{ct_i} - y_i)^2.$$

Theorem 1. *Suppose that the data (w_i, t_i, y_i), $i = 1,\dots,K$, $K > 3$, satisfy the conditions (5) and (6). Then the LS estimate for the Bass adoption curve (3) exists if and only if there is a point $(m_0, p_0, q_0) \in \mathcal{P}$ such that $F(m_0, p_0, q_0) \le E^\star$.*

In other words, under the assumptions of the theorem, the LS estimate exists if and only if there is at least one regression curve defined by (3) which is in an LS sense as good as 'or better than' the best exponential curve of type $t \mapsto b\,\mathrm{e}^{ct}$, where $b, c > 0$.

It is clear that, regardless of how much effort is put into marketing, there is a certain upper bound, say M, for the market potential m (i.e., the maximum number of adopters). In most cases management has a judgement, a strong intuitive feel, about the upper bound M, but if not, the upper bound M can be the size of the relevant population. The following theorem tells us that if parameter m is bounded above, then the LS estimate will exist. First, let us introduce the following notation: Given any real number $M > 0$, let

$$\mathcal{P}_M := \{(m,p,q) : 0 < m \le M, p > 0,\ q \ge 0\}.$$

Theorem 2. *Suppose that the data (w_i, t_i, y_i), $i = 1,\dots,K$, $K > 3$, satisfy the conditions (5) and (6). Then functional F defined by (7) attains its infimum on $\mathcal{P}_M$, i.e., there exists a point $(m^\star, p^\star, q^\star) \in \mathcal{P}_M$ such that $F(m^\star, p^\star, q^\star) = \inf_{(m,p,q)\in\mathcal{P}_M} F(m,p,q)$.*

The proof of this theorem is the same for respective parts of the proof of Theorem 1, with the exception that we do not have to prove that $m^\star < \infty$. Hence, it is omitted.

The LS problem is a nonlinear l_2-norm one. During the last few decades an increased interest in alternative l_s-norm has become apparent (see, e.g., Atieg and Watson, 2004; Gonin and Money, 1989). For example, l_1-norm criteria are more suitable if there are wild points (outliers) in the data. Thus, instead of minimizing functional F, sometimes a more adequate criterion for estimation of unknown parameters m, p and q of the function (3) is to minimize the following functional:

$$F_s(m,p,q) = \sum_{i=1}^{K} w_i \big| n(t_i; m,p,q) - y_i \big|^s, \qquad (8)$$

where s $(1 \le s < \infty)$ is an arbitrary fixed number. To state the corresponding l_s-norm $(1 \le s < \infty)$ generalizations of Theorems 1 and 2, we need an additional notation. Let

$$E_s^\star := \inf_{(b,c)\in\mathbb{R}_+^2} E_s(b,c),$$

where

$$E_s(b,c) = \sum_{i=1}^{K} w_i |b\,\mathrm{e}^{ct_i} - y_i|^s.$$

Obviously, $E^\star = E_2^\star$ and $F = F_2$.

Theorem 3. *If the data (w_i, t_i, y_i), $i = 1,\dots,K$, $K > 3$, satisfy the conditions (5) and (6), then functional F_s defined by (8) attains its infimum on $\mathcal{P}$ if and only if there is a point $(m_0, p_0, q_0) \in \mathcal{P}$ such that $F_s(m_0, p_0, q_0) \le E_s^\star$.*

The proof of the following theorem is also omitted; it is the same for the respective parts of the proof of Theorem 3, with the exception that we do not have to prove that $m^\star < \infty$.

Theorem 4. *If the data (w_i, t_i, y_i), $i = 1,\dots,K$, $K > 3$, satisfy the conditions (5) and (6), then there exists a point $(m^\star, p^\star, q^\star) \in \mathcal{P}_M$ such that*

$$F_s(m^\star, p^\star, q^\star) = \inf_{(m,p,q)\in\mathcal{P}_M} F_s(m,p,q).$$

2.3. Proofs of Theorems 1 and 3. The following lemma will be used in proofs of both Theorems 1 and 3.

Lemma 1. *Suppose that the data (w_i, t_i, y_i), $i = 1,\dots,K$, $K > 3$, satisfy the conditions (5) and (6). Then given any $i_0 \in \{1,\dots,K-1\}$ there exists a point in $\mathcal{P}$ at which functional F_s defined by (8) attains a value less than*

$$\sum_{\substack{i=1\\ i\ne i_0, i_0+1}}^{K} w_i |y_i|^s.$$

Proof. Let us first write

$$x_0 := \frac{1}{t_{i_0+1} - t_{i_0}} \max\Big\{ \ln(\frac{y_{i_0+1}}{y_{i_0}}), \ln\Big(\frac{y_{i_0}}{y_{i_0+1}}\Big)\Big\},$$

and then define functions $\alpha, m, p, q : (x_0, \infty) \to (0, \infty)$ as follows:

$$\alpha(x) := \frac{1 - \sqrt{\frac{y_{i_0}}{y_{i_0+1}}}\,\mathrm{e}^{-\frac{x}{2}(t_{i_0+1}-t_{i_0})}}{\sqrt{\frac{y_{i_0}}{y_{i_0+1}}} - \mathrm{e}^{-\frac{x}{2}(t_{i_0+1}-t_{i_0})}},$$

$$m(x) := \frac{y_{i_0+1}[1+\alpha(x)\,\mathrm{e}^{-\frac{x}{2}(t_{i_0+1}-t_{i_0})}]^2\,\mathrm{e}^{xt_{i_0+1}}}{x[1+\alpha(x)\,\mathrm{e}^{\frac{x}{2}(t_{i_0}+t_{i_0+1})}]},$$

$$p(x) := \frac{x}{1+\alpha(x)\,\mathrm{e}^{\frac{x}{2}(t_{i_0}+t_{i_0+1})}},$$

$$q(x) := \frac{x\alpha(x)\,\mathrm{e}^{\frac{x}{2}(t_{i_0}+t_{i_0+1})}}{1+\alpha(x)\,\mathrm{e}^{\frac{x}{2}(t_{i_0}+t_{i_0+1})}}.$$

By using the definition of x_0, it is easy to show that function α is well defined and strictly positive on (x_0, ∞). Thus for all $x \in (x_0, \infty)$ we have that $(m(x), p(x), q(x)) \in \mathcal{P}$. Furthermore, it is easy to verify that

$$\begin{aligned} &n(t; m(x), p(x), q(x)) \\ &= y_{i_0+1}\big[1+\alpha(x)\,\mathrm{e}^{-\frac{x}{2}(t_{i_0+1}-t_{i_0})}\big]^2 \\ &\times \frac{\mathrm{e}^{-x(t-t_{i_0+1})}}{\big[1+\alpha(x)\,\mathrm{e}^{-x(t-\frac{t_{i_0}+t_{i_0+1}}{2})}\big]^2}. \end{aligned}$$

Now, by a straightforward but tedious calculation, one can verify that, for all $x \in (x_0, \infty)$,

$$n(t_{i_0}; m(x), p(x), q(x)) = y_{i_0},$$
$$n(t_{i_0+1}; m(x), p(x), q(x)) = y_{i_0+1},$$

and

$$\lim_{x\to\infty} n(t; m(x), p(x), q(x)) = \begin{cases} 0 & \text{if } t \in (-\infty, t_{i_0}) \cup (t_{i_0+1}, \infty), \\ \infty & \text{if } t \in (t_{i_0}, t_{i_0+1}). \end{cases}$$

In Fig. 3 we plot the graph of the function $t \mapsto n(t; m(x), p(x), q(x))$.

Let $x > x_0$ be sufficiently large, so that

$$0 < n(t_i; m(x), p(x), q(x)) \le y_i, \qquad i = 1, \dots, K,$$

whereby the equality holds only if $i = i_0$ or $i = i_0 + 1$. Due to the above mentioned facts, such x exists. Then

$$\begin{aligned} &F_s(m(x), p(x), q(x)) \\ &= \sum_{i=1}^{K} w_i |n(t_i; m(x), p(x), q(x)) - y_i|^s \\ &< \sum_{\substack{i=1 \\ i\ne i_0, i_0+1}}^{K} w_i |y_i|^s. \end{aligned}$$

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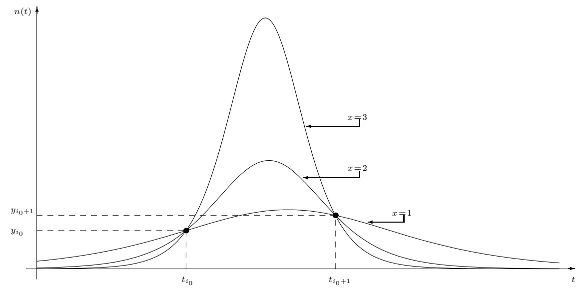

Fig. 3. Plots of the Bass adoption curve $n(t; m(x), p(x), q(x))$ for some values of x.

Proof. (Theorem 1) Assume first that $(m^\star, p^\star, q^\star) \in \mathcal{P}$ is the LS estimate, and then show that $F(m^\star, p^\star, q^\star) \le E^\star$. In order to do this, first note that, for all $b, c, x > 0$,

$$\begin{aligned} F(m^\star, p^\star, q^\star) &\le F\Big(\frac{bx}{c}, \frac{c}{x+1}, \frac{cx}{x+1}\Big) \\ &= \sum_{i=1}^{K} w_i \Big[xb\frac{(1+x)\,\mathrm{e}^{-ct_i}}{(1+x\,\mathrm{e}^{-ct_i})^2} - y_i\Big]^2, \end{aligned}$$

from where taking the limit as $x \to \infty$ it follows that (see the proof of Proposition 1)

$$F(m^\star, p^\star, q^\star) \le \sum_{i=1}^{K} w_i (b\,\mathrm{e}^{ct_i} - y_i)^2.$$

From the last inequality and the definition of $E^\star$ we obtain that $F(m^\star, p^\star, q^\star) \le E^\star$.

Let us show the converse of the theorem. Suppose that there is a point $(m_0, p_0, q_0) \in \mathcal{P}$ such that $F(m_0, p_0, q_0) \le E^\star$. Since the functional F is nonnegative, there exists $F^\star := \inf_{(m,p,q)\in\mathcal{P}} F(m, p, q)$. It should be shown that the LS estimate exists, i.e., that there exists a point $(m^\star, p^\star, q^\star) \in \mathcal{P}$ such that $F(m^\star, p^\star, q^\star) = F^\star$. To do this, first note that

$$F^\star \le F(m_0, p_0, q_0) \le E^\star.$$

If $F^\star = F(m_0, p_0, q_0)$, to complete the proof it is enough to set $(m^\star, p^\star, q^\star) = (m_0, p_0, q_0)$. Hence, we can further assume that

$$F^\star < F(m_0, p_0, q_0) \le E^\star. \tag{9}$$

Let (m_k, p_k, q_k) be a sequence in $\mathcal{P}$, such that

$$\begin{aligned} F^\star &= \lim_{k\to\infty} F(m_k, p_k, q_k) \\ &= \lim_{k\to\infty} \sum_{i=1}^{K} w_i [n(t_i; m_k, p_k, q_k) - y_i]^2. \end{aligned} \tag{10}$$

Without loss of generality, in further deliberations we may assume that sequences (m_k), (p_k) and (q_k) are monotone.

This is possible because the sequence (m_k, p_k, q_k) has a subsequence $(m_{l_k}, p_{l_k}, q_{l_k})$, such that all its component sequences (m_{l_k}), (p_{l_k}) and (q_{l_k}) are monotone, and since $\lim_{k\to\infty} F(m_{l_k}, p_{l_k}, q_{l_k}) = \lim_{k\to\infty} F(m_k, p_k, q_k) = F^\star$.

Since each monotone sequence of real numbers converges in the extended real number system $\overline{\mathbb{R}}$, define

$$m^\star := \lim_{k\to\infty} m_k, \quad p^\star := \lim_{k\to\infty} p_k, \quad q^\star := \lim_{k\to\infty} q_k.$$

Note that $0 \le m^\star, p^\star, q^\star \le \infty$, because $(m_k, p_k, q_k) \in \mathcal{P}$.

To complete the proof, it is enough to show that $(m^\star, p^\star, q^\star) \in \mathcal{P}$, i.e., that $0 < m^\star < \infty$, $0 < p^\star < \infty$ and $0 \le q^\star < \infty$. The continuity of the functional F will then imply that $F^\star = \lim_{k\to\infty} F(m_k, p_k, q_k) = F(m^\star, p^\star, q^\star)$.

It remains to show that $(m^\star, p^\star, q^\star) \in \mathcal{P}$. The proof will be derived in three steps. In Step 1, we will show that $0 < m^\star < \infty$. After that, in Step 2, we will show that $0 < p^\star + q^\star < \infty$. The proof that $p^\star > 0$ will be given in Step 3.

Step 1. Let us first show that $0 < m^\star < \infty$. We prove this by contradiction. Suppose on the contrary that $m^\star = 0$ or $m^\star = \infty$. Then only one of the following three cases can occur: (i) $p^\star + q^\star = 0$, (ii) $0 < p^\star + q^\star < \infty$, or (iii) $p^\star + q^\star = \infty$. Now, we are going to show that the functional F cannot attain its infimum in either of these three cases, which will prove that $0 < m^\star < \infty$.

Case (i): $p^\star + q^\star = 0$. Let $L := \lim_{k\to\infty} m_k p_k$. First note that $0 \le L \le \infty$.

By using the inequality

$$1 < \frac{1 + \frac{q_k}{p_k}}{1 + \frac{q_k}{p_k}\,\mathrm{e}^{-(p_k+q_k)t}} < \mathrm{e}^{(p_k+q_k)t} \text{ for all } t \ge 0,$$

it follows readily that

$$\lim_{k\to\infty} \frac{1 + \frac{q_k}{p_k}}{1 + \frac{q_k}{p_k}\,\mathrm{e}^{-(p_k+q_k)t}} = 1 \text{ for all } t \ge 0.$$

Hence,

$$\begin{aligned}&\lim_{k\to\infty} n(t_i; m_k, p_k, q_k)\\ &= \lim_{k\to\infty} \Big[m_k p_k \Big(\frac{1 + \frac{q_k}{p_k}}{1 + \frac{q_k}{p_k}\,\mathrm{e}^{-(p_k+q_k)t_i}} \Big)^2 \mathrm{e}^{-(p_k+q_k)t_i} \Big]\\ &= L,\end{aligned}$$

$i = 1, \dots, K$. If $L = \infty$, then it would follow that $F^\star = \infty$, which is impossible. If $L < \infty$, then we would obtain that

$$F^\star = \sum_{i=1}^{K} w_i (L - y_i)^2. \tag{11}$$

Since, by the definition of $E^\star$,

$$\sum_{i=1}^{K} w_i (L\,\mathrm{e}^{ct_i} - y_i)^2 \ge E^\star \quad \text{for all } c > 0,$$

taking the limit as $c \to 0+$ it follows that $\sum_{i=1}^{K} w_i (L - y_i)^2 \ge E^\star$. From this and (11), we would have that $F^\star \ge E^\star$, which contradicts the assumption (9). This means that in this case the functional F cannot attain its infimum.

Case (ii): $0 < p^\star + q^\star < \infty$. Note that

$$\begin{aligned}&n(t_i; m_k, p_k, q_k)\\ &\quad = m_k p_k \left(\frac{1 + \frac{q_k}{p_k}}{1 + \frac{q_k}{p_k}\,\mathrm{e}^{-(p_k+q_k)t_i}} \right)^2 \mathrm{e}^{-(p_k+q_k)t_i},\end{aligned} \tag{12}$$

$i = 1, \dots, K$. It readily follows, regardless of whether q_k/p_k converges to a finite number or diverges to infinity, that there exist all limits

$$\alpha_i^\star := \lim_{k\to\infty} \left(\frac{1 + \frac{q_k}{p_k}}{1 + \frac{q_k}{p_k}\,\mathrm{e}^{-(p_k+q_k)t_i}} \right)^2, \quad i = 1, \dots, K, \tag{13}$$

and that $0 < \alpha_i^\star < \infty$.

Let us first show that $m^\star \ne 0$. We prove this by contradiction. Suppose on the contrary that $m^\star = 0$. Then, by using (12) and (13), it is easy to show that $\lim_{k\to\infty} n(t_i; m_k, p_k, q_k) = 0$ for all $i = 1, \dots, K$, and therefore it would follow that $F^\star = \sum_{i=1}^{K} w_i y_i^2$. Since, according to Lemma 1, there exists a point in $\mathcal{P}$ at which the functional F attains a value smaller than $\sum_{i \ne i_0} w_i y_i^2$, this means that in this way ($m^\star = 0$) the functional F cannot attain its infimum.

It remains to show that $m^\star \ne \infty$. The proof will be given also by contradiction. Assume that $m^\star = \infty$. Then by using (12) and (13) one can easy to show that there must be $\lim_{k\to\infty} p_k = p^\star = 0$, because otherwise we would have that $\lim_{k\to\infty} n(t_i; m_k, p_k, q_k) = \infty$ for all $i = 1, \dots, K$, and consequently, $F^\star = \infty$, which is, as we know, impossible. Therefore, because $p^\star = 0$ and $0 < p^\star + q^\star < \infty$, there must be $q^\star > 0$, which would imply $\lim_{k\to\infty} q_k/p_k = \infty$ and, consequently,

$$\alpha_i^\star = \mathrm{e}^{2q^\star t_i}, \quad i = 1, \dots, K.$$

Now, by using (12) it is easy to show that

$$\lim_{k\to\infty} n(t_i; m_k, p_k, q_k) = L\,\mathrm{e}^{q^\star t_i}, \quad i = 1, \dots, K,$$

where, as in Case (i), $L = \lim_{k\to\infty} m_k p_k$. If $L = \infty$, then it would follow that $F^\star = \infty$, which is impossible. If $L < \infty$, then we would obtain that

$$F^\star = \lim_{k\to\infty} F(m_k, p_k, q_k) = \sum_{i=1}^{K} w_i (L\,\mathrm{e}^{q^\star t_i} - y_i)^2 \ge E^\star,$$

which contradicts the assumption (9). This proves that $m^\star \neq \infty$.

Case (iii): $p^\star + q^\star = \infty$. Note that

$$\begin{aligned} &n(t_i; m_k, p_k, q_k) \\ &= \frac{m_k(p_k+q_k)}{1+\frac{q_k}{p_k}\,\mathrm{e}^{-(p_k+q_k)t_i}} \frac{\left(1+\frac{q_k}{p_k}\right)\mathrm{e}^{-(p_k+q_k)t_i}}{1+\frac{q_k}{p_k}\,\mathrm{e}^{-(p_k+q_k)t_i}}, \\ &\qquad i = 1,\dots,K. \end{aligned} \tag{14}$$

Before continuing the proof, let us also note that

$$\begin{aligned} \frac{q_k}{p_k}\,\mathrm{e}^{-(p_k+q_k)t_1} &\geq \frac{q_k}{p_k}\,\mathrm{e}^{-(p_k+q_k)t_2} \\ &\geq \cdots \geq \frac{q_k}{p_k}\,\mathrm{e}^{-(p_k+q_k)t_K}. \end{aligned}$$

Now, let us first show that $\lim_{k\to\infty} q_k/p_k\,\mathrm{e}^{-(p_k+q_k)t_i}$ is either 0 or ∞, for all $i = 1,\dots,K$. Assume on the contrary that there exists an index i_0 such that

$$0 < \lim_{k\to\infty} \frac{q_k}{p_k}\,\mathrm{e}^{-(p_k+q_k)t_{i_0}} := L_{i_0} < \infty.$$

Then

$$\lim_{k\to\infty} \frac{\left(1+\frac{q_k}{p_k}\right)\mathrm{e}^{-(p_k+q_k)t_{i_0}}}{1+\frac{q_k}{p_k}\,\mathrm{e}^{-(p_k+q_k)t_{i_0}}} = \frac{L_{i_0}}{1+L_{i_0}} > 0,$$

and therefore from (14) it would follow that

$$\begin{aligned} &\lim_{k\to\infty} n(t_{i_0}; m_k, p_k, q_k) \\ &\qquad = \lim_{k\to\infty} m_k(p_k+q_k)\frac{L_{i_0}}{(1+L_{i_0})^2}. \end{aligned}$$

In order to have a bounded functional, the limit $\lim_{k\to\infty} m_k(p_k+q_k)$ must be finite (regardless of the value of $m^\star$). Due to this and the equality

$$\frac{q_k}{p_k}\,\mathrm{e}^{-(p_k+q_k)t_i} = \frac{q_k}{p_k}\,\mathrm{e}^{-(p_k+q_k)t_{i_0}}\,\mathrm{e}^{-(p_k+q_k)(t_i-t_{i_0})},$$

it follows readily that

$$\lim_{k\to\infty} \frac{q_k}{p_k}\,\mathrm{e}^{-(p_k+q_k)t_i} = \begin{cases} \infty, & i < i_0 \\ 0, & i > i_0. \end{cases}$$

Now, by using (14) it is easy to check that $\lim_{k\to\infty} n(t_i; m_k, p_k, q_k) = 0$ for each $i \neq i_0$. That would imply that $F^\star \geq \sum_{i\neq i_0} w_i y_i^2$. But since according to Lemma 1 there exists a point in $\mathcal{P}$ at which the functional F attains a value smaller than $\sum_{i\neq i_0} w_i y_i^2$, we have proved that $\lim_{k\to\infty}(q_k/p_k)\,\mathrm{e}^{-(p_k+q_k)t_i}$ is either 0 or ∞, for all $i = 1,\dots,K$.

Note that only one of the following three subcases can occur: (a) $\lim_{k\to\infty} \frac{q_k}{p_k}\,\mathrm{e}^{-(p_k+q_k)t_i} = \infty$ for all $i = 1,\dots,K$, (b) $\lim_{k\to\infty} \frac{q_k}{p_k}\,\mathrm{e}^{-(p_k+q_k)t_i} = 0$ for all $i = 1,\dots,K$, or (c) there exists $i_0 \neq K$ such that

$$\lim_{k\to\infty} \frac{q_k}{p_k}\,\mathrm{e}^{-(p_k+q_k)t_i} = \begin{cases} \infty, & i \leq i_0 \\ 0, & i > i_0. \end{cases}$$

Subcase (a): If $\lim_{k\to\infty} \frac{q_k}{p_k}\,\mathrm{e}^{-(p_k+q_k)t_i} = \infty$ for all $i = 1,\dots,K$, then

$$\lim_{k\to\infty} \frac{\left(1+\frac{q_k}{p_k}\right)\mathrm{e}^{-(p_k+q_k)t_i}}{1+\frac{q_k}{p_k}\,\mathrm{e}^{-(p_k+q_k)t_i}} = 1, \quad i = 1,\dots,K$$

and therefore

$$\begin{aligned} &\lim_{k\to\infty} n(t_i; m_k, p_k, q_k) \\ &\quad = \lim_{k\to\infty} m_k p_k\,\mathrm{e}^{(p_k+q_k)t_i} \left[\frac{\left(1+\frac{q_k}{p_k}\right)\mathrm{e}^{-(p_k+q_k)t_i}}{1+\frac{q_k}{p_k}\,\mathrm{e}^{-(p_k+q_k)t_i}}\right]^2 \\ &\quad = \lim_{k\to\infty} m_k p_k\,\mathrm{e}^{(p_k+q_k)t_i}, \quad i = 1,\dots,K. \end{aligned}$$

To keep the functional F bounded, the limit $\lim_{k\to\infty} m_k p_k\,\mathrm{e}^{(p_k+q_k)t_K}$ must be finite. Because of that, by using the equality

$$m_k p_k\,\mathrm{e}^{(p_k+q_k)t_i} = m_k p_k\,\mathrm{e}^{(p_k+q_k)t_K}\,\mathrm{e}^{(p_k+q_k)(t_i-t_K)}$$

it follows directly that

$$\lim_{k\to\infty} m_k p_k\,\mathrm{e}^{(p_k+q_k)t_i} = 0,$$

i.e.,

$$\lim_{k\to\infty} n(t_i; m_k, p_k, q_k) = 0$$

for all $i = 1,\dots,K-1$. In this subcase we would have

$$F^\star = \lim_{k\to\infty} F(m_k, p_k, q_k) \geq \sum_{i=1}^{K-1} w_i y_i^2.$$

Subcase (b): Let us assume that

$$\lim_{k\to\infty} \frac{q_k}{p_k}\,\mathrm{e}^{-(p_k+q_k)t_i} = 0$$

for all $i = 1,\dots,K$. Then

$$\lim_{k\to\infty} \frac{\frac{p_k}{q_k}\,\mathrm{e}^{(p_k+q_k)t_i}}{1+\frac{p_k}{q_k}\,\mathrm{e}^{(p_k+q_k)t_i}} = 1, \quad i = 1,\dots,K,$$

and thus

$$\begin{aligned} &\lim_{k\to\infty} n(t_i; m_k, p_k, q_k) \\ &= \lim_{k\to\infty} m_k \frac{(p_k+q_k)^2}{p_k}\,\mathrm{e}^{-(p_k+q_k)t_i} \left[\frac{\frac{p_k}{q_k}\,\mathrm{e}^{(p_k+q_k)t_i}}{1+\frac{p_k}{q_k}\,\mathrm{e}^{(p_k+q_k)t_i}}\right]^2 \\ &= \lim_{k\to\infty} m_k \frac{(p_k+q_k)^2}{p_k}\,\mathrm{e}^{-(p_k+q_k)t_i}, \qquad i = 1,\dots,K. \end{aligned}$$

Proceeding similarly as in the previous subcase, it can be shown that

$$\lim_{k\to\infty} n(t_i; m_k, p_k, q_k) = 0, \quad i = 2, \dots, K,$$

and therefore in this subcase we would have

$$F^\star = \lim_{k\to\infty} F(m_k, p_k, q_k) \geq \sum_{i=2}^{K} w_i y_i^2.$$

Subcase (c): Arguing similarly as in subcases (a) and (b), it can be shown that

$$\lim_{k\to\infty} n(t_i; m_k, p_k, q_k) = 0, \qquad i \in \{1, \dots, K\} \setminus \{i_0, i_0 + 1\},$$

and therefore in this subcase we would have

$$F^\star = \lim_{k\to\infty} F(m_k, p_k, q_k) \geq \sum_{\substack{i=1\\ i\neq i_0, i_0+1}}^{K} w_i y_i^2.$$

Since according to Lemma 1 in each of Subcases (a)–(c) there exists a point in $\mathcal{P}$ at which the functional F attains a value smaller than $\lim_{k\to\infty} F(m_k, p_k, q_k)$, our functional F cannot attain its infimum in either of these three subcases, regardless of whether $m^\star = 0$ or $m^\star = \infty$.

So far we have shown that $0 < m^\star < \infty$, and this will be used in the sequel.

Step 2. Let us first show that $p^\star + q^\star > 0$. We prove this by contradiction. Suppose on the contrary that $p^\star + q^\star = 0$. Then from the inequalities

$$1 < \frac{1 + \frac{q_k}{p_k}}{1 + \frac{q_k}{p_k}\,\mathrm{e}^{-(p_k+q_k)t_i}} < \mathrm{e}^{(p_k+q_k)t_i}, \quad i = 1, \dots, K$$

it follows that

$$\lim_{k\to\infty} \frac{1 + \frac{q_k}{p_k}}{1 + \frac{q_k}{p_k}\,\mathrm{e}^{-(p_k+q_k)t_i}} = 1, \quad i = 1, \dots, K.$$

Accordingly, now it is easy to show that

$$\lim_{k\to\infty} n(t_i; m_k, p_k, q_k) = \lim_{k\to\infty} m_k p_k \Big[\frac{1 + \frac{q_k}{p_k}}{1 + \frac{q_k}{p_k}\,\mathrm{e}^{-(p_k+q_k)t_i}}\Big]^2 \mathrm{e}^{-(p_k+q_k)t_i} = 0,$$

$i = 1, \dots, K$, and therefore from (10) it would follow that $F^\star = \sum_{i=1}^{K} w_i y_i^2$. Since according to Lemma 1 there exists a point in $\mathcal{P}$ at which functional F attains a value smaller than $\sum_{i=1}^{K} w_i y_i^2$, this means that in this way functional F cannot attain its infimum. Thus, we have proved that $p^\star + q^\star > 0$.

The proof that $p^\star + q^\star < \infty$ can be given by contradiction. To do this, it is sufficient to proceed as in Case (iii) from Step 1.

In this way we have completed the proof that $0 < p^\star + q^\star < \infty$.

Step 3. It remains to show that $p^\star > 0$. Suppose on the contrary that $p^\star = 0$. Then from the inequalities $0 < p^\star + q^\star < \infty$ it follows that $q^\star > 0$, and consequently $\lim_{k\to\infty} \frac{q_k}{p_k} = \infty$. Now it is easy to conclude that

$$\lim_{k\to\infty} n(t_i; m_k, p_k, q_k) = \lim_{k\to\infty} m_k p_k \Big[\frac{1 + \frac{q_k}{p_k}}{1 + \frac{q_k}{p_k}\,\mathrm{e}^{-(p_k+q_k)t_i}}\Big]^2 \mathrm{e}^{-(p_k+q_k)t_i} = 0,$$

$i = 1, \dots, K$. As shown in Step 2, in this way functional F cannot attain its infimum. Thus, we proved that $p^\star > 0$ and herewith we have completed the proof. ■

Proof. (Theorem 3) The proof of Theorem 3 is similar to that of Theorem 1; just replace the l_2 norm with the l_s norm. Thereby all parts of the proof remain the same.

3. Numerical experiments

In the following examples, the obtained estimates of the optimal parameters $(m^\star, p^\star, q^\star)$ will be denoted by $(\bar{m}^\star, \bar{p}^\star, \bar{q}^\star)$.

Example 1. To illustrate the accuracy of the parameter estimate approach for the Bass model based on the LS fitting adoption curve, we start with the following data which satisfy the exact solution $(m = 1000, p = 0.001, q = 0.2)$ of the differential equation (1):

$$\tau_i = i, \quad N_i = N(\tau_i; 1000, 0.001, 0.2),$$

$i = 1, \dots, 53$. These data have a point of inflection where

$$t_I = \frac{1}{p+q} \ln(q/p) = 26.3598.$$

We analyzed three sets of data: the data up to the point just before the point of inflection ($K = 26$), the data up to the point just after the point of inflection ($K = 27$), and the data up to the ceiling ($K = 53$). The results of the LS fitting Bass adoption curve to corresponding data obtained by using the centered finite approximation are given in Table 1. For all weights w_i we took 1. ♦

Example 2. Let (τ_i, N_i), $i = 1, \dots, K$, be the data where

$$K = 50, \quad \tau_i = i, \quad i = 1, \dots, K,$$
$$N_i = N(\tau_i; 1000, 0.001, 0.2) + \varepsilon_i, \quad \varepsilon_i = \mathcal{N}(0, \sigma^2).$$

As measures of the quality of fitting, we will use the Mean Absolute Relative Error (MARE) and the Root

Table 1. Accuracy of parameter estimates.

	$\bar{m}^\star$	$\left\lvert\frac{\bar{m}^\star - m}{m}\right\rvert \cdot 100$	$\bar{p}^\star$	$\left\lvert\frac{\bar{p}^\star - p}{p}\right\rvert \cdot 100$	$\bar{q}^\star$	$\left\lvert\frac{\bar{q}^\star - q}{q}\right\rvert \cdot 100$
$K = 26$	999.486	0.0514	0.00100308	0.3080	0.199901	0.0495
$K = 27$	999.636	0.0364	0.00100313	0.3130	0.199886	0.0570
$K = 53$	1000.060	0.0060	0.00100355	0.3550	0.199830	0.0850

Mean Squared Relative Error (RMSRE):

$$MARE = \frac{1}{K}\sum_{i=1}^{K}\Big\lvert\frac{N_i - \hat{N}_i}{N_i}\Big\rvert,$$

$$RMSRE = \sqrt{\frac{1}{K}\sum_{i=1}^{K}\Big(\frac{N_i - \hat{N}_i}{N_i}\Big)^2},$$

where K, N_i and $\hat{N}_i$ denote the number of data points, the observed values and the estimated values, respectively.

In a large number of numerical experiments it was confirmed that, in terms of the MARE and RMSRE, minimization of the functional F defined by (7) (with data obtained by the centered finite difference approximation method) provides a much better fit than minimization of functional S defined by (4). For example, the results of one experiment with $\sigma^2 = 0.3$ are shown in Table 2. ♦

Example 3. We are going to fit the Bass model to real diffusion data for seven products: room air conditioners, color televisions, clothes dryers, ultrasound, mammography, foreign language, and accelerated program (Table 3). These data, taken from Mahajan *et al.* (1986), have been used extensively in the diffusion modeling literature to illustrate the efficiency of estimation procedures (see Mahajan *et al.*, 1986; Schmittlein and Mahajan, 1982; Scitovski and Meler, 2002; Srinivasan and Mason, 1986).

For each product or service, we estimated the unknown parameters m, p and q by minimizing functional F defined by (7), as well as by minimizing functional S defined by (4). For all weights w_i we took 1. Parameter estimates and fit statistics are reported in Tables 4 and 5.

As can be seen, in terms of the MARE, minimization of functional F provides a better fit for five products (room air conditioners, color televisions, mammography, foreign language, and accelerated program). In terms of the RMSRE, minimization of the functional F provides a better fit for two products (a foreign language, and an accelerated program). ♦

4. Conclusions

The best-known and widely used model in diffusion research is the Bass model. It has three parameters: the coefficient of innovation ($p > 0$), the coefficient of imitation ($q \geq 0$), and the total market potential ($m > 0$). In practice, the unknown parameters p, q and m are not known in advance and must be estimated from the actual adoption data. There is no unique way to estimate the unknown parameters and many different methods have been proposed in the literature.

In this paper, we have considered the parameter estimation approach for the Bass model based on the nonlinear weighted least squares fitting of the Bass adoption curve. We have shown that the best least squares estimate for the Bass adoption curve does not necessarily exist (Proposition 1). As our main results, we present two theorems (Theorems 1 and 2) on the existence of the least squares estimate, as well as their generalizations in the l_s norm ($1 \leq s < \infty$). Theorem 1 gives a necessary and sufficient condition for the existence of the least squares estimate. For practical purposes, Theorem 2 is extremely important, as it guarantees the existence of the least squares estimate in the case when parameter m is bounded above. Some numerical experiments are included to illustrate the efficiency of our estimation approach.

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Table 2. Accuracy of parameter estimates.

	by minimizing F	by minimizing S
	$(\bar{m}^\star, \bar{p}^\star, \bar{q}^\star) = (999.99, 0.001005, 0.199791)$	$(\hat{m}, \hat{p}, \hat{q}) = (999.94, 0.001001, 0.199962)$
MARE	0.008468	0.049424
RMSRE	0.026829	0.077858

Table 3. Data listing.

	Product													
	Room air conditioners		Color televisions		Clothes dryers		Ultrasound		Mammography		Foreign language		Accelerated program	
τ_i	Year	$N_i(10^3)$	Year	$N_i(10^3)$	Year	$N_i(10^3)$	Year	N_i	Year	N_i	Year	N_i	Year	N_i
1	1949	96	1963	747	1949	106	1965	5	1965	2	1952	1.25	1952	0.67
2	1950	291	1964	2227	1950	425	1966	8	1966	4	1953	2.02	1953	1.15
3	1951	529	1965	4873	1951	917	1967	10	1967	6	1954	2.88	1954	3.26
4	1952	909	1966	9991	1952	1552	1968	15	1968	9	1955	3.36	1955	3.55
5	1953	1954	1967	15768	1953	2289	1969	22	1969	13	1956	4.70	1956	6.14
6	1954	3184	1968	21750	1954	3179	1970	34	1970	22	1957	8.26	1957	8.35
7	1955	4451	1969	27712	1955	4576	1971	40	1971	29	1958	11.62	1958	25.15
8	1956	6279	1970	32343	1956	6099	1972	56	1972	45	1959	17.86	1959	36.19
9	1957	7865			1957	7393	1973	72	1973	68	1960	23.81	1960	50.59
10	1958	9538			1958	8633	1974	100	1974	92	1961	30.05	1961	57.02
11	1959	11338			1959	10058	1975	128	1975	107	1962	34.94	1962	63.17
12	1960	12918			1960	11318	1976	149	1976	113	1963	36.19	1963	64.32
13	1961	14418			1961	12554	1977	162	1977	118				
14							1978	168	1978	119				

Table 4. Parameter estimates and fit statistics obtained by minimizing functional F (our approach).

Product	$\bar{m}^\star$	$\bar{p}^\star$	$\bar{q}^\star$	MARE	RMSRE
Room air conditioners	18.72×10^6	0.00953	0.37328	0.30810	0.50741
Color televisions	39.69×10^6	0.01889	0.60920	0.10506	0.16738
Clothes dryers	16.50×10^6	0.01367	0.32565	0.19744	0.43997
Ultrasound	167.44	0.00136	0.61627	0.43294	0.53512
Mammography	111.51	0.00045	0.84864	0.43308	0.55568
Foreign language	37.62	0.00199	0.68890	0.35119	0.47026
Accelerated program	64.61	0.00084	0.90948	0.28898	0.42592

Table 5. Parameter estimates and fit statistics obtained by minimizing functional S (NLS approach).

Product	$\hat{m}$	$\hat{p}$	$\hat{q}$	MARE	RMSRE
Room air conditioners	18.71×10^6	0.00944	0.37476	0.30984	0.47103
Color televisions	39.66×10^6	0.01847	0.61586	0.11028	0.15061
Clothes dryers	16.50×10^6	0.01360	0.32670	0.19525	0.42240
Ultrasound	167.38	0.00132	0.62060	0.41165	0.50388
Mammography	111.39	0.00041	0.86065	0.44583	0.55339
Foreign language	37.56	0.00189	0.69676	0.40617	0.52545
Accelerated program	64.43	0.00074	0.92828	0.76973	1.15919

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Darija Marković holds a B.Sc. degree in mathematics and computer science from the University of Osijek, as well as an M.Sc. in mathematics from the University of Zagreb. In 2009 she obtained a Ph.D. in mathematics from the University of Zagreb. Presently she is an assistant professor at the Department of Mathematics, University of Osijek. Her research interests cover applied and numerical mathematics, specifically parameter estimation, mathematical modelling, nonlinear least squares and smoothing methods. She has co-authored several papers in journals and conference proceedings on these subjects.

Dragan Jukić received a B.Sc. degree in mathematics and physics from the University of Osijek, Croatia, in 1986, as well as an M.Sc. and a Ph.D. degree in mathematics from the University of Zagreb, Croatia, in 1990 and 1996, respectively. In 2000 he became a professor and in 2004 a full professor at the Department of Mathematics, University of Osijek. His current research interests include mathematical modelling, parameter estimation, curve fitting, nonlinear least squares and smoothing methods. He is the author or coauthor of more than 50 journal and conference papers.

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