A VERIFIED METHOD FOR SOLVING PIECEWISE SMOOTH INITIAL VALUE PROBLEMS

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In many applications, there is a need to choose mathematical models that depend on non-smooth functions. The task of simulation becomes especially difficult if such functions appear on the right-hand side of an initial value problem. Moreover, solution processes from usual numerics are sensitive to roundoff errors so that verified analysis might be more useful if a guarantee of correctness is required or if the system model is influenced by uncertainty. In this paper, we provide a short overview of possibilities to formulate non-smooth problems and point out connections between the traditional non-smooth theory and interval analysis. Moreover, we summarize already existing verified methods for solving initial value problems with non-smooth (in fact, even not absolutely continuous) right-hand sides and propose a way of handling a certain practically relevant subclass of such systems. We implement the approach for the solver VALENCIA-IVP by introducing into it a specialized template for enclosing the first-order derivatives of non-smooth functions. We demonstrate the applicability of our technique using a mechanical system model with friction and hysteresis. We conclude the paper by giving a perspective on future research directions in this area.

Keywords: interval methods, non-smooth systems, initial value problems.

1. Introduction

A large number of applications from the theory of automatic control, mechanics, or electrical engineering are represented by mathematical models that depend on discontinuous or non-differentiable functions. Such situations occur, for example, when engineers describe systems with friction, take into account saturation effects in quantities of interest or simply express naturally arising conditions such as non-positivity of variables (Patton *et al.*, 2012; Myśliński, 2012; Barboteu *et al.*, 2013). The task becomes especially complicated if non-smooth Initial Value Problems (IVPs) are considered. Here, even the definition of the solution might depend on the application at hand. Solving such problems is often additionally impeded by uncertainty in parameters. Besides, the solution might be sensitive to numerical errors. One possibility to deal with these difficulties is the use of verified methods both at the modeling and simulation stages.

Verified methods (Moore, 1966) provide a guarantee that results obtained on a computer are consistent with the formal model developed for the real-life system considered. The application of existing interval methods to real-life scenarios is challenging since they might provide overly conservative enclosures of exact solutions[1]. Even in the case of jump discontinuities, where the solution is not differentiable at just several switching points, the accuracy after encountering such a point might be poor and, consequently, the resulting enclosures might be too wide (Rihm, 1992). This is probably the reason for the relatively little attention non-smooth problems have received in the last decades whereas the verified solution of smooth IVPs has been extensively explored (Lohner, 1988; Nedialkov, 2002; Eble, 2007; Rauh and Auer, 2011).

Verified works on this topic can be roughly divided into two groups according to the description of the

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[1]*Exact* in a predefined sense.

discontinuous system in question. The first group assumes that the system is given in terms of analytical expressions (see the work of Rihm (1992) and the references therein). The second group assumes that the system is represented by a graph containing different Ordinary Differential Equations (ODEs) as vertices and logical conditions for jumps as edges (Rauh *et al.*, 2006; Eggers *et al.*, 2009; Nedialkov and von Mohrenschildt, 2002). Additionally, a lot of research has been done on generalizing the notion of the derivative for non-smooth functions in the area of verified optimization. The goal was to allow using more efficient derivative-type methods also for the practically highly relevant class of non-smooth cost functions. Here, concepts such as slope intervals, the generalized gradient, and the slant derivative were developed and compared with each other, at least, for non-smooth Lipschitz problems (see the works of Kearfott (1996), Munoz and Kearfott (2004) or Schnurr (2007) and the references therein).

In traditional theory, there exist many possibilities to formulate a non-smooth problem. However, the community concentrating on this approach to handling such systems is in general not aware of interval-based or similar methods for solving differential equations with convex and closed set-valued right-hand sides. Therefore, it is interesting not only to establish equivalences and connections between different concepts, but also to point out which of the cases can be covered by interval analysis.

In this paper, we identify important types of non-smooth applications along with their corresponding problem and solution definitions in Section 2. After that, we provide an overview of existing techniques for the calculation of verified enclosures of solutions to non-smooth IVPs and point out possible application areas for them in Section 3. Next, we focus our discussion on a special case in which the switching points are known *a priori* in a certain sense. For this situation, we describe a simple method to solve non-smooth IVPs using basically the same techniques as in the smooth case. Here, we combine a generalized derivative definition with the algorithm of the verified solver VALENCIA-IVP[2] (Rauh and Auer, 2011) to obtain enclosures of the solutions for non-smooth systems given algorithmically, that is, as a certain piece of code. In Section 4, the problem we consider is stated, its solution defined, a suitable derivative definition introduced. Moreover, we show that the guarantee of correctness is preserved inside the changed algorithm of VALENCIA-IVP. Finally, we demonstrate the applicability of the method using a mechanical system with friction and hysteresis (Rauh *et al.*, 2011) in Section 5. The results are compared with those from the cited paper. Conclusions and an outlook on

our future work are given in Section 6.

2. Problem formulations

In this section, we summarize types of non-smooth problems arising in practice. After that, we describe possible problem formulations and appropriate solution definitions from the point of view of the traditional theory. Where applicable, we provide interval-based reformulations.

2.1. Non-smoothness in practice. Non-smooth or discontinuous IVPs arise naturally in many practical applications (Kunze, 2000; Acary and Brogliato, 2008). In mechanics, they describe, e.g., systems with friction, with impacts, with piecewise contact laws or with hysteresis; in electrical engineering, electrical circuits with (ideal) diodes; in control engineering, sliding or switching control systems as well as a number of optimal control laws; in biology, systems with instantaneous switches or with hysteresis. Besides, non-smooth representations are useful in economics, hydraulic circuits, material science, and many other areas. The mathematical formalisms and solution methods for the problems in different fields might be very similar. As an illustration, consider the mathematical system given by Magnus and Popp (2005, p. 43):

$$m\ddot{x} + h \cdot \text{sign}(x) = 0, \qquad (1)$$

with $h = \text{const}$ and $\text{sign}(\cdot)$ the sign function, which can be interpreted as a nonlinear sliding pendulum (mechanics) or a relay oscillator (electrical engineering). Another example is that the complementarity condition between the current across an ideal diode and its voltage (electrical circuits) is similar to the relation between the contact force and the distance between the system and an obstacle in unilateral mechanics (Acary and Brogliato, 2008). The two characteristics in question should be non-negative and orthogonal to each other in the geometrical sense. That is, one characteristic can only be positive if the other is equal to zero, which can be described by a multivalued function.

What is often overlooked or neglected is the fact that non-smoothness might arise in every research area relying on numerical computations when scientists try to ensure numerical stability in their programs. From the point of view of implementation, non-smoothness can be caused simply by the presence of `IF-THEN-ELSE` or `SWITCH` statements on variables in a program code. This kind of discontinuity is not as obvious as that arising in, for example, systems with impacts since it is often hidden in the code. Although floating-point based IVP solvers might be rather accurate in treating such problems as smooth ones, there is no guarantee of correctness for

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[2] VALidation of state ENClosures using Interval Arithmetic for Initial Value Problems.

the obtained results. Since verified IVP solvers rely on derivatives in their algorithms, they would encounter a conceptual problem immediately in this situation.

Depending on the application area and the background of researchers, non-smooth systems can be formally represented in two general ways: as a kind of an automaton or, mathematically, in the form of differential and/or algebraic expressions. In this paper, we will focus on the mathematical representation mainly. However, we will also touch upon automaton-based verified methods where appropriate.

The automaton group assumes that the system is represented by a kind of graph containing different ODEs as vertices and logical conditions for jumps as edges. The modeling formalisms can be roughly divided into those based on hybrid automata, hybrid bond graphs and hybrid discrete event system specification (Lunze and Lamnabhi-Lagarrigue, 2009; Kofman, 2004). Note that floating-point based numerical algorithms are used in the mentioned references.

In the case of the mathematical representation, researchers assume that the system is given in terms of analytical expressions, e.g.,

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{2}$$

with a possibly discontinuous function f. Discontinuities might be expressed in different ways, e.g., as jumps depending on zeros of a certain switching function (cf. Section 2.2). A very common formalism to describe and understand such systems in the traditional theory is that of *Differential Inclusions* (DIs) (Filippov, 1988; Acary and Brogliato, 2008; Orlov, 2004),

$$\dot{x}(t) \in F(t, x(t)), \quad t \in [t_0, t_{\text{end}}], \quad x(t_0) = x_0, \tag{3}$$

where F is a set-valued map $\mathbb{R} \times \mathbb{R}^n \mapsto S(\mathbb{R}^n)$ with $S(\mathbb{R}^n)$ being a set of subsets of $\mathbb{R}^n$. Geometrically, F might represent convex (or non-convex) bounded (or unbounded) sets. A requirement is that the inclusion in (3) be satisfied almost everywhere on $[t_0, t_{\text{end}}]$. Different applications dictate different requirements for the solution $x(t)$, e.g., for it to be absolutely continuous. Equally, the map F might be defined differently according to the application at hand, and so require a different kind of analysis in each case. Note that if F maps to the set of intervals $\mathbb{IR}^n$ and can be expressed as a sufficiently (at least once) continuously differentiable function with interval parameters, the task is related to a continuous IVP in its interval formulation. This situation was extensively explored, for example, by Lohner (1988), Nedialkov (2002) and Eble (2007). The numerical algorithms in these works provide an enclosure guaranteed to contain the exact solution. For example, the problem

$$\dot{x} \in [-1, 1], \quad x(0) = 0,$$

can be associated with the interval analysis formulation

$$\dot{x} = c, \quad c = [-1, 1], \quad x(0) = 0,$$

so that the smooth solution $x(t)$ lies inside $\boldsymbol{x}(t) = [-1, 1]t$ (in Section 2.3.2, a further example of the relation between these two concepts is given).

In addition to bounded/unbounded DIs, formalisms such as linear complementarity systems, evolution variational inequalities, and piecewise continuous representations are common. As stated by Acary and Brogliato (2008), introducing large general classes of descriptions is useful only in a limited way, and narrow classes have to be defined to obtain accurate results. An important direction of research should be "the study of the relationships between the existing formalisms, like possible equivalences". In this paper, we shall focus on convex closed DIs and point out where an interval-based reformulation of the problem description is possible.

In the following, we summarize several types of problem descriptions without the claim of being exhaustive. The general goal of such descriptions (or problem models) is to reformulate the problem (2) in a way which allows proving the existence (and in some cases the uniqueness) of the solution. In each of the subsections below, we cover the following topics: problem formulation, solution definition, existence (and uniqueness) of the solution, relationship between the solution(s) of the concept considered and the original IVP, connections to interval analysis, and application areas.

2.2. ODEs with discontinuities. The following type of problems is usually denoted as ODEs with discontinuities:

$$\dot{x}(t) = f(t, x(t)) = \begin{cases} f_1(t, x(t)), & g(t, x(t)) < 0, \\ f_2(t, x(t)), & g(t, x(t)) > 0, \end{cases} \tag{4}$$

with $x(t_0) = x_0$, where $f_1(t, x)$, $f_2(t, x)$, and $g(t, x)$ are smooth. Points x^* such that $g(t, x^*) = 0$ for some t are called switching points. Both f_1 and f_2 are considered to be bounded in x^*, that is, we examine only jump discontinuities on the right-hand side. Such systems are a special case of the so-called Filippov systems (Filippov, 1988) described in more detail in Section 2.3.1. This situation is well-explored theoretically (Mannshardt, 1978; Stewart, 1990), in particular, under the presence of bounded additive uncertainty (Orlov, 2004) and in the interval case (Rihm, 1993). The main tool here is to compute the left and right derivatives of g according to

$$\dot{g}_1(t, x) := \frac{\partial g}{\partial t} + \frac{\partial g}{\partial x} f_1(t, x), \tag{5}$$

$$\dot{g}_2(t, x) := \frac{\partial g}{\partial t} + \frac{\partial g}{\partial x} f_2(t, x). \tag{6}$$

If the *transversality condition* holds, i.e., there exists a $G > 0$ such that $\dot{g}_1(t, x) \geq G$ and $\dot{g}_2(t, x) \geq G$, then the solution crosses over the switching surface $\{(t, x) : g(t, x) = 0\}$ from the area $G_1 = \{g(t, x) < 0\}$ to the area $G_2 = \{g(t, x) > 0\}$. If there is a $G < 0$ such that $\dot{g}_1(t, x) \leq G$ and $\dot{g}_2(t, x) \leq G$, then the solution crosses over the switching surface in the opposite direction (from G_2 to G_1). If $\dot{g}_1(t, x) < 0$ and $\dot{g}_2(t, x) > 0$, then the solutions in G_1 and G_2 run away from the switching surface and the solution of (4) is not unique. This situation is not interesting from the practical point of view, since the motion along the switching surface "is unstable and does not occur in real systems" Filippov (1988, p. 52). The points (t, x) where this happens are called end points. If $\dot{g}_1(t, x) > 0$ and $\dot{g}_2(t, x) < 0$, then the solutions in G_1 and G_2 both run into the switching surface and the solution to (4) has to stay there. Solution definitions allowing sliding along such switching surfaces are given, e.g., by Filippov (1988); see also Section 2.3.1. Rihm (1993; 1998) gives a verified algorithm for the problem (4) (cf. Section 3.2), where the solution is defined as shown below.

Definition 1. A continuous function $x(t) : I \subset [t_0, t_{\text{end}}] \mapsto \mathbb{R}^n$ is a *solution* to (4) if the zeros of the switching function $g(t, x(t))$ are isolated, $\dot{x}(t) = f(t, x(t))$ everywhere except possibly on the set of all zeros of the switching function, and the initial condition $x(t_0) = x_0$ holds.

This definition coincides with the so-called Carathéodory or classical definition except for the fact that the latter do not require the zeros to be isolated (Filippov, 1988). Carathéodory studied the systems (2) whose right-hand sides were defined and continuous in x for almost all t, measurable in t and bounded for all x. He defined the solution to such a system as an absolutely continuous function.

Note that Definition 1 does not cover sliding solutions where $\dot{g}_1(t, x) > 0$, $\dot{g}_2(t, x) < 0$ and the zeros of the switching function are not isolated. To take into account such situations, Rihm resorts to the concept of Filippov's DIs described in more detail in the next subsection. The approach is to define a solution for which the DI (3) holds except possibly on a set of isolated points. Here, the set-valued F is supposed to coincide with f_1 in the area G_1, f_2 in the area G_2, and is allowed to contain sets of higher cardinality at the switching surfaces where $g(t, x(t)) = 0$.

This problem and solution formulation are useful for mechanical systems with friction and hysteresis as well as in non-smooth control tasks.

2.3. Differential inclusions. The concept of a DI is described by the relation (3). Depending on the chosen F and the definition of the solution, different DI types can

be specified. A detailed description of DIs and similar (or equivalent) concepts is to be found in the work of Acary and Brogliato (2008). The concept can be also used to model the behavior of systems on switching surfaces. In this subsection, we summarize briefly the ideas necessary for the further understanding of the material in this paper.

2.3.1. Filippov's inclusions. The assumption of absolute continuity of the solution to (3) implies that only the solution's derivative may contain jumps. The intention behind Filippov's inclusions is to formulate the problem in such a way as to ensure the existence of solutions and their compliance with solutions to IVPs with continuous right-hand sides (in the areas G_1 and G_2 for the problem formulation (4)). In the work of Filippov (1988), one of the possibilities to define a solution on the switching surface is introduced as follows.

Definition 2. (*Simplest convex definition*) For each (t, x), x a point of discontinuity, $F(t, x)$ is the smallest convex closed set containing all the limit values of the function $f(t, x^*)$ where t is fixed, (t, x^*) is not a point of discontinuity, and $x^* \to x$.

For the problem (4), this definition can be reduced to

$$F(t, x) = \{\alpha f_1(t, x) + (1 - \alpha)f_2(t, x) \mid \alpha \in [0, 1]\} \quad (7)$$

for (t, x) on the switching surface. The solution to (2) is then the solution to (3) with F defined as above. In this definition, the map $F(\cdot)$ is upper semicontinuous, and there always exists an absolute continuous solution to (3) (Acary and Brogliato, 2008). As pointed out there, this model does not reflect to the full extent the behavior on the switching surfaces.

The remaining two definitions of the DI and the solution by Filippov (1988) concern a special class of control-motivated equations and are beyond the scope of this paper.

2.3.2. Lipschitzian DI. Consider an autonomous DI with F depending only on the states $x \in \mathbb{R}^n$.

Definition 3. A DI of the form (3) is called *Lipschitzian* if the sets $F(x)$ are closed and convex and the map $F(x)$ is Lipschitz, i.e.,

$$F(x_1) \subset F(x_2) + l||x_1 - x_2|| \cdot \{y \mid ||y|| \leq 1\} \quad (8)$$

for all $x_1, x_2 \in \mathbb{R}^n$ and l a positive constant (Acary and Brogliato, 2008).

Each function $x(t)$ satisfying (3) almost everywhere with the mapping F defined as above is a solution of (3). Each absolutely continuous function $x(t)$ for which (2) holds is a solution of the Lipschitzian DI. Vice versa, there always exists a solution of the DI which is also the solution of (2).

Note that in the interval setting, that is, if the sets $F(x)$ for each x are (or can be enclosed by) axis-parallel interval vectors, the first condition in Definition 3 always holds and the second one can be replaced by the interval Lipschitz condition

$$F(x_1) - F(x_2) \subset L(x_1 - x_2), \quad L \in \mathbb{IR}^n . \quad (9)$$

The solution to the interval problem

$$\dot{\boldsymbol{x}}(t) = F(\boldsymbol{x}(t)), \quad \boldsymbol{x}(0) = x_0 \quad (10)$$

can then be interpreted in the sense usual for interval analysis (see, e.g., Lohner, 1988) as a set-valued map satisfying the equation above. If F is (at least) differentiable[3], such a function can be computed using methods of interval analysis. If it is only Lipschitzian, the method proposed in Section 4 can be used.

The connections between the solutions to (2), to (3), and to (10) can be illustrated using an example inspired by Acary and Brogliato (2008). Let the problem be given as

$$\dot{x}(t) = x(t)u(t), \quad x(0) = x_0, \quad (11)$$

where $u(t)$ is an unknown smooth function taking all its values in the interval $[-1, 1]$. The first possibility is to solve the IVP itself according to

$$x(t) = x_0 \exp \left(\int\limits_0^t u(s)\,\mathrm{d}s \right),$$

that is, if $x(0) = 0$, then $x(t) = 0$ is the problem solution. By considering the corresponding DI according to Definition 3,

$$\dot{x}(t) \in [-x(t), x(t)], \quad x(0) = 0 , \quad (12)$$

we obtain the same solution $x(t) = 0$. However, the function

$$x(t) = \begin{cases} 0, & 0 \leq t \leq 2, \\ t^2, & t \geq 2, \end{cases} \quad (13)$$

which fails to be absolutely continuous, is also a solution of the DI above. Finally, if we consider the interval problem

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{x}(t) \cdot [-1, 1], \boldsymbol{x}(0) = x_0, \quad (14)$$

the exact solution is the set-valued function $\boldsymbol{x}(t) = x_0 \exp([-1, 1]t)$ and therefore $\boldsymbol{x}(t) = 0$ for $x_0 = 0$.

Lipschitzian DIs cover a smaller class of problems than Filippov's DIs. It is shown by Smirnov (2002) that there exists a solution to a Lipschitzian DI with $x(0) = x_0$ on $\mathbb{R}^+$ for any x_0. In addition, if the one-sided Lipschitz condition is fulfilled, then the solution is unique. Application areas for such types of DIs are usually control and systems theories.

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[3] In general, better enclosures can be obtained if the right-hand side is differentiable up to higher orders.

2.4. Further concepts. Many more types of problem descriptions for non-smooth systems can be encountered in the literature. In the works of Acary and Brogliato (2008) or Bernardo *et al.* (2007), the following are mentioned: Moreau's sweeping process, unilateral DIs, evolution variational inequalities, differential variational inequalities, projected dynamical systems, dynamical complementarity systems, switched systems, and impulsive differential equations. They emerged mainly out of the needs of certain applications, so that some of them are equivalent (or at least interconnected) theoretically. Traditional approaches for handling them come from the areas of convex analysis, non-smooth analysis, complementarity theory, or the theory of variational inequalities. Some of the concepts can be simplified by allowing the convex sets considered to be intervals (or by enclosing them in intervals). Then we can replace them with interval initial value (or boundary) problems for ODEs or Differential-Algebraic Equations (DAEs) as well as interval inequalities in some of the cases, similarly to the example in Section 2.3.2. IVPs for ODEs were covered rather extensively from the verified point of view (Nedialkov, 2002; Eble, 2007; Makino, 1998). There are also several works on interval IVPs for DAEs (Rauh *et al.*, 2009). Certain complementarity problems can be solved by verified optimization methods (Hansen and Walster, 2004; Kearfott, 1996; Jaulin *et al.*, 2001). Note that this substitution does not mean that the solution set would be exactly the same as for non-interval problem formulations, because, strictly speaking, we change the model. However, the interval solution might provide enough information about the behavior of the technical system considered.

The number of problems that can be reformulated directly from the point of view of interval analysis is relatively limited. To study the equivalences and precise connections of the above mentioned concepts to interval analysis remains a large and interesting topic for future research. Our focus in this paper will be to consider in detail the class of problems described in Sections 2.2 and 2.3. However, some of the other mentioned concepts can also be covered since they are related: consider Example 2.43 for evolution variational inequalities of Acary and Brogliato (2008) as an illustration of such a correlation. There, the dynamics of a system with the Coloumb and viscous friction is modeled initially by the inclusion

$$m\ddot{q}(t) + c\dot{q}(t) + kq(t) \in -\partial\varphi(\dot{q}(t))$$
$$= -\mu \cdot \mathrm{sign}(\dot{q}) \quad (15)$$

with $\varphi(\dot{q}) = \mu|\dot{q}|$, m, c, k, μ positive coefficients, and the

set-valued sign function

$$\operatorname{sign}(x) = \begin{cases} -1, & x < 0, \\ [-1, 1], & x = 0, \\ 1, & x > 0. \end{cases} \quad (16)$$

The DI above is then reformulated as a variational inequality and an evolution variational inequality. In Section 4 of this paper, we will suggest a method to solve the following interval formulation of this problem:

$$\begin{cases} \dot{\boldsymbol{x}}_1 = \boldsymbol{x}_2, \\ \dot{\boldsymbol{x}}_2 = -\dfrac{\mu}{m} \cdot \operatorname{sign}(\boldsymbol{x}_2) - \dfrac{c}{m}\boldsymbol{x}_2 - \dfrac{k}{m}\boldsymbol{x}_1, \end{cases} \quad (17)$$

with the sign function defined in the same way as above and $x_1 = q$, $x_2 = \dot{q}$. Here, the solution is understood as a function tube $(\boldsymbol{x}_1(t)\ \boldsymbol{x}_2(t))^{\mathsf{T}}$ containing a/the exact solution to the problem in the sense of Definitions 1 and 2. We restrict our discussion to interval enclosures at discrete time points $\{t_k = t_{k-1} + h_k \mid k = 1, \dots, n\}$, where $t_n = t_{\text{end}}$, h_k the stepsize. The corresponding (continuous) tube is usually obtained using the Taylor expansion with suitable interval Taylor coefficients and the enclosure of the error term (Lohner, 1988, p. 45). In our case, this is not possible in general, since the solution is potentially non-differentiable. However, we can enclose the solution at points $t \neq t_k$ using the mean value theorem with the derivative definition given in Section 4.1. We will focus on absolutely continuous solutions, i.e., those without jumps in the states.

In traditional theory, there are two general research directions for describing non-smooth systems. Some researchers choose to focus on qualitative analytical characteristics such as (non-smooth) Lyapunov exponents, Conley's index, the Kolmogorov–Arnold–Moser theory, or Melnikov's theory (Kunze, 2000; Bernardo *et al.*, 2007). Another group of researchers is explicitly interested in devising accurate numerical algorithms for characterizing the solutions (Stewart, 1990; Mannshardt, 1978; Acary and Brogliato, 2008). Direct numerical simulation methods (time-stepping or event-driven) can (or should) be supplemented by the so-called path-following ones in order to accurately compute bifurcation points and unstable invariant sets (Bernardo *et al.*, 2007). In Section 4 of this paper, we suggest a verified numerical approach for a certain subclass of non-smooth systems, that is, the computed enclosure is proved to contain a true solution to the original problem.

3. Overview of the existing verified methods for non-smooth IVPs

As pointed out before, interval methods (Moore, 1966) offer a natural way of taking into account bounded,

purely epistemic uncertainty. Additionally, in the case of smooth dynamics, they provide a guarantee that the resulting numerical enclosure contains the exact solution to the system model considered. Their main drawback is possible overestimation, that is, conservative enclosures which are too wide to give any information about the system's behavior. Note that overestimation is an inherent feature of interval arithmetic. Geometrically speaking, it arises from the fact that sets which are not axis-parallel must be enclosed by axis-parallel interval boxes. A set-theoretic reason is that intervals contain no information on the dependency of variables, so that, for example, the expression $x - x$ is treated in the same way as $x - y$ in interval arithmetic. To deal with this drawback, further kinds of set-valued arithmetics were developed based on, for instance, affine forms (de Figueiredo and Stolfi, 2004) or Taylor models (Makino, 1998). This class of methods is called verified[4] since they assert the correspondence between the computed result and the solution to the chosen formal model. Presently (to our knowledge), there exist verified algorithms for non-smooth systems based only on interval analysis. More advanced affine or Taylor model methods require, in general, differentiability up to higher orders, making it challenging to account for non-smoothness. In the verified case, the methods fall into two categories based on either mathematical or automaton description again.

We open the section with a brief overview of the main notions of interval analysis. After that, we outline the method proposed by Rihm (1992) and an automaton-based technique of Rauh *et al.* (2011). We mention further approaches (some of them only partially verified) in Section 3.4. This survey is not supposed to be exhaustive; we outline several characteristic verified approaches to finding solutions to (2).

3.1. Basics on interval analysis. An interval $\boldsymbol{x} = [\underline{x}, \overline{x}]$, where $\underline{x} \in \mathbb{R}$, $\overline{x} \in \mathbb{R}$ are the lower and upper bounds, respectively, is defined as

$$\boldsymbol{x} = \{x \in \mathbb{R} \mid \underline{x} \leq x \leq \overline{x}\}. \quad (18)$$

For any operation $\circ = \{+, -, \cdot, /\}$ and intervals $\boldsymbol{x}, \boldsymbol{y}$, the corresponding interval operation can be defined as

$$\begin{aligned} \boldsymbol{x} \circ \boldsymbol{y} &= [\min\{\underline{x} \circ \underline{y}, \underline{x} \circ \overline{y}, \overline{x} \circ \underline{y}, \overline{x} \circ \overline{y}\}, \\ &\quad \max\{\underline{x} \circ \underline{y}, \underline{x} \circ \overline{y}, \overline{x} \circ \underline{y}, \overline{x} \circ \overline{y}\}]. \end{aligned} \quad (19)$$

It can be shown that the result of an interval operation is also an interval. Every possible combination $x \circ y$ with $x \in \boldsymbol{x}$ and $y \in \boldsymbol{y}$ lies inside this interval. (For division of intervals, usually $0 \notin \boldsymbol{y}$ is assumed.)

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[4]Here, the term refers to result verification and not code or formal verification.

If a real function f is given, then its *range* over an interval $\boldsymbol{x}$ from its domain is defined as the set $f(\boldsymbol{x}) := \{f(x) \mid x \in \boldsymbol{x}\}$. The *natural interval extension* of f is obtained by replacing all constants, variables, and operations in its analytical expression by the corresponding (point) interval constants, variables, and operations. The natural interval extension of a function encloses its range over the same interval. Other kinds of interval extensions (evaluations) are possible, e.g., mean value forms based on the mean value theorem. Here, the range is enclosed in the manner

$$f(\boldsymbol{x}) \subseteq f(x_0) + f'(\boldsymbol{x})(\boldsymbol{x} - x_0) , \qquad (20)$$

where x_0 is the reference point. This kind of interval extension produces better enclosures of the range than the natural interval extension if the diameter of $\boldsymbol{x}$ tends to zero (Alefeld and Herzberger, 1983, p. 28). Many complex verified algorithms, such as those for optimization or solution of IVPs, make use of this property for obtaining tighter function range enclosures, which either speeds the algorithm up or leads to tighter enclosures of the solution.

To be able to work with these definitions on a computer using a finite precision arithmetic, the concept of machine intervals is necessary. They are represented by floating point numbers for the lower and upper bounds. To obtain the corresponding machine interval for the real interval $\boldsymbol{x}$, the lower bound is rounded down to the largest representable machine number equal to or less than $\underline{x}$, and the upper bound is rounded up to the smallest machine number equal to or greater than $\overline{x}$ (outward rounding). The real interval operations are also rounded outward. These notions can be extended to define interval vectors $\boldsymbol{x} \in \mathbb{R}^n$ and matrices $\boldsymbol{X} \in \mathbb{R}^{n \times m}$.

A common technique for obtaining rough enclosures of the solution to (smooth) IVPs is based on transforming the problem into an equivalent integral equation and applying a certain fixed point theorem to it (e.g., Banach's). If the assumptions of the theorem used can be checked on a computer, then the enclosure of the solution obtained by an interval equivalent of the Picard iteration is certain to contain the exact solution (see, e.g., Lohner, 1988; Eble, 2007).

There is a number of software libraries implementing this theory in different programming languages such as C++ or FORTRAN and computer algebra packages such as MAPLE or MATLAB.

3.2. Rihm's method. The goal is to find an enclosure of the exact solution of (4) specified in Definition 1. Note that the solution is supposed to be an absolutely continuous function that runs through the switching point at t^* (the point where the solution $x(t)$ intersects the switching surface $\{g(t, x(t)) = 0\}$). First, we describe

the method for the case when t^* satisfies the transversality conditions and can therefore be uniquely continued from G_1 to G_2.

Let $t^* \in [t^-, t^+]$ be an enclosure of the current switching point obtained, e.g., using a variant of the verified Newton method (Moore, 1966). If $\boldsymbol{x}^-$ denoting the enclosure of the exact solution $x(t^-)$ in the area of smoothness G_1 is given[5], then the goal is to compute an enclosure of the solution at t^+, which is situated to the right of the switching point t^*. Rihm (1992) proposes the following approach.

With the abbreviations $x^* := x(t^*)$, $f^- := f_1(t^-, x(t^-))$, $f^+ := f_2(t^*, x^*)$, $h^- := t^* - t^-$, $h^+ := t^+ - t^*$, we obtain the two formulas below by applying the Euler method with the exact local errors $z^- \in \boldsymbol{z}^-$ and $z^+ \in \boldsymbol{z}^+$:

$$x^* = x(t^-) + h^- \cdot f^- + z^-, \qquad (21)$$
$$x(t^+) = x^* + h^+ \cdot f^+ + z^+ . \qquad (22)$$

Note that if we denote by s the width of the interval $[t^-, t^+]$, $s = t^+ - t^-$, then the following relations hold for the unknown stepsizes h^-, h^+:

$$h^- + h^+ = s, \qquad h^- = s - h^+, \qquad h^+ \in [0, s].$$

Substituting (21) for x^* in Eqn. (22) and using the relations above, we obtain an enclosure $\boldsymbol{x}^+$ of the exact solution $x(t^+)$ as

$$\begin{aligned} x(t^+) &= x(t^-) + h^+(f^+ - f^-) + sf^- + z^- + z^+ \\ &\in \boldsymbol{x}^- + s\boldsymbol{f}^- + [0, s](\boldsymbol{f}^+ - \boldsymbol{f}^-) + \boldsymbol{z}^- + \boldsymbol{z}^+ \\ &=: \boldsymbol{x}^+, \end{aligned}$$

where $\boldsymbol{x}^* := \boldsymbol{x}^- + [0, s]\boldsymbol{f}^- + \boldsymbol{z}^-$ and $\boldsymbol{f}^- := f_1(t^-, \boldsymbol{x}^-)$, $\boldsymbol{f}^+ := f_2([t^-, t^+], \boldsymbol{x}^*)$ are interval evaluations of the smooth functions f_1 and f_2. In particular, the width of $\boldsymbol{x}^+$ depends on how big the gap $|\boldsymbol{f}^+ - \boldsymbol{f}^-|$ between f_1 and f_2 is.

The algorithm can be extended (Rihm, 1993) to cover the solutions in the sense of Definition 2 if the corresponding condition for sliding holds. We consider DIs (3) with the right-hand sides of the form (7), where

$$\alpha(t, x) := \frac{\dot{g}_2(t, x)}{\dot{g}_2(t, x) - \dot{g}_1(t, x)} \in [0, 1] \qquad (23)$$

is a continuous function. The function

$$f_0 := \alpha(t, x)f_1(t, x) + (1 - \alpha(t, x))f_2(t, x) \qquad (24)$$

is also continuous, that is, there exists at least one classical solution to $\dot{x} = f_0(t, x)$, $x(t^*) = x^*$ which is also a solution to (3). Under the assumption that

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[5]Or obtained by a smooth IVP method.

Table 1. Summary of different verified methods for non-smooth IVPs: **R** (Rihm, 1993), **RS** (Rauh *et al.*, 2011), **NM** (Nedialkov and von Mohrenschildt, 2002), **H** (Henzinger *et al.*, 2000), **I** (Ishii, 2010), **MC** (Mahmoud and Chen, 2008), **EE** (Eggers *et al.*, 2009), and **RA** (Ratschan, 2012). Our method from Section 4 is denoted by **A**. The letters *m* and *a* denote mathematical and automaton-based representations, respectively, + and − indicate the availability or absence of a feature, respectively, *o* shows that a property does not apply, *p* stands for piecewise continuous, *h* and *l* stand for *high* and *low*, respectively.

Property	Method								A
	R	RS	NM	H	I	MC	EE	RA	
representation	m	a	a	a	a	m	a	a	m
conditions for f	p	p	p	p	p	Lipschitz	p	p	p^{**}
conditions for f_i	$\mathcal{C}^q$	$\mathcal{C}^q$	Lipschitz*/$\mathcal{C}^q$	$\mathcal{C}^{1*}$	Lipschitz	o	$\mathcal{C}^q$	$\mathcal{C}$	$\mathcal{C}^1$
use of transversality conditions	+	−	+	−	−	−	−	−	−
analytical solution	−	−	+/−	+/−	−	−	−	−	−
continuousness of solution	+	p	p	p	p	+	o	o	+
consideration of sliding solutions	+	−	−	−	−	−	−	−	−
use of verified IVPS	+	+	+	+	+	o	+	+	+
analytic/verified constraint prop.	−	−	−	+	+	−	+	+	−
verified equation solvers	+	−	+	−	+	+	+	−	−
formal model checkers	o	−	o	−	o	o	+	+	o
verified solution enclosure	+	+	+	+	+	−	−	−	+
user involvement	l	h	l	l	l	l	l	l	l

* f_i should be chosen in such a way as for the IVP to have an analytical solution if no verified integration is used (e.g., linear).
** f_i should be representable in the form (27).

f_0 is continuously differentiable, the same approach as described above can be applied to the modified IVP.

Nedialkov and von Mohrenschildt (2002) apply this approach to non-smooth hybrid systems in combination with the smooth IVP solver VNODE (Nedialkov, 2002).

3.3. Automaton-based method.
There are several authors who consider result verification of non-smooth models given in the automaton representation (Nedialkov and von Mohrenschildt, 2002; Rauh *et al.*, 2011; Ratschan, 2012; Henzinger *et al.*, 2000). Rauh *et al.* (2011) study dynamical systems consisting of l different smooth models $\mathcal{S} = \{S_1, S_2, \ldots, S_l\}$ given in the state-space representation

$$\dot{x}(t) = f_{S_i}(x(t), p, u(t), t), \quad i = 1, \ldots, l. \quad (25)$$

Here, $x \in \mathbb{R}^n$ denotes the solution, $p \in \mathbb{R}^q$ the vector of uncertain system parameters, and $u \in \mathbb{R}^r$ the vector of control variables. The different models S_i, $i = 1, \ldots, l$, are interpreted as discrete states of the overall non-smooth model for the real world system. A transition from the currently active state S_i to the state S_j, $i, j = 1, \ldots, l$, takes place if the condition $T_i^j(x, u)$, which depends on the solution and the control input, becomes active. The condition T_i^i denotes that the state S_i remains active. T_i^j are supposed to be mutually exclusive.

The goal is to compute guaranteed enclosures of solutions to the non-smooth model at discrete time points. For this purpose, the authors suggest to extend smooth IVP solving routines from interval analysis with a technique to detect all possible points of time at which transition conditions T_i^j are activated. To provide tight enclosures of the state variables, it is necessary to detect as soon as possible that one of the states from $\mathcal{S}$ is not active at a given point of time. The proposed algorithm consists of the following four stages.

1. Calculation of a coarse bounding box b_k^a for all solutions in the time interval $[t_k, t_{k+1}]$. The Picard iteration (Lohner, 1988) is applied formally for each $t = t_k$, the given stepsize $h = t_{k+1} - t_k$, and the enclosure at the previous step x_k with

$$b_k^a := \underline{\bigcup}_{i \in \mathcal{I}_a} (x_0 + [0, h] \cdot f_{S_i}(x_k, p, u(t_k), t_k)),$$

where $\underline{\cup}$ denotes the convex hull and

$$\mathcal{I}_a = \{i \mid T_i^i = \texttt{true}\} \quad \text{for} \quad t = t_k.$$

Note that there is always at least one index in $\mathcal{I}_a$. The sets $f_{S_i}(x_k, p, u(t_k), t_k)$ can be obtained, for example, by interval evaluation of the right-hand sides of the smooth problems S_i. Interval bounds of admissible, piecewise constant control inputs $u(t_k)$ are assumed to be available during the computation.

2. Activation of additional transition conditions. The bounding box b_k^a serves as a basis for checking whether additional transition conditions become active for at least one of the active models S_i, $i \in \mathcal{I}_a$. If transition conditions to new models are activated, the bounding box b_k^a from the previous step is adjusted to contain the additional information. That is, if $\tilde{\mathcal{I}}_a \neq \mathcal{I}_a$, where

$$\tilde{\mathcal{I}}_a = \mathcal{I}_a \cup \left\{ j \mid \exists i \in \mathcal{I}_a : T_i^j(b_k^a, u([t_k, t_{k+1}])) = \texttt{true} \right\},$$

then

$$\tilde{\boldsymbol{b}}_k^a := \boldsymbol{b}_k^a \cup \underline{\bigcup_{i \in \tilde{\mathcal{I}}_a \setminus \mathcal{I}_a}} \Big(\boldsymbol{x}_0 + [0, h]$$

$$\cdot f_{S_i}(\boldsymbol{b}_k^a, \boldsymbol{p}, \boldsymbol{u}([t_k, t_{k+1}]), [t_k, t_{k+1}]) \Big).$$

This procedure is repeated until no new indices can be added to $\tilde{\mathcal{I}}_a$.

3. Computation of a guaranteed enclosure of the solution $\boldsymbol{x}_{k+1}$ at t_{k+1}. The enclosure $\boldsymbol{x}_{k+1}$ at $t = t_{k+1}$ is computed using an interval Taylor series expansion method (Lohner, 1988). If only one discrete state is activated ($\sharp \tilde{\mathcal{I}}_a$=1), an order $q > 0$ of the Taylor method is used. If several discrete states S_i are active, the order $q = 0$ of the Taylor expansion is used for a convex hull f_a of functions f_{S_i}, $i \in \tilde{\mathcal{I}}_a$. For more details, see the work of Rauh *et al.* (2006).

4. Deactivation of discrete states for t_{k+1}. The indices of discrete states S_i inadmissible at $t = t_{k+1}$ have to be deleted from $\tilde{\mathcal{I}}_a$ using predefined deactivation conditions $D_i^j(\boldsymbol{x}_{k+1}, u(t_{k+1}))$.

In the current implementation by Rauh *et al.* (2011), the transition/deactivation conditions T_i^j, D_i^j as well as the hulls f_a have to be specified manually, which, in general, need not be so (cf. Eggers *et al.*, 2009; Ratschan, 2012). The approach is implemented in MATLAB.

3.4. Further methods. Ratschan (2012), Eggers *et al.* (2009) and Ishii (2010) rely on constraint solving techniques and formal verification methods in combination with interval analysis to study hybrid systems given as automata. In the first two references, the main goal is to formally verify that a hybrid system does not reach a set of states marked as unsafe, whereas the latter seeks to provide "a proof of the reachability of a model, and [...] a guide in the over-approximation refinement procedure". In the works of Eggers *et al.* (2009) and Ishii (2010), the problem is formulated as a hybrid constraint system which consists of instantaneous constraints, continuous constraints on trajectories, and guard constraints on states causing discrete changes. Ratschan (2012) provides a generalized specification framework. Another method from this class is described by Henzinger *et al.* (2000). There, a hybrid system model checker is developed which can handle dynamics expressed as a combination of polynomials, exponentials, and trigonometric functions. It conservatively overapproximates the set of reachable states by using interval methods as a tool for computing the flow successors of a given region.

A different approach that combines verified and traditional algorithms for IVPs (2)[6] with Lipschitz continuous right-hand sides is that by Mahmoud and Chen

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[6]With $\mathrm{d}(M \cdot x(t))/\mathrm{d}t$ instead of just $\mathrm{d}x(t)/\mathrm{d}t$.

(2008). The authors adapt an Implicit Runge–Kutta (IRK) method of order s to non-smooth problems. First, they solve the nonlinear system of equations associated with IRK using a generalized Krawczyk algorithm to obtain an enclosure $\boldsymbol{x}_k$ of its exact solution. After that, they prove that the error of the approximation to the IVP solution obtained in the second step of IRK is smaller than a certain weighted sum of the maximal widths of $\boldsymbol{x}_k$. Here, the midpoints of $\boldsymbol{x}_k$ are used to compute the approximation.

The above-mentioned techniques demonstrate their close connection to the algorithms from non-smooth optimization. Kearfott (1996) as well as Munoz and Kearfott (2004) study from the verified point of view the relationships between different generalizations of the derivative for non-smooth functions and their application to optimization problems and, in particular, to solving systems of equations. In the work of Schnurr (2007), the concept of slopes is implemented and extended to the second order.

Further interesting references are the works of Goldsztejn *et al.* (2010), Ramdani and Nedialkov (2011), Ishii *et al.* (2011), Zgliczynski and Kapela (2009), or Galias (2012).

In Table 1, the methods mentioned in this section are summarized with respect to their main components and requirements. A simple verified approach we propose in the next section is shown in the last column. From the table, it can be observed that this method needs less strict requirements than most of the other methods to produce a verified enclosure, at the cost of representing the right-hand side in the form (27).

4. Simple verified approach

As pointed out by Kearfott (1996) with respect to non-smooth optimization, "simplicity is a major advantage of treating [...] non-smooth problems with the same techniques as smooth problems". This is also true in the area of solving IVPs and makes a difference between our approach and those outlined in the previous section. To be able to treat non-smooth problems as smooth, a generalized derivative definition is necessary since derivative-free techniques can be overly conservative. While devising such a definition, we have to ensure that the intervals obtained similarly to (20) enclose the range of the function so that no part of the solution is lost. The generalization of the derivative proposed here is not the same as in the formulas by Kearfott (1996) since it combines symbolic and automatic differentiation techniques while factoring at the development point of the mean value theorem similarly to slopes to improve the resulting enclosures. Moreover, our technique is different from the implementation proposed by Schnurr (2007) since we do not overload all variables with the data type for slopes (or derivatives, in our case) to obtain

an enclosure of a function in a bottom-up approach, but compute it in a top-down way instead.

In this section, we formulate the IVP under consideration, specify the type of the right-hand side exactly and define the corresponding solution. After that, we propose a way of obtaining an interval evaluation of this right-hand side and a generalized first derivative for it. Finally, we show that using the above mentioned generalized derivative in VALENCIA-IVP does not change the verified nature of its results while extending its application domain to a certain class of non-smooth IVPs.

4.1. Problem formulation and a derivative definition.
Let the right-hand side $f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$, where D is open, of the autonomous IVP with uncertain initial values from the interval $\boldsymbol{x}_0 \in \mathbb{IR}^n$

$$\dot{x} = f(x), \quad x(0) \in \boldsymbol{x}_0, \qquad (26)$$

be available in its algorithmic representation (Stauning, 1997),

$$\begin{cases} \tau_i(x) = x_i, \quad i = 1, \ldots, n, \\ \tau_i(x) = g_i(\tau_{i_1}(x)) \text{ or } g_i(\tau_{i_1}(x), \tau_{i_2}(x)), \\ \qquad\qquad i = n+1, \ldots, l, \\ g_i \in S_{EO} \cup S_{PW}, \end{cases} \qquad (27)$$

where all functions τ_i and g_i are scalar, g_i may be unary or binary, and $i_k \in \{1, \ldots, i-1\}$, $k = 1, 2$. The last n variables $\tau_{l-n+1}, \ldots, \tau_l$ correspond to the output variables $f_1, \ldots, f_n$. The usual definition for g_i is to belong to the set S_{EO} of *operations* $+, -, *, /$ and *elementary functions* such as trigonometric ones. In this paper, we allow g_i to be part of the set of piecewise smooth functions (denoted by S_{PW}). A function from this set depends on one variable and can be defined by a separate algorithmic scheme. Let $y = \tau_\nu(x)$ be an input or an intermediate variable with the index $\nu \in \{1, \ldots, i-1\}$ and each $\varphi_j(y)$, $j = 0, \ldots, L$, be composed according to (27) with all $g_{j_i} \in S_{EO}$ and the single input variable y. Then $\tau_i(y) = \varphi_{L+1}(y) \in S_{PW}$ is defined as

$$\varphi_{L+1}(y) \qquad\qquad (28)$$
$$= \begin{cases} \varphi_0(y) & \text{for } c_{-1} = -\infty < y \begin{Bmatrix} \leq \\ < \end{Bmatrix} c_0, \\ \varphi_1(y) & \text{for } c_0 \begin{Bmatrix} \leq \\ < \end{Bmatrix} y \begin{Bmatrix} \leq \\ < \end{Bmatrix} c_1, \\ \vdots & \vdots \\ \varphi_{L-1}(y) & \text{for } c_{L-2} \begin{Bmatrix} \leq \\ < \end{Bmatrix} y \begin{Bmatrix} \leq \\ < \end{Bmatrix} c_{L-1}, \\ \varphi_L(y) & \text{for } c_{L-1} \begin{Bmatrix} \leq \\ < \end{Bmatrix} y < c_L = +\infty. \end{cases}$$

Here, $c = (c_0, \ldots, c_{L-1})$ are constant values for which the function changes its behavior (switching points in

terms of y). The subfunctions $\varphi_0, \ldots, \varphi_L$ should be continuous, differentiable, and bounded in the definition domain. The arity of $\tau_i \in S_{PW}$ can be assumed to equal one (similarly to elementary functions such as the sine).

Note that the fact that the piecewise function defined in (28) depends on a single variable does not imply that right-hand sides of ODEs with several variables cannot be represented. For example, the function $f(x_1, x_2) = |x_1| + x_1 \cdot \text{sign}(x_2)$ can be easily constructed. It is also possible to use nested piecewise operations such as $|\text{sign}(x_1)|$. However, operations depending on more than one variable cannot be represented as, for example, in

$$f(x_1, x_2) = \begin{cases} 1, & x_2 < x_1, \\ 2, & x_2 > x_1. \end{cases} \qquad (29)$$

The formulation (4) can cover this situation by introducing functions $g(x_1, x_2) = x_2 - x_1$, $f_1 = 1$, $f_2 = 2$. Yet it is difficult to represent, for example, the dead time

$$f(x) = \begin{cases} -h, & x < -x_+, \\ 0, & -x_+ < x < x_+, \\ h, & x_+ < x \end{cases} \qquad (30)$$

(with h, x_+ constants) in the form (4), whereas (28) fits.

If a function φ_i is smooth, we understand its interval extension $\varphi_i(\boldsymbol{x})$ over $\boldsymbol{x} \in \mathbb{IR}$ as the natural interval extension (cf. Section 3.1). An *interval extension* of a function $\varphi := \varphi_{L+1}$ from (28) over $\boldsymbol{x}$ is constructed as

$$\varphi(\boldsymbol{x}) \qquad\qquad (31)$$
$$= \begin{cases} \varphi_i(\boldsymbol{x}) \text{ for } \boldsymbol{x} \subset (c_{i-1}, c_i), \\ \left(\bigcup_{k=i+1}^{j-1} \varphi_k([c_{k-1}, c_k]) \right) \underline{\cup} \varphi_i([\underline{x}, c_i]) \\ \quad \underline{\cup} \varphi_j([c_{j-1}, \overline{x}]) \quad \text{for } \boldsymbol{x} \subset (c_{i-1}, c_j), \end{cases}$$

with (c_{i-1}, c_j) the smallest interval enclosing $\boldsymbol{x}$, $0 \leq i < j \leq L$ and $\underline{\bigcup}$ denoting the convex hull again. Note that the interval evaluation in (31) always encloses both left and right values if a jump discontinuity occurs within $\boldsymbol{x}$ or on its border regardless of which condition ($<$ or $\leq$) is chosen. This regulation has its origin in practice and differs from the definition we offered in our previous work (Auer *et al.*, 2011). For example, if we consider the function

$$f(v) = \begin{cases} -\lambda + \mu \cdot v, & v < 0 =: c_0, \\ +\lambda + \mu \cdot v, & v > 0, \end{cases} \qquad (32)$$

where λ and μ are positive scalars, which often serves as a model for friction, the favored way to define the function value in the point of the discontinuity $v = 0$ (static friction) is as varying between $-\lambda$ and λ (because the static friction force is undefined but known to be bounded by λ). With the help of the definition from Eqn. (31), we compute $f([0, 0]) = [-\lambda, \lambda]$ because

$[0,0] \subset (c_{-1}, c_1) = (-\infty, +\infty)$ and, therefore, both branches are active. In the same way, $f([-0.5, 0]) = [-\lambda - \mu/2, \lambda]$ or $f([0, 0.5]) = [-\lambda, \lambda + \mu/2]$. Note that the continuity of the right-hand side $f(x)$ defined as in (27) does not necessarily follow from the fact that each $g_i \in S_{PW}$ is continuous ($\varphi_j(c_j) = \varphi_{j+1}(c_j)$, $0 \le j < L$), since S_{EO} contains operations discontinuous everywhere (e.g., division or the reciprocal $\frac{1}{x}$). The following property holds.

Property 1. *If we assume that all g_i are continuous (or Lipschitz continuous) everywhere, then the right-hand side f of (26) is also continuous (Lipschitz continuous).*

This result follows from the theorem about the composition of continuous (Lipschitz continuous) functions. Note, however, that this excludes the division. To allow this operation in the framework of algorithmic procedures, it is decomposed into multiplication and the reciprocal. In this case, all binary operations are smooth, and the unary reciprocal $1/x$ is handled as an elementary function undefined everywhere (the plane $x = 0$ has to be excluded).

A possible *generalization of the derivative* of φ over the interval $\boldsymbol{x}$ is

$$\varphi^!(\boldsymbol{x}) \qquad (33)$$
$$= \begin{cases} \varphi_i'(\boldsymbol{x}) \text{ for } \boldsymbol{x} \subset (c_{i-1}, c_i), \\ \bigcup_{k=i+1}^{j-1} \varphi_k'([c_{k-1}, c_k]) \underline{\cup} \varphi_i'([\underline{x}, c_i]) \underline{\cup} \varphi_j'([c_{j-1}, \overline{x}]) \\ \qquad\qquad \text{for } \boldsymbol{x} \subset (c_{i-1}, c_j) . \end{cases}$$

Defined in this way, it encloses both left and right derivatives at a switching point c_i. Note that the following property holds for this definition.

Property 2. *If φ' exists for an $x \in \boldsymbol{x}$, then $\varphi'(x) \in \varphi^!(\boldsymbol{x})$. That is, the interval derivative from Eqn. (33) always contains the correct derivative value in $\boldsymbol{x}$.*

If the mean value form is to enclose the function range in the manner of Eqn. (20) in the one-dimensional case, a derivative generalization has to produce Lipschitz matrices (cf. Kearfott, 1996, Chapter 6). This is true for the definition in Eqn. (33) for continuous functions (McLeod, 1964/1965). However, this definition cannot be used for discontinuous functions. Consider, for example,

$$f(x) = \begin{cases} x, & x < 0, \\ 2x + 2 & x > 0. \end{cases} \qquad (34)$$

The enclosure of its range over the interval $\boldsymbol{x} = [-1, 2]$ is $[-1, 6]$. The definition in Eqn. (33) delivers $[1, 2]$ as the derivative enclosure. If we apply the mean value theorem with this definition and $x_0 = -1/2$, we obtain

$$f(-\frac{1}{2}) + f^!(\boldsymbol{x})(\boldsymbol{x} + \frac{1}{2}) = -\frac{1}{2} + [1, 2]([-1, 2] + \frac{1}{2})$$
$$= [-1.5, 4.5] \not\supseteq [-1, 6].$$

Without loss of generality, we restrict our discussion in the following to a function of the form

$$\varphi(x) = \begin{cases} \varphi_0(x), & x \begin{Bmatrix} \le \\ < \end{Bmatrix} c_0, \\ \varphi_1(x), & \text{otherwise,} \end{cases}$$

in the interval $\boldsymbol{x} \ni c_0$, where φ_i, $i = 0, 1$, are continuous, differentiable, and bounded (a classical IF-THEN-ELSE operator). An *interval extension of the derivative* of a continuous function $\varphi_{\text{cont}}^!(\boldsymbol{x})$ can be defined as in Eqn. (33). If φ is discontinuous in c_0, then we have to account for the gap $|\varphi_1(c_0) - \varphi_0(c_0)|$. To avoid the necessity to return intervals $(-\infty, +\infty)$ as in the work of Kearfott (1996), we need to know in which part of $\boldsymbol{x}$ the reference point x_0 from the mean value theorem is contained. If this information is available, the definition for discontinuous functions is

$$\varphi_{\text{dis}}^!(\boldsymbol{x}) \qquad (35)$$
$$= \begin{cases} \varphi_0'([\underline{x}, c_0]) \underline{\cup} \left(\dfrac{\varphi_1(c_0) - \varphi_0(c_0)}{[c_0, \overline{x}] - x_0} + \right. \\ \quad \left. \varphi_0'([\underline{x}, c_0]) \underline{\cup} \varphi_1'([c_0, \overline{x}]) \right) & \text{if } x_0 \in [\underline{x}, c_0), \\[2mm] \varphi_1'([c_0, \overline{x}]) \underline{\cup} \left(\dfrac{\varphi_0(c_0) - \varphi_1(c_0)}{[\underline{x}, c_0] - x_0} + \right. \\ \quad \left. \varphi_0'([\underline{x}, c_0]) \underline{\cup} \varphi_1'([c_0, \overline{x}]) \right) & \text{if } x_0 \in (c_0, \overline{x}], \\[2mm] \varphi_0'([\underline{x}, c_0]) \underline{\cup} \varphi_1'([c_0, \overline{x}]) & \text{if } x_0 = c_0. \end{cases}$$

Property 3. *The interval extension of the derivative of φ over $\boldsymbol{x}$ defined as in (35) satisfies the mean value theorem if φ is discontinuous.*

Proof. Let the reference point of the mean value theorem x_0 lie in the interval $[\underline{x}, c_0)$ and $x \in \boldsymbol{x}$. There are two non-degenerate possibilities: $x \in (c_0, \overline{x}]$ or $x \in [\underline{x}, c_0)$.

Situation (a): $x \in [\underline{x}, c_0)$, $x_0 \in [\underline{x}, c_0)$. Here, $\varphi(x) \equiv \varphi_0(x)$ and $\varphi(x_0) \equiv \varphi_0(x_0)$. That is, the relation

$$\varphi(x) \equiv \varphi_0(x) = \varphi_0(x_0) + \varphi_0'(\xi_a)(x - x_0)$$

holds for a $\xi_a \in [\underline{x}, c_0)$ according to the usual mean value theorem. That is,

$$\varphi(x) \in \varphi_0(x_0) + \varphi_0'([c_0, \overline{x}])(x - x_0). \qquad (36)$$

Situation (b): $x \in (c_0, \overline{x}]$, $x_0 \in [\underline{x}, c_0)$. In this case, $\varphi(x) \equiv \varphi_1(x)$ and $\varphi(x_0) \equiv \varphi_0(x_0)$. The function is discontinuous, that is, $\varphi_0(c_0) \ne \varphi_1(c_0)$. Let $g(c_0) := \varphi_1(c_0) - \varphi_0(c_0)$. We apply the mean value theorem to functions $\varphi_0(x)$ and $\varphi_1(x_0)$ with the reference point c_0, $\underline{x} \le x_0 < c_0 < x \le \overline{x}$:

$$\varphi_0(x_0) = \varphi_0(c_0) + \varphi_0'(\xi_{b_1})(x_0 - c_0),$$
$$\varphi_1(x) = \varphi_1(c_0) + \varphi_1'(\xi_{b_2})(x - c_0),$$

with $\xi_{b_1} \in (\underline{x}, c_0)$ and $\xi_{b_2} \in (c_0, \overline{x})$. That is,

$$\varphi_1(x) - \varphi_0(x_0) = g(c_0) + \varphi_1'(\xi_{b_2}) \cdot (x - c_0)$$
$$- \varphi_0'(\xi_{b_1}) \cdot (x_0 - c_0).$$

Note that $(x - c_0) > 0$ and $(x_0 - c_0) < 0$, i.e., $c_0 - x_0 > 0$ in our setting. Moreover, the relation $(a + b)\boldsymbol{s} = a\boldsymbol{s} + b\boldsymbol{s}$ holds for $a, b \in \mathbb{R}$ such that $ab > 0$ and an interval $\boldsymbol{s} \in \mathbb{IR}$. Since $\varphi_0'(\xi_{b_1})$, $\varphi_1'(\xi_{b_2})$ are contained in the interval $\boldsymbol{s} := \varphi_0'([\underline{x}, c_0]) \sqcup \varphi_1'([c_0, \overline{x}])$, the following enclosure can be derived:

$$\varphi_1(x) - \varphi_0(x_0) \in g(c_0) + \boldsymbol{s}(x - c_0) + \boldsymbol{s}(c_0 - x_0)$$
$$= \left(\frac{g(c_0)}{(x - x_0)} + \boldsymbol{s} \right) (x - x_0). \quad (37)$$

In the situation when $x_0 \in (c_0, \overline{x}]$, $x \in \boldsymbol{x}$, we can prove analogously that $\varphi(x)$ is either in

$$\varphi_1(x_0) + \varphi_1'([\underline{x}, c_0]) \cdot (x - x_0), \quad (38)$$

if $x \in (c_0, \overline{x}]$, or in

$$\varphi_1(x_0) + (x - x_0)$$
$$\cdot \left(\frac{-g(c_0)}{(x - x_0)} + \varphi_0'([\underline{x}, c_0]) \sqcup \varphi_1'([c_0, \overline{x}]) \right), \quad (39)$$

if $x \in (c_0, \overline{x}]$. Taking the convex hull of the multiplicands of $(x - x_0)$ in the relations (36)–(39) and keeping in mind the interval parts that x comes from in each case, we get the mean value form enclosure for the range of $\varphi(x)$ with the derivative defined as in (35). $\blacksquare$

If there is more than one switching point c_i, but $\boldsymbol{x}$ contains only one of them, the definition turns into

$$\varphi^!(\boldsymbol{x}) \quad (40)$$
$$= \begin{cases} \varphi_i'(\boldsymbol{x}) & \text{for } \boldsymbol{x} \subset (c_{i-1}, c_i), \\ \varphi_{\text{cont}}^!(\boldsymbol{x}) & \text{for } \boldsymbol{x} \subset (c_{i-1}, c_{i+1}), \\ & \varphi \text{ is continuous at } c_i, \\ \varphi_{\text{dis}}^!(\boldsymbol{x}) & \text{for } \boldsymbol{x} \subset (c_{i-1}, c_{i+1}), \\ & \varphi \text{ is discontinuous at } c_i. \end{cases}$$

Note that it is necessary to consider domain intervals both left and right of c_i whenever the boundary of $\boldsymbol{x}$ coincides with it, regardless of the particular inequality sign.

This definition can be generalized for intervals containing more than one switching point. However, we can usually assume that an interval $\boldsymbol{x}$ containing more than two switching points is too wide in practice. For $c_0, c_1 \in \boldsymbol{x}$ and φ discontinuous in both c_0, c_1 with

$f_j = \varphi_{j+1}(c_j) - \varphi_j(c_j)$, $j = 0, 1$, the following holds:

$$\varphi_{\text{dis}}^!(\boldsymbol{x}) \quad (41)$$
$$= \begin{cases} \varphi_0' \sqcup \left(\frac{f_0}{[c_0, c_1] - x_0} + (\varphi_0' \sqcup \varphi_1') \right) \\ \sqcup \left(\frac{f_0 + f_1}{[c_1, \overline{x}] - x_0} + (\varphi_0' \sqcup \varphi_1' \sqcup \varphi_2') \right) & \text{if } x_0 \in [\underline{x}, c_0), \\ \varphi_1' \sqcup \left(\frac{-f_0}{[\underline{x}, c_0] - x_0} + (\varphi_0' \sqcup \varphi_1') \right) \\ \sqcup \left(\frac{f_1}{[c_1, \overline{x}] - x_0} + (\varphi_2' \sqcup \varphi_1') \right) & \text{if } x_0 \in (c_0, c_1), \\ \varphi_2' \sqcup \left(\frac{-f_1}{[c_0, c_1] - x_0} + (\varphi_2' \sqcup \varphi_1') \right) \\ \sqcup \left(\frac{f_0 + f_1}{[c_1, \overline{x}] - x_0} + (\varphi_0' \sqcup \varphi_1' \sqcup \varphi_2') \right) & \text{if } x_0 \in (c_1, \overline{x}], \\ \varphi_0' \sqcup \varphi_1' \sqcup \left(\frac{f_1}{[c_1, \overline{x}] - x_0} + (\varphi_2' \sqcup \varphi_1') \right) & \text{if } x_0 = c_0, \\ \varphi_1' \sqcup \varphi_2' \sqcup \left(\frac{f_0}{[c_0, c_1] - x_0} + (\varphi_0' \sqcup \varphi_1') \right) & \text{if } x_0 = c_1. \end{cases}$$

for φ_i, $i = 0, 1, 2$, evaluated over their respective domains $[\underline{x}, c_0], [c_0, c_1], [c_1, \overline{x}]$. Situations in which φ is continuous in c_0 and discontinuous in c_1 or vice versa can be handled by setting f_0 or f_1 to zero, respectively.

The overestimation we get by using the definitions above is considerable. The reason is that we divide a constant (corresponding to the gap in c_0, c_1) by an interval (which might have a large width, or be close to zero). This necessity has its source in the intention to produce a general derivative definition obeying the mean value theorem.

4.2. VALENCIA-IVP for non-smooth IVPs.

We use the definitions from Eqns. (31) and (40) in combination with the IVP solver VALENCIA-IVP to solve non-smooth problems of the form (26). We chose this particular software because it needs only the Jacobian matrix of the right-hand side of the ODE to provide guaranteed solution enclosures. VALENCIA-IVP computes an enclosure of the true solution to a smooth IVP as the functional tube $\boldsymbol{x}(t)$ consisting of the non-verified approximation $\tilde{x}(t)$ and the verified error bound $\boldsymbol{R}(t)$ according to

$$x^*(t) \in \boldsymbol{x}(t) := \tilde{x}(t) + \boldsymbol{R}(t). \quad (42)$$

Here, bounds for $\boldsymbol{R}(t)$ can be derived by integration of its derivative, which is in turn obtained formally using the Picard iteration

$$\dot{\boldsymbol{R}}^{(k+1)}(t) = -\dot{\tilde{x}}(t) + f(\boldsymbol{x}^{(k)}). \quad (43)$$

The enclosure of the range of f over $\boldsymbol{x}^{(k)}$ need to be as tight as possible. For this purpose, VALENCIA-IVP uses mean value forms (among other techniques), which needs derivatives satisfying (20) in order to produce verified

results. For more details about the algorithm of the solver, see the results of Dötschel *et al.* (2013), Rauh and Auer (2011), or Rauh *et al.* (2009).

We understand the term *solution* in the sense defined by Rihm (cf. Definition 1), if the right-hand side of the IVP is non-smooth in x. Such solutions are known to exist and even to be unique in the general theory (cf. Section 2.2). To demonstrate that the algorithm of VALENCIA-IVP works in the discontinuous case, we need to show that a certain fixed point theorem can be applied to the operator

$$A(x)(t) := x_0 + \int_0^t f(x(s)) \, \mathrm{d}s$$

and to the equivalent fixed point formulation of the IVP $x(t) = A(x(t))$ in order to compute the enclosure of the true solution of the IVP also for discontinuous right-hand sides f.

To apply the algorithm of VALENCIA-IVP, we have to use the appropriate definition of the derivative as explained above. We showed that $f(x) \in f(x_0) + f^!(\boldsymbol{x})(\boldsymbol{x} - x_0)$ for all $x, x_0 \in \boldsymbol{x}$ (Property 3) so that the exchange in derivative definitions is valid. Next, we need to prove that the fixed point iteration, on which VALENCIA-IVP is based, is true for this class of functions. In particular, we have to find a fixed point theorem which can be applied in this case. If the right-hand side $f(x)$ of the IVP (26) is Lipschitz continuous for $[0, T]$, then Banach's theorem can be applied as usual to the integral operator $A(\boldsymbol{x})$ which can be shown to be contracting (cf. Walter, 1972). Then the solution exists and is unique. If f is continuous, then Schauder's theorem ensures that the computed enclosure contains a solution to the IVP.

In the case of discontinuous functions, a generalization of Kakutani's fixed point theorem (Granas and Dugundji, 2003) for infinite dimensions (e.g., the Fan–Glicksberg theorem) can be applied: If X is a non-empty, compact and convex subset of a locally convex space $\mathbb{E}$ and $\varphi : X \mapsto S(X)$ is a compact and convex set-valued function with a closed graph from X to the set of its non-empty subsets, then φ has a fixed point. Intervals are non-empty, compact and convex. The interval evaluation of a discontinuous right-hand side f according to the definition in Eqn. (31) corresponds to a continuous set-valued function with (point) intervals as (convex) sets (that is, it is at least upper semicontinuous). Since integration preserves the continuity properties, the integral operator $A(\boldsymbol{x})$ is also upper semicontinuous and possesses a fixed point if the inclusion property holds.

5. System with friction and hysteresis

As an example of the applicability of the method described in Section 4, we consider the following mechanical system with friction and hysteresis (Rauh *et al.*, 2011):

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ \frac{1}{m} \left(F_a \left(t \right) - F_f \left(x_2 \right) \right) \end{pmatrix}, \quad (44)$$

where $x = (x_1 \ x_2)^{\mathrm{T}}$ describes the motion of a mass m subject to accelerating forces F_a and friction F_f. The sliding friction force is given by

$$F_f \left(x_2 \right) = \begin{cases} -F_s + \mu \cdot x_2 & \text{for} \quad S_1 = \texttt{true} \\ & \text{or} \quad S_2 = \texttt{true}, \\ +F_s + \mu \cdot x_2 & \text{for} \quad S_4 = \texttt{true} \\ & \text{or} \quad S_5 = \texttt{true}, \end{cases}$$

where $F_s \in \boldsymbol{F}_s$ is the static friction coefficient, with the static friction force

$$F_f \left(x_2 \right) \in F_s^{\max} := \left[-\overline{F}_s \,, \, \overline{F}_s \right] \text{ if } S_3 = \texttt{true}.$$

Here, the conditions S_i are defined as follows:

$$S_1 = \{x_2 < 0, \omega \geq 0\},$$
$$S_2 = \{x_2 < 0, \omega < 0\},$$
$$S_3 = \{x_2 = 0\},$$
$$S_4 = \{x_2 > 0, \omega \geq 0\},$$
$$S_5 = \{x_2 > 0, \omega < 0\} \,.$$

The accelerating force is specified as

$$F_a \left(t \right) := u \left(t \right) - \varphi \left(x_1 \left(t \right), \omega \left(t \right) \right),$$

where $u \left(t \right)$ is the control variable provided by an actuator, $\varphi \left(x_1 \left(t \right), \omega \left(t \right) \right) = \kappa_x x_1 + \kappa_\omega \omega$ is a restoring spring force, and $\omega(t)$ is determined through the Bouc–Wen model,

$$\dot{\omega} \left(t \right) = \rho \cdot \left(x_2 \left(t \right) - \sigma \cdot |x_2 \left(t \right)| \cdot |\omega \left(t \right)|^{\nu-1} \cdot \omega \left(t \right) \right.$$
$$\left. + \left(\sigma - 1 \right) \cdot x_2 \left(t \right) \cdot |\omega \left(t \right)|^{\nu} \right)$$

with time invariant parameters ρ, σ and ν. This hysteresis model can be used to describe velocity-dependent spring forces. It can be easily extended to represent hysteretic damping elements.

Rauh *et al.* (2011) solved this problem by transforming it into a series of state transitions (cf. the above-mentioned reference for details) and applying a Taylor series verified enclosure method described in Section 3.3 to the transition graph. In this application, the transition conditions T_i^j describe those state and control input dependent relations which lead to the activation of the discrete model state S_j from the currently active state S_i as shown in Fig. 1 from the paper by Rauh *et al.* (2011).

We simulated the same problem using VALENCIA-IVP in combination with a C++ class implementing the derivative definition from Section 4.1. In our case, we can write down the equations for the system as

$$\begin{cases} \dot{\boldsymbol{x}}_1 &= \boldsymbol{x}_2 \\ \dot{\boldsymbol{x}}_2 &= \frac{1}{\boldsymbol{m}}(u(\boldsymbol{x}_4) - \kappa_x \boldsymbol{x}_1 - \kappa_\omega \boldsymbol{x}_3 - \varphi_1(\boldsymbol{x}_2)), \\ \dot{\boldsymbol{x}}_3 &= \rho \cdot \big(\boldsymbol{x}_2 - \sigma \cdot \varphi_2(\boldsymbol{x}_2) \cdot \varphi_2(\boldsymbol{x}_3)^{\nu-1} \cdot \boldsymbol{x}_3 \\ &\quad + (\sigma - 1) \cdot \boldsymbol{x}_2 \cdot \varphi_2(\boldsymbol{x}_3)^\nu\big), \\ \dot{\boldsymbol{x}}_4 &= 1, \end{cases}$$

$$(45)$$

where φ_1 is the friction force (including static and sliding parts) and φ_2 is the absolute value function specified in terms of the definition in Eqn. (28) simply as

$$\varphi_1(x_2) = F_f(x_2)$$
$$= \begin{cases} -F_s + \mu \cdot x_2, & x_2 < 0, \\ F_s + \mu \cdot x_2, & x_2 > 0, \end{cases} \quad (46)$$

$$\varphi_2(x_i) = |x_i|$$
$$= \begin{cases} -x_i, & x_i < 0, \\ x_i, & x_i > 0 \end{cases} \quad (47)$$

for a new interval state vector $\boldsymbol{x} = (\boldsymbol{x}_1\ \boldsymbol{x}_2\ \boldsymbol{x}_3\ \boldsymbol{x}_4)^{\mathsf{T}}$, $x_i \in \boldsymbol{x}_i$, $i = 2, 3$ and $F_s \in \boldsymbol{F}_s$.

We considered the two sets of parameters shown in Table 2. The first set eventually activated all conditions S_i except S_3 whereas the second one caused activation of S_3, that is, the velocity turned to zero after some time. We compared the results with those obtained with the MAT-LAB simulation by Rauh *et al.* (2011). In Fig. 1, the velocity $\boldsymbol{x}_2$ of the mass is shown for the parameter sets one (top) and two (bottom). Verified results obtained with the approach proposed in this paper are represented by thick black dashed curves and those from the MATLAB simulation by thick solid gray ones. In the bottom figure, the results for the second simulation are shown in a similar way except that the additional thin black lines represent verified outcomes obtained with point-valued parameters. We included solutions for the lower and upper bounds of parameters, respectively, to give an idea about the true solution range (that need not necessarily be the correct enclosure of the solution).

As expected, the figures demonstrate that the results produced by the approach proposed in this paper are consistent with those by Rauh *et al.* (2011) in both the cases. In the first setting, the interval widths are the same for simulations with VALENCIA-IVP. The second, degenerate situation, where the velocity turns to zero after some time, that is, remains on the switching surface, is also properly reflected by our approach. The true solution set is contained in the resulting intervals. However, the approach overestimates the enclosures after the point of time where the velocity initially turns to zero since it possesses no special handling for such situations. The simulation in VALENCIA-IVP with the

Table 2. Parameter values for the two simulation scenarios considered.

Parameter	Value	
$\boldsymbol{x}_1(0)$	0 m	
$\boldsymbol{x}_3(0)$	−0.001	
κ_ω	0.001	
$\boldsymbol{m}$	[1.1,1.21] kg	
$\boldsymbol{F}_s$	[0.15,0.03] N	
σ	0.001	
ρ	0.001	
μ	0.001	
ν	1	
	Simulation 1	Simulation 2
$u(t)$	$2\sin(3t)$ N	0.010 N
$\boldsymbol{x}_2(0)$	0 m/s	−0.010 m/s
κ_x	0.001	0

step size 0.001 takes approximately 26 seconds CPU time on an Intel Xeon 2GHz multicore processor under Linux 2.6.23.14-115.fc8. The MATLAB[7] simulation needs 376 seconds on an Intel Core2 Duo E8400 3GHz computer under Windows 7 Professional. Another advantage of our approach is that we do not need to track down all possible transitions T_i^j manually. Equations (45)–(47) suffice for describing the system.

6. Conclusions and an outlook

We presented a simple approach for computing interval evaluations of non-smooth functions and their first derivatives. We implemented it as a template class and combined it with the solver VALENCIA-IVP for smooth problems extending the area of applicability of this solver. We tested the approach using the example of a physically motivated system with friction and hysteresis and compared the results to the automaton-based method by Rauh *et al.* (2011). The enclosures were consistent, and the computing times were lower. (Note that the comparison between MATLAB and C++ gives us only a reference.) The main advantage of the proposed implementation is, however, a simple way to introduce the goal system into the IVP solver. Another advantage (e.g., over slopes or more complex methods) is that the proposed approach needs less work per step. This might lead to better computing times, which can be exploited in situations where verified enclosures have to be obtained with fixed stepsizes in real-time. However, we need to perform a more in-depth comparison to show the improvement in speed, which is a topic for our future work.

The approach can find its application in many areas. One scenario is stance stabilization from the field of

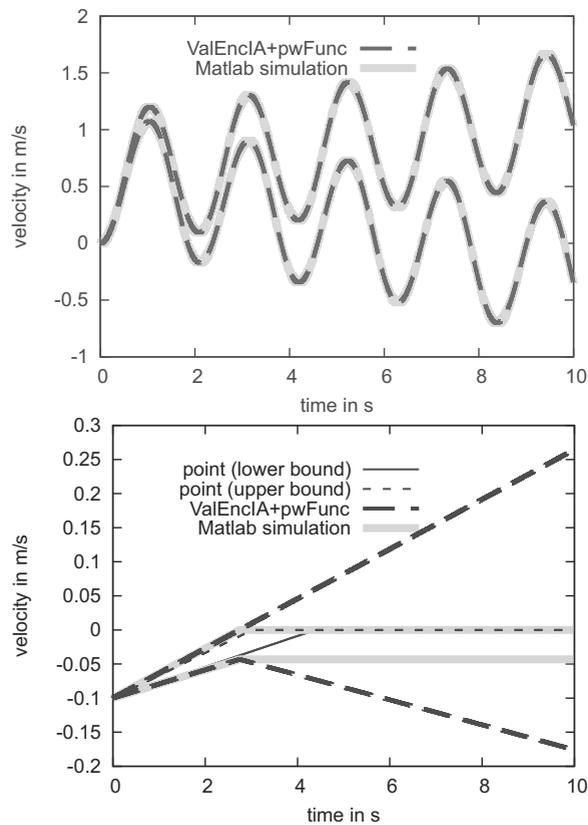---

[7]MATLAB R2011a (64 bit) with INTLAB V6.

Fig. 1. Velocity of the mass for the parameter sets one (top) and two (bottom) compared with the automaton-based method.

biomechanics. Generally, many models in biomechanics employ non-smooth functions, and, in this particular case, there is one for the reaction force of the impact between the foot and the ground and one for the rotation of the forefoot (for more information, see the work of Auer *et al.* (2011)). On the one hand, the data for this problem are affected by uncertainty, for example, because they are obtained by averaging different sets of features. On the other hand, the results need to be especially reliable for a surgeon to suggest a therapy for a patient. Therefore, the use of verified methods to handle uncertainty, at least partially, is of particular importance. An additional advantage is that of enhanced modeling. For example, the contact between a cylinder (foot) and a plane (ground) is not a point but a whole area for small angles between the corresponding normals. The center of this area is usually projected into a point, whereas verified methods would offer a possibility to work with the original contact set as an interval.

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