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DEGRADATION TOLERANT OPTIMAL CONTROL DESIGN FOR STOCHASTIC LINEAR SYSTEMS

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Safety-critical and mission-critical systems are often sensitive to functional degradation at the system or component level. Such degradation dynamics are often dependent on system usage (or control input), and may lead to significant losses and a potential system failure. Therefore, it becomes imperative to develop control designs that are able to ensure system stability and performance whilst mitigating the effects of incipient degradation by modulating the control input appropriately. In this context, this paper proposes a novel approach based on an optimal control theory framework wherein the degradation state of the system is considered in the augmented system model and estimated using sensor measurements. Further, it is incorporated within the optimal control paradigm leading to a control law that results in deceleration of the degradation rate at the cost of system performance whilst ensuring system stability. To that end, the speed of degradation and the state of the system in discrete time are considered to develop a linear quadratic tracker (LQT) and regulator (LQR) over a finite horizon in a mathematically rigorous manner. Simulation studies are performed to assess the proposed approach.

Keywords: linear quadratic Gaussian control, optimal control, Kalman filter, stochastic linear system, degradation.

1. Introduction

Traditional control system designs (Stengel, 1986; Åström and Wittenmark, 1995) focus only on the stability and performance without taking into consideration the effects of aging, fatigue, and damage of the concerned components and without minimizing the risk of failure. However, safety-critical systems (Knight, 2002) arise in several application areas, such as transportation and air-traffic control systems, space systems, nuclear plants and automated industrial processes. The evolution of such complex systems calls for development of new control technologies that maintain system stability and performance specifications, and also address the progressive incipient degradation.

In this context, recent works include approaches such as adaptive or robust control to address issues where the degree of failure may be unknown. In the work of Bole *et al*. (2010), a fault adaptive control is proposed for incipient fault modes growing into catastrophic failure conditions. The methodology is developed for a finite constrained optimization problem where the model of the system and the degradation is supposed to be known. Zhang *et al*. (2022) develop a reconfiguration control method using a multiple-model based adaptive control. The proposed control law allows handling component faults while maintaining the performance of the electro-hydraulic position servo system. Moreover, fault tolerant control design (Noura *et al*., 2009; Blanke *et al*., 2006) has been developed for various industrial, mission critical and safety critical systems that operate in closed loop, in order to compensate for fault occurrence. In the work of Hamdi *et al*. (2021), a fault tolerant control was introduced for delayed linear parameter varying systems including disturbances and actuator faults.

In recent years, new methods have been developed such that the useful life of critical systems can be enhanced. In this context, health aware control has emerged as a significant domain where control laws are designed while taking into consideration the state of health (SoH) and/or remaining useful life (RUL) prognostics of critical components. Some prominent works have proposed methods for developing control laws

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that attempt to extend the RUL of a component/system (e.g., Lipiec *et al*., 2022; Pour *et al*., 2021; Rodriguez *et al*., 2018; Salazar *et al*., 2017). Moreover, in the framework of model predictive control (MPC), several works have been adopted to design a controller that ensures robustness to particular failures, thus reducing their impact on the system (Brown *et al.*, 2010; 2021).

However, in the case where the states of a system are not measurable, it can be challenging to design effective controllers that can maintain good performance in the presence of these uncertainties. In such situations, various techniques can be employed to estimate the system states and the degradation state such as the Kalman filter (Durrant-Whyte, 2006), the extended Kalman filter (Kanso *et al*., 2022; Obando *et al*., 2021; Bressel *et al*., 2016), particle filtering (Jha *et al*., 2016), etc. In the framework of linear systems, Kalman filtering is often used for real-time control applications due to its low computational complexity and convergence guarantee. Combining the Kalman filter and linear quadratic control (LQC) yields a powerful control approach known as linear quadratic Gaussian (LQG) control (Lewis *et al*., 2012; Söderström, 2002). It is commonly applied to optimize the performance of linear systems in the presence of additive white Gaussian noise. It is widely used in a variety of applications to maintain good control performance in the presence of noise (Athans, 1971). The Kalman filter allows estimating the states of the system based on noisy measurements; then the estimated states are used to compute the optimal control input, ensuring optimal system behavior and performance.

In the context of incipient degradation, incorporating the degradation state into control design is a non-trivial task (Söderström, 2002). This challenge is further compounded when the degradation states are not directly measurable (Félix *et al*., 2022). Consequently, it can be difficult to accurately assess the extent of degradation and to design controllers that can adapt to changing degradation levels over time or decelerate the degradation speed.

Most of the existing work focuses on integrating fault tolerance within the control design without addressing the incipient functional degradation phenomena that lead to such faults and, consequently, a system failure. On the other hand, very few works have addressed the problem emanating due to degradation that is often not measurable and incipient in nature. In this context, this paper proposes a novel approach based on an optimal control theory framework wherein the degradation state of the system is considered in the augmented system model and estimated using sensor measurements. Further, it is incorporated within the optimal control paradigm leading to a control law that results in deceleration of the degradation rate at the cost of system performance whilst ensuring system stability. While the importance of MPC control and its practical benefits are recognized, our specific research focus centers on exploring the potential of the LQR and preparing the foundation for future research on reinforcement learning approaches in control design for nonlinear systems.

This work is an extension of the previous one (Kanso *et al*., 2023), wherein a linear quadratic regulator (LQR) and tracker (LQT) were designed for a deterministic discrete-time linear system in the presence of a linear degradation, and the full information about the state and the degradation was considered available.

This paper aims to extend the previous work and address the cases of incomplete states information, thereby addressing stochastic systems using LQG control. The main scientific contribution is the proposition of a novel degradation tolerant approach based on optimal control theory for a stochastic discrete-time linear system with partially measurable states and degradation.

This paper is organized as follows. Section 2 introduces the problem statement. Section 3 presents the proposed reconfiguration approach for deterministic systems. Section 4 develops LQG control for incomplete state information. Section 5 examines the feasibility of the proposed approach using an academic example. Also, it highlights the distinction between the formulation of this article and the approach of extending the state vector to include both the system state and the degradation state for the control design. Finally, the conclusions summarize significant advances and presents future perspectives.

2. Problem formulation

The degradation of a system's components affects its performance and stability. The state of degradation or deterioration, considered as a health indicator, also affects directly the remaining useful life of the active system, consequently reducing the usability and productivity of the system. Moreover, the SoH is predominantly influenced by the states of the system, and implicitly affected by the action of the controller. Therefore, the development of an optimal approach for performing a control action that takes into account the performance requirements, the stability and also the SoH of the system gains paramount importance for such systems undergoing component degradation.

This paper focuses on linear MIMO (multiple inputs multiple outputs) discrete-time systems represented by the state transition, control and observation matrices, $A_1 \in \mathbb{R}^{n\times n}$, $A_2 \in \mathbb{R}^{n\times l}$, $B_1 \in \mathbb{R}^{n\times m}$ and $C_1 \in \mathbb{R}^{p\times n}$,

$$x_{k+1} = A_1 x_k + A_2 d_k + B_1 u_k, \tag{1}$$

$$y_k = C_1 x_k, \tag{2}$$

where $u \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, $d \in \mathbb{R}^l$ and $y \in \mathbb{R}^p$ correspond respectively to the input, state of the system, state of degradation and measurement vectors. The system is affected by the degradation in an affine manner and the degradation evolution is described by the following state-space representation:

$$d_{k+1} = A_3 x_k + A_4 d_k \tag{3}$$

with $A_3 \in \mathbb{R}^{l\times n}$ and $A_4 \in \mathbb{R}^{l\times l}$. In most cases, the evolution of the degradation is monotonic and irreversible; moreover, it is generally unknown. In this work, the current state of degradation is assumed to be dependent on the previous state of degradation and also the previous state of the system. Moreover, it is assumed that the degradation variable d_k has a maximum value $d_{\max}$, such that, if the degradation level at any time k is less than $d_{\max}$, the system remains stable and can be asymptotically stabilized on $\mathbb{R}^n$.

In order to maintain the system performance while minimizing the energy and the speed of evolution of degradation, a quadratic utility function is defined by

$$\begin{aligned}\mathscr{U}_k = {} & (C_1 x_k - r_k)^T Q (C_1 x_k - r_k) + u_k^T R u_k \\ & + \Delta d_k^T Q_1 \Delta d_k,\end{aligned} \tag{4}$$

where r_k is the desired reference trajectory and Δd_k is the rate of evolution of degradation described by

$$\Delta d_k = d_{k+1} - d_k = (A_4 - I)d_k + A_3 x_k. \tag{5}$$

The utility function (4) is used to develop the performance index of a linear quadratic tracker problem, which gives the following quadratic cost function:

$$\begin{aligned}J_0 = {} & \frac{1}{2}[(C_1 x_N - r_N)^T \bar{S}_N (C_1 x_N - r_N) \\ & + \Delta d_N^T \bar{P}_N \Delta d_N] \\ & + \frac{1}{2}\sum_{k=0}^{N-1}[(C_1 x_k - r_k)^T Q (C_1 x_k - r_k) \\ & + u_k^T R u_k + \Delta d_k^T Q_1 \Delta d_k],\end{aligned} \tag{6}$$

where Q, Q_1, R, $\bar{S}_N$ and $\bar{P}_N$ are symmetric positive definite cost-weighting matrices and $|R| \neq 0$. The initial plant and degradation state are given as x_0 and d_0, respectively.

In the following section, the control problem will be addressed for the case of deterministic systems while minimizing the rate of evolution of degradation.

3. Optimal reconfiguration control of deterministic systems

In this section, the developed optimal control based approach allows the synthesis of a state feedback control law using the minimization of a quadratic criterion involving the state, control and rate of evolution of degradation. The problem posed is to bring the state to any reference track, which is equivalent to bringing the state to equilibrium (zero) starting from a non-zero initial condition. Hence, in Section 3.1, the solution of the problem is developed for an LQT, and then the solution is deduced for a LQR problem. The constructed controller is for deterministic systems with fully measurable states.

3.1. Linear quadratic tracker. This section synthesizes an optimal control law that forces the system to track a desired reference trajectory r_k over a specified time interval $[0, N]$. The cost function (6) is sensitive to the tracking error, the input and Δd to force the state to reach the reference and to decelerate the speed of evolution of degradation. Using (5) to eliminate Δd in (6) gives

$$\begin{aligned}J_0 = {} & \frac{1}{2}[x_N^T (C^T \bar{S}_N C + A_3^T \bar{P}_N A_3) x_N \\ & + d_N^T (A_4 - I)^T \bar{P}_N (A_4 - I) d_N + r_N^T \bar{S}_N r_N \\ & - x_N^T C^T \bar{S}_N r_N - r_N^T \bar{S}_N C x_N \\ & + d_N^T (A_4 - I)^T \bar{P}_N A_3 x_N + x_N^T A_3^T \bar{P}_N (A_4 - I) d_N] \\ & + \frac{1}{2}\sum_{k=0}^{N-1}[x_k^T (C^T Q C + A_3^T Q_1 A_3) x_k + u_k^T R u_k \\ & + d_k^T (A4 - I)^T Q_1 (A_4 - I) d_k + r_k^T Q r_k \\ & - x_k^T C^T Q r_k - r_k^T Q C x_k + d_k^T (A_4 - I)^T Q_1 A_3 x_k \\ & + x_k^T A_3^T Q_1 (A_4 - I) d_k].\end{aligned} \tag{7}$$

To solve the LQT problem, the Hamiltonian is first considered in order to derive the necessary conditions. The Hamiltonian function is defined by the following equation:

$$\begin{aligned}H_k = {} & \frac{1}{2}[x_k^T (C_1^T Q C_1 + A_3^T Q_1 A_3) x_k + u_k^T R u_k \\ & + d_k^T (A_4 - I)^T Q_1 (A_4 - I) d_k + r_k^T Q r_k \\ & - x_k^T C_1^T Q r_k - r_k^T Q C_1 x_k \\ & + d_k^T (A_4 - I)^T Q_1 A_3 x_k + x_k^T A_3^T Q_1 (A_4 - I) d_k] \\ & + \lambda_{k+1}[A_1 x_k + A_2 d_k + B_1 u_k],\end{aligned} \tag{8}$$

where $\lambda_k \in \mathbb{R}^n$ is the costate of the system and is given by

$$\begin{aligned}\lambda_k & = \frac{\partial H_k}{\partial x_k} \\ & = (C_1^T Q C_1 + A_3^T Q_1 A_3) x_k + A_1^T \lambda_{k+1} \\ & \quad + A_3^T Q_1 (A_4 - I) d_k - C_1^T Q r_k.\end{aligned} \tag{9}$$

Solving the stationarity condition $\frac{\partial H_k}{\partial u_k} = 0$ yields

$$u_k = -R^{-1} B_1^T \lambda_{k+1}. \tag{10}$$

If the optimal λ_k can be found, (10) can be used to find the optimal control. Moreover, the boundary condition is given by

$$\begin{aligned}\lambda_N &= \frac{\partial \Phi_N}{\partial x_N} \\ &= (C_1^T \bar{S}_N C_1 + A_3^T \bar{P}_N A_3) x_N \\ &\quad + A_3^T \bar{P}_N (A_4 - I) d_N - C_1^T \bar{S}_N r_N\end{aligned} \tag{11}$$

with

$$\begin{aligned}\Phi_N = \frac{1}{2}[& x_N^T (C_1^T \bar{S}_N C_1 + A_3^T \bar{P}_N A_3) x_N \\ &+ d_N^T (A_4 - I)^T \bar{P}_N (A_4 - I) d_N + r_N^T \bar{S}_N r_N \\ &- x_N^T C_1^T \bar{S}_N r_N - r_N^T \bar{S}_N C_1 x_N \\ &+ d_N^T (A_4 - I)^T \bar{P}_N A_3 x_N \\ &+ x_N^T A_3^T \bar{P}_N (A_4 - I) d_N].\end{aligned} \tag{12}$$

Thus, assuming that a linear relation like (11) holds for all times $k \leq N$, the costate equation can be written as

$$\lambda_k = S_k x_k + P_k d_k - q_k. \tag{13}$$

Using (13) in the state equation (1), we get

$$\begin{aligned}x_{k+1} = {} & (I + B_1 R^{-1} B1^T S_{k+1})^{-1} \\ & \times [(A_1 - B_1 R^{-1} B_1^T P_{k+1} A_3) x_k \\ & + (A_2 - B_1 R^{-1} B_1^T P_{k+1} A_4) d_k \\ & + B_1 R^{-1} B_1^T q_{k+1}].\end{aligned} \tag{14}$$

Using (14) and (13) in the costate equation (9) gives

$$\begin{aligned}& S_k x_k + P_k d_k - q_k \\ &= [C_1^T Q C_1 + A_3^T Q_1 A_3] x_k \\ &\quad + A_1^T S_{k+1} [I + B_1 R^{-1} B_1^T S_{k+1}]^{-1} \\ &\quad \times [A_1 - B_1 R^{-1} B_1^T P_{k+1} A_3] x_k \\ &\quad + A_1^T S_{k+1} [I + B_1 R^{-1} B_1^T S_{k+1}]^{-1} \\ &\quad \times [A_2 - B_1 R^{-1} B_1^T P_{k+1} A_4] d_k \\ &\quad + A_1^T S_{k+1} [I + B_1 R^{-1} B_1^T S_{k+1}]^{-1} B_1 R^{-1} B_1^T q_{k+1} \\ &\quad + A_1^T P_{k+1} A_4 d_k + A_1^T P_{k+1} A_3 x_k \\ &\quad + A_3^T Q_1 (A_4 - I) d_k - A_1 q_{k+1} - C_1^T Q r_k.\end{aligned} \tag{15}$$

This equation must hold for all state sequences x_k and d_k given any x_0 and d_0, leading to

$$\begin{aligned}S_k = {} & C_1^T Q C_1 + A_3^T Q_1 A_3 \\ & + A_1^T S_{k+1} (I + B_1 R^{-1} B_1^T S_{k+1})^{-1} \\ & \times (A_1 - B_1 R^{-1} B_1^T P_{k+1} A_3) + A_1^T P_{k+1} A_3,\end{aligned} \tag{16}$$

$$\begin{aligned}P_k = {} & A_1^T S_{k+1} (I + B_1 R^{-1} B_1^T S_{k+1})^{-1} \\ & \times (A_2 - B_1 R^{-1} B_1^T P_{k+1} A_4) \\ & + A_1^T P_{k+1} A_4 + A_3^T Q_1 (A_4 - I),\end{aligned} \tag{17}$$

$$\begin{aligned}q_k = {} & A_1^T q_{k+1} + C_1^T Q r_k - A_1^T S_{k+1} \\ & \times (I + B_1 R^{-1} B_1^T S_{k+1})^{-1} B_1 R^{-1} B_1^T q_{k+1}.\end{aligned} \tag{18}$$

By comparing (11) and (13), the boundary conditions for these recursions are

$$\begin{aligned}S_N &= C_1^T \bar{S}_N C_1 + A_3^T \bar{P}_N A_3, \\ P_N &= A_3^T \bar{P}_N (A_4 - I), \\ v_N &= -C_1^T \bar{S}_N r_N.\end{aligned} \tag{19}$$

Since the sequences S_k, P_k and q_k can be computed, the assumption (13) is valid and the optimal control is

$$u_k = -R^{-1} B_1^T (S_{k+1} x_{k+1} + P_{k+1} d_{k+1} - q_{k+1}). \tag{20}$$

Substituting (1) and (3) in (20) yields

$$u_k = -K_k^x x_k - K_k^d d_k + K_k^q q_{k+1} \tag{21}$$

with

$$K_k^x = (R + B_1^T S_{k+1} B_1)^{-1} B^T (S_{k+1} A_1 + P_{k+1} A_3), \tag{22}$$

$$K_k^d = (R + B_1^T S_{k+1} B_1)^{-1} B^T (S_{k+1} A_2 + P_{k+1} A_4), \tag{23}$$

$$K_k^q = (R + B_1^T S_{k+1} B_1)^{-1} B^T. \tag{24}$$

Equations (22), (23) and (24) are solved off-line and backwards in time, starting from time $N \to 0$.

The solution for the LQR is reached by determining the control sequence $u_0, u_1, \dots, u_{N-1}$ that minimizes J_0 in

$$\begin{aligned}J_0 = {} & \frac{1}{2}(x_N^T \bar{S}_N x_N + \Delta d_N^T \bar{P}_N \Delta d_N) \\ & + \frac{1}{2} \sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k + \Delta d_k^T Q_1 \Delta d_k.\end{aligned} \tag{25}$$

However, the regulation problem is nothing but a tracking problem where the reference is the equilibrium (zero). Thus, in this case $q_N = 0$, which implies that the optimal control takes the form

$$u_k = -K_k^x x_k - K_k^d d_k, \tag{26}$$

where K_k^x and K_k^d are computed using Eqns. (22) and (23), respectively.

The controllers in (21) and (26) are full state and degradation feedback, so they require complete information on the system and degradation states. In many real-world systems, it is not possible to measure all of the states directly, especially the degradation state. This can be due to a variety of factors, such

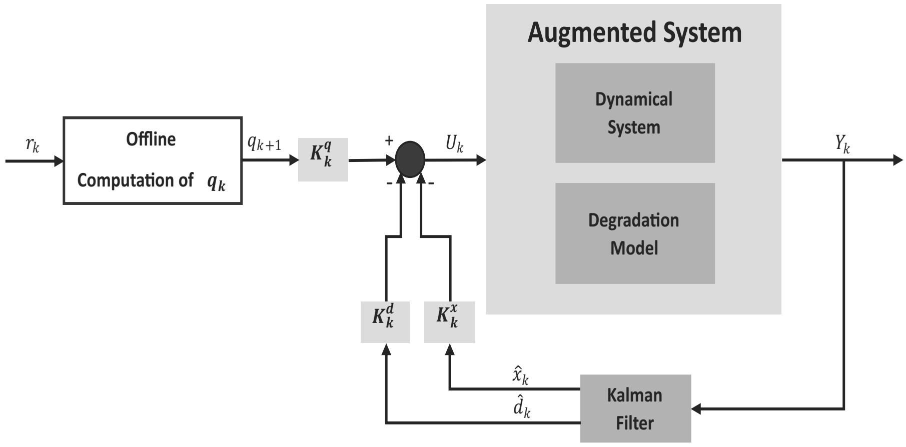


Fig. 1. Regulator design using state feedback and the Kalman filter as the observer.

as the system complexity, the cost or difficulty of obtaining measurements, or the inherent limitations of the measurement devices. As a result, it is often necessary to estimate the system states based on partial or noisy measurements, using techniques such as Kalman filtering. In the following section, the control problem of stochastic linear systems with incomplete state information will be addressed.

4. Optimal control with incomplete state information

In the previous section, the system was assumed to be exactly known, with no modeling inaccuracies, disturbances, or noises. In control design, often not all states are available for feedback purposes, only measurements are accessible. This can be due to various factors, such as the cost or complexity of measurements and the limitations of available sensors. In this section, incomplete state information is assumed to be available and the measurements are considered noisy. To solve this problem, a Kalman filter observer will be used to estimate the state and the degradation from noisy measurements. LQT control combined with the Kalman filter constitute together the linear quadratic Gaussian (LQG) control. It provides a powerful method for controlling linear systems in the presence of noise.

Suppose we have the following systems described by the stochastic dynamical equations:

$$\begin{aligned} x_{k+1} &= A_1 x_k + A_2 d_k + B_1 u_k + w_{1,k}, \\ d_{k+1} &= A_3 x_k + A_4 d_k + w_{2,k}, \\ y_k &= C_1 x_k + v_k. \end{aligned} \tag{27}$$

The signals $w_{1,k}$ and $w_{2,k}$ are unknown process noise that acts to disturb respectively the dynamical system and the degradation, and it could represent unmodeled high-frequency plant dynamics or the effects of wind gusts, for instance. The signal v_k is unknown measurement noise that impairs the measurements and it represents sensor noise. The signals $w_{1,k}$, $w_{2,k}$ and v_k are uncorrelated.

Consider the following augmented system composed of the dynamical system and the degradation with $X_k = \begin{bmatrix} x_k & d_k \end{bmatrix}^T \in \mathbb{R}^{n+l}$ and $U_k = u_k$:

$$\begin{aligned} X_{k+1} &= AX_k + BU_k + w_k, \\ Y_k &= CX_k + v_k, \end{aligned} \tag{28}$$

where

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \quad B = \begin{bmatrix} B1 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} C_1 & 0 \end{bmatrix}, \quad w_k = \begin{bmatrix} w_{1,k} \\ w_{2,k} \end{bmatrix}.$$

Here $w_k \sim (0, Q_{\text{obs}})$, $v_k \sim (0, R_{\text{obs}})$ are white noise processes orthogonal to each other.

Suppose we have the full state-feedback control

$$\begin{aligned} u_k &= -K_k^x x_k - K_k^d d_k + K_k^q q_{k+1} \\ &= -K_k^X X_k + K_k^q q_{k+1} \end{aligned} \tag{29}$$

with $K_k^X = \begin{bmatrix} K_k^x & K_k^d \end{bmatrix}$. The same feedback vector is used as when the system was deterministic and the states were known (Stengel, 1986).

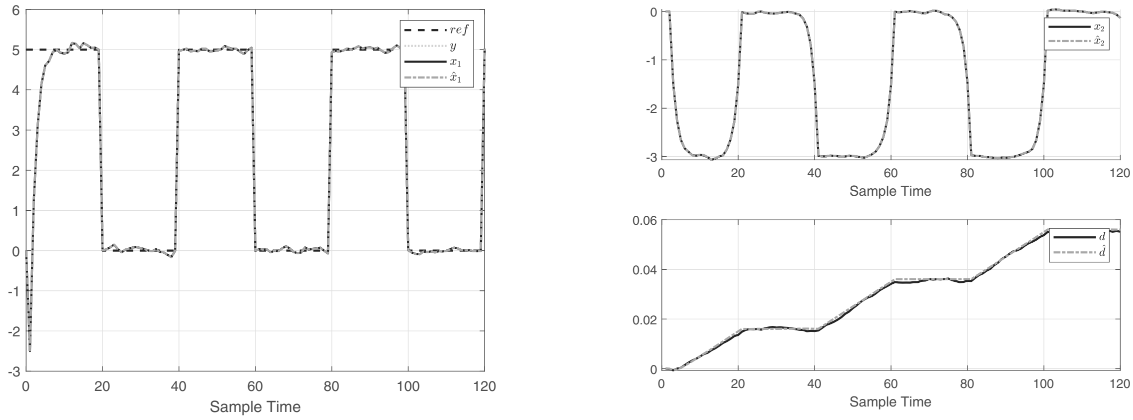
The closed-loop system becomes

$$X_{k+1} = (A - BK_k^X)X_k + BK_k^q q_{k+1} + w_k. \tag{30}$$

The control law (29) cannot be implemented since not all the states are usually measurable. Now, consider a Kalman filter designed as

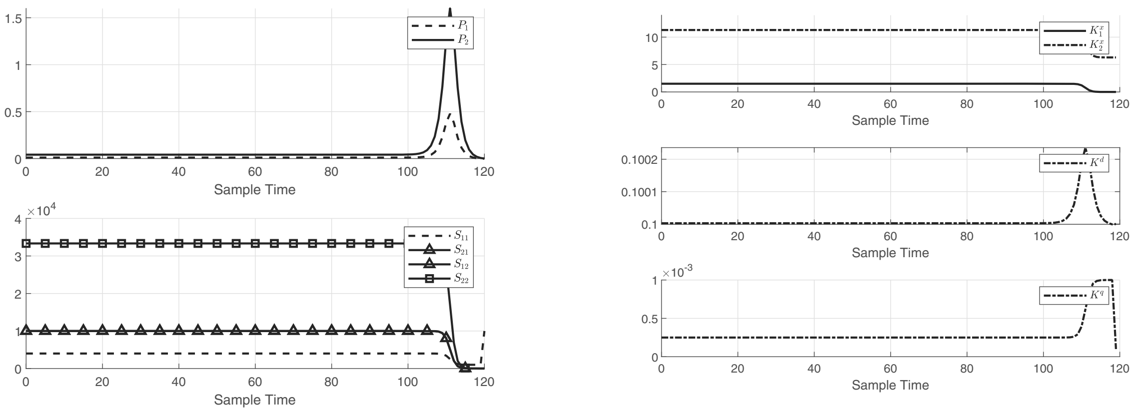
$$\hat{X}_{k+1} = (A - L_{k+1}C)\hat{X}_k + BU_k + L_{k+1}Y_k, \tag{31}$$

where the filter gain L_{k+1} is obtained using the Kalman filter algorithm (Durrant-Whyte, 2006).



(a) (b)

Fig. 2. Trajectory of estimated and real states and degradation in closed loop: $x_{1,k}$, $\hat{x}_{1,k}$ and the measurement y_k (a), the second state $x_{2,k}$, $\hat{x}_{2,k}$ and the degradation d_k and $\hat{d}_k$ (b).



(a) (b)

Fig. 3. Convergence of S_k, P_k , K_k^x, K_k^d and K_k^q: evolution of matrices P_k and S_k with respect to time (a), evolution of the controller gains K_k^x, K_k^d and K_k^q with respect to time (b).

The feedback of the estimated states $\hat{X}_k$ is used instead of the actual state X_k. Hence, the feedback control law becomes

$$u_k = -K_k^X \hat{X}_k + K_k^q q_{k+1}. \tag{32}$$

The closed-loop structure using this controller is illustrated in Fig. 1.

The state feedback gains and the observer gain can be developed separately to obtain the desired observer behavior and closed-loop plant behavior. This leads to the separation theorem (Lewis *et al*., 2012), which is the core of modern control design. To verify the effectiveness of the developed control schemes, a finite horizon tracker is implemented on an academic example in the next section.

5. Simulation results

Consider the following unstable stochastic discrete-time linear system:

$$x_{k+1} = \begin{bmatrix} 0 & 6.3 \\ 0.6 & 2 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_k + 0.1 d_k + w_{1,k},$$
$$y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k + v_k.$$

The dynamic of the evolution of degradation is described by the following equation:

$$d_{k+1} = \begin{bmatrix} 2 \times 10^{-3} & 0 \end{bmatrix} x_k + d_k + w_{2,k}.$$

The weighting matrices of the tracker are chosen as

$$Q = \begin{bmatrix} 10^3 & 0 \\ 0 & 10^3 \end{bmatrix}, \quad Q_1 = 10^3, \quad R = 0.01,$$
$$\bar{S}_N = Q, \quad \bar{P}_N = Q_1.$$

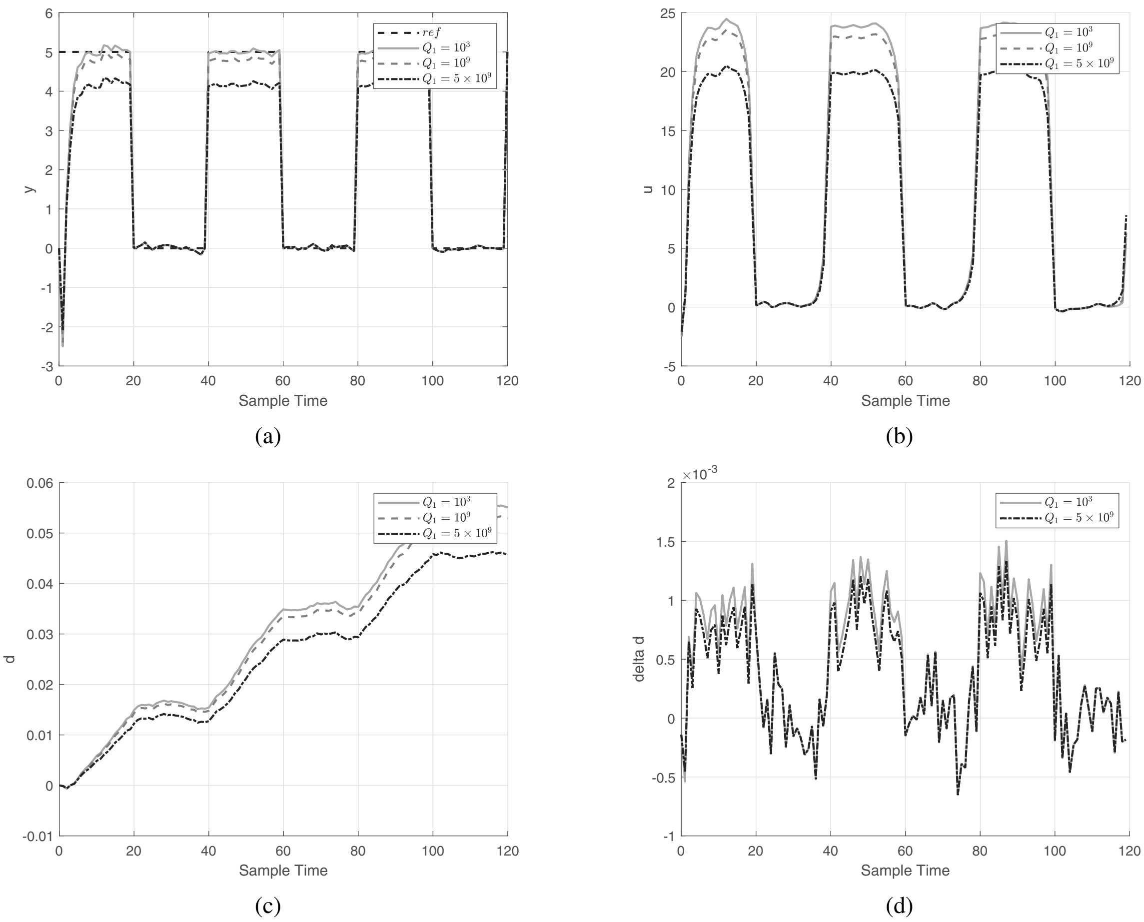


Fig. 4. System performance and degradation for different values of Q_1: system output (a), control input of the system (b), the degradation (c) and the rate of the evolution of the degradation (d).

The augmented system is given by

$$X_{k+1} = \begin{bmatrix} 0 & 6.3 & 0.1 \\ 0.6 & 2 & 0 \\ 2 \times 10^3 & 0 & 1 \end{bmatrix} X_k + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} U_k + w_k.$$

$$Y_k = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} X_k + v_k$$

The first step in the Kalman filter algorithm is to initialize the states and to adjust the covariance matrices to make the filter work properly. The noises w_k and v_k are assumed to be Gaussian with zero mean and variances Q_{obs} and R_{obs}, respectively, with

$$Q_{\text{obs}} = \begin{bmatrix} \sigma_{w_1}^2 & 0 & 0 \\ 0 & \sigma_{w_1}^2 & 0 \\ 0 & 0 & \sigma_{w_2}^2 \end{bmatrix}, \quad R_{\text{obs}} = \sigma_v^2,$$

where $\sigma_{w_1} = 10^{-3}$, $\sigma_{w_2} = 10^{-3}$ and $\sigma_v = 10^{-2}$.

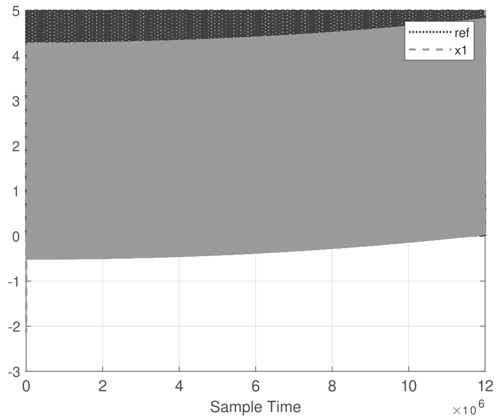
Figure 2(a) shows the trajectory of $x_{1,k}$, $\hat{x}_{1,k}$ and the measurements y_k for $N = 120$. A strong correlation is observed between the three curves, which implies that the Kalman filter is able to estimate x_1 since $\hat{x}_1$ and the measured values are overlapped. Moreover, the output y_k tracks the reference r_k. This indicates that the controller is able to force the output y_k to reach the desired trajectory and to stabilize the system while using the estimated states' feedback.

Consistent results are obtained in Fig. 2(b) since the trajectories of $x_{2,k}$ and $\hat{x}_{2,k}$ coincide. The second graph in Fig. 2(b) shows the evolution of the estimated and the real value of the degradation in closed loop; the two curves are correlated with a small error between d_k and $\hat{d}_k$.

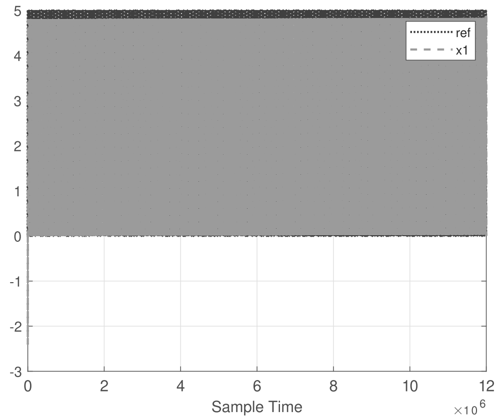
Figure 3(a) displays the evolution of the matrices P_k and S_k, which form respectively the solutions of Eqns. (17) and (16). The values are computed offline backward in time from N to 0. It can be seen from these two figures that the matrices' parameters converge respectively to P_0 and S_0, for any S_N and P_N, as k approaches 0.

Similar performance is obtained in Fig. 3(b), respectively, which is reasonable, as K_k^x,K_k^d and K_k^q are computed with respect to P_{k+1} and S_{k+1} (22)–(24). As $N \to \infty$, P_k and S_k converge to P_∞ and S_∞, respectively, which implies that K_k^x, K_k^d and K_k^q reach steady-state values K_∞^x, K_∞^d and K_∞^q.

Thus, in this case, the optimal control can be written



(a)



(b)

Fig. 5. Trajectory of $x_{1,k}$ in closed loop for $Q_1 = 10^9$: $x_{1,k}$ for the formulation of the augmented state vector (a), $x_{1,k}$ for the formulation developed in the article.

as follows:

$$u_k = -K^x_\infty x_k - K^d_\infty d_k + K^q_\infty q_{k+1}.$$

A disadvantage of this formulation is that q_k needs to be computed offline using the backward recursion (18).

The main objective of the developed work is to decelerate the speed of evolution of degradation Δd_k. Looking at Eqn. (6), it can be seen that the weighting matrix Q_1 strongly affects the progression of the degradation. Thus, the impact of Q_1 on the system behaviour will be studied in the following results.

For different values of Q_1 =[10^3, 10^9, 5×10^9] and for a finite horizon $N = 120$, the trajectory of the output y_k is represented in Fig. 4(a). It can be seen that, by increasing the value of Q_1, the steady state error between the output and the reference increases; thus, the input u_k reduces in its turn, as shown in Fig. 4(b). Moreover, Fig. 4(c) displays the evolution of the degradation for different values of Q_1. It shows that the final value of the degradation d_N decreases when Q_1 increases. This means that the controller tries to find a trade-off between the performance and the speed of degradation. Thus by augmenting the value of Q_1, the controller will prioritize reducing the rate of evolution of degradation over the system performance.

Figure 4(d) shows results consistent with the previous ones, as it displays the rate of evolution of degradation for different Q_1 and confirms that the speed of degradation is slower when Q_1 is large.

The choice of the formulation presented in this article was carefully considered based on several factors and issues. One of the primary reasons for selecting this specific formulation was to address the problem of decelerating the speed of degradation. One intuitive and simplistic formulation that can be employed was the augmentation of the system state vector with d_k and Δd_k. Using such a formulation leads to an undesired behavior in the trajectory of x_1, where it exhibits negative values for a large horizon (Fig. 5(a)). Conversely, the approved formulation, which is the focus of this article, demonstrated a more desirable behavior in the trajectory of x_1, without exhibiting such unconventional values (Fig. 5(b)). This observation reinforces the effectiveness of the chosen formulation in controlling the system dynamics and achieving the desired objectives.

6. Conclusions and perspectives

This paper proposed a degradation tolerant control (DTC) design based on the LQG approach, where the degradation is hidden. A finite horizon optimization approach was developed for linear systems, where these were supposed to be affected by a linear degradation in an affine manner. The degradation and state of the system were considered not fully measurable. Thus, the Kalman filter was employed to estimate the system states and the degradation state in order to implement the state feedback controller. This allows fast estimation with a negligible residual using noisy measurements. Moreover, using the output of the observer, the LQT was able to force the measured state to follow the reference and to stabilise the system. In order to slow down the speed of degradation, the value of the matrix Q_1 can be increased so that the controller prioritizes reducing the rate of evolution of degradation over the system performance, thus preventing a system breakdown.

The convergence of the gains K^x_k, K^d_k and K^q_k, as well as the matrices P_{k+1} and S_{k+1}, needs to be investigated. Therefore, future work will focus on proving the convergence of the controller gains. Furthermore, the integration of the remaining useful life in the cost function to extend its value will be examined. However, it is also important to extend the framework to address more complex and real-world cases. This involves considering system dynamics with nonlinearities and handling scenarios that involve higher-dimensional and realistic environments. By addressing these challenges, the future work aims to provide practical and effective solutions that can be applied to real systems. To this end, the future work will also involve developing reinforcement learning algorithms and techniques that are specifically designed to handle the challenges posed by nonlinear systems.

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