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CONVOLUTION TYPE STOCHASTIC VOLTERRA EQUATIONS

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PREFACE

This volume is the habilitation dissertation of the author written at the Faculty of Mathematics, Computer Science and Econometrics of the University of Zielona Góra.

The aim of this work is to present, in self-contained form, results concerning fundamental and the most important questions related to linear stochastic Volterra equations of convolution type. The paper is devoted to study the existence and some kind of regularity of solutions to stochastic Volterra equations in Hilbert space and the space of tempered distributions, as well.

In recent years the theory of Volterra equations, particularly fractional ones, has undergone a big development. This is an emerging area of research with interesting mathematical questions and various important applications. The increasing interest in these equations comes from their applications to problems from physics and engeenering, particularly from viscoelasticity, heat conduction in materials with memory or electrodynamics with memory.

The paper is divided into four chapters. The first two of them have an introductory character and provide deterministic and stochastic tools needed to study existence of solutions to the equations considered and their regularity. Chapter 1 is devoted to stochastic Volterra equations in a separable Hilbert space. In Chapter 4 stochastic linear evolution equations in the space of distributions are studied.

The work is based on some earlier papers of the author, however part of results is not yet published.

I wish to express my sincere gratitude to prof. Jerzy Zabczyk for introducing me into the world of stochastic Volterra equations and inspiring mathematical discussions. I am particularly grateful to prof. Carlos Lizama for fruitful joint research, exclusively through internet connection, which results is a big part of this monograph.

Zielona Góra, June 2007 *Anna Karczewska*

CONTENTS

INTRODUCTION

The main results in brief

In the paper, two general problems concerning linear stochastic evolution equations of convolution type are studied: existence of strong solutions to such stochastic Volterra equations in a Hilbert space and regularity of solutions to two classes of stochastic Volterra equations in spaces of distributions.

First, we consider Volterra equations in a separable Hilbert space H of the form

$$X(t) = X(0) + \int_0^t a(t-\tau)\, A\, X(\tau)\, d\tau + \int_0^t \Psi(\tau)\, dW(\tau), \quad t \geq 0, \tag{0.1}$$

where $X(0) \in H$, $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ is a scalar kernel function and A is a closed linear unbounded operator with the dense domain $D(A)$ equipped with the graph norm. In (0.1), $\Psi(t)$, $t \geq 0$ is an appropriate stochastic process and $W(t)$, $t \geq 0$ is a cylindrical Wiener process; both processes are defined on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$.

Equation (0.1) arises, in the deterministic case, in a variety of applications as model problems, see e.g. [70], [5] and references therein. Well-known techniques like localization, perturbation and coordinate transformation allow to transfer results for such problems to integro-differential equations. In these applications, the operator A typically is a differential operator acting in spatial variables, like the Laplacian, the Stokes operator, or the elasticity operator. The kernel function $a(t)$ should be thought as a kernel like $a(t) = e^{-\eta t} t^{\beta-1}/\Gamma(\beta)$; $\eta \geq 0$, $\beta \in (0, 2)$. The stochastic approach to integral equations has been recently used due to the fact that the level of accuracy for a given model not always seems to be significantly changed with increasing model complexity.

Our main results concerning (0.1), rely essentially on techniques using a strongly continuous family of operators $S(t)$, $t \geq 0$, defined on the space H and called the *resolvent*. Hence, in what follows, we assume that the deterministic version of equation (0.1) is *well-posed*, that is, admits a resolvent $S(t)$, $t \geq 0$.

The stochastic Volterra equations of the form (0.1) have been treated by many authors, see e.g. [14]–[17] or [72], [73]. In the first three papers stochastic Volterra equations are studied in connection with viscoelasticity and heat conduction in materials with memory. The paper due to Clément and Da Prato [14] is particularly significant because the authors have extended the well-known semigroup approach, applied to stochastic differential equations, to a subclass of the equation (0.1). In the next papers, weak and mild solutions to the equation (0.1) have been studied and some results like regularity of solutions or large deviations of equations have been given. The resolvent approach to stochastic Volterra equations, introduced in [14], enables us to obtain new results in an elegant way, analogously like in semigroup case. In resolvent case, new difficulties appear because the family $S(t)$, $t \geq 0$, in general do not create a semigroup.

Our main results concerning the equations (0.1) in the space H are the existence theorems of strong solutions to some classes of such equations. We provide existence of strong solutions to (0.1) under different conditions on the kernel function. In some cases, we arrive at stochastic versions of fractional Volterra equations with corresponding α-times resolvent families $S_\alpha(t)$, $t \geq 0$. The key role in our proofs is played by convergence of resolvent or α-times resolvent families corresponding to deterministic versions of Volterra equations. These convergence theorems are resolvent analogies of the well-known Hille–Yosida theorem in semigroup case. Our convergence results generalize theorems due to Clément and Nohel [18] obtained for contraction semigroups. Having such resolvent analogies of the Hille–Yosida theorem, we proved that the stochastic convolutions arising in Volterra equations (0.1) are strong solutions to (0.1).

In the remaining part of the paper we study two classes of equations of convolution type: the equation

$$X(t,\theta) = X_0(\theta) + \int_0^t b(t-\tau) A X(\tau,\theta)\, d\tau + W_\Gamma(t,\theta), \quad t \geq 0,\ \theta \in \mathbb{R}^d, \tag{0.2}$$

and the integro-differential stochastic equation with infinite delay

$$X(t,\theta) = \int_{-\infty}^t b(t-s)[\Delta X(s,\theta) + \dot{W}_\Gamma(s,\theta)] ds, \quad t \geq 0,\ \theta \in T^d, \tag{0.3}$$

where T^d is a d-dimensional torus. The kernel function b is integrable on $\mathbb{R}_+$ and the class of operators A contains the Laplace operator and its fractional powers. In both equations (0.2) and (0.3), W_Γ denotes a spatially homogeneous Wiener process, which takes values in the space of tempered distributions $S'(\mathbb{R}^d)$, $d \geq 1$.

Equations (0.2) and (0.3) are generalizations of stochastic heat and wave equations studied by many authors. Particularly, regularity problems of these equations have attracted many authors. For an exhaustive bibliography we refer to [57].

In the paper, we consider existence of the solutions to (0.2) and (0.3) in the space $S'(\mathbb{R}^d)$ and next we derive conditions under which the solutions to (0.2) and (0.3) take values in function spaces.

In the case of equation (0.2), the results have been obtained by using the resolvent operators corresponding to Volterra equations. The regularity results have been expressed in terms of the spectral measure μ and the covariance kernel Γ of the Wiener process W_Γ. Moreover, we give necessary and sufficient conditions for the existence of a limit measure to the equation (0.2).

In the case of the equation (0.3), we study a particular case of weak solutions under the basis of an explicit representation of the solution to (0.3). The regularity results have been expressed in terms of the Fourier coefficients of the space covariance Γ of the process W_Γ.

A guided tour through the paper

Chapter 1 has a preliminary character. Its goal is to introduce the reader to the theory of deterministic Volterra equations in Banach space and to provide facts used in the paper. Section 1.1 gives notations used in the paper and Section 1.2 gives basic definitions connected with resolvent operators. Sections 1.3 and 1.4 contain definitions and facts concerning kernel functions, particularly regular ones for parabolic Volterra equations. Some ideas are illustrated by examples. Section 1.5 provides new results due to Karczewska and Lizama [51], that is, the approximation theorems, Theorems 1.19 and 1.20, not yet published. These results are resolvent analogies of the Hille–Yosida theorem in semigroup case and play the same role like the Hille–Yosida theorem does. These results concerning convergence of resolvents for the deterministic version of the equation (0.1) in Banach space play the key role for existence theorems for strong solutions and they are used in Chapter 3.

Chapter 2 contains concepts and results from the infinite dimensional stochastic analysis recalled from well-known monographs [19], [23] and [38]. Among others, we recall an infinite dimensional versions of the Fubini theorem and the Itô formula. Additionally, we recall a construction, published in [46], of stochastic integral with respect to cylindrical Wiener process. The construction bases on the Ichikawa idea for the stochastic integral with respect to classical infinite dimensional Wiener process and it is an alternative to the construction given in [23].

Chapter 3 contains the main results for stochastic Volterra equations in Hilbert space. In Section 3.1 we introduce the definitions of solutions to the equation (0.1) and formulate auxiliary results being a framework for the main theorems. In Section 3.2 we prove existence of strong solutions for two classes of equation (0.1). Basing on convergence of resolvents obtained in Chapter 1, we can formulate Lemma 3.15 and Theorem 3.16 giving sufficient conditions under which stochastic convolution corresponding to (0.1) is strong solution to the equation (0.1). Section 3.3 deals with the so-called fractional Volterra equations. First, we prove using other tools than in Section 1.5, approximation results, that is, Theorems 3.20 and 3.21, for α-times resolvents corresponding to the fractional equations. Next, we prove the existence of strong solutions to fractional equations. These results are formulated in Lemma 3.29 and Theorem 3.30. In Section 3.4 we give several examples illustrating the class of equations fulfilling conditions of theorems providing existence of strong solutions.

In Chapter 4 we study regularity of two classes of stochastic Volterra equations in the space of tempered distributions. Section 4.1 has an introductory character. It contais notions and facts concerning generalized and classical homogeneous Gaussian random fields needed in the sequel. In Section 4.2 we introduce the stochastic integral in the space of distributions. Then we formulate Theorem 4.4 which characterizes the stochastic convolution corresponding to (0.2). The main regularity results obtained in Section 4.2 are collected in Theorems 4.6–4.8. These theorems give sufficient conditions under which solutions to the equation (0.2) are function-valued and even continuous with respect to the space variable. These conditions are given in terms of the covariance kernel Γ of the Wiener process W_Γ and the spectral measure of W_Γ, as well. The results obtained in this section are illustrated by several examples. Section 4.3 is a natural continuation of the previous one. In this section we give necessary and sufficient conditions for the existence of a limit measure to the equation (0.2) and then we describe all limit measures to (0.2). The main results of this section, that is Lemmas 4.13, 4.14 and Theorems 4.15, 4.16, are in a sense analogous to those formulated in [24, Chapter 6], obtained for semigroup case.

Section 4.4 is devoted to regularity of solutions to the equation (0.3). Here we study a particular case of weak solutions basing on an explicit representation of the solution to (0.3). We find the expression for the solution in terms of the kernel b and next we reduce the questions of regularity of solutions to problems arising in harmonic analysis. Our main results of this section, that is Theorem 4.21 and Proposition 4.22, provide necessary and sufficient conditions under which solutions to (0.3) are function-valued. These conditions are given in terms of the Fourier coefficients of the covariance Γ of the Wiener process W_Γ. Additionally, Section 4.4 contains some corollaries which are consequences of the main results.

Bibliographical notes

Sections 1.1–1.4 contain introductory material which in a similar form can be found in the mongraph [70]. Section 1.5 originates from [50] and [51], the latter not yet published.

Sections 2.1, 2.2 and 2.4 contain classical results recalled from [19], [42], [23] and [38]. Section 2.3 originates from [46].

The results of Section 3.1 come from [48]. Section 3.2 originates from [50] and [51]. The content of Sections 3.3 and 3.4 can be found in [52].

Section 4.1 contains material coming from [34], [36], [2] and [66]. The results of Section 4.2 come from [54]. Section 4.3 originates from [47]. The results of Section 4.4 can be found in [49].

CHAPTER 1

DETERMINISTIC VOLTERRA EQUATIONS

This chapter contains notations and concepts concerning Volterra equations used throughout the monograph and collects some results necessary to make the work self-contained. The notations are standard and follow the book by Prüss [70].

Section 1.1 has a preliminary character. In Section 1.2 we recall the definition of the resolvent family to the deterministic Volterra equation and the concept of well-posedness connected with the resolvent. Kernel functions, particularly k-regular ones, are recalled in Sections 1.3 and 1.4. The above mentioned definitions and results are described and commented in detail in Prüss' monograph [70] and appropriate references therein.

Section 1.5 originates from [51]. Theorems 1.19 and 1.20, yet non-published, are deterministic approximation theorems which play a key role for existence of strong solutions to stochastic Volterra equations. These approximation theorems are resolvent analogies to Hille–Yosida's theorem for semigroup and play the same role as that theorem.

1.1. Notations and preliminaries

Let B be a complex Banach space with the norm $|\cdot|$. We consider in B the Volterra equation of the form

$$u(t) = \int_0^t a(t-\tau)\, Au(\tau)\, d\tau + f(t). \tag{1.1}$$

In (1.1), $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R})$ is a non-zero scalar kernel; for abbreviation we will write $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$. A is a closed unbounded linear operator in B with a dense domain $D(A)$ and f is a continuous B-valued function. In the sequel we assume that the domain $D(A)$ is equipped with the graph norm $|\cdot|_A$ of A, i.e. $|x|_A := |x| + |Ax|$ for $x \in D(A)$. Then $(D(A), |\cdot|_A)$ is a Banach space because A is closed (see e.g. [27]) and it is continuously and densely embedded into $(B, |\cdot|)$.

By $\sigma(A)$ and $\varrho(A)$ we shall denote spectrum and resolvent set of the operator A, respectively.

The equation (1.1) includes a big class of equations and is an abstract version of several deterministic problems, see e.g. [70]. For example, if $a(t) = 1$ and f is a function of C^1-class, the equation (1.1) is equivalent to the Cauchy problem

$$\dot{u}(t) = Au(t) + \dot{f}(t) \quad \text{with } u(0) = f(0).$$

Analogously, in the case $a(t) = t$ and f of C^2-class, the equation (1.1) is equivalent to

$$\ddot{u}(t) = Au(t) + \ddot{f}(t) \quad \text{with initial conditions} \quad u(0) = f(0) \quad \text{and} \quad \dot{u}(0) = \dot{f}(0).$$

Several other examples of problems which lead to Volterra equation (1.1) can be found in [70, Section 5].

In the first three chapters of the monograph we shall use the abbreviation

$$(g \star h)(t) = \int_0^t g(t-\tau)h(\tau)\, d\tau, \quad t \geq 0,$$

for the convolution of two functions g and h.

In the paper we write $a(t)$ for the kernel function a. The notation $a(t)$ will mean the function and not the value of the function a at t. Such notation will allow to distinguish the function $a(t)$ and the article a.

If a function $v \in L^1_{\mathrm{loc}}(\mathbb{R}_+; B)$ is of *exponential growth*, i.e. $\int_0^\infty e^{-\omega t}|v(t)|dt < \infty$ for some $\omega \in \mathbb{R}$, we can define the *Laplace transform* of the function v

$$\widehat{v}(\lambda) = \int_0^\infty e^{-\lambda t} v(t)\, dt, \quad \operatorname{Re}\lambda \geq \omega.$$

In the whole paper we shall denote by $\widehat{v}$ the Laplace transform of the function v.

In the whole paper the operator norm will be denoted by $||\cdot||$.

1.2. Resolvents and well-posedness

The concept of the resolvent is very important for the theory of linear Volterra equations. The so-called resolvent approach to the Volterra equation (1.1) has been introduced many years ago, probably by Friedman and Shinbrot [31]; recently the approach has been presented in detail in the great monograph by Prüss [70]. The resolvent approach is a generalization of the semigroup approach.

By $S(t)$, $t \geq 0$, we shall denote the family of resolvent operators corresponding to the Volterra equation (1.1) and defined as follows.

Definition 1.1 (see e.g. [70]). A family $(S(t))_{t\geq 0}$ of bounded linear operators in the space B is called *resolvent* for (1.1) if the following conditions are satisfied:

(a) $S(t)$ is strongly continuous on $\mathbb{R}_+$ and $S(0) = I$;

(b) $S(t)$ commutes with the operator A, that is, $S(t)(D(A)) \subset D(A)$ and $AS(t)x = S(t)Ax$ for all $x \in D(A)$ and $t \geq 0$;

(c) the following *resolvent equation* holds

$$S(t)x = x + \int_0^t a(t-\tau)AS(\tau)x\,d\tau \tag{1.2}$$

for all $x \in D(A)$, $t \geq 0$.

We shall assume that the equation (1.1) is *well-posed* in the sense that (1.1) admits the resolvent $S(t)$, $t \geq 0$. (Precise definition of well-posedness is given in [70]). That defintion is a direct extension of well-posedness of Cauchy problems. The lack of well-posedness of (1.1) leads to distribution resolvents, see e.g. [22].

Proposition 1.2 ([70, Proposition 1.1]). *The equation* (1.1) *is well-posed if and only if* (1.1) *admits a resolvent* $S(t)$. *If this is the case then, in addition, the range* $R(a \star S(t)) \subset D(A)$ *for all* $t \geq 0$ *and*

$$S(t)x = x + A\int_0^t a(t-\tau)S(\tau)x\,d\tau \quad \textit{for all } x \in H,\ t \geq 0. \tag{1.3}$$

Comment. Let us emphasize that the resolvent $S(t), t \geq 0$, is determined by the operator A and the function $a(t)$. Moreover, as a consequence of the strong continuity of $S(t)$ we have for any $T > 0$

$$\sup_{t \leq T} ||S(t)|| < \infty. \tag{1.4}$$

Suppose $S(t)$ is the resolvent for (1.1) and let $-\mu \in \sigma(A)$ be an eigenvalue of A with eigenvector $x \neq 0$. Then

$$S(t)x = s(t;\mu)x, \quad t \geq 0, \tag{1.5}$$

where $s(t;\mu)$ is the solution of the one-dimensional Volterra equation

$$s(t;\mu) + \mu\int_0^t a(t-\tau)s(\tau;\mu)d\tau = 1, \quad t \geq 0. \tag{1.6}$$

By $W^{1,p}_{\mathrm{loc}}(\mathbb{R}_+;B)$ we denote the Sobolev space of order $(1,p)$ of Bochner locally p-integrable functions acting from $\mathbb{R}_+$ into the space B, see e.g. [27].

Definition 1.3. A resolvent $S(t)$, for the equation (1.1), is called *differentiable* if $S(\,\cdot\,)x \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+;B)$ for any $x \in D(A)$ and there exists a function $\varphi \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ such that $|\dot{S}(t)x| \leq \varphi(t)|x|_A$ a.e. on $\mathbb{R}_+$, for every $x \in D(A)$.

Similarly, if $S(t)$ is differentiable then

$$\dot{S}(t)x = \mu r(t;\mu)x, \quad t \geq 0, \tag{1.7}$$

where $r(t;\mu)$ is the solution of the one-dimensional equation

$$r(t;\mu) + \mu \int_0^t a(t-\tau) r(\tau;\mu)\, d\tau = a(t), \quad t \geq 0. \tag{1.8}$$

In some special cases the functions $s(t;\mu)$ and $r(t;\mu)$ may be found explicitely. For example, for $a(t) = 1$, we have $s(t;\mu) = r(t;\mu) = e^{-\mu t}$, $t \geq 0$, $\mu \in \mathbb{C}$. For $a(t) = t$, we obtain $s(t;\mu) = \cos(\sqrt{\mu}t)$, $r(t;\mu) = \sin(\sqrt{\mu}t)/\sqrt{\mu}$, $t \geq 0$, $\mu \in \mathbb{C}$.

Definition 1.4. Suppose $S(t)$, $t \geq 0$, is a resolvent for (1.1). $S(t)$ is called *exponentially bounded* if there are constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$||S(t)|| \leq M\, e^{\omega t}, \quad \text{for all } t \geq 0.$$

(M, ω) is called a *type* of $S(t)$.

Let us note that in contrary to the case of semigroups, not every resolvent needs to be exponentially bounded even if the kernel function $a(t)$ belongs to $L^1(\mathbb{R}_+)$. The Volterra equation version of the Hille–Yosida theorem (see e.g. [70, Theorem 1.3]) provides the class of equations that admit exponentially bounded resolvents. An important class of kernels providing such class of resolvents are $a(t) = t^{\alpha-1}/\Gamma(\alpha)$, $\alpha \in (0,2)$. For details, counterexamples and comments we refer to [26].

1.3. Kernel functions

Two classes of kernel functions defined below play a prominent role in the theory of Volterra equations.

Definition 1.5. A C^∞-function $a\colon (0,\infty) \to \mathbb{R}$ is called *completely monotonic* if $(-1)^n a^{(n)}(t) \geq 0$ for all $t > 0$, $n \in \mathbb{N}$.

Definition 1.6. We say that function $a \in L^1(0,T)$ is *completely positive* on $[0,T]$ if for any $\mu \geq 0$, the solutions of the convolution equations (1.6) and (1.8) satisfy $s(t;\mu) \geq 0$ and $r(t;\mu) \geq 0$ on $[0,T]$, respectively.

We recall that if $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ is completely positive, then $s(t;\mu)$, the solution to (1.6), is nonnegative and nonincreasing for any $t \geq 0$, $\mu \geq 0$. In the consequence, one has $0 \leq s(t;\mu) \leq 1$. This is a special case of the result due to [29].

Kernels with this property have been introduced by Clément and Nohel [18]. We note that the class of completely positive kernels appears quite naturally in applications, particularly in the theory of viscoelasticity. Several properties and examples of such kernels appear in [70, Section 4.2].

Examples. (a) Let $a(t) = t^{\alpha-1}/\Gamma(\alpha)$, $\alpha > 0$, where Γ is the gamma function. For $\alpha \in (0,1]$, function $a(t)$ is completely monotonic and completely positive.

(b) Another example of completely positive function is $a(t) = e^{-t}$, $t \geq 0$. An easy computation shows that then $s(t;\mu) = (1+\mu)^{-1}[1+\mu\, e^{-(1+\mu)t}]$, for $t, \mu > 0$.

1.4. Parabolic equations and regular kernels

This section is devoted to the so-called parabolic Volterra equations defined by Prüss [69].

Let B be a complex Banach space and

$$\sum(\omega,\theta) := \{\lambda \in \mathbb{C} : |\arg(\lambda - \omega)| < \theta\}.$$

Definition 1.7 ([70, Definition 2.1]). A resolvent $S(t)$ for (1.1) is called *analytic*, if the function $S(\,\cdot\,)\colon \mathbb{R}_+ \to L(B)$ admits analytic extension to a sector $\sum(0,\theta_0)$ for some $0 < \theta_0 < \pi/2$. An analytic resolvent $S(t)$ is said to be of *analyticity type* (ω_0, θ_0) if for each $\theta < \theta_0$ and $\omega > \omega_0$ there is $M = M(\omega,\theta)$ such that

$$||S(z)|| \leq M\, e^{\omega \mathrm{Re} z}, \quad z \in \sum(0,\theta_0). \tag{1.9}$$

Corollary 1.8 ([70, Corollary 2.1]). *Suppose $S(t)$ is an analytic resolvent for (1.1) of analyticity type (ω_0,θ_0). Then for each $\omega > \omega_0$ and $\theta < \theta_0$ there is $M = M(\omega,\theta)$ such that*

$$||S^{(n)}(t)|| \leq M\, n!\, e^{\omega t(1+\alpha)} (\alpha t)^{-n}, \quad t > 0,\ n \in \mathbb{N}, \tag{1.10}$$

where $\alpha = \sin\theta$.

Analytic resolvents, the analog of analytic semigroups for Volterra equations, have been introduced by Da Prato and Iannelli [21]. Analogously like in the theory of analytic semigroups, a characterization of analytic resolvents in terms of the spectrum of the operator A and the Laplace transform of the kernel function $a(t)$ is possible.

Theorem 1.9 ([70, Theorem 2.1]). *Let A be a closed unbounded operator in B with dense domain $D(A)$ and let $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ satisfy $\int_0^t |a(t)|\, e^{-\omega_a t}\, dt < \infty$ for some $\omega_a \in \mathbb{R}$. Then (1.1) admits an analytic resolvent $S(t)$ of analyticity type (ω_0,θ_0) if and only if the following conditions hold:*

(a) *$\widehat{a}(\lambda)$ admits meromorphic extension to $\sum(\omega_0, \theta_0 + \pi/2)$;*
(b) *$\widehat{a}(\lambda) \neq 0$, and $1/\widehat{a}(\lambda) \in \varrho(A)$ for all $\lambda \in \sum(\omega_0,\theta_0+\pi/2)$;*
(c) *For each $\omega > \omega_0$ and $\theta < \theta_0$ there is a constant $C = C(\omega,\theta)$ such that $H(\lambda) := (1/\widehat{a}(\lambda) - A)^{-1}/(\lambda\,\widehat{a}(\lambda))$ satisfies estimate*

$$||H(\lambda)|| \leq C/|\lambda - \omega| \quad \text{for all } \lambda \in \sum(\omega, \theta + \pi/2). \tag{1.11}$$

Typical examples of the kernel functions $a(t)$ and the operator A fulfilling conditions of Theorem 1.9 are the following.

Examples. (a) Let kernels be $a(t) = t^{\beta-1}/\Gamma(\beta)$, $t > 0$, where $\beta \in (0,2)$ and Γ denotes the gamma function. The pair $(t^{\beta-1}/\Gamma(\beta), A)$ generates a bounded analytic resolvent if and only if $\varrho(A) \supset \sum(0, \beta\pi/2)$ and $||\mu(\mu - A)^{-1}|| \leq M$ for all $\mu \in \sum(0, \beta\pi/2)$.

(b) An important class of kernels $a(t)$ which satisfy the above conditions is the class of completely monotonic kernels. By [70, Corollary 2.4] if additionally $a \in C(0,\infty) \cap L^1(0,1)$ and A generates an analytic semigroup $T(t)$ such that $||T(t)|| \leq M$ on $\Sigma(0,\theta)$ then (1.1) admits an analytic resolvent $S(t)$ of type $(0,\theta)$.

Parabolic Volterra equations appear in a context of Volterra equations admiting analytical resolvents.

Definition 1.10. Equation (1.1) is called *parabolic*, if the following conditions hold:

(a) $\widehat{a}(\lambda) \neq 0$, $1/\widehat{a}(\lambda) \in \varrho(A)$ for all $\operatorname{Re}\lambda > 0$.
(b) There is a constant $M \geq 1$ such that $H(\lambda) = (I - \widehat{a}(\lambda)A)^{-1}/\lambda$ satisfies $||H(\lambda)|| \leq M/|\lambda|$ for all $\operatorname{Re}\lambda > 0$.

From the resolvent point of view, the concept of parabolicity is between the bounded and the analytic resolvents: if (1.1) admits an analytic resolvent $S(t)$ then (1.1) is parabolic. On the other hand, if the equation (1.1) is parabolic and the kernel function $a(t)$ has some properties, like convexity, then the resolvent corresponding to (1.1) has, roughly speaking, similar properties like analytic resolvent.

Definition 1.11. Let $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ be of subexponential growth and suppose $\widehat{a}(\lambda) \neq 0$ for all $\operatorname{Re}\lambda > 0$. The function $a(t)$ is called *sectorial with angle* $\theta > 0$ (or merely *θ-sectorial*) if $|\arg \widehat{a}(\lambda)| \leq \theta$ for all $\operatorname{Re}\lambda > 0$.

The standard situation leading to parabolic equations is provided by sectorial kernels and some closed linear densely defined operators A.

The following criteria provide parabolic equations.

Proposition 1.12 ([70, Proposition 3.1])**.** *Let $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ be θ-sectorial for some $\theta < \pi$, suppose A is closed linear densely defined, such that $\varrho(A) \supset \sum(0,\theta)$, and $||(\mu - A)^{-1}|| \leq M/|\mu|$ for all $\mu \in \sum(0,\theta)$. Then* (1.1) *is parabolic.*

The particular case is when A is the generator of a bounded analytic C_0-semigroup and the function $a(t)$ is $\pi/2$-sectorial. Because $a(t)$ is $\pi/2$-sectorial if and only if $a(t)$ is of positive type, we obtain the following class of parabolic equations.

Corollary 1.13 ([70, Corollary 3.1]). *Let $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ be of subexponential growth and of positive type, and let A generate a bounded analytical C_0-semigroup in B. Then* (1.1) *is parabolic.*

In the sequel we will need some regular kernels.

Definition 1.14. Let $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ be of subexponential growth and $k \in \mathbb{N}$. The function $a(t)$ is called *k-regular* if there is a constant $c > 0$ such that $|\lambda^n\, a^{(n)}(\lambda)| \le c|\widehat{a}(\lambda)|$ for all $\operatorname{Re}\lambda > 0$, $0 \le n \le k$.

Comment. Any k-regular kernel $a(t)$, $k \ge 1$ has the property that $\widehat{a}(\lambda)$ has no zeros in the open right halfplane.

We would like to emphasize that convolutions of k-regular kernels are again k-regular what follows from the product rule of convolutions. The integration and differentiation preserve k-regularity, as well. Unfortunately, sums and differences of k-regular kernels need not be k-regular. We may check it taking $a(t) = 1$ and $b(t) = t^2$. However, if $a(t)$ and $b(t)$ are k-regular and $|\arg\widehat{a}(\lambda) - \arg\widehat{b}(\lambda)| \le \theta \le \pi$, $\operatorname{Re}\lambda > 0$ then $a(t) + b(t)$ is k-regular.

If $a(t)$ is real-valued and 1-regular then $a(t)$ is sectorial. The converse of this is not true. As the counterexample we can take $a(t) = 1$ for $t \in [0,1]$, $a(t) = 0$ for $t > 1$.

Proposition 1.15 ([70, Proposition 3.2]). *Suppose $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ is such that $\widehat{a}(\lambda)$ admits analytic extension to $\sum(0,\varphi)$, where $\varphi > \pi/2$, and there is $\theta \in (0,\infty)$ such that $|\arg\widehat{a}(\lambda)| \le \theta$ for all $\lambda \in \sum(0,\varphi)$. Then $a(t)$ is k-regular for every $k \in \mathbb{N}$.*

So, nonnegative and nonicreasing kernels are in general not 1-regular but if the kernel is also convex, then it is 1-regular.

Definition 1.16. Let $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ and $k \ge 2$. The function $a(t)$ is called *k-monotone* if $a \in C^{k-2}(0,\infty)$, $(-1)^n\, a^{(n)}(t) \ge 0$ for all $t > 0, 0 \le n \le k-2$, and $(-1)^{k-2}\, a^{(k-2)}(t)$ is nonincreasing and convex.

By definition, a 2-monotone kernel $a(t)$ is nonnegative, nonicreasing and convex, and $a(t)$ is completely monotonic if and only if $a(t)$ is k-monotone for all $k \ge 2$.

Proposition 1.17 ([70, Proposition 3.3]). *Suppose $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ is $(k+1)$-monotone, $k \ge 1$. Then $a(t)$ is k-regular and of positive type.*

Now, we recall the main theorem on resolvents for parabolic Volterra equations.

Theorem 1.18 ([70, Theorem 3.1]). *Let B be a Banach space, A a closed linear operator in B with dense domain $D(A)$, $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$. Assume* (1.1) *is parabolic, and $a(t)$ is k-regular, for some $k \ge 1$. Then there is a resolvent*

$S \in C^{k-1}((0,\infty); L(B))$ for (1.1), and there is a constant $M \geq 1$ such that estimates

$$||t^n\, S^{(n)}(t)|| \leq M, \quad \textit{for all } t \geq 0,\ n \leq k-1, \tag{1.12}$$

and

$$||t^k S^{(k-1)}(t) - s^k S^{(k-1)}(s)|| \leq M|t-s|\left[1+\log\frac{t}{t-s}\right], \quad 0 \leq s < t < \infty, \tag{1.13}$$

are valid.

1.5. Approximation theorems

In this paper the following results contained in [51] concerning convergence of resolvents for the equation (1.1) in Banach space B will play the key role. They extend some results of Clément and Nohel obtained in [18] for contraction semigroups. Theorems 1.19, 1.20 and Proposition 1.21 are not yet published.

Theorem 1.19. *Let A be the generator of a C_0-semigroup in B and suppose the kernel function $a(t)$ is completely positive. Then (A, a) admits an exponentially bounded resolvent $S(t)$. Moreover, there exist bounded operators A_n such that (A_n, a) admit resolvent families $S_n(t)$ satisfying $||S_n(t)|| \leq Me^{w_0 t}$ ($M \geq 1$, $w_0 \geq 0$) for all $t \geq 0$, $n \in \mathbb{N}$, and*

$$S_n(t)x \to S(t)x \quad \textit{as } n \to \infty \tag{1.14}$$

for all $x \in B$, $t \geq 0$. Additionally, the convergence is uniform in t on every compact subset of $\mathbb{R}_+$.

Proof. The first assertion follows directly from [68, Theorem 5] (see also [70, Theorem 4.2]). Since A generates a C_0-semigroup $T(t)$, $t \geq 0$, the resolvent set $\rho(A)$ of A contains the ray $[w,\infty)$ and

$$||R(\lambda, A)^k|| \leq \frac{M}{(\lambda - w)^k} \quad \text{for } \lambda > w,\ k \in \mathbb{N},$$

where $R(\lambda, A) = (\lambda I - A)^{-1}$, $\lambda \in \rho(A)$.

Define

$$A_n := nAR(n, A) = n^2 R(n, A) - nI, \quad n > w \tag{1.15}$$

the *Yosida approximation* of A. Then

$$\begin{aligned} ||e^{tA_n}|| &= e^{-nt}||e^{n^2 R(n,A)t}|| \\ &\leq e^{-nt}\sum_{k=0}^{\infty}\frac{n^{2k}t^k}{k!}||R(n,A)^k|| \leq Me^{(-n+n^2/(n-w))t} = Me^{nwt/(n-w)}. \end{aligned}$$

Hence, for $n > 2w$ we obtain

$$||e^{A_n t}|| \leq Me^{2wt}. \tag{1.16}$$

Taking into account the above estimate and the complete positivity of the kernel function $a(t)$, we can follow the same steps as in [68, Theorem 5] to obtain that there exist constants $M_1 > 0$ and $w_1 \in \mathbb{R}$ (independent of n, due to (1.16)) such that

$$||[H_n(\lambda)]^{(k)}|| \leq \frac{M_1}{\lambda - w_1} \quad \text{for } \lambda > w_1,$$

where $H_n(\lambda) := (\lambda - \lambda\widehat{a}(\lambda)A_n)^{-1}$. Here and in the sequel the hat indicates the Laplace transform. Hence, the generation theorem for resolvent families implies that for each $n \in \mathbb{N}$, the pair (A_n, a) admits resolvent family $S_n(t)$ such that

$$||S_n(t)|| \leq M_1 e^{w_1 t} \quad \text{for all } n \in \mathbb{N}. \tag{1.17}$$

In particular, the Laplace transform $\widehat{S}_n(\lambda)$ exists and satisfies

$$\widehat{S}_n(\lambda) = H_n(\lambda) = \int_0^\infty e^{-\lambda t} S_n(t)\, dt, \quad \lambda > w_1.$$

Now recall from semigroup theory that for all μ sufficiently large we have

$$R(\mu, A_n) = \int_0^\infty e^{-\mu t} e^{A_n t}\, dt$$

as well as,

$$R(\mu, A) = \int_0^\infty e^{-\mu t} T(t)\, dt.$$

Since $\widehat{a}(\lambda) \to 0$ as $\lambda \to \infty$, we deduce that for all λ sufficiently large, we have

$$H_n(\lambda) = \frac{1}{\lambda\widehat{a}(\lambda)} R\left(\frac{1}{\widehat{a}(\lambda)}, A_n\right) = \frac{1}{\lambda\widehat{a}(\lambda)} \int_0^\infty e^{(-1/\widehat{a}(\lambda))t} e^{A_n t}\, dt,$$

and

$$H(\lambda) = \frac{1}{\lambda\widehat{a}(\lambda)} R\left(\frac{1}{\widehat{a}(\lambda)}, A\right) = \frac{1}{\lambda\widehat{a}(\lambda)} \int_0^\infty e^{(-1/\widehat{a}(\lambda))t} T(t)\, dt.$$

Hence, from the identity

$$H_n(\lambda) - H(\lambda) = \frac{1}{\lambda\widehat{a}(\lambda)} \left[R\left(\frac{1}{\widehat{a}(\lambda)}, A_n\right) - R\left(\frac{1}{\widehat{a}(\lambda)}, A\right) \right]$$

and the fact that $R(\mu, A_n) \to R(\mu, A)$ as $n \to \infty$ for all μ sufficiently large (see e.g. [65, Lemma 7.3], we obtain that

$$H_n(\lambda) \to H(\lambda) \quad \text{as } n \to \infty. \tag{1.18}$$

Finally, due to (1.17) and (1.18) we can use the Trotter–Kato theorem for resolvent families of operators (cf. [58, Theorem 2.1]) and the conclusion follows. □

Let us recall, e.g. from [28], that a family $\mathcal{C}(t)$, $t \geq 0$, of linear bounded operators on H is called *cosine family* if $\mathcal{C}(t+s) + \mathcal{C}(t-s) = 2\mathcal{C}(t)\mathcal{C}(s)$ for every $t, s \geq 0$, $t > s$.

Theorem 1.19 may be reformulated in the following version.

Theorem 1.20. *Let A generate a cosine family $T(t)$ in B such that $||T(t)|| \leq Me^{\omega t}$ for $t > 0$ holds, and suppose the kernel function $a(t)$ is completely positive. Then (A, a) admits an exponentially bounded resolvent $S(t)$. Moreover, there exist bounded operators A_n such that (A_n, a) admit resolvent families $S_n(t)$ satisfying $||S_n(t)|| \leq \widetilde{M}e^{w_0 t}$ ($\widetilde{M} \geq 1$, $w_0 \geq 0$) for all $t \geq 0$, $n \in \mathbb{N}$ and*

$$S_n(t)x \to S(t)x \quad \text{as } n \to \infty$$

for all $x \in B$, $t \geq 0$. Additionally, the convergence is uniform in t on every compact subset of $\mathbb{R}_+$.

Remarks. (a) By [70, Theorem 4.3] or [68, Theorem 6] Theorem 1.20 holds also in two other cases:

(a1) $a(t)$ is a creep function with the function $a_1(t)$ log-convex;
(a2) $a = c \star c$ with some completely positive $c \in L^1_{\text{loc}}(\mathbb{R}_+)$.

(Let us recall the definition [70, Definition 4.4]: A function $a: \mathbb{R}_+ \mapsto \mathbb{R}$ is called a *creep function* if $a(t)$ is nonnegative, nondecreasing, and concave. A creep function $a(t)$ has a *standard form*

$$a(t) = a_0 + a_\infty t + \int_0^t a_1(\tau)\, d\tau, \quad t > 0,$$

where $a_0 = a(0+) \geq 0$, $a_\infty = \lim_{t\to\infty} a(t)/t = \inf_{t>0} a(t)/t \geq 0$, and $a_1(t) = \dot{a}(t) - a_\infty$ is nonnegative, nonincreasing, $\lim_{t\to\infty} a_1(t) = 0$.)

(b) Other examples of the convergence (1.14) for the resolvents are given, e.g. in [18] and [30]. In the first paper, the operator A generates a linear continuous contraction semigroup. In the second one, A belongs to some subclass of sectorial operators and the kernel $a(t)$ is an absolutely continuous function fulfilling some technical assumptions.

Comment. The above theorem gives a partial answer to the following open problem for a resolvent family $S(t)$ generated by a pair (A, a): do exist bounded linear operators A_n generating resolvent families $S_n(t)$ such that $S_n(t)x \to S(t)x$? Note that in case $a(t) \equiv 1$ the answer is yes, namely A_n are provided by the Hille–Yosida approximation of A and $S_n(t) = e^{tA_n}$.

The following result will be used in the sequel.

Proposition 1.21. *Let A, A_n and $S_n(t)$ be given as in Theorem* 1.19 *or Theorem* 1.20*. Then the operators $S_n(t)$ commute with the operator A, for every n sufficiently large and $t \geq 0$.*

Proof. For each n sufficiently large the bounded operators A_n admit a resolvent family $S_n(t)$, so by the complex inversion formula for the Laplace transform we have

$$S_n(t) = \frac{1}{2\pi i} \int_{\Gamma_n} e^{\lambda t} H_n(\lambda)\, d\lambda$$

where Γ_n is a simple closed rectifiable curve surrounding the spectrum of A_n in the positive sense.

On the other hand, $H_n(\lambda) := (\lambda - \lambda\widehat{a}(\lambda)A_n)$ where $A_n := nA(n-A)^{-1}$, so each A_n commutes with A on $D(A)$ and then each $H_n(\lambda)$ commutes with A, on $D(A)$, too.

Finally, because A is closed and all the following integrals are convergent (exist), for all n sufficiently large and $x \in D(A)$ we have

$$\begin{aligned} AS_n(t)x &= A\int_{\Gamma_n} e^{\lambda t} H_n(\lambda)x \, d\lambda \\ &= \int_{\Gamma_n} e^{\lambda t} AH_n(\lambda)x \, d\lambda = \int_{\Gamma_n} e^{\lambda t} H_n(\lambda)Ax \, d\lambda = S_n(t)Ax. \qquad \square \end{aligned}$$

CHAPTER 2

PROBABILISTIC BACKGROUND

In this chapter we recall from [19], [42], [23] and [38] basic and important concepts and results of the infinite dimensional stochastic analysis needed in the sequel. In particular, we present construction of stochastic integral with respect to a cylindrical Wiener process, published in [46].

2.1. Notations and conventions

Assume that $(\Omega, \mathcal{F}, P)$ is a probability space equipped with an increasing family of σ-fields $(\mathcal{F}_t)$, $t \in I$, where $I = [0,T]$ or $I = [0,\infty)$, called *filtration*. We shall denote by $\mathcal{F}_{t^+}$ the intersection of all σ-fields $\mathcal{F}_s$, $s > t$. We say that filtration is *normal* if $\mathcal{F}_0$ contains all sets $B \in \mathcal{F}$ with measure $P(B) = 0$ and if $\mathcal{F}_t = \mathcal{F}_{t^+}$ for any $t \in I$, that is, the filtration is right continuous.

In the paper we assume that filtration $(\mathcal{F}_t)_{t\in I}$ is normal. This assumption enables to choose modifications of considered stochastic processes with required measurable properties.

Let H and U be two separable Hilbert spaces. In the whole paper we write explicitely indexes indicating the appropriate space in norms $|\cdot|_{(\,\cdot\,)}$ and inner products $\langle\,\cdot\,,\,\cdot\,\rangle_{(\,\cdot\,)}$.

Definition 2.1. The H-valued process $X(t)$, $t \in I$, is *adapted* to the family $(\mathcal{F}_t)_{t\in I}$, if for arbitrary $t \in I$ the random variable $X(t)$ is $\mathcal{F}_t$-measurable.

Definition 2.2. The H-valued process $X(t)$, $t \in [0,T]$, is *progressively measurable* if for every $t \in [0,T]$ the mapping $[0,t] \times \Omega \to H$, $(s,w) \mapsto X(s,w)$ is $\mathcal{B}([0,t]) \times \mathcal{F}_t$-measurable.

We will use the following well-known result, see e.g. [23].

Proposition 2.3 ([23, Proposition 3.5]). *Let $X(t)$, $t \in [0,T]$, be a stochastically continuous and adapted process with values in H. Then X has a progressively measurable modification.*

By $\mathcal{P}_\infty$ we denote a σ-field of subsets of $[0,\infty)\times\Omega$ defined as follows: $\mathcal{P}_\infty$ is the σ-field generated by sets of the form: $(s,t]\times F$, where $0\le s\le t<\infty$, $F\in\mathcal{F}_s$ and $\{0\}\times F$, when $F\in\mathcal{F}_0$. The restriction of the σ-field $\mathcal{P}_\infty$ to $[0,T]\times\Omega$ will be denoted by $\mathcal{P}_T$.

Definition 2.4. An arbitrary measurable mapping from $([0,\infty)\times\Omega,\mathcal{P}_\infty)$ or $([0,T]\times\Omega,\mathcal{P}_T)$ into $(H;\mathcal{B}(H))$ is called a *predictable process.*

Comment. A predictable process is an adapted one.

Proposition 2.5 ([23, Proposition 3.6])**.** *Assume that $X(t)$, $t\in[0,T]$, is an adapted and stochastically continuous process. Then the process X has a predictable version on $[0,T]$.*

By $L(U,H)$, $L(U)$ we denote spaces of linear bounded operators from U into H and in U, respectively. As previously, the operator norm is denoted by $||\cdot||$.

An important role will be played by the space of Hilbert–Schmidt operators. Let us recall the following definition.

Definition 2.6 ([4] or [23])**.** Assume that $\{e_k\}\subset U$ and $\{f_j\}\subset H$ are orthonormal bases of U and H, respectively. A linear bounded operator $T\colon U\to H$ is called *Hilbert–Schmidt operator* if $\sum_{k=1}^\infty |Te_k|_H^2<\infty$.

Because

$$\sum_{k=1}^\infty |Te_k|_H^2=\sum_{k=1}^\infty\sum_{j=1}^\infty (Te_k,f_j)_H^2=\sum_{j=1}^\infty |T^*f_j|_U^2,$$

where T^* denotes the operator adjoint to T, then the definition of Hilbert-Schmidt operator and the number $||T||_{HS}=\left(\sum_{k=1}^\infty |Te_k|_H^2\right)^{1/2}$ do not depend on the basis $\{e_k\}$, $k\in\mathbb{N}$. Moreover $||T||_{HS}=||T^*||_{HS}$.

Additionally, $L_2(U,H)$ — the set of all Hilbert–Schmidt operators from U into H, endowed with the norm $||\cdot||_{HS}$ defined above, is a separable Hilbert space.

2.2. Classical infinite dimensional Wiener process

Here we recall from [19] and [42] the definition of Wiener process with values in a real separable Hilbert space U and the stochastic integral with respect to this process.

Definition 2.7. Let $Q\colon U\to U$ be a linear symmetric non-negative nuclear operator ($\operatorname{Tr} Q<\infty$). A square integrable U-valued stochastic process $W(t)$, $t\ge 0$, defined on a probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\ge0},P)$, where $\mathcal{F}_t$ denote σ-fields such that $\mathcal{F}_t\subset\mathcal{F}_s\subset\mathcal{F}$ for $t<s$, is called classical or genuine *Wiener process* with covariance operator Q if:

(a) $W(0)=0$,
(b) $EW(t)=0$, $\operatorname{Cov}[W(t)-W(s)]=(t-s)Q$ for all $s,t\ge 0$,

(c) W has independent increments, that is $W(s_4) - W(s_3)$ and $W(s_2) - W(s_1)$ are independent whenever $0 \le s_1 \le s_2 \le s_3 \le s_4$,
(d) W has continuous trajectories,
(e) W is adapted with respect to the filtration $(\mathcal{F}_t)_{t\ge 0}$.

If we choose $\mathcal{F}_t$ to be the σ-field generated by $\{W(s) : 0 \le s \le t\}$, then $W(t) - W(s)$ is independent of $\mathcal{F}_s$ for all $t > s$ from condition (c) of the above definition. Then $\mathbb{E}\{W(t) - W(s)|\mathcal{F}_s\} = \mathbb{E}\{W(t) - W(s)\} = 0$ by condition (b). Hence, $\mathbb{E}\{W(t)|\mathcal{F}_s\} = W(s)$ w.p.1 and $\{W(t), \mathcal{F}_t\}$ is a martingale on $[0, \infty)$.

We remark that an alternative definition is to replace condition (d) of Definition 2.7 by assuming that $W(t)$ is Gaussian for all $t \ge 0$, see [19] for details.

In the light of the above, Wiener process is Gaussian and has the following expansion (see e.g. [19, Lemma 5.23]). Let $\{e_i\} \subset U$ be an orthonormal set of eigenvectors of Q with corresponding eigenvalues ζ_i (so $\operatorname{Tr} Q = \sum_{i=1}^{\infty} \zeta_i$), then

$$W(t) = \sum_{i=1}^{\infty} \beta_i(t) e_i,$$

where β_i are mutually independent real Wiener processes with $E(\beta_i^2(t)) = \zeta_i t$.

Remark. If $W(t)$ is a Wiener process in U with covariance operator Q, then $\mathbb{E}|W(t) - W(s)|_U^{2n} \le (2n-1)!(t-s)^n(\mathrm{Tr\,Q})^n$, where the equality holds for $n = 1$.

The above type of structure of Wiener process will be used in the definition of the stochastic integral.

For any Hilbert space H we denote by $M(H)$ the space of all stochastic processes $g\colon [0,T] \times \Omega \to L(U,H)$ such that

$$E\biggl(\int_0^T \|g(t)\|_{L(U,H)}^2 \, dt\biggr) < \infty$$

and for all $u \in U$, $(g(t)u)$, $t \in [0,T]$ is an H-valued and $\mathcal{F}_t$-adapted stochastic process.

For each $t \in [0,T]$, the stochastic integral $\int_0^t g(s)\, dW(s) \in H$ is defined for all $g \in M(H)$ by

$$\int_0^t g(s)\, dW(s) = \lim_{m\to\infty} \sum_{i=1}^{m} \int_0^t g(s) e_i \, d\beta_i(s)$$

in $L^2(\Omega; H)$.

We shall show that the series in the above formula is convergent. Let $W^{(m)}(t) = \sum_{i=1}^{m} e_i \beta_i(t)$. Then, the integral

$$\int_0^t g(s)\, dW^{(m)}(s) = \sum_{i=1}^{m} \int_0^t g(s) e_i \, d\beta_i(s)$$

is well defined for $g \in M(H)$ and additionally

$$\int_0^t g(s)\, dW^{(m)}(s) \xrightarrow{m\to\infty} \int_0^t g(s)\, dW(s)$$

in $L^2(\Omega; H)$.

This convergence comes from the fact that the sequence

$$y_m = \int_0^t g(s)\, dW^{(m)}(s), \quad m \in \mathbb{N}$$

is Cauchy sequence in the space of square integrable random variables. Using properties of stochastic integrals with respect to $\beta_i(s)$, for any $m, n \in \mathbb{N}$, $m < n$, we have:

$$\begin{aligned} E(|y_n - y_m|_H^2) &= \sum_{i=m+1}^{n} \zeta_i E \int_0^t (g(s)e_i, g(s)e_i)_H \, ds \\ &\leq \Big(\sum_{i=m+1}^{n} \zeta_i\Big) E \int_0^t \|g(s)\|^2_{L(U,H)}\, ds \xrightarrow{m,n\to\infty} 0. \end{aligned} \tag{2.1}$$

Hence, there exists a limit of the sequence (y_m) which defines the stochastic integral $\int_0^t g(s)\, dW(s)$.

The stochastic integral defined above has the following properties (see [42]).

Proposition 2.8. *Let $g \in M(H)$. Then*

(a)
$$\mathbb{E}\left(\int_0^T g(t) dW(t)\right) = 0;$$

(b)
$$\begin{aligned} \mathbb{E}\left|\int_0^T g(t)dW(t)\right|_H^2 &= \int_0^T \mathbb{E}\left(\mathrm{Tr}(g(t)Qg^*(t))\right) dt \\ &= \int_0^T \mathbb{E}\left(\mathrm{Tr}(g^*(t)g(t)Q)\right) dt \leq \mathrm{Tr}\, Q \int_0^T \mathbb{E}\, |g(t)|_H^2\, dt; \end{aligned}$$

(c)
$$\begin{aligned} \mathbb{E}\left[\sup_{t\in[0,T]} \left|\int_0^t g(s)\, dW(s)\right|_H^2\right] &\leq 4\,\mathbb{E}\left|\int_0^T g(s)\, dW(s)\right|_H^2 \\ &\leq 4\mathrm{Tr}\, Q \int_0^T \mathbb{E}\, |g(s)|_H^2\, ds; \end{aligned}$$

(d)
$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|\int_0^t g(s)\, dW(s)\right|_H\right] \leq 3\,\mathbb{E}\left[\int_0^T \mathrm{Tr}(g(s)Qg^*(s))\, ds\right]^{1/2}.$$

(Since $|\int_0^t g(s)\, dW(s)|_H^2$ is a submartinagale, (c) follows from Doob's inequality. Property (d) is also a consequence of a general inequality for martingales.)

2.3. Stochastic integral with respect to cylindrical Wiener process

The construction of the stochastic integral in Section 2.2 required that Q was a nuclear operator. In some cases, this assumption seems to be artificial. For instance, all processes stationary with respect to space variable, have non-nuclear covariance operator. So, we shall extend the definition of the stochastic integral to the case of general bounded self-adjoint, non-negative operator Q on Hilbert space U. In this section we provide a construction, published in [46], of stochastic integral with respect to an infinite dimensional cylindrical Wiener process alternative to that given in [23]. The construction is based on the stochastic integrals with respect to real-valued Wiener processes. The advantage of using of such a construction is that we can use basic results and arguments of the finite dimensional case. To avoid trivial complications we shall assume that Q is strictly positive, that is: Q is non-negative and $Qx \neq 0$ for $x \neq 0$.

Let us introduce the subspace U_0 of the space U defined by $U_0 = Q^{1/2}(U)$ with the norm

$$|u|_{U_0} = |Q^{-1/2}u|_U, \quad u \in U_0.$$

Assume that U_1 is an arbitrary Hilbert space such that U is continuously embedded into U_1 and the embedding of U_0 into U_1 is a Hilbert–Schmidt operator.

In particular, when $Q = I$, then $U_0 = U$ and the embedding of U into U_1 is Hilbert–Schmidt operator. When Q is a nuclear operator, that is, $\operatorname{Tr} Q < \infty$, then $U_0 = Q^{1/2}(U)$ and we can take $U_1 = U$. Because in this case $Q^{1/2}$ is Hilbert–Schmidt operator then the embedding $U_0 \subset U$ is Hilbert–Schmidt operator.

We denote by $L_2^0 = L_2(U_0, H)$ the space of Hilbert–Schmidt operators acting from U_0 into H.

Let us consider the norm of the operator $\psi \in L_2^0$:

$$\|\psi\|_{L_2^0}^2 = \sum_{h,k=1}^{\infty} (\psi g_h, f_k)_H^2 = \sum_{h,k=1}^{\infty} \lambda_h (\psi e_h, f_k)_H^2 = \|\psi Q^{1/2}\|_{HS}^2 = \operatorname{Tr}(\psi Q \psi^*),$$

where $g_j = \sqrt{\lambda_j} e_j$, and $\{\lambda_j\}$, $\{e_j\}$ are eigenvalues and eigenfunctions of the operator Q; $\{g_j\}$, $\{e_j\}$ and $\{f_j\}$ are orthonormal bases of spaces U_0, U and H, respectively.

The space L_2^0 is a separable Hilbert space with the norm $\|\psi\|_{L_2^0}^2 = \operatorname{Tr}(\psi Q \psi^*)$.

Particular cases:

(1) If $Q = I$ then $U_0 = U$ and the space L_2^0 becomes $L_2(U, H)$.

(2) When Q is a nuclear operator then $L(U, H) \subset L_2(U_0, H)$. Assume that $K \in L(U, H)$ and let us consider the operator $\psi = K|_{U_0}$, that is the restriction of operator K to the space U_0. Because Q is nuclear operator, then $Q^{1/2}$ is Hilbert–Schmidt operator. So, the embedding J of the space U_0 into U is Hilbert–Schmidt operator. We have to compute the

norm $\|\psi\|_{L_2^0}$ of the operator $\psi: U_0 \to H$. We obtain $\|\psi\|_{L_2^0}^2 \equiv \|KJ\|_{L_2^0}^2 = \operatorname{Tr} KJ(KJ)^*$, where $J: U_0 \to U$.

Because J is Hilbert–Schmidt operator and K is linear bounded operator then, basing on the theory of Hilbert–Schmidt operators (e.g. [35, Chapter I]), KJ is Hilbert–Schmidt operator, too. Next, $(KJ)^*$ is Hilbert–Schmidt operator. In consequence, $KJ(KJ)^*$ is nuclear operator, so $\operatorname{Tr} KJ(KJ)^* < \infty$. Hence, $\psi = K|_{U_0}$ is Hilbert–Schmidt operator on the space U_0, that is $K \in L_2(U_0, H)$.

Although Propositions 2.9 and 2.10 introduced below are known (see e.g. Proposition 4.11 in the monograph [23]), because of their importance we formulate them again. In both propositions, $\{g_j\}$ denotes an orthonormal basis in U_0 and $\{\beta_j\}$ is a family of independent standard real-valued Wiener processes.

Proposition 2.9. *The formula*

$$W_c(t) = \sum_{j=1}^{\infty} g_j \beta_j(t), \tag{2.2}$$

for $t \geq 0$, defines Wiener process in U_1 with the covariance operator Q_1 such that $\operatorname{Tr} Q_1 < \infty$.

Proof. This comes from the fact that the series (2.2) is convergent in space $L^2(\Omega, \mathcal{F}, P; U_1)$. We have

$$\begin{aligned}
E\left(\left|\sum_{j=1}^{n} g_j\beta_j(t) - \sum_{j=1}^{m} g_j\beta_j(t)\right|_{U_1}^2\right) &= E\left(\left|\sum_{j=m+1}^{n} g_j\beta_j(t)\right|_{U_1}^2\right) \\
&= E\left(\sum_{j=m+1}^{n} g_j\beta_j(t), \sum_{k=m+1}^{n} g_k\beta_k(t)\right)_{U_1} = E\sum_{j=m+1}^{n} (g_j\beta_j(t), g_j\beta_j(t))_{U_1} \\
&= E\left(\sum_{j=m+1}^{n} (g_j, g_j)_{U_1} \beta_j^2(t)\right) = t\sum_{j=m+1}^{n} |g_j|_{U_1}^2,
\end{aligned}$$

for $n \geq m \geq 1$. From the assumption, the embedding $J: U_0 \to U_1$ is Hilbert–Schmidt operator, then for the basis $\{g_j\}$, complete and orthonormal in U_0, we have $\sum_{j=1}^{\infty} |Jg_j|_{U_1}^2 < \infty$. Because $Jg_j = g_j$ for any $g_j \in U_0$, then $\sum_{j=1}^{\infty} |g_j|_{U_1}^2 < \infty$ which means $\sum_{j=m+1}^{n} |g_j|_{U_1}^2 \to 0$ when $m, n \to \infty$.

Conditions (a)–(c) and (e) of the Definition 2.7 of Wiener process are obviously satisfied. The process defined by (2.2) is Gaussian because $\beta_j(t)$, $j \in \mathbb{N}$, are independent Gaussian processes. By Kolmogorov test theorem (see e.g. [23, Theorem 3.3]), trajectories of the process $W_c(t)$ are continuous (condition (d) of the definition of Wiener process) because $W_c(t)$ is Gaussian.

Let $Q_1 \colon U_1 \to U_1$ denote the covariance operator of the process $W_c(t)$ defined by (2.2). From the definition of covariance, for $a, b \in U_1$ we have:

$$\begin{aligned}(Q_1 a, b)_{U_1} &= E(a, W_c(t))_{U_1} (b, W_c(t))_{U_1} = E\Big(\sum_{j=1}^{\infty} (a, g_j)_{U_1} (b, g_j)_{U_1} \beta_j^2(t) \Big) \\ &= t \sum_{j=1}^{\infty} (a, g_j)_{U_1} (b, g_j)_{U_1} = t \Big(\sum_{j=1}^{\infty} g_j (a, g_j)_{U_1}, b \Big)_{U_1} .\end{aligned}$$

Hence $Q_1 a = t \sum_{j=1}^{\infty} g_j (a, g_j)_{U_1}$.

Because the covariance operator Q_1 is non-negative, then (by Proposition C.3 in [23]) Q_1 is a nuclear operator if and only if $\sum_{j=1}^{\infty} (Q_1 h_j, h_j)_{U_1} < \infty$, where $\{h_j\}$ is an orthonormal basis in U_1.

From the above considerations

$$\sum_{j=1}^{\infty} (Q_1 h_j, h_j)_{U_1} \leq t \sum_{j=1}^{\infty} |g_j|_{U_1}^2$$

and then

$$\sum_{j=1}^{\infty} (Q_1 h_j, h_j)_{U_1} \equiv \operatorname{Tr} Q_1 < \infty. \qquad \square$$

Proposition 2.10. *For any $a \in U$ the process*

$$(a, W_c(t))_U = \sum_{j=1}^{\infty} (a, g_j)_U \beta_j(t) \tag{2.3}$$

is real-valued Wiener process and

$$E(a, W_c(s))_U (b, W_c(t))_U = (s \wedge t)(Qa, b)_U \quad \textit{for } a, b \in U.$$

Additionally, $\operatorname{Im} Q_1^{1/2} = U_0$ *and* $|u|_{U_0} = |Q_1^{-1/2} u|_{U_1}$.

Proof. We shall prove that the series (2.3) defining the process $(a, W_c(t))_U$ is convergent in the space $L^2(\Omega; \mathbb{R})$.

Let us notice that the series (2.3) is the sum of independent random variables with zero mean. Then the series does converge in $L^2(\Omega; \mathbb{R})$ if and only if the following series $\sum_{j=1}^{\infty} E((a, g_j)_U \beta_j(t))^2$ converges.

Because J is Hilbert–Schmidt operator, we obtain

$$\sum_{j=1}^{\infty} E((a, g_j)_U^2 \beta_j^2(t)) = \sum_{j=1}^{\infty} (a, g_j)_U^2 \leq |a|_U^2 \sum_{j=1}^{\infty} |g_j|_U^2 \leq C |a|_U^2 \sum_{j=1}^{\infty} |J g_j|_{U_1}^2 < \infty.$$

Hence, the series (2.3) does converge. Moreover, when $t \geq s \geq 0$, we have

$$\begin{aligned} &E((a,W_c(t))_U(b,W_c(s))_U)\\ &\quad = E((a,W_c(t)-W_c(s))_U(b,W_c(s))_U) + E((a,W_c(s))_U(b,W_c(s))_U)\\ &\quad = E((a,W_c(s))_U(b,W_c(s))_U) = E\bigg(\Big[\sum_{j=1}^{\infty}(a,g_j)_U\beta_j(s)\Big]\Big[\sum_{k=1}^{\infty}(b,g_k)_U\beta_k(s)\Big]\bigg). \end{aligned}$$

Let us introduce

$$S^a := \sum_{j=1}^{\infty}(a,g_j)_U\beta_j(t), \quad S^b := \sum_{k=1}^{\infty}(b,g_k)_U\beta_k(t), \quad \text{for } a,b\in U.$$

Next, let S^a_N and S^b_N denote the partial sums of the series S^a and S^b, respectively. From the above considerations the series S^a and S^b are convergent in $L^2(\Omega;\mathbb{R})$. Hence $E(S^aS^b) = \lim_{N\to\infty} E(S^a_NS^b_N)$. In fact,

$$\begin{aligned} &E|S^a_NS^b_N - S^aS^b|\\ &\qquad = E|S^a_NS^b_N - S^a_nS^b + S^bS^a_n - S^bS^a| \leq E|S^a_N||S^b_N - S^b| + E|S^b||S^a_N - S^a|\\ &\qquad \leq (E|S^a_N|^2)^{1/2}(E|S^b_N - S^b|^2)^{1/2} + (E|S^b|^2)^{1/2}(E|S^a_N - S^a|^2)^{1/2} \xrightarrow{N\to\infty} 0 \end{aligned}$$

because S^a_N converges to S^a and S^b_N converges to S^b in quadratic mean. Additionally,

$$E(S^a_NS^b_N) = t\sum_{j=1}^{N}(a,g_j)_U(b,g_j)_U$$

and, when $N\to\infty$,

$$E(S^aS^b) = t\sum_{j=1}^{\infty}(a,g_j)_U(b,g_j)_U.$$

Let us notice that

$$\begin{aligned} (Q_1a,b)_{U_1} &= E(a,W_c(1))_{U_1}(b,W_c(1))_{U_1} = \sum_{j=1}^{\infty}(a,g_j)_{U_1}(b,g_j)_{U_1}\\ &= \sum_{j=1}^{\infty}(a,Jg_j)_{U_1}(b,Jg_j)_{U_1} = \sum_{j=1}^{\infty}(J^*a,g_j)_{U_0}(J^*b,g_j)_{U_0}\\ &= \bigg(J^*a,\sum_{j=1}^{\infty}(J^*b,g_j)g_j\bigg)_{U_0} = (J^*a,J^*b)_{U_0} = (JJ^*a,b)_{U_1}. \end{aligned}$$

That gives $Q_1 = JJ^*$. In particular

$$|Q_1^{1/2}a|^2_{U_1} = (JJ^*a,a)_{U_1} = |J^*a|^2_{U_0}, \quad a\in U_1. \tag{2.4}$$

Having (2.4), we can use theorems on images of linear operators, e.g. [23, Appendix B.2, Proposition B.1(ii)].

By that proposition $\operatorname{Im} Q_1^{1/2} = \operatorname{Im} J$. But for any $j \in \mathbb{N}$, and $g_j \in U_0$, $Jg_j = g_j$, that is $\operatorname{Im} J = U_0$. Then $\operatorname{Im} Q_1^{1/2} = U_0$.

Moreover, the operator $G = Q_1^{-1/2} J$ is a bounded operator from U_0 on U_1. From (2.4) the adjoint operator $G^* = J^* Q_1^{-1/2}$ is an isometry, so G is isometry, too. Thus

$$|Q_1^{-1/2} u|_{U_1} = |Q_1^{-1/2} J u|_{U_1} = |u|_{U_0}. \qquad \square$$

In the case when Q is nuclear operator, $Q^{1/2}$ is Hilbert-Schmidt operator. Taking $U_1 = U$, the process $W_c(t)$, $t \geq 0$, defined by (2.2) is the classical Wiener process introduced in Definition 2.7.

Definition 2.11. The process $W_c(t)$, $t \geq 0$, defined in (2.2), is called *cylindrical Wiener process on U* when $\operatorname{Tr} Q = \infty$.

The stochastic integral with respect to cylindrical Wiener process is defined as follows.

As we have already written above, the process $W_c(t)$ defined by (2.2) is a Wiener process in the space U_1 with the covariance operator Q_1 such that $\operatorname{Tr} Q_1 < \infty$. Then the stochastic integral $\int_0^t g(s)\, dW_c(s) \in H$ is well defined in U_1, where $g(s) \in L(U_1, H)$.

Let us notice that U_1 is not uniquely determined. The space U_1 can be an arbitrary Hilbert space such that U is continuously embedded into U_1 and the embedding of U_0 into U_1 is a Hilbert–Schmidt operator. We would like to define the stochastic integral with respect to cylindrical Wiener proces $W_c(t)$ (given by (2.2)) in such a way that the integral is well defined on the space U and does not depend on the choice of the space U_1.

We denote by $\mathcal{N}^2(0,T;L_2^0)$ the space of all stochastic processes

$$\Phi\colon [0,T] \times \Omega \to L_2(U_0, H) \tag{2.5}$$

such that

$$||\Phi||_T^2 := E\left(\int_0^T ||\Phi(t)||^2_{L_2(U_0,H)}\, dt \right) < \infty \tag{2.6}$$

and for all $u \in U_0$, $(\Phi(t)u)$, $t \in [0,T]$, is an H-valued and $\mathcal{F}_t$-adapted stochastic process.

The stochastic integral $\int_0^t \Phi(s)\, dW_c(s) \in H$ with respect to cylindrical Wiener process, given by (2.2) for any process $\Phi \in \mathcal{N}^2(0,T;L_2^0)$, can be defined as the limit

$$\int_0^t \Phi(s)\, dW_c(s) = \lim_{m\to\infty} \sum_{j=1}^m \int_0^t \Phi(s) g_j\, d\beta_j(s) \tag{2.7}$$

in $L^2(\Omega; H)$.

Comment. Before we prove that the stochastic integral given by the formula (2.7) is well defined, let us recall properties of the operator Q_1. From Proposition 2.9, cylindrical Wiener process $W_c(t)$ given by (2.2) has the covariance operator $Q_1 \colon U_1 \to U_1$, which is a nuclear operator in the space U_1, that is $\operatorname{Tr} Q_1 < \infty$. Next, basing on Proposition 2.10, $Q_1^{1/2} \colon U_1 \to U_0$, $\operatorname{Im} Q_1^{1/2} = U_0$ and $|u|_{U_0} = |Q_1^{-1/2} u|_{U_1}$ for $u \in U_0$.

Moreover, from the above considerations and properties of the operator Q_1 we may deduce that $L(U_1, H) \subset L_2(U_0, H)$. This means that each operator $\Phi \in L(U_1, H)$, that is linear and bounded from U_1 into H, is Hilbert–Schmidt operator acting from U_0 into H, that is $\Phi \in L_2(U_0, H)$ when $\operatorname{Tr} Q_1 < \infty$ in U_1. This means that conditions (2.5) and (2.6) for the family $\mathcal{N}^2(0, T; L_2^0)$ of integrands are natural assumptions for the stochastic integral given by (2.7).

Now, we shall prove that the series from the right hand side of (2.7) is convergent. Denote

$$W_c^{(m)}(t) := \sum_{j=1}^{m} g_j \beta_j(t) \quad \text{and} \quad Z_m := \int_0^t \Phi(s) W_c^{(m)}(s), \quad t \in [0, T].$$

Then, for $n \geq m \geq 1$, we have

$$\begin{aligned} E(|Z_n - Z_m|_H^2) &= E \bigg| \sum_{j=m+1}^{n} \int_0^t \Phi(s) g_j \, d\beta_j(s) \bigg|_H^2 \\ &\leq E \sum_{j=m+1}^{n} \int_0^t |\Phi(s) g_j|_H^2 \, ds \xrightarrow{m,n\to\infty} 0, \end{aligned}$$

because from the assumption (2.6)

$$E \int_0^t \left(\sum_{j=1}^{\infty} |\Phi(s) g_j|_H^2 \right) ds < \infty.$$

Then, the sequence (Z_m) is Cauchy sequence in the space of square-integrable random variables. So, the stochastic integral with respect to cylindrical Wiener process given by (2.7) is well defined.

As we have already mentioned, the space U_1 is not uniquely determined. Hence, the cylindrical Wiener proces $W_c(t)$ defined by (2.2) is not uniquely determined either.

Let us notice that the stochastic integral defined by (2.7) does not depend on the choice of the space U_1. Firstly, in the formula (2.7) there are not elements of the space U_1 but only $\{g_j\}$-basis of U_0. Additionally, in (2.7) there are not eigenfunctions of the covariance operator Q_1. Secondly, the class $\mathcal{N}^2(0, T; L_2^0)$ of integrands does not depend on the choice of the space U_1 because (by Proposition 2.10) the spaces $Q_1^{1/2}(U_1)$ are identical for any spaces U_1:

$$Q_1^{1/2} \colon U_1 \to U_0 \quad \text{and} \quad \operatorname{Im} Q_1^{1/2} = U_0.$$

2.3.1. Properties of the stochastic integral. In this subsection we recall from [23] some properties of stochastic integral used in the paper.

Proposition 2.12. *Assume that* $\Phi \in \mathcal{N}^2(0,T;L_0^2)$. *Then the stochastic integral* $\Phi \bullet W(t) := \int_0^t \Phi(s)\,dW(s)$ *is a continuous square integrable martingale and its quadratic variation is of the form*

$$\langle\!\langle \Phi \bullet W(t) \rangle\!\rangle = \int_0^t Q_\Phi(s)\,ds,$$

where $Q_\Phi(s) = (\Phi(s)Q^{1/2})(\Phi(s)Q^{1/2})^*$ *for* $s,t \in [0,T]$.

Proposition 2.13. *If* $\Phi \in \mathcal{N}^2(0,T;L_0^2)$, *then*

$$\mathbb{E}(\Phi \bullet W(t)) = 0 \quad \textit{and} \quad \mathbb{E}|\Phi \bullet W(t)|_H^2 < \infty \quad \textit{for } t \in [0,T].$$

Proposition 2.14. *Assume that* $\Phi_1, \Phi_2 \in \mathcal{N}^2(0,T;L_0^2)$. *Then the correlation operators*

$$V(s,t) := \mathrm{Cor}(\Phi_1 \bullet W(s), \Phi_2 \bullet W(t)),$$

for $s,t \in [0,T]$ *are given by the formula*

$$V(s,t) = \mathbb{E}\int_0^{s\wedge t} (\Phi_2(r)Q^{1/2})(\Phi_1(r)Q^{1/2})^*\,dr.$$

Corollary 2.15. *From the definition of the correlation operator we have*

$$\mathbb{E}(\Phi_1 \bullet W(s), \Phi_2 \bullet W(t))_H = \mathbb{E}\int_0^{s\wedge t} \mathrm{Tr}[(\Phi_2(r)Q^{1/2})(\Phi_1(r)Q^{1/2})^*]\,dr.$$

2.4. The stochastic Fubini theorem and the Itô formula

The below theorems are recalled directly from the book by Da Prato and Zabczyk [23].

Assume that $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ is a probability space, $\Omega_T := [0,T]\times\Omega$ and recall that $\mathcal{P}_T$ is the σ-field defined in Section 2.1, that is $\mathcal{P}_T$ is the σ-field generated by sets of the form: $(s,t]\times F$, where $0 \leq s \leq t \leq T$, $F \in \mathcal{F}_s$ and $\{0\}\times F$, when $F \in \mathcal{F}_0$.

Let $(E,\mathcal{E})$ be a measurable space and let

$$\text{(2.8)} \quad \Phi\colon (t,\omega,x) \to \Phi(t,\omega,x) \text{ be a measurable mapping}$$
$$\text{from } (\Omega_T \times E, \mathcal{P}_T \times \mathcal{B}(E)) \text{ into } (L_2^0, \mathcal{B}(L_2^0)),$$

where $\mathcal{B}(E)$ and $\mathcal{B}(L_2^0)$ denote Borel σ-fields on E and L_2^0, respectively. Thus, in particular, for arbitrary $x \in E$, $\Phi(\,\cdot\,,\,\cdot\,,x)$ is a predictable L_2^0-valued process. Let in addition μ be a finite positive measure on $(E,\mathcal{E})$.

Theorem 2.16 (The stochastic Fubini theorem)**.** *Assume* (2.8) *and that*

$$\int_E ||\Phi(\,\cdot\,,\,\cdot\,,x)||_T\,\mu(dx) < \infty.$$

Then P-a.s.

$$\int_E \left[\int_0^T \Phi(t,x)\,dW(t)\right]\mu(dx) = \int_0^T \left[\int_E \Phi(t,x)\,\mu(dx)\right] dW(t).$$

Assume that Φ is an L_2^0-valued process stochastically integrable in $[0,T]$, ϕ is an H-valued predictable process Bochner integrable on $[0,T]$, P-a.s. and $X(0)$ is an $\mathcal{F}_0$-measurable H-valued random variable. Then the following process

$$X(t) = X(0) + \int_0^t \phi(s)\,ds + \int_0^t \Phi(s)\,dW(s), \quad t\in[0,T],$$

is well defined.

Assume that a function $F\colon[0,T]\times H\to R^1$ and its partial derivatives F_t, F_x, F_{xx}, are uniformly continuous on bounded subsets of $[0,T]\times H$.

Theorem 2.17 (The Itô formula)**.** *Under the above conditions*

$$\begin{aligned} F(t,X(t)) = F(0,X(0)) &+ \int_0^t \langle F_x(s,X(s)),\Phi(s)\,dW(s)\rangle_H \\ &+ \int_0^t \{F_t(s,X(s)) + \langle F_x(s,X(s)),\phi(s)\rangle_H \\ &+ \frac{1}{2}\mathrm{Tr}\,[F_{xx}(s,X(s))(\Phi(s)Q^{1/2})(\Phi(s)Q^{1/2})^*]\}\,ds \end{aligned}$$

holds P-a.s. for all $t\in[0,T]$.

CHAPTER 3

STOCHASTIC VOLTERRA EQUATIONS IN HILBERT SPACE

The aim of this chapter is to study some fundamental questions related to the linear convolution type stochastic Volterra equations of the form

$$X(t) = X(0) + \int_0^t a(t-\tau)AX(\tau)\,d\tau + \int_0^t \Psi(\tau)\,dW(\tau), \quad t \geq 0, \tag{3.1}$$

in a separable Hilbert space H. Particularly, we provide sufficient conditions for the existence of strong solutions to some classes of the equation (3.1), which is a stochastic version of the equation (1.1).

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a probability space. In (3.1), the kernel function $a(t)$ and the operator A are the same as previously, $X(0)$ is an H-valued $\mathcal{F}_0$-measurable random variable, W is a cylindrical Wiener process on a separable Hilbert space U and Ψ is an appropriate process defined below.

This chapter is organized as follows. In Section 3.1 we give definitions of solutions to (3.1) and some introductory results concerning stochastic convolution arising in (3.1). Additionally, we show that under some conditions a weak solution to (3.1) is a mild solution and vice versa. These results have been recalled from [48].

Section 3.2 deals with strong solution to (3.1). We formulate sufficient conditions for a stochastic convolution to be a strong solution to (3.1). The above results come from the paper [51], not yet published.

In Section 3.3, based on [52], we study particular class of equations (3.1), that is, so-called fractional Volterra equations. We decided to consider that class of equations separately because of specific problems appearing during the study of such equations. First, we formulate the deterministic results which play the key role for stochastic results. We study in detail α-times resolvent families corresponding to fractional Volterra equations. Next, we consider mild, weak and strong solutions to those equations.

In the whole chapter we shall use the following:

Volterra Assumptions (abbr. (VA))**.**

(a) $A\colon D(A) \subset H \to H$, is a closed linear operator with the dense domain $D(A)$ equipped with the graph norm $|\cdot|_{D(A)}$;
(b) $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ is a scalar kernel;
(c) $S(t)$, $t \geq 0$, are resolvent operators for the Volterra equation (1.1) determined by the operator A and the function $a(t)$, $t \geq 0$.

The domain $D(A)$ is equipped with the graph norm defined as follows: $|h|_{D(A)} := (|h|_H^2 + |Ah|_H^2)^{1/2}$ for $h \in D(A)$, where $|\cdot|_H$ denotes a norm in H. Because H is a separable Hilbert space and A is a closed operator, the space $(D(A), |\cdot|_{D(A)})$ is a separable Hilbert space, too.

$W(t)$, $t \geq 0$, is a cylindrical Wiener process on U with the covariance operator Q and $\operatorname{Tr} Q = \infty$.

By $L_2^0 := L_2(U_0, H)$, as previously, we denote the set of all Hilbert–Schmidt operators acting from U_0 into H, where $U_0 = Q^{1/2}(U)$.

For shortening, we introduce:

Probability Assumptions (abbr. (PA))**.**

(a) $X(0)$ is an H-valued, $\mathcal{F}_0$-measurable random variable;
(b) Ψ belongs to the space $\mathcal{N}^2(0, T; L_2^0)$, where the finite interval $[0, T]$ is fixed.

3.1. Notions of solutions to stochastic Volterra equations

In this section we introduce the definitions of solutions to the stochastic Volterra equation (3.1) and then formulate some results, not yet published, setting a framework for further research.

Definition 3.1. Assume that conditions (VA) and (PA) hold. An H-valued predictable process $X(t)$, $t \in [0, T]$, is said to be a *strong solution* to (3.1), if X has a version such that $P(X(t) \in D(A)) = 1$ for almost all $t \in [0, T]$; for any $t \in [0, T]$

$$\int_0^t |a(t-\tau) A X(\tau)|_H \, d\tau < \infty, \quad P\text{-a.s.} \tag{3.2}$$

and for any $t \in [0, T]$ the equation (3.1) holds P-a.s.

Comment. Because the integral $\int_0^t \Psi(\tau)\, dW(\tau)$ is a continuous H-valued process then the above definition yields continuity of the strong solution.

Let A^* denotes the adjoint of the operator A, with dense domain $D(A^*) \subset H$ and the graph norm $|\cdot|_{D(A^*)}$ defined as follows: $|h|_{D(A^*)} := (|h|_H^2 + |A^* h|_H^2)^{1/2}$, for $h \in D(A^*)$. The space $(D(A^*), |\cdot|_{D(A^*)})$ is a separable Hilbert space.

Definition 3.2. Let conditions (VA) and (PA) hold. An H-valued predictable process $X(t)$, $t \in [0,T]$, is said to be a *weak solution* to (3.1), if $P(\int_0^t |a(t-\tau)X(\tau)|_H \, d\tau < \infty) = 1$ and if for all $\xi \in D(A^*)$ and all $t \in [0,T]$ the following equation holds

$$\langle X(t),\xi\rangle_H = \langle X(0),\xi\rangle_H + \left\langle \int_0^t a(t-\tau)X(\tau)\,d\tau, A^*\xi \right\rangle_H + \left\langle \int_0^t \Psi(\tau)\,dW(\tau),\xi \right\rangle_H, \quad P\text{-a.s.}$$

Definition 3.3. Assume that (VA) are satisfied and $X(0)$ is an H-valued $\mathcal{F}_0$-measurable random variable. An H-valued predictable process $X(t)$, $t \in [0,T]$, is said to be a *mild solution* to the stochastic Volterra equation (3.1), if

$$\mathbb{E}\left(\int_0^t ||S(t-\tau)\Psi(\tau)||^2_{L_2^0}\,d\tau\right) < \infty \quad \text{for } t \le T \tag{3.3}$$

and, for arbitrary $t \in [0,T]$,

$$X(t) = S(t)X(0) + \int_0^t S(t-\tau)\Psi(\tau)\,dW(\tau), \quad P\text{-a.s.} \tag{3.4}$$

We will show that in some cases weak solution to the equation (3.1) coincides with mild solution to (3.1) (see, Subsection 3.1.2). In consequence, having results for the convolution

$$W^\Psi(t) := \int_0^t S(t-\tau)\Psi(\tau)\,dW(\tau), \quad t \in [0,T], \tag{3.5}$$

where $S(t)$ and Ψ are the same as in (3.4), we will obtain results for weak solution to (3.1), too.

3.1.1. Introductory results. In this section we collect some properties of the stochastic convolution of the form

$$W^B(t) := \int_0^t S(t-\tau)B\,dW(\tau) \tag{3.6}$$

in the case when $B \in L(U,H)$ and W is a cylindrical Wiener process.

Lemma 3.4. *Assume that the operators $S(t)$, $t \ge 0$, and B are as above, $S^*(t)$, B^* are their adjoints, and*

$$\int_0^T ||S(\tau)B||^2_{L_2^0}\,d\tau = \int_0^T \mathrm{Tr}[S(\tau)BQB^*S^*(\tau)]\,d\tau < \infty. \tag{3.7}$$

Then we have:

(a) *the process W^B is Gaussian, mean-square continuous on $[0,T]$ and then has a predictable version;*

(b)

$$\mathrm{Cov} W^B(t) = \int_0^t [S(\tau)BQB^*S^*(\tau)]\,d\tau, \quad t \in [0,T]; \tag{3.8}$$

(c) *trajectories of the process* W^B *are* P*-a.s. square integrable on* $[0,T]$.

Proof. (a) Gaussianity of the process W^B follows from the definition and properties of stochastic integral. Let us fix $0 \le t < t+h \le T$. Then

$$W^B(t+h) - W^B(t) = \int_0^t [S(t+h-\tau) - S(t-\tau)]B\,dW(\tau) + \int_t^{t+h} S(t+h-\tau)B\,dW(\tau).$$

Let us note that the above integrals are stochastically independent. Using the extension of the process W (mentioned in Section 3.1) and properties of stochastic integral with respect to real Wiener processes (see e.g. [42]), we have

$$\mathbb{E}\,|W^B(t+h) - W^B(t)|_H^2 = \sum_{k=1}^{\infty} \lambda_k \int_0^t |[S(t+h-\tau) - S(t-\tau)]Be_k|_H^2\,d\tau + \sum_{k=1}^{\infty} \lambda_k \int_t^{t+h} |S(t+h-\tau)Be_k|_H^2\,d\tau := I_1(t,h) + I_2(t,h).$$

Then, invoking (1.4), the strong continuity of $S(t)$ and the Lebesgue dominated convergence theorem, we can pass in $I_1(t,h)$ with $h \to 0$ under the sum and integral signs. Hence, we obtain $I_1(t,h) \to 0$ as $h \to 0$.

Observe that

$$I_2(t,h) = \int_t^{t+h} ||S(t+h-\tau)BQ^{1/2}||_{HS}^2\,d\tau,$$

where $||\cdot||_{HS}$ denotes the norm of Hilbert–Schmidt operator. By the condition (3.7) we have

$$\int_0^T ||S(t)BQ^{1/2}||_{HS}^2\,dt < \infty,$$

what implies that $\lim_{h\to 0} I_2(t,h) = 0$.

The proof for the case $0 \le t-h < t \le T$ is similar. Existence of a predictable version is a consequence of the continuity and Proposition 2.5.

(b) The shape of the covariance (3.8) follows from theory of stochastic integral, see e.g. [23].

(c) From the definition (3.6) and assumption (3.7) we have the following estimate

$$\begin{aligned}\mathbb{E}\int_0^T |W^B(\tau)|_H^2\, d\tau &= \int_0^T \mathbb{E}\,|W^B(\tau)|_H^2\, d\tau\\ &= \int_0^T \mathbb{E}\left|\int_0^\tau S(\tau-r)B\,dW(r)\right|_H^2 d\tau = \int_0^T\int_0^\tau ||S(r)B||_{L_2^0}^2\, dr\, d\tau < \infty.\end{aligned}$$

Hence, the function $W^B(\,\cdot\,)$ may be regarded like random variable with values in the space $L^2(0,T;H)$. $\square$

Now, we formulate an auxiliary result which will be used in the next section.

Lemma 3.5. *Let Volterra assumptions hold with the function $a \in W^{1,1}(\mathbb{R}_+)$. Assume that X is a weak solution to* (3.1) *in the case when $\Psi(t) = B$, where $B \in L(U,H)$ and trajectories of X are integrable P-a.s. on $[0,T]$. Then, for any function $\xi \in C^1([0,t];D(A^*))$, $t \in [0,T]$, the following formula holds*

$$\begin{aligned}\text{(3.9)}\qquad \langle X(t),\xi(t)\rangle_H =\, &\langle X(0),\xi(0)\rangle_H\\ &+\int_0^t \langle (\dot a \star X)(\tau) + a(0)X(\tau), A^*\xi(\tau)\rangle_H\, d\tau\\ &+\int_0^t \langle \xi(\tau), BdW(\tau)\rangle_H + \int_0^t \langle X(\tau),\dot\xi(\tau)\rangle_H d\tau,\end{aligned}$$

where dots above a and ξ mean time derivatives and $\star$ means the convolution.

Proof. First, we consider functions of the form $\xi(\tau) := \xi_0\varphi(\tau)$, $\tau \in [0,T]$, where $\xi_0 \in D(A^*)$ and $\varphi \in C^1[0,T]$. For simplicity we omit index $_H$ in the inner product. Let us denote $F_{\xi_0}(t) := \langle X(t),\xi_0\rangle$, $t \in [0,T]$.

Using Itô's formula to the process $F_{\xi_0}(t)\varphi(t)$, we have

$$\text{(3.10)}\qquad d[F_{\xi_0}(t)\varphi(t)] = \varphi(t)dF_{\xi_0}(t) + \dot\varphi(t)F_{\xi_0}(t)\,dt, \quad t\in[0,T].$$

Because X is weak solution to (3.1), we have

$$\begin{aligned}\text{(3.11)}\quad dF_{\xi_0}(t) &= \left\langle \int_0^t \dot a(t-\tau)X(\tau)\,d\tau + a(0)X(t), A^*\xi_0\right\rangle dt + \langle BdW(t),\xi_0\rangle\\ &= \langle (\dot a\star X)(t) + a(0)X(t), A^*\xi_0\rangle\, dt + \langle BdW(t),\xi_0\rangle.\end{aligned}$$

From (3.10) and (3.11), we obtain

$$\begin{aligned}F_{\xi_0}(t)\varphi(t) =\, &F_{\xi_0}(0)\varphi(0) + \int_0^t \varphi(s)\langle(\dot a\star X)(s) + a(0)X(s), A^*\xi_0\rangle\, ds\\ &+\int_0^t \langle \varphi(s)B\,dW(s),\xi_0\rangle + \int_0^t \dot\varphi(s)\langle X(s),\xi_0\rangle\, ds\end{aligned}$$

$$= \langle X(0), \xi(0)\rangle_H + \int_0^t \langle (\dot{a} \star X)(s) + a(0)X(s), A^*\xi(s)\rangle\, ds$$
$$+ \int_0^t \langle B\, dW(s), \xi(s)\rangle + \int_0^t \langle X(s), \dot{\xi}(s)\rangle\, ds.$$

Hence, we proved the formula (3.9) for functions ξ of the form $\xi(s) = \xi_0\varphi(s)$, $s \in [0,T]$. Because such functions form a dense subspace in $C^1([0,T]; D(A^*))$, the proof is completed. $\square$

3.1.2. Results in general case. In this subsection we consider weak and mild solutions to the equation (3.1).

First we study the stochastic convolution defined by (3.5), that is

$$W^\Psi(t) := \int_0^t S(t-\tau)\Psi(\tau)\, dW(\tau), \quad t \in [0,T].$$

Proposition 3.6. *Assume that $S(t)$, $t \geq 0$, are (as earlier) the resolvent operators corresponding to the Volterra equation* (1.1)*. Then, for arbitrary process $\Psi \in \mathcal{N}^2(0,T; L_2^0)$, the process $W^\Psi(t)$, $t \geq 0$, given by* (3.5) *has a predictable version.*

Proof. Because proof of Proposition 3.6 is analogous to some schemes in the theory of stochastic integrals (see e.g. [59, Chapter 4]) we provide only an outline of the proof.

First, let us notice that the process $S(t-\tau)\Psi(\tau)$, where $\tau \in [0,t]$, belongs to $\mathcal{N}^2(0,T; L_2^0)$, because $\Psi \in \mathcal{N}^2(0,T; L_2^0)$. Then we may use the apparently well-known estimate (see e.g. Proposition 4.16 in [23]): for arbitrary $a > 0$, $b > 0$ and $t \in [0,T]$

$$P(|W^\Psi(t)|_H > a) \leq \frac{b}{a^2} + P\bigg(\int_0^t ||S(t-\tau)\Psi(\tau)||^2_{L_2^0} d\tau > b\bigg). \tag{3.12}$$

Because the resolvent operators $S(t)$, $t \geq 0$, are uniformly bounded on compact itervals (see [70]), there exists a constant $C > 0$ such that $||S(t)|| \leq C$ for $t \in [0,T]$. So, we have $||S(t-\tau)\Psi(\tau)||^2_{L_2^0} \leq C^2 ||\Psi(\tau)||^2_{L_2^0}$, $\tau \in [0,T]$.

Then the estimate (3.12) may be rewritten as

$$P(|W^\Psi(t)|_H > a) \leq \frac{b}{a^2} + P\bigg(\int_0^t ||\Psi(\tau)||^2_{L_2^0} d\tau > \frac{b}{C^2}\bigg). \tag{3.13}$$

Let us consider predictability of the process W^Ψ in two steps. In the first step we assume that Ψ is an elementary process understood in the sense given in Section 4.2 in [23]. In this case the process W^Ψ has a predictable version by Proposition 2.5.

In the second step Ψ is an arbitrary process belonging to $\mathcal{N}^2(0,T; L_2^0)$. Since elementary processes form a dense set in the space $\mathcal{N}^2(0,T; L_2^0)$, there exists a sequence (Ψ_n) of elementary processes such that for arbitrary $c > 0$

$$P\bigg(\int_0^T ||\Psi(\tau) - \Psi_n(\tau)||^2_{L_2^0} \, d\tau > c \bigg) \xrightarrow{n\to\infty} 0. \tag{3.14}$$

By the previous part of the proof the sequence W_n^Ψ of convolutions

$$W_n^\Psi(t) := \int_0^t S(t-\tau)\Psi_n(\tau)\, dW(\tau)$$

converges in probability. Hence, it has a subsequence converging almost surely. This implies the predictability of the convolution $W^\Psi(t)$, $t\in[0,T]$. □

Proposition 3.7. *Assume that $\Psi \in \mathcal{N}^2(0,T;L_2^0)$. Then the process $W^\Psi(t)$, $t\in[0,T]$, defined by* (3.5) *has square integrable trajectories.*

Proof. We have to prove that $\mathbb{E}\int_0^T |W^\Psi(t)|_H^2\, dt < \infty$. From Fubini's theorem and properties of stochastic integral

$$\mathbb{E}\int_0^T \left| \int_0^t S(t-\tau)\Psi(\tau)\, dW(\tau)\right|_H^2 dt = \int_0^T \left[\mathbb{E}\left| \int_0^t S(t-\tau)\Psi(\tau)\, dW(\tau)\right|_H^2 \right] dt$$
$$= \int_0^T \int_0^t ||S(t-\tau)\Psi(\tau)||^2_{L_2^0}\, d\tau\, dt \le M \int_0^T \int_0^t ||\Psi(\tau)||^2_{L_2^0}\, d\tau\, dt < \infty$$

(from boundness of operators $S(t)$ and, because $\Psi(\tau)$ are Hilbert–Schmidt). □

Proposition 3.8. *Assume that $a\in BV(\mathbb{R}_+)$,* (VA) *are satisfied and, additionally $S\in C^1(0,\infty;L(H))$. Let X be a predictable process with integrable trajectories. Assume that X has a version such that $P(X(t)\in D(A))=1$ for almost all $t\in[0,T]$ and* (3.3) *holds. If for any $t\in[0,T]$ and $\xi\in D(A^*)$*

$$\langle X(t),\xi\rangle_H = \langle X(0),\xi\rangle_H + \int_0^t \langle a(t-\tau)X(\tau), A^*\xi\rangle_H\, d\tau + \int_0^t \langle \xi, \Psi(\tau)\, dW(\tau)\rangle_H, \quad P\text{-a.s.}, \tag{3.15}$$

then

$$X(t) = S(t)X(0) + \int_0^t S(t-\tau)\Psi(\tau)\, dW(\tau), \quad t\in[0,T]. \tag{3.16}$$

Proof. For simplicity we omit index $_H$ in the inner product. Since $a\in BV(\mathbb{R}_+)$, we see, analogously like in Lemma 3.5, that if (3.15) is satisfied, then

$$\langle X(t),\xi(t)\rangle = \langle X(0),\xi(0)\rangle + \int_0^t \langle (\dot a \star X)(\tau) + a(0)X(\tau), A^*\xi(\tau)\rangle\, d\tau + \int_0^t \langle \Psi(\tau)\, dW(\tau), \xi(\tau)\rangle + \int_0^t \langle X(\tau), \dot\xi(\tau)\rangle\, d\tau, \quad P\text{-a.s.} \tag{3.17}$$

holds for any $\xi\in C^1([0,t],D(A^*))$ for any $t\in[0,T]$.

Now, let us take $\xi(\tau) := S^*(t-\tau)\zeta$ with $\zeta \in D(A^*)$, $\tau \in [0,t]$. The equation (3.17) may be written like

$$\langle X(t), S^*(0)\zeta\rangle = \langle X(0), S^*(t)\zeta\rangle + \int_0^t \langle (\dot a \star X)(\tau) + a(0)X(\tau), A^*S^*(t-\tau)\zeta\rangle\, d\tau$$
$$+ \int_0^t \langle \Psi(\tau)\, dW(\tau), S^*(t-\tau)\zeta\rangle + \int_0^t \langle X(\tau), (S^*(t-\tau)\zeta)'\rangle\, d\tau,$$

where derivative $(\,\cdot\,)'$ in the last term is taken over τ.

Next, using $S^*(0) = I$, we rewrite

$$\begin{aligned} \langle X(t),\zeta\rangle &= \langle S(t)X(0),\zeta\rangle \\ &+ \int_0^t \Big\langle S(t-\tau)A\Big[\int_0^\tau \dot a(\tau-\sigma)X(\sigma)\, d\sigma + a(0)X(\tau)\Big], \zeta\Big\rangle\, d\tau \\ &+ \int_0^t \langle S(t-\tau)\Psi(\tau)\, dW(\tau),\zeta\rangle + \int_0^t \langle \dot S(t-\tau)X(\tau),\zeta\rangle\, d\tau. \end{aligned} \tag{3.18}$$

To prove (3.16) it is enough to show that the sum of the first integral and the third one in the equation (3.18) gives zero.

Because $S \in C^1(0,\infty; L(H))$ we can use properties of resolvent operators and the derivative $\dot S(t-\tau)$ with respect to τ. Then

$$\begin{aligned} I_3 &:= \Big\langle \int_0^t \dot S(t-\tau)X(\tau)\, d\tau, \zeta\Big\rangle = \Big\langle -\int_0^t \dot S(\tau)X(t-\tau)\, d\tau,\zeta\Big\rangle \\ &= \Big\langle -\int_0^t \Big[\int_0^\tau a(\tau-s)AS(s)\, ds\Big]' X(t-\tau)\, d\tau, \zeta\Big\rangle \\ &= \Big\langle \Big(\int_0^t \Big[\int_0^\tau \dot a(\tau-s)AS(s)\, ds\Big] X(t-\tau)\, d\tau \int_0^t a(0)AS(\tau)X(t-\tau)\, d\tau\Big), \zeta\Big\rangle \\ &= \langle (-[A(\dot a \star S)(\tau)\star X](t) - a(0)A(S\star X)(t)), \zeta\rangle. \end{aligned}$$

Note that $a \in BV(\mathbb{R}_+)$ and hence the convolution $(a\star S)(\tau)$ has sense (see [70, Section 1.6] or [3]). Since

$$\int_0^t \langle a(0)AS(t-\tau)X(\tau),\zeta\rangle\, d\tau = \int_0^t \langle a(0)AS(\tau)X(t-\tau),\zeta\rangle\, d\tau$$

and

$$\begin{aligned} I_1 &:= \int_0^t \Big\langle S(t-\tau)A\Big[\int_0^\tau \dot a(\tau-\sigma)X(\sigma)\, d\sigma\Big], \zeta\Big\rangle\, d\tau \\ &= \int_0^t \langle AS(t-\tau)(\dot a\star X)(\tau),\zeta\rangle\, d\tau \\ &= \langle A(S\star(\dot a\star X)(\tau))(t),\zeta\rangle = \langle A((S\star \dot a)(\tau)\star X)(t),\zeta\rangle \end{aligned}$$

for any $\zeta \in D(A^*)$, so $I_1 = -I_3$.

This means that (3.16) holds for any $\zeta \in D(A^*)$. Since $D(A^*)$ is dense in H^*, then (3.16) holds. $\square$

Remark. If (3.1) is parabolic and the kernel $a(t)$ is 3-monotone, understood in the sense defined by Prüss [70, Section 3], then $S \in C^1(0,\infty;L(H))$, and $a \in BV(\mathbb{R}_+)$, respectively.

Comment. Proposition 3.8 shows that under particular conditions a weak solution to (3.1) is a mild solution to the equation (3.1).

Proposition 3.9. *Let Volterra assumptions be satisfied. If* $\Psi \in \mathcal{N}^2(0,T;L_2^0)$, *then the stochastic convolution* W^Ψ *fulfills the equation* (3.15).

Proof. Let us notice that the process W^Ψ has integrable trajectories. From the defintion of convolution (3.5), using Dirichlet's formula and stochastic Fubini's theorem, for any $\xi \in D(A^*)$ we have

$$
\begin{aligned}
&\int_0^t \langle a(t-\tau) W^\Psi(\tau), A^*\xi\rangle_H \, d\tau \\
&\qquad \equiv \int_0^t \left\langle a(t-\tau) \int_0^\tau S(\tau-\sigma)\Psi(\sigma)\, dW(\sigma), A^*\xi\right\rangle_H d\tau \\
&\qquad = \int_0^t \left\langle \left[\int_\sigma^t a(t-\tau) S(\tau-\sigma)\, d\tau\right] \Psi(\sigma)\, dW(\sigma), A^*\xi \right\rangle_H \\
&\qquad = \int_0^t \left\langle A\left[\int_0^{t-\sigma} a(t-\sigma-z) S(z)\, dz\right] \Psi(\sigma)\, dW(\sigma), \xi \right\rangle_H .
\end{aligned}
$$

Next, using definition of convolution and the resolvent equation (1.3), as $A(a\star S)(t-\sigma)x = (S(t-\sigma)-I)x$, for $x \in H$, we can write

$$
\begin{aligned}
&\int_0^t \langle a(t-\tau) W^\Psi(\tau), A^*\xi\rangle_H \, d\tau \\
&\qquad = \left\langle \int_0^t A[(a\star S)(t-\sigma)]\Psi(\sigma)\, dW(\sigma), \xi \right\rangle_H \\
&\qquad = \left\langle \int_0^t [S(t-\sigma)-I]\Psi(\sigma)\, dW(\sigma), \xi \right\rangle_H \\
&\qquad = \left\langle \int_0^t S(t-\sigma)\Psi(\sigma)\, dW(\sigma), \xi \right\rangle_H - \left\langle \int_0^t \Psi(\sigma)\, dW(\sigma), \xi \right\rangle_H .
\end{aligned}
$$

Hence, we obtained the following equation

$$
\langle W^\Psi(t), \xi\rangle_H = \int_0^t \langle a(t-\tau) W^\Psi(\tau), A^*\xi\rangle_H d\tau + \int_0^t \langle \xi, \Psi(\tau)\, dW(\tau)\rangle_H
$$

for any $\xi \in D(A^*)$. $\square$

Corollary 3.10. *Let Volterra assumptions hold with a bounded operator A. If Ψ belongs to $\mathcal{N}^2(0,T;L_2^0)$ then*

$$W^\Psi(t) = \int_0^t a(t-\tau)AW^\Psi(\tau)\,d\tau + \int_0^t \Psi(\tau)\,dW(\tau), \quad P\text{-}a.s. \tag{3.19}$$

Comment. The formula (3.19) says that the convolution W^Ψ is a strong solution to (3.1) if the operator A is bounded.

The below theorem is a consequence of the results obtained up to now.

Theorem 3.11. *Suppose that* (VA) *and* (PA) *hold. Then a strong solution* (*if exists*) *is always a weak solution of* (3.1)*. If, additionally, assumptions of Proposition* 3.8 *are satisfied, a weak solution is a mild solution to the Volterra equation* (3.1)*. Conversely, under conditions of Proposition* 3.9*, a mild solution X is also a weak solution to* (3.1)*.*

Now, we provide two estimates for stochastic convolution (3.5).

Theorem 3.12. *If $\Psi \in \mathcal{N}^2(0,T;L_2^0)$ then the following estimate holds*

$$\sup_{t\le T} \mathbb{E}\,(|W^\Psi(t)|_H) \le C\, M_T\, \mathbb{E} \left(\int_0^T ||\Psi(t)||^2_{L_2^0}\, dt \right)^{1/2}, \tag{3.20}$$

where C is a constant and $M_T = \sup_{t\le T} ||S(t)||$.

Comment. The estimate (3.20) seems to be rather coarse. It comes directly from the definition of stochastic integral. Since (3.20) reduces to the Davis inequality for martingales if $S(\,\cdot\,) = I$, the constant C appears on the right hand side. Unfortunately, we can not use more refined tools, for instance Itô formula (see e.g. [76] for Tubaro's estimate), because the process W^Ψ is not enough regular.

The next result is a consequence of Theorem 3.12.

Theorem 3.13. *Assume that $\Psi \in \mathcal{N}^2(0,T;L_2^0)$. Then*

$$\sup_{t\le T} \mathbb{E}\,(|W^\Psi(t)|_H) \le \widetilde{C}(T)\, ||\Psi||_{\mathcal{N}^2(0,T;L_2^0)},$$

where a constant $\widetilde{C}(T)$ depends on T.

Proof. From (3.5) and property of stochastic integral, writing out the Hilbert–Schmidt norm, we obtain

$$\begin{aligned} \mathbb{E}\,(|W^\Psi(t)|_H) &= \mathbb{E} \left(\left| \int_0^t S(t-\tau)\Psi(\tau)\,dW(\tau) \right|_H \right) \\ &\le C\, \mathbb{E} \left(\int_0^t ||S(t-\tau)\Psi(\tau)||^2_{L_2^0}\, d\tau \right)^{1/2} \end{aligned}$$

$$\le C\,\mathbb{E}\left(\int_0^t ||S(t-\tau)||^2\, ||\Psi(\tau)||^2_{L_2^0}\, d\tau\right)^{1/2}$$
$$\le C M_T \mathbb{E}\left(\int_0^t ||\Psi(\tau)||^2_{L_2^0}\, d\tau\right)^{1/2}$$
$$\le C M_T \left(\mathbb{E}\int_0^t ||\Psi(\tau)||^2_{L_2^0}\, d\tau\right)^{1/2} \le \widetilde{C}(T)\, ||\Psi||_{\mathcal{N}^2(0,T;L_2^0)},$$

where M_T is as above and $\widetilde{C}(T) = C\, M_T$. □

3.2. Existence of strong solution

In this section W is a cylindrical Wiener process, that is, $\mathrm{Tr}\, Q = \infty$ and the spaces U_0, L_2^0, $\mathcal{N}^2(0,T;L_2^0)$ are the same like previously (see definitions in Chapter 2). The results from this section originate from [51] and are not yet published.

Let us recall the stochastic convolution introduced in (3.5)

$$W^\Psi(t) := \int_0^t S(t-\tau)\Psi(\tau)\, dW(\tau),$$

where Ψ belongs to the space $\mathcal{N}^2(0,T;L_2^0)$. In consequence, because resolvent operators $S(t)$, $t \ge 0$, are bounded, then $S(t - \cdot\,)\Psi(\,\cdot\,) \in \mathcal{N}^2(0,T;L_2^0)$, too.

In the sequel, by $A\Phi(t)$, $t \ge 0$, we will denote the composition of the operators $\Phi(t)$ and A.

We will use the following well-known result, where the operator A is, as previously, a closed linear operator with the dense domain $D(A)$ equipped with the graph norm $|\cdot|_{D(A)}$ and $\Phi(t)$, $t \in [0,T]$ is an $L_2(U_0,H)$-predictable process.

Proposition 3.14 (see e.g. [23, Proposition 4.15]). *If $\Phi(t)(U_0) \subset D(A)$, P-a.s. for all $t \in [0,T]$ and*

$$P\biggl(\int_0^T ||\Phi(t)||^2_{L_2^0}\, dt < \infty\biggr) = 1, \qquad P\biggl(\int_0^T ||A\Phi(t)||^2_{L_2^0}\, dt < \infty\biggr) = 1,$$

then

$$P\biggl(\int_0^T \Phi(t)\, dW(t) \in D(A)\biggr) = 1$$
$$\textit{and}\quad A\int_0^T \Phi(t)\, dW(t) = \int_0^T A\Phi(t)\, dW(t), \quad P\textit{-a.s.}$$

Let us recall assumptions of approximation theorems (Theorems 1.19 and 1.20) formulated for Hilbert space H:

(AS1) The operator A is the generator of a C_0-semigroup in H and the kernel function $a(t)$ is completely positive.

(AS2) A generates an exponentialy bounded cosine family in H and the function $a(t)$ is completely positive (or fulfills one of two other cases listed in Remark (a) on page 22).

Lemma 3.15. *Let assumptions* (VA) *be satisfied. Suppose* (AS1) *or* (AS2) *hold. If* Ψ *and* $A\Psi$ *belong to* $\mathcal{N}^2(0,T;L_2^0)$ *and in addition* $\Psi(t)(U_0) \subset D(A)$, P*-a.s., then the following equality holds*

$$W^\Psi(t) = \int_0^t a(t-\tau) A\, W^\Psi(\tau)\, d\tau + \int_0^t \Psi(\tau)\, dW(\tau), \quad P\text{-}a.s. \tag{3.21}$$

Comment. Let us emphasize that assumptions concerning the operators $\Psi(t)$, $t \geq 0$, particularly requirement that $\Psi(t)(U_0) \subset D(A)$, P-a.s., are the same like in semigroup case, see e.g. [23, Proposition 6.4].

Proof. Because formula (3.19) holds for any bounded operator, then it holds for the Yosida approximation A_n of the operator A, too, that is

$$W_n^\Psi(t) = \int_0^t a(t-\tau) A_n W_n^\Psi(\tau)\, d\tau + \int_0^t \Psi(\tau)\, dW(\tau),$$

where

$$W_n^\Psi(t) := \int_0^t S_n(t-\tau)\Psi(\tau)\, dW(\tau)$$

and

$$A_n W_n^\Psi(t) = A_n \int_0^t S_n(t-\tau)\Psi(\tau)\, dW(\tau).$$

Recall that by assumption $\Psi \in \mathcal{N}^2(0,T;L_2^0)$. Because the operators $S_n(t)$ are deterministic and bounded for any $t \in [0,T]$, $n \in \mathbb{N}$, then the operators $S_n(t-\,\cdot\,)\Psi(\,\cdot\,)$ belong to $\mathcal{N}^2(0,T;L_2^0)$, too. In consequence, the difference

$$\Phi_n(t-\,\cdot\,) := S_n(t-\,\cdot\,)\Psi(\,\cdot\,) - S(t-\,\cdot\,)\Psi(\,\cdot\,) \tag{3.22}$$

belongs to $\mathcal{N}^2(0,T;L_2^0)$ for any $t \in [0,T]$ and $n \in \mathbb{N}$. This means that

$$\mathbb{E}\left(\int_0^t ||\Phi_n(t-\tau)||^2_{L_2^0}\, d\tau\right) < \infty \tag{3.23}$$

for any $t \in [0,T]$.

Let us recall that the cylindrical Wiener process $W(t)$, $t \geq 0$, can be written in the form

$$W(t) = \sum_{j=1}^{\infty} g_j\, \beta_j(t), \tag{3.24}$$

where $\{g_j\}$ is an orthonormal basis of U_0 and $\beta_j(t)$ are independent real Wiener processes. From (3.24) we have

$$\int_0^t \Phi_n(t-\tau)\, dW(\tau) = \sum_{j=1}^{\infty} \int_0^t \Phi_n(t-\tau)\, g_j\, d\beta_j(\tau). \tag{3.25}$$

From (3.23), we obtain

$$\mathbb{E}\left[\int_0^t \left(\sum_{j=1}^{\infty} |\Phi_n(t-\tau)\, g_j|_H^2\right) d\tau\right] < \infty \tag{3.26}$$

for any $t \in [0,T]$. Next, from (3.25), properties of stochastic integral and (3.26) we obtain for any $t \in [0,T]$,

$$\begin{aligned}
\mathbb{E}\left|\int_0^t \Phi_n(t-\tau)\, dW(\tau)\right|_H^2 &= \mathbb{E}\left|\sum_{j=1}^{\infty}\int_0^t \Phi_n(t-\tau)\, g_j\, d\beta_j(\tau)\right|_H^2 \\
\le \mathbb{E}\left[\sum_{j=1}^{\infty}\int_0^t |\Phi_n(t-\tau)\, g_j|_H^2\, d\tau\right] &\le \mathbb{E}\left[\sum_{j=1}^{\infty}\int_0^T |\Phi_n(T-\tau)\, g_j|_H^2\, d\tau\right] < \infty.
\end{aligned}$$

By Theorem 1.19 or 1.20, the convergence (1.14) of resolvent families is uniform in t on every compact subset of $\mathbb{R}_+$, particularly on the interval $[0,T]$. Now, we use (1.14) in the Hilbert space H, so (1.14) holds for every $x \in H$. Then, for any fixed j,

$$\int_0^T |[S_n(T-\tau) - S(T-\tau)]\Psi(\tau) g_j|_H^2\, d\tau \to 0 \tag{3.27}$$

for $n \to \infty$. Summing up our considerations, particularly using (3.26) and (3.27) we can write

$$\begin{aligned}
&\sup_{t\in[0,T]} \mathbb{E}\left|\int_0^t \Phi_n(t-\tau)\, dW(\tau)\right|_H^2 \\
&\qquad \equiv \sup_{t\in[0,T]} \mathbb{E}\left|\int_0^t [S_n(t-\tau) - S(t-\tau)]\Psi(\tau)\, dW(\tau)\right|_H^2 \\
&\qquad \le \mathbb{E}\left[\sum_{j=1}^{\infty}\int_0^T |[S_n(T-\tau) - S(T-\tau)]\Psi(\tau)\, g_j|_H^2\, d\tau\right] \to 0
\end{aligned}$$

as $n \to \infty$. Hence, by the Lebesgue dominated convergence theorem

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \mathbb{E}\left|W_n^{\Psi}(t) - W^{\Psi}(t)\right|_H^2 = 0. \tag{3.28}$$

By assumption, $\Psi(t)(U_0) \subset D(A)$, P-a.s. Because $S(t)(D(A)) \subset D(A)$, then $S(t-\tau)\Psi(\tau)(U_0) \subset D(A)$, P-a.s., for any $\tau \in [0,t]$, $t \ge 0$. Hence, by Proposition 3.14, $P(W^{\Psi}(t) \in D(A)) = 1$.

For any $n \in \mathbb{N}$, $t \ge 0$, we have

$$|A_n W_n^{\Psi}(t) - AW^{\Psi}(t)|_H \le N_{n,1}(t) + N_{n,2}(t),$$

where

$$\begin{aligned}
N_{n,1}(t) &:= |A_n W_n^{\Psi}(t) - A_n W^{\Psi}(t)|_H, \\
N_{n,2}(t) &:= |A_n W^{\Psi}(t) - AW^{\Psi}(t)|_H = |(A_n - A)W^{\Psi}(t)|_H.
\end{aligned}$$

Then

$$|A_n W_n^\Psi(t) - AW^\Psi(t)|_H^2 \le N_{n,1}^2(t) + 2N_{n,1}(t)N_{n,2}(t) + N_{n,2}^2(t) < 3[N_{n,1}^2(t) + N_{n,2}^2(t)]. \tag{3.29}$$

Let us study the term $N_{n,1}(t)$. Note that the unbounded operator A generates a semigroup. Then we have for the Yosida approximation the following properties:

$$A_n x = J_n A x \quad \text{for any } x \in D(A), \quad \sup_n ||J_n|| < \infty \tag{3.30}$$

where $A_n x = nAR(n,A)x = AJ_n x$ for any $x \in H$, with $J_n := nR(n,A)$, Moreover (see [27, Chapter II, Lemma 3.4]):

$$\begin{aligned} \lim_{n\to\infty} J_n x &= x \qquad \text{for any } x \in H, \\ \lim_{n\to\infty} A_n x &= Ax \quad \text{for any } x \in D(A). \end{aligned} \tag{3.31}$$

By Proposition 1.21, $AS_n(t)x = S_n(t)Ax$ for every n sufficiently large and for all $x \in D(A)$. So, by Propositions 1.21 and 3.14 and the closedness of A we can write

$$\begin{aligned} A_n W_n^\Psi(t) &\equiv A_n \int_0^t S_n(t-\tau)\Psi(\tau)\,dW(\tau) \\ &= J_n \int_0^t AS_n(t-\tau)\Psi(\tau)\,dW(\tau) = J_n\left[\int_0^t S_n(t-\tau)A\Psi(\tau)\,dW(\tau)\right]. \end{aligned}$$

Analogously,

$$A_n W^\Psi(t) = J_n\left[\int_0^t S(t-\tau)A\Psi(\tau)\,dW(\tau)\right].$$

By (3.30) we have

$$\begin{aligned} N_{n,1}(t) &= \left| J_n \int_0^t [S_n(t-\tau) - S(t-\tau)]A\Psi(\tau)\,dW(\tau)\right|_H \\ &\le \left| \int_0^t [S_n(t-\tau) - S(t-\tau)]A\Psi(\tau)\,dW(\tau)\right|_H. \end{aligned}$$

Since from assumptions $A\Psi \in \mathcal{N}^2(0,T;L_2^0)$, then the term apearing above, $[S_n(t-\tau)-S(t-\tau)]A\Psi(\tau)$ may be treated like the difference Φ_n defined by (3.22).

Hence, from (3.30) and (3.28), for the first term of the right hand side of (3.29) we have

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \mathbb{E}\,(N_{n,1}^2(t)) \to 0.$$

For the second term of (3.29), that is $N^2_{n,2}(t)$, we can follow the same steps as above for proving (3.28).

$$\begin{aligned} N_{n,2}(t) &= |A_n W^\Psi(t) - AW^\Psi(t)|_H \\ &\equiv \left| A_n \int_0^t S(t-\tau)\Psi(\tau)\, dW(\tau) - A\int_0^t S(t-\tau)\Psi(\tau)\, dW(\tau)\right|_H \\ &= \left| \int_0^t [A_n - A] S(t-\tau)\Psi(\tau)\, dW(\tau)\right|_H . \end{aligned}$$

From assumptions, $\Psi, A\Psi \in \mathcal{N}^2(0,T;L_2^0)$. Because A_n, $S(t)$, $t \geq 0$ are bounded, then $A_n S(t-\,\cdot\,)\Psi(\,\cdot\,) \in \mathcal{N}^2(0,T;L_2^0)$, too.

Analogously, $AS(t-\,\cdot\,)\Psi(\,\cdot\,) = S(t-\,\cdot\,)A\Psi(\,\cdot\,) \in \mathcal{N}^2(0,T;L_2^0)$.

Let us note that the set of all Hilbert–Schmidt operators acting from one separable Hilbert space into another one, equipped with the operator norm defined on page 26, is a separable Hilbert space. Particularly, sum of two Hilbert–Schmidt operators is a Hilbert–Schmidt operator, see e.g. [4]. Therefore, we can deduce that the operator $(A_n - A)\ S(t-\,\cdot\,)\Psi(\,\cdot\,) \in \mathcal{N}^2(0,T;L_2^0)$, for any $t \in [0,T]$. Hence, the term $[A_n - A]S(t-\tau)\Psi(\tau)$ may be treated like the difference Φ_n defined by (3.22). So, for any $t \in [0,T]$, we obtain

$$\begin{aligned} \mathbb{E}\left(N^2_{n,2}(t)\right) &= \mathbb{E}\left(\int_0^t \left[\sum_{j=1}^\infty \left|[A_n - A]S(t-\tau)\Psi(\tau)\, g_j\right|_H^2\right] d\tau\right) \\ &\leq \mathbb{E}\left(\sum_{j=1}^\infty \int_0^T \left|[A_n - A]S(t-\tau)\Psi(\tau)\, g_j\right|_H^2 d\tau\right) < \infty. \end{aligned}$$

By the convergence (3.31), for any fixed j,

$$\int_0^T |[A_n - A]S(t-\tau)\Psi(\tau)\, g_j|_H^2\, d\tau \to 0 \quad \text{for } n \to \infty.$$

Summing up our considerations, we have

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \mathbb{E}\,(N^2_{n,2}(t)) \to 0.$$

So, we can deduce that

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \mathbb{E}\,|A_n W^\Psi_n(t) - AW^\Psi(t)|_H^2 = 0,$$

and then (3.21) holds. □

These considerations give rise to the following result.

Theorem 3.16. *Suppose that assumptions of Lemma* 3.15 *hold. Then the equation* (3.1) *has a strong solution. Precisely, the convolution* W^Ψ *defined by* (3.5) *is the strong solution to* (3.1).

Proof. Since Proposition 3.6 and Lemma 3.15 hold, we have to show only the condition (3.2). Let us note that by Proposition 3.7, the convolution $W^\Psi(t)$ has integrable trajectories. Because the closed unbounded linear operator A becomes bounded on $(D(A), |\cdot|_{D(A)})$, see e.g. [78], we obtain that $AW^\Psi(\cdot) \in L^1([0,T];H)$, P-a.s. Next, properties of convolution provide integrability of the function $a(T-\cdot)AW^\Psi(\cdot)$, what finishes the proof. □

3.3. Fractional Volterra equations

Assume, as previously, that H is a separable Hilbert space with a norm $|\cdot|_H$ and A is a closed linear operator with dense domain $D(A) \subset H$ equipped with the graph norm $|\cdot|_{D(A)}$. The purpose of this section is to study the existence of strong solutions for a class of stochastic Volterra equations of the form

$$X(t) = X(0) + \int_0^t a_\alpha(t-\tau)AX(\tau)\,d\tau + \int_0^t \Psi(\tau)\,dW(\tau), \quad t \geq 0, \tag{3.32}$$

where $a_\alpha(t) := t^{\alpha-1}/\Gamma(\alpha)$, $\alpha > 0$, $\Gamma(\alpha)$ is the gamma function and W, Ψ are appropriate stochastic processes. There are several situations that can be modeled by stochastic Volterra equations, see e.g. [40, Section 3.4] and references therein. A similar equation was studied in [12], too. Here we are interested in the study of strong solutions when equation (3.32) is driven by a cylindrical Wiener process W. We give sufficient conditions for stochastic convolution to be a strong solution to (3.32).

The equation (3.32) is a stochastic version of the deterministic Volterra equation

$$u(t) = \int_0^t a_\alpha(t-\tau)Au(\tau)\,d\tau + f(t), \tag{3.33}$$

where f is an H-valued function.

In the case when $a_\alpha(t)$ is a completely positive function, sufficient conditions for existence of strong solutions for (3.32) may be obtained like in Section 3.2, that is, using a method which involves the use of a resolvent family associated to the deterministic version of equation (3.32).

However, there are two kinds of problems that arise when we study (3.32). On the one hand, the kernels $t^{\alpha-1}/\Gamma(\alpha)$ are α-regular and $(\alpha\pi/2)$-sectorial but not completely positive functions for $\alpha > 1$, so e.g. the results in [51] can not be used directly for $\alpha > 1$. On the other hand, for $\alpha \in (0,1)$, we have a singularity of the kernel in $t = 0$. This fact strongly suggests the use of α-times resolvent families associated to equation (3.33). These new tools appeared in [5] as well as their relationship with fractional derivatives. For convenience of the reader, we provide below the main results on α-times resolvent families to be used in this paper.

Our second main ingredient to obtain strong solutions of (3.32) relies on approximation of α-times resolvent families. This kind of result was very recently formulated by Li and Zheng [60]. It enables us to prove a key result on convergence of α-times resolvent families (see Theorem 3.20 below). Then we can follow the methods employed in [51] to obtain existence of solutions — particularly strong — for the stochastic equation (3.32).

3.3.1. Convergence of α-times resolvent families. In this section we formulate the main deterministic results on convergence of resolvents.

By $S_\alpha(t)$, $t \geq 0$, we denote the family of α-times resolvent families corresponding to the Volterra equation (3.33), if it exists, and defined analogously like resolvent family, see Definition 1.1.

Definition 3.17 (see [5]). A family $(S_\alpha(t))_{t\geq 0}$ of bounded linear operators in a Banach space B is called *α-times resolvent family* for (3.33) if the following conditions are satisfied:

(a) $S_\alpha(t)$ is strongly continuous on $\mathbb{R}_+$ and $S_\alpha(0) = I$;
(b) $S_\alpha(t)$ commutes with the operator A, that is, $S_\alpha(t)(D(A)) \subset D(A)$ and $AS_\alpha(t)x = S_\alpha(t)Ax$ for all $x \in D(A)$ and $t \geq 0$;
(c) the following *resolvent equation* holds

$$S_\alpha(t)x = x + \int_0^t a_\alpha(t-\tau)AS_\alpha(\tau)x\,d\tau \tag{3.34}$$

for all $x \in D(A)$, $t \geq 0$.

Necessary and sufficient conditions for existence of the α-times resolvent family have been studied in [5]. Observe that the α-times resolvent family corresponds to a C_0-semigroup in case $\alpha = 1$ and a cosine family in case $\alpha = 2$. In consequence, when $1 < \alpha < 2$ such resolvent families interpolate C_0-semigroups and cosine functions. In particular, for $A = \Delta$, the integrodifferential equation corresponding to such resolvent family interpolates the heat equation and the wave equation, see [32] or [75].

Definition 3.18. An α-times resolvent family $(S_\alpha(t))_{t\geq 0}$ is called *exponentially bounded* if there are constants $M \geq 1$ and $\omega \geq 0$ such that

$$\|S_\alpha(t)\| \leq Me^{\omega t}, \quad t \geq 0. \tag{3.35}$$

If there is the α-times resolvent family $(S_\alpha(t))_{t\geq 0}$ for A and satisfying (3.35), we write $A \in \mathcal{C}^\alpha(M,\omega)$. Also, set

$$\mathcal{C}^\alpha(\omega) := \bigcup_{M\geq 1} \mathcal{C}^\alpha(M,\omega) \quad \text{and} \quad \mathcal{C}^\alpha := \bigcup_{\omega\geq 0} \mathcal{C}^\alpha(\omega).$$

Remark. It was proved by Bazhlekova [5, Theorem 2.6] that if $A \in \mathcal{C}^\alpha$ for some $\alpha > 2$, then A is bounded.

The following subordination principle is very important in the theory of α-times resolvent families (see [5, Theorem 3.1]).

Theorem 3.19. *Let $0 < \alpha < \beta \le 2$, $\gamma = \alpha/\beta$, $\omega \ge 0$. If $A \in \mathcal{C}^\beta(\omega)$ then $A \in \mathcal{C}^\alpha(\omega^{1/\gamma})$ and the following representation holds*

$$S_\alpha(t)x = \int_0^\infty \varphi_{t,\gamma}(s) S_\beta(s)x\, ds, \quad t > 0, \tag{3.36}$$

where $\varphi_{t,\gamma}(s) := t^{-\gamma}\Phi_\gamma(st^{-\gamma})$ and $\Phi_\gamma(z)$ is the Wright function defined as

$$\Phi_\gamma(z) := \sum_{n=0}^\infty \frac{(-z)^n}{n!\,\Gamma(-\gamma n + 1 - \gamma)}, \quad 0 < \gamma < 1. \tag{3.37}$$

Remarks. (a) We recall that the Laplace transform of the Wright function corresponds to $E_\gamma(-z)$ where E_γ denotes the Mittag–Leffler function. In particular, $\Phi_\gamma(z)$ is a probability density function.

(b) Also we recall from [5, (2.9)] that the continuity in $t \ge 0$ of the Mittag–Leffler function together with the asymptotic behavior of it, imply that for $\omega \ge 0$ there exists a constant $C > 0$ such that

$$E_\alpha(\omega t^\alpha) \le C e^{\omega^{1/\alpha} t}, \quad t \ge 0,\ \alpha \in (0,2). \tag{3.38}$$

As we have already written, in this paper the results concerning convergence of α-times resolvent families in a Banach space B will play the key role. Using a very recent result due to Li and Zheng [60] we are able to prove the following theorem.

Theorem 3.20. *Let A be the generator of a C_0-semigroup $(T(t))_{t\ge 0}$ in a Banach space B such that*

$$\|T(t)\| \le M e^{\omega t}, \quad t \ge 0. \tag{3.39}$$

Then, for each $0 < \alpha < 1$ we have $A \in \mathcal{C}^\alpha(M, \omega^{1/\alpha})$. Moreover, there exist bounded operators A_n and α-times resolvent families $S_{\alpha,n}(t)$ for A_n satisfying $||S_{\alpha,n}(t)|| \le MCe^{(2\omega)^{1/\alpha} t}$, for all $t \ge 0$, $n \in \mathbb{N}$ and

$$S_{\alpha,n}(t)x \to S_\alpha(t)x \quad \text{as } n \to \infty \tag{3.40}$$

for all $x \in B$, $t \ge 0$. Moreover, the convergence is uniform in t on every compact subset of $\mathbb{R}_+$.

Proof. Since A is the generator of a C_0 semigroup satisfying (3.39), we have $A \in C^1(\omega)$. Hence, the first assertion follows directly from Theorem 3.19, that is, for each $0 < \alpha < 1$ there is an α-times resolvent family $(S_\alpha(t))_{t\geq 0}$ for A given by

$$S_\alpha(t)x = \int_0^\infty \varphi_{t,\alpha}(s)T(s)x\,ds, \quad t > 0. \tag{3.41}$$

Since A generates a C_0-semigroup, the resolvent set $\rho(A)$ of A contains the ray $[w, \infty)$ and

$$||R(\lambda, A)^k|| \leq \frac{M}{(\lambda - w)^k} \quad \text{for } \lambda > w,\ k \in \mathbb{N}.$$

Define

$$A_n := nAR(n, A) = n^2 R(n, A) - nI, \quad n > w, \tag{3.42}$$

the *Yosida approximation* of A. Then

$$\begin{aligned} ||e^{tA_n}|| = e^{-nt}||e^{n^2R(n,A)t}|| &\leq e^{-nt}\sum_{k=0}^{\infty}\frac{n^{2k}t^k}{k!}||R(n, A)^k|| \\ &\leq Me^{(-n+n^2/(n-w))t} = Me^{nwt/(n-w)}. \end{aligned}$$

Hence, for $n > 2w$ we obtain

$$||e^{A_nt}|| \leq Me^{2wt}. \tag{3.43}$$

Next, since each A_n is bounded, it follows also from Theorem 3.19 that for each $0 < \alpha < 1$ there exists an α-times resolvent family $(S_{\alpha,n}(t))_{t\geq 0}$ for A_n given as

$$S_{\alpha,n}(t) = \int_0^\infty \varphi_{t,\alpha}(s)e^{sA_n}\,ds, \quad t > 0. \tag{3.44}$$

By (3.43) and Remark (a), page 54, it follows that

$$\begin{aligned} \|S_{\alpha,n}(t)\| &\leq \int_0^\infty \varphi_{t,\alpha}(s)\|e^{sA_n}\|\,ds \leq M\int_0^\infty \varphi_{t,\alpha}(s)e^{2\omega s}\,ds \\ &= M\int_0^\infty \Phi_\alpha(\tau)e^{2\omega t^\alpha \tau}\,d\tau = ME_\alpha(2\omega t^\alpha), \quad t \geq 0. \end{aligned}$$

This together with Remark (b), page 54 gives

$$\|S_{\alpha,n}(t)\| \leq MCe^{(2\omega)^{1/\alpha}t}, \quad t \geq 0. \tag{3.45}$$

Now, we recall the fact that $R(\lambda, A_n)x \to R(\lambda, A)x$ as $n \to \infty$ for all λ sufficiently large (see e.g. [65, Lemma 7.3]), so we can conclude from [60, Theorem 4.2] that

$$S_{\alpha,n}(t)x \to S_\alpha(t)x \quad \text{as } n \to \infty \tag{3.46}$$

for all $x \in B$, uniformly for t on every compact subset of $\mathbb{R}_+$. □

An analogous result can be proved in the case when A is the generator of a strongly continuous cosine family.

Theorem 3.21. *Let A be the generator of a C_0-cosine family $(T(t))_{t\geq 0}$ in a Banach space B. Then, for each $0<\alpha<2$ we have $A\in\mathcal{C}^\alpha(M,\omega^{2/\alpha})$. Moreover, there exist bounded operators A_n and α-times resolvent families $S_{\alpha,n}(t)$ for A_n satisfying $||S_{\alpha,n}(t)||\leq MCe^{(2\omega)^{1/\alpha}t}$, for all $t\geq 0$, $n\in\mathbb{N}$ and*

$$S_{\alpha,n}(t)x\to S_\alpha(t)x \quad \textit{as } n\to\infty$$

for all $x\in B$, $t\geq 0$. Moreover, the convergence is uniform in t on every compact subset of $\mathbb{R}_+$.

Let us note that formulae (3.41) and (3.44) still hold when A is the C_0-cosine family and $0<\alpha<2$.

In the following, we denote by $\Sigma_\theta(\omega)$ the open sector with vertex $\omega\in\mathbb{R}$ and opening angle 2θ in the complex plane which is symmetric with respect to the real positive axis, i.e.

$$\Sigma_\theta(\omega):=\{\lambda\in\mathbb{C}: |\arg(\lambda-\omega)|<\theta\}.$$

We recall from [5, Definition 2.13] that an α-times resolvent family $S_\alpha(t)$ is called *analytic* if $S_\alpha(t)$ admits an analytic extension to a sector Σ_{θ_0} for some $\theta_0\in(0,\pi/2]$. An α-times analytic resolvent family is said to be of *analyticity type* (θ_0,ω_0) if for each $\theta<\theta_0$ and $\omega>\omega_0$ there is $M=M(\theta,\omega)$ such that

$$\|S_\alpha(t)\|\leq Me^{\omega Ret}, \quad t\in\Sigma_\theta.$$

The set of all operators $A\in\mathcal{C}^\alpha$ generating α-times analytic resolvent families $S_\alpha(t)$ of type (θ_0,ω_0) is denoted by $\mathcal{A}^\alpha(\theta_0,\omega_0)$. In addition, denote

$$\mathcal{A}^\alpha(\theta_0):=\bigcup\{\mathcal{A}^\alpha(\theta_0,\omega_0);\omega_0\in\mathbb{R}_+\}, \quad \mathcal{A}^\alpha:=\bigcup\{\mathcal{A}^\alpha(\theta_0);\theta_0\in(0,\pi/2]\}.$$

For $\alpha=1$ we obtain the set of all generators of analytic semigroups.

Remark. We note that the spatial regularity condition $\mathcal{R}(S_\alpha(t))\subset D(A)$ for all $t>0$ is satisfied by α-times resolvent families whose generator A belongs to the set $\mathcal{A}^\alpha(\theta_0,\omega_0)$ where $0<\alpha<2$ (see [5, Proposition 2.15]). In particular, setting $\omega_0=0$ we have that $A\in\mathcal{A}^\alpha(\theta_0,0)$ if and only if $-A$ is a positive operator with spectral angle less or equal to $\pi-\alpha(\pi/2+\theta)$. Note that such condition is also equivalent to the following

$$\Sigma_{\alpha(\pi/2+\theta)}\subset\rho(A) \text{ and } \|\lambda(\lambda I-A)^{-1}\|\leq M, \quad \lambda\in\Sigma_{\alpha(\pi/2+\theta)}. \tag{3.47}$$

The above considerations give us the following remarkable corollary.

Corollary 3.22. *Suppose A generates an analytic semigroup of angle $\pi/2$ and $\alpha\in(0,1)$. Then A generates an α-times analytic resolvent family.*

Proof. Since A generates an analytic semigroup of angle $\pi/2$ we have

$$\|\lambda(\lambda I-A)^{-1}\|\leq M, \quad \lambda\in\Sigma_{\pi-\varepsilon}.$$

Then the condition (3.47) (see also [5, Corollary 2.16]) implies

$$A \in \mathcal{A}^\alpha(\min\{(2-\alpha)\pi/2\alpha, \pi/2\}, 0), \quad \alpha \in (0,2),$$

that is A generates an α-times analytic resolvent family. □

In the sequel we will use the following assumptions concerning Volterra equations:

(A1) A is the generator of C_0-semigroup in H and $\alpha \in (0,1)$; or
(A2) A is the generator of a strongly continuous cosine family in H and $\alpha \in (0,2)$.

Observe that (A2) implies (A1) but not vice versa.

3.3.2. Strong solution. As previously H and U are separable Hilbert spaces and W is a cylindrical Wiener process defined on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, P)$, with the positive symmetric covariance operator $Q \in L(U)$, $\operatorname{Tr} Q = \infty$. The spaces $U_0, L_2^0 = L_2(U_0, H)$ and $\mathcal{N}(0,T;L_2^0)$ are the same like in previous sections.

For the reader's convenience we formulate definitions of solutions to the equation (3.32). We define solutions to the equation (3.32) analogously like in Section 3.1.

Definition 3.23. Assume that (PA) hold. An H-valued predictable process $X(t)$, $t \in [0,T]$, is said to be a *strong solution* to (3.32), if X has a version such that $P(X(t) \in D(A)) = 1$, for almost all $t \in [0,T]$; for any $t \in [0,T]$

$$\int_0^t |a_\alpha(t-\tau)AX(\tau)|_H \, d\tau < \infty, \quad P\text{-a.s.}, \ \alpha > 0, \tag{3.48}$$

and for any $t \in [0,T]$ the equation (3.32) holds P-a.s.

Definition 3.24. Let (PA) hold. An H-valued predictable process $X(t)$, $t \in [0,T]$, is said to be a *weak solution* to (3.32), if

$$P\left(\int_0^t |a_\alpha(t-\tau)X(\tau)|_H \, d\tau < \infty\right) = 1, \quad \alpha > 0,$$

and if for all $\xi \in D(A^*)$ and all $t \in [0,T]$ the following equation holds

$$\begin{aligned}\langle X(t), \xi\rangle_H = \langle X(0), \xi\rangle_H + \left\langle \int_0^t a_\alpha(t-\tau)X(\tau)\, d\tau, A^*\xi \right\rangle_H \\ + \left\langle \int_0^t \Psi(\tau)\, dW(\tau), \xi \right\rangle_H, \quad P\text{-a.s.}\end{aligned}$$

Definition 3.25. Assume that $X(0)$ is $\mathcal{F}_0$-measurable random variable. An H-valued predictable process $X(t)$, $t \in [0,T]$, is said to be a *mild solution* to the stochastic Volterra equation (3.32), if

$$\mathbb{E}\left(\int_0^t ||S_\alpha(t-\tau)\Psi(\tau)||^2_{L_2^0}\,d\tau\right) < \infty, \quad \alpha > 0,$$

for $t \le T$ and, for arbitrary $t \in [0,T]$,

$$X(t) = S_\alpha(t)X(0) + \int_0^t S_\alpha(t-\tau)\Psi(\tau)\,dW(\tau), \quad P\text{-a.s.} \tag{3.49}$$

where $S_\alpha(t)$ is the α-times resolvent family.

We define the stochastic convolution

$$W_\alpha^\Psi(t) := \int_0^t S_\alpha(t-\tau)\Psi(\tau)\,dW(\tau), \tag{3.50}$$

where $\Psi \in \mathcal{N}^2(0,T;L_2^0)$. Because α-times resolvent families $S_\alpha(t)$, $t \ge 0$, are bounded, then $S_\alpha(t-\cdot)\Psi(\cdot) \in \mathcal{N}^2(0,T;L_2^0)$, too.

Analogously like in Section 3.1, we can formulate the following results.

Proposition 3.26. *Assume that $S_\alpha(t)$, $t \ge 0$, are the resolvent operators to* (3.33)*. Then, for any process $\Psi \in \mathcal{N}^2(0,T;L_2^0)$, the convolution $W_\alpha^\Psi(t)$, $t \ge 0$, $\alpha > 0$, given by* (3.5) *has a predictable version. Additionally, the process $W_\alpha^\Psi(t)$, $t \ge 0$, $\alpha > 0$, has square integrable trajectories.*

Under some conditions every mild solution to (3.32) is a weak solution to (3.32).

Proposition 3.27. *If $\Psi \in \mathcal{N}^2(0,T;L_2^0)$, then the stochastic convolution W_α^Ψ fulfills the equation*

$$\langle W_\alpha^\Psi(t),\xi\rangle_H = \int_0^t \langle a_\alpha(t-\tau)W_\alpha^\Psi(\tau), A^*\xi\rangle_H + \int_0^t \langle \xi, \Psi(\tau)dW(\tau)\rangle_H, \tag{3.51}$$

$\alpha > 0$ for any $t \in [0,T]$ and $\xi \in D(A^*)$.

Immediately from the equation (3.51) we deduce the following result.

Corollary 3.28. *If A is a bounded operator and $\Psi \in \mathcal{N}^2(0,T;L_2^0)$, then the following equality holds*

$$W_\alpha^\Psi(t) = \int_0^t a_\alpha(t-\tau)AW_\alpha^\Psi(\tau)\,d\tau + \int_0^t \Psi(\tau)\,dW(\tau), \tag{3.52}$$

for $t \in [0,T]$, $\alpha > 0$.

Remark. The formula (3.52) says that the convolution $W_\alpha^\Psi(t)$, $t \ge 0$, $\alpha > 0$, is a strong solution to (3.32) if the operator A is bounded.

We can formulate following result which plays a key role in this subsection.

Lemma 3.29. *Let assumptions* (VA) *be satisfied. Suppose that* (A1) *or* (A2) *holds. If* Ψ *and* $A\Psi$ *belong to* $\mathcal{N}^2(0,T;L_2^0)$ *and in addition* $\Psi(t)(U_0) \subset D(A)$ *P-a.s., then the following equality holds*

$$W_\alpha^\Psi(t) = \int_0^t a_\alpha(t-\tau)A\,W_\alpha^\Psi(\tau)\,d\tau + \int_0^t \Psi(\tau)\,dW(\tau), \quad P\text{-}a.s. \tag{3.53}$$

Remark. Let us emphasize that in (A1), $\alpha \in (0,1)$ and in (A2), $\alpha \in (0,2)$.

Although the proof is analogous to that given in Section 3.1, we formulate it for the reader's convenience.

Proof. Because formula (3.52) holds for any bounded operator, then it holds for the Yosida approximation A_n of the operator A, too, that is,

$$W_{\alpha,n}^\Psi(t) = \int_0^t a_\alpha(t-\tau)A_n W_{\alpha,n}^\Psi(\tau)\,d\tau + \int_0^t \Psi(\tau)\,dW(\tau),$$

where

$$W_{\alpha,n}^\Psi(t) := \int_0^t S_{\alpha,n}(t-\tau)\Psi(\tau)\,dW(\tau)$$

and

$$A_n W_{\alpha,n}^\Psi(t) = A_n \int_0^t S_{\alpha,n}(t-\tau)\Psi(\tau)\,dW(\tau).$$

By assumption $\Psi \in \mathcal{N}^2(0,T;L_2^0)$. Because the operators $S_{\alpha,n}(t)$ are deterministic and bounded for any $t \in [0,T]$, $\alpha > 0$, $n \in \mathbb{N}$, then the operators $S_{\alpha,n}(t-\,\cdot\,)\Psi(\,\cdot\,)$ belong to $\mathcal{N}^2(0,T;L_2^0)$, too. In consequence, the difference

$$\Phi_{\alpha,n}(t-\,\cdot\,) := S_{\alpha,n}(t-\,\cdot\,)\Psi(\,\cdot\,) - S_\alpha(t-\,\cdot\,)\Psi(\,\cdot\,) \tag{3.54}$$

belongs to $\mathcal{N}^2(0,T;L_2^0)$ for any $t \in [0,T]$, $\alpha > 0$ and $n \in \mathbb{N}$. This means that

$$\mathbb{E}\left(\int_0^t ||\Phi_{\alpha,n}(t-\tau)||_{L_2^0}^2\,d\tau\right) < \infty \tag{3.55}$$

for any $t \in [0,T]$.

The cylindrical Wiener process $W(t)$, $t \geq 0$, can be expanded in the series

$$W(t) = \sum_{j=1}^\infty g_j\,\beta_j(t), \tag{3.56}$$

where $\{g_j\}$ is an orthonormal basis of U_0 and $\beta_j(t)$ are independent real Wiener processes. From (3.56) we have

$$\int_0^t \Phi_{\alpha,n}(t-\tau)\,dW(\tau) = \sum_{j=1}^\infty \int_0^t \Phi_{\alpha,n}(t-\tau)\,g_j\,d\beta_j(\tau). \tag{3.57}$$

In consequence, from (3.55)

$$\mathbb{E}\left[\int_0^t \left(\sum_{j=1}^{\infty} |\Phi_{\alpha,n}(t-\tau)\, g_j|_H^2\right) d\tau\right] < \infty \tag{3.58}$$

for any $t \in [0,T]$. Next, from (3.57), properties of stochastic integral and (3.58) we obtain for any $t \in [0,T]$,

$$\mathbb{E}\left|\int_0^t \Phi_{\alpha,n}(t-\tau)\, dW(\tau)\right|_H^2 = \mathbb{E}\left|\sum_{j=1}^{\infty}\int_0^t \Phi_{\alpha,n}(t-\tau)\, g_j\, d\beta_j(\tau)\right|_H^2$$
$$\leq \mathbb{E}\left[\sum_{j=1}^{\infty}\int_0^t |\Phi_{\alpha,n}(t-\tau)\, g_j|_H^2\, d\tau\right] \leq \mathbb{E}\left[\sum_{j=1}^{\infty}\int_0^T |\Phi_{\alpha,n}(T-\tau)\, g_j|_H^2\, d\tau\right] < \infty.$$

By Theorem 3.20 or Theorem 3.21, the convergence of α-times resolvent families is uniform in t on the interval $[0,T]$. So, for any fixed α and j,

$$\int_0^T |[S_{\alpha,n}(T-\tau) - S_\alpha(T-\tau)]\, \Psi(\tau)\, g_j|_H^2 d\tau \to 0 \quad \text{for } n \to \infty. \tag{3.59}$$

Then, using (3.58) and (3.59) we can write

$$\sup_{t\in[0,T]} \mathbb{E}\left|\int_0^t \Phi_{\alpha,n}(t-\tau)\, dW(\tau)\right|_H^2$$
$$\equiv \sup_{t\in[0,T]} \mathbb{E}\left|\int_0^t [S_{\alpha,n}(t-\tau) - S_\alpha(t-\tau)]\Psi(\tau)\, dW(\tau)\right|_H^2$$
$$\leq \mathbb{E}\left[\sum_{j=1}^{\infty}\int_0^T |[S_{\alpha,n}(T-\tau) - S_\alpha(T-\tau)]\Psi(\tau)\, g_j|_H^2\, d\tau\right] \to 0$$

as $n \to \infty$ for any fixed $\alpha > 0$.

Hence, by the Lebesgue dominated convergence theorem we obtained

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \mathbb{E}\, |W_{\alpha,n}^{\Psi}(t) - W_\alpha^{\Psi}(t)|_H^2 = 0. \tag{3.60}$$

By Proposition 3.14, $P(W_\alpha^\Psi(t) \in D(A)) = 1$.

For any $n \in \mathbb{N}$, $\alpha > 0$, $t \geq 0$, we have

$$|A_n W_{\alpha,n}^{\Psi}(t) - AW_\alpha^{\Psi}(t)|_H \leq N_{n,1}(t) + N_{n,2}(t),$$

where

$$N_{n,1}(t) := |A_n W_{\alpha,n}^{\Psi}(t) - A_n W_\alpha^{\Psi}(t)|_H,$$
$$N_{n,2}(t) := |A_n W_\alpha^{\Psi}(t) - AW_\alpha^{\Psi}(t)|_H = |(A_n - A)W_\alpha^{\Psi}(t)|_H.$$

Then

$$|A_n W_{\alpha,n}^{\Psi}(t) - AW_\alpha^{\Psi}(t)|_H^2 < 3[N_{n,1}^2(t) + N_{n,2}^2(t)]. \tag{3.61}$$

Let us study the term $N_{n,1}(t)$. Note that, either in cases (A1) or (A2) the unbounded operator A generates a semigroup. Then we have from the Yosida approximation the following properties:

$$A_n x = J_n A x \quad \text{for any } x \in D(A), \quad \sup_n ||J_n|| < \infty, \tag{3.62}$$

where $A_n x = nAR(n,A)x = AJ_n x$ for any $x \in H$ with $J_n := nR(n,A)$. Moreover (see [27, Chapter II, Lemma 3.4]):

$$\lim_{n\to\infty} n\,R(n,A)x = x \qquad \text{for any } x \in H, \tag{3.63}$$

$$\lim_{n\to\infty} A_n x = Ax \quad \text{for any } x \in D(A). \tag{3.64}$$

Note that $AS_{\alpha,n}(t)x = S_{\alpha,n}(t)Ax$ for all $x \in D(A)$, since e^{tA_n} commutes with A and A is closed (see (3.44)). So, by Proposition 3.14 and again the closedness of A, we can write

$$\begin{aligned} A_n W_{\alpha,n}(t) &\equiv A_n \int_0^t S_{\alpha,n}(t-\tau)\Psi(\tau)\,dW(\tau) \\ &= nR(n,A)\left[\int_0^t S_{\alpha,n}(t-\tau)A\Psi(\tau)\,dW(\tau)\right]. \end{aligned}$$

Analogously,

$$A_n W_\alpha(t) = nR(n,A)\left[\int_0^t S_\alpha(t-\tau)A\Psi(\tau)\,dW(\tau)\right].$$

By (3.63) we have

$$\begin{aligned} N_{n,1}(t) &= \left| J_n \int_0^t [S_{\alpha,n}(t-\tau) - S_\alpha(t-\tau)]A\Psi(\tau)\,dW(\tau)\right|_H \\ &\le \left| \int_0^t [S_{\alpha,n}(t-\tau) - S_\alpha(t-\tau)]A\Psi(\tau)\,dW(\tau)\right|_H. \end{aligned}$$

From assumptions, $A\Psi \in \mathcal{N}^2(0,T;L_2^0)$. Then $[S_{\alpha,n}(t-\tau) - S_\alpha(t-\tau)]A\Psi(\tau)$ may be estimated exactly like the difference $\Phi_{\alpha,n}$ defined by (3.54).

Hence, from (3.63) and (3.60) for the first term of the right hand side of (3.61) we obtain

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \mathbb{E}\,(N_{n,1}^2(t)) \to 0.$$

For the second and third terms of (3.61) we can follow the same steps as above for proving (3.60). We have to use the properties of Yosida approximation, particularly the convergence (3.64). So, we can deduce that

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \mathbb{E}\,|A_n W_{\alpha,n}^{\Psi}(t) - AW_\alpha^{\Psi}(t)|_H^2 = 0,$$

what gives (3.5). $\square$

Now, we are able to formulate the main result of this section.

Theorem 3.30. *Suppose that assumptions of Lemma* 3.29 *hold. Then the equation* (3.32) *has a strong solution. Precisely, the convolution* W_α^Ψ *defined by* (3.50) *is the strong solution to* (3.32).

Proof. We have to show only the condition (3.48). The convolution $W_\alpha^\Psi(t)$ has integrable trajectories (see Section 3.1), that is, $W_\alpha^\Psi(\cdot) \in L^1([0,T];H)$, P-a.s. The closed linear unbounded operator A becomes bounded on $(D(A), |\cdot|_{D(A)})$, see [78, Chapter 5]. So, we obtain $AW_\alpha^\Psi(\cdot) \in L^1([0,T];H)$, P-a.s. Hence, the function $a_\alpha(T-\tau)AW_\alpha^\Psi(\tau)$ is integrable with respect to τ, what finishes the proof. □

The following result is an immediate consequence of Corollary 3.22 and Theorem 3.30.

Corollary 3.31. *Assume that* (VA) *hold, A generates an analytic semigroup of angle $\pi/2$ and $\alpha \in (0,1)$. If Ψ and $A\Psi$ belong to $\mathcal{N}^2(0,T;L_2^0)$ and in addition $\Psi(t)(U_0) \subset D(A)$, P-a.s., then the equation* (3.32) *has a strong solution.*

3.4. Examples

In this short section we give several examples fulfilling conditions of theorems providing existence of strong solutions. The class of such equations depends on where the operator A is defined, in particular, the domain of A depends on each considered problem, and also depends on the properties of the kernel function $a(t)$, $t \geq 0$.

Let G be a bounded domain in $\mathbb{R}^n$ with smooth boundary ∂G. Consider the differential operator of order $2m$:

$$A(x,D) = \sum_{|\alpha| \leq 2m} b_\alpha(x) D^\alpha \tag{3.65}$$

where the coefficients $b_\alpha(x)$ are sufficiently smooth complex-valued functions of x in $\overline{G}$. The operator $A(x,D)$ is called *strongly elliptic* if there exists a constant $c > 0$ such that

$$Re(-1)^m \sum_{|\alpha|=2m} b_\alpha(x)\xi^\alpha \geq c|\xi|^{2m}$$

for all $x \in \overline{G}$ and $\xi \in \mathbb{R}^n$.

Let $A(x,D)$ be a given strongly elliptic operator on a bounded domain $G \subset \mathbb{R}^n$ and set $D(A) = H^{2m}(G) \cap H_0^m(G)$. For every $u \in D(A)$ define

$$Au = A(x,D)u.$$

Then the operator $-A$ is the infinitesimal generator of an analytic semigroup of operators on $H = L^2(G)$ (cf. [65, Theorem 7.2.7]). We note that if the operator A has constant coefficients, the result remains true for the domain $G = \mathbb{R}^n$.

The next example is the Laplacian

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2},$$

since $-\Delta$ is clearly strongly elliptic. It follows that Δu on $D(A) = H^2(G) \cap H_0^1(G)$ is the infinitesimal generator of an analytic semigroup on $L^2(G)$.

In particular, by [70, Corollary 2.4] the operator A given by (3.65) generates an analytic resolvent $S(t)$ whenever $a \in C(0,\infty) \cap L^1(0,1)$ is completely monotonic.

This example fits in our results if $a(t)$ is also completely positive. For example: $a(t) = t^{\alpha-1}/\Gamma(\alpha)$ is both, completely positive and completely monotonic for $0 < \alpha \leq 1$ (but not for $\alpha > 1$).

Another class of examples is provided by the following: suppose $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ is of subexponential growth and $\pi/2$-sectorial, and let A generate a bounded analytic C_0-semigroup in a complex Hilbert space H. Then it follows from [70, Corollary 3.1] that the Volterra equation of scalar type $u = a*Au+f$ is parabolic. If, in addition, $a(t)$ is k-regular for all $k \geq 1$ we obtain from [70, Theorem 3.1] the existence of a resolvent $S \in C^{k-1}((0,\infty), \mathcal{B}(H))$ such that $\mathcal{R}(S(t)) \subset D(A)$ for $t > 0$ (see [70, (f), p. 82]).

CHAPTER 4

STOCHASTIC VOLTERRA EQUATIONS IN SPACES OF DISTRIBUTIONS

In this chapter we study two classes of linear Volterra equations driven by spatially homogeneous Wiener process. We consider existence of solutions to these equations in the space of tempered distributions and then derive conditions under which the solutions are function-valued or even continuous. The conditions obtained are expressed in terms of spectral measure and the space correlation of the noise process, as well. Moreover, we give description of asymptotic properties of solutions.

The chapter is organized as follows. In Section 4.1 we introduce generalized and classical homogeneous Gaussian random fields basing on [36], [2] and [66]. We recall some facts which connect the generalized random fields with their space corralations and spectral measures. Moreover, we recall some results used in the proofs of the main theorems. Section 4.2 originates from [54]. Here we study regularity of solutions to the equation (4.1) and provide some applications of these results. Section 4.3 is a natural continuation of Section 4.2. In this section we give necessary and sufficient conditions for the existence of a limit measure to the stochastic equation under consideration. Results of Section 4.3 come from [47]. In Section 4.4 we study an integro-differential stochastic equation with infinite delay. We provide necessary and sufficient conditions under which weak solution to that equation takes values in a Sobolev space. Section 4.4 originates from [49].

4.1. Generalized and classical homogeneous Gaussian random fields

We start from recalling several concepts needed in this chapter. Let $S(\mathbb{R}^d)$, $S_c(\mathbb{R}^d)$, denote respectively the spaces of all infinitely differentiable rapidly decreasing real and complex functions on $\mathbb{R}^d$ and $S'(\mathbb{R}^d)$, $S_c'(\mathbb{R}^d)$ denote the spaces of real and complex, tempered distributions. The value of a distribution $\xi \in S_c'(\mathbb{R}^d)$ on a test function ψ will be written as $\langle \xi, \psi \rangle$. For $\psi \in S(\mathbb{R}^d)$ we set $\psi_{(s)}(\theta) = \overline{\psi(-\theta)}$, $\theta \in \mathbb{R}^d$. Denote by $S_{(s)}(\mathbb{R}^d)$ the space of all $\psi \in S(\mathbb{R}^d)$ such that $\psi = \psi_{(s)}$, and by $S'_{(s)}(\mathbb{R}^d)$ the space of all $\xi \in S'(\mathbb{R}^d)$ such that $\langle \xi, \psi \rangle = \langle \xi, \psi_{(s)} \rangle$ for every $\psi \in S(\mathbb{R}^d)$.

We define the derivative $\dot{\xi}$ of the distribution $\xi \in S'(\mathbb{R}^d)$ by the formula $\langle \dot{\xi}, \varphi \rangle = -\langle \xi, \dot{\varphi} \rangle$ for $\varphi \in S(\mathbb{R}^d)$, see [34].

In the chapter we denote by $\mathcal{F}$ the Fourier transform both on $S_c(\mathbb{R}^d)$, and on $S'_c(\mathbb{R}^d)$. In particular,

$$\mathcal{F}\psi(\theta) = \int_{\mathbb{R}^d} e^{-2\pi i \langle \theta, \eta \rangle} \psi(\eta)\, d\eta, \quad \psi \in S_c(\mathbb{R}^d),$$

and for the inverse Fourier transform $\mathcal{F}^{-1}$,

$$\mathcal{F}^{-1}\psi(\theta) = \int_{\mathbb{R}^d} e^{2\pi i \langle \theta, \eta \rangle} \psi(\eta)\, d\eta, \quad \psi \in S_c(\mathbb{R}^d).$$

Moreover, if $\xi \in S'_c(\mathbb{R}^d)$,

$$\langle \mathcal{F}\xi, \psi \rangle = \langle \xi, \mathcal{F}^{-1}\psi \rangle$$

for all $\psi \in S_c(\mathbb{R}^d)$. Let us note that $\mathcal{F}$ transforms the space of tempered distributions $S'(\mathbb{R}^d)$ into $S'_{(s)}(\mathbb{R}^d)$.

For any $h \in \mathbb{R}^d$, $\psi \in S(\mathbb{R}^d)$, $\xi \in S'(\mathbb{R}^d)$, the *translations* $\tau_h\psi$, $\tau'_h\xi$ are defined by the formulas

$$\tau_h\psi(x) = \psi(x-h), \quad \langle \tau'_h\xi, \psi \rangle = \langle \xi, \tau_h\psi \rangle, \quad x \in \mathbb{R}^d.$$

By $\mathcal{B}(S'(\mathbb{R}^d))$ and $\mathcal{B}(S'_c(\mathbb{R}^d))$ we denote the smallest σ-algebras of subsets of $S'(\mathbb{R}^d)$ and $S'_c(\mathbb{R}^d)$, respectively, such that for any test function φ the mapping $\xi \to \langle \xi, \varphi \rangle$ is measurable.

The below notions of generalized random fields, their space correlations and spectral measures are recalled directly from [36].

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. Any measurable mapping $Y\colon \Omega \to S'(\mathbb{R}^d)$ is called a *generalized random field*. A generalized random field Y is called *Gaussian* if $\langle Y, \varphi \rangle$ is a Gaussian random variable for any $\varphi \in S(\mathbb{R}^d)$. The definition implies that for any functions $\varphi_1, \dots, \varphi_n \in S(\mathbb{R}^d)$ the random vector $(\langle Y, \varphi_1 \rangle, \dots, \langle Y, \varphi_n \rangle)$ is also Gaussian. One says that a generalized random field Y is *homogeneous* or *stationary* if for all $h \in \mathbb{R}^d$, the translation $\tau'_h(Y)$ of Y has the same probability law as Y.

A distribution Γ on the space $S(\mathbb{R}^d)$ is called *positive-definite* if $\langle \Gamma, \varphi \star \varphi_{(s)} \rangle \geq 0$ for every $\varphi \in S(\mathbb{R}^d)$, where $\varphi \star \varphi_{(s)}$ denotes the convolution of the functions φ and $\varphi_{(s)}$.

If Y is a homogeneous, Gaussian random field then for each $\psi \in S(\mathbb{R}^d)$, $\langle Y, \psi \rangle$ is a Gaussian random variable and the bilinear functional $q\colon S(\mathbb{R}^d) \times S(\mathbb{R}^d) \to \mathbb{R}$ defined by the formula,

$$q(\varphi, \psi) = \mathbb{E}\,(\langle Y, \varphi \rangle \langle Y, \psi \rangle) \quad \text{for } \varphi, \psi \in S(\mathbb{R}^d),$$

is continuous and positive definite. Since $q(\varphi,\psi) = q(\tau_h\varphi,\tau_h\psi)$ for all φ, $\psi \in S(\mathbb{R}^d)$, $h \in \mathbb{R}^d$, there exists, see e.g. [36, Chapter II], a unique positive-definite distribution $\Gamma \in S'(\mathbb{R}^d)$ such that for all $\varphi, \psi \in S(\mathbb{R}^d)$, one has

$$q(\varphi,\psi) = \langle \Gamma, \varphi * \psi_{(s)} \rangle.$$

The distribution Γ is called the *space correlation* of the field Y. By Bochner–Schwartz theorem the positive-definite distribution Γ is the inverse Fourier transform of a unique positive, symmetric, tempered measure μ on $\mathbb{R}^d$: $\Gamma = \mathcal{F}^{-1}(\mu)$. The measure μ is called the *spectral measure* of Γ and of the field Y.

Summing up a generalized homogeneous Gaussian random field Y is characterized by the following properties:

(1) for any $\psi \in S(\mathbb{R}^d)$, $\langle Y, \psi \rangle$ is a real-valued Gaussian random variable,
(2) there exists a positve-definite distribution $\Gamma \in S'(\mathbb{R}^d)$ such that for all $\varphi, \psi \in S(\mathbb{R}^d)$

$$\mathbb{E}\left(\langle Y,\varphi\rangle\langle Y,\psi\rangle\right) = \langle \Gamma, \varphi * \psi_{(s)} \rangle,$$

(3) the distribution Γ is the inverse Fourier transform of a positive and symmetric *tempered measure* μ on $\mathbb{R}^d$, that is, such that

$$\int_{\mathbb{R}^d} (1+|\lambda|)^r \mu(d\lambda) < \infty, \quad \text{for some } r < 0.$$

Let $Y\colon \Omega \to S'(\mathbb{R}^d)$ be a generalized random field. When the values of Y are functions, with probability 1, then Y is called a *classical random field* or shortly *random field*. In this case, by Fubini's theorem, for any $\theta \in \mathbb{R}^d$ the function $Y(\theta)$ is well-defined and

$$\langle Y, \varphi \rangle = \int_{\mathbb{R}^d} Y(\theta)\varphi(\theta)\, d\theta,$$

for any $\varphi \in S(\mathbb{R}^d)$. Thus any random field Y may be identified with the family of random variables $\{Y_\theta\}_{\theta \in \mathbb{R}^d}$ parametrized by $\theta \in \mathbb{R}^d$. In particular a homogeneous (stationary), Gaussian random field is a family of Gaussian random variables $Y(\theta)$, $\theta \in \mathbb{R}^d$, with Gaussian laws invariant with respect to all translations. That is, for any $\theta_1, \ldots, \theta_n \in \mathbb{R}^d$ and $h \in \mathbb{R}^d$, the law of $(Y(\theta+h), \ldots, Y(\theta_n+h))$ does not depend on $h \in \mathbb{R}^d$.

For the sake of completness we sketch now the proof of the following result, (see also [66]).

Proposition 4.1. *A generalized, homogeneous, Gaussian random field Y is classical if and only if the space correlation Γ of Y is a bounded function and if and only if the spectral measure μ of Y is finite.*

Proof. First, let us prove that if a positive definite distribution Γ is a bounded function then it is continuous and its spectral measure μ is finite. It is enough to show only that the spectral measure μ of the distribution Γ is finite.

Let $p_t(\,\cdot\,)$ denote the normal density with the Fourier transform $e^{-t|\lambda|^2}$, $t>0$, $\lambda\in\mathbb{R}^d$. Define measures μ_t, $t>0$, by the formula

$$\mu_t(B)=\int_B e^{-t|\lambda|^2}\mu(d\lambda),\quad B\subset\mathbb{R}^d.$$

Since the measure μ is tempered, the measures μ_t are finite.

The Fourier transform $\mathcal{F}(\mu_t)$ of μ_t, for any $t>0$, is a continuous function and

$$\Gamma_t(\theta)=\mathcal{F}^{-1}(\mu_t)(\theta)=(\Gamma*p_t)(\theta),\quad \theta\in\mathbb{R}^d.$$

But

$$\Gamma_t(\theta)=\int_{\mathbb{R}^d}\Gamma(\theta-\eta)p_t(\eta)\,d\eta$$

and

$$|\Gamma_t(\theta)|\le\Big[\sup_{\zeta\in\mathbb{R}^d}|\Gamma(\zeta)|\Big]\int_{\mathbb{R}^d}p_t(\eta)\,d\eta\le\Big[\sup_{\zeta\in\mathbb{R}^d}|\Gamma(\zeta)|\Big].$$

In particular

$$|\Gamma_t(0)|=\int_{\mathbb{R}^d}e^{-t|\lambda|^2}\mu(d\lambda)\le\Big[\sup_{\zeta\in\mathbb{R}^d}|\Gamma(\zeta)|\Big].$$

Letting $t\downarrow 0$ one obtains that

$$\int_{\mathbb{R}^d}\mu(d\lambda)\le\Big[\sup_{\zeta\in\mathbb{R}^d}|\Gamma(\zeta)|\Big]<\infty,$$

so the measure μ is finite as required.

Let now Y be a classical, homogeneous, Gaussian random field. It means that the field Y is function-valued. Moreover, $\mathbb{E}\,(Y(\theta_1)Y(\theta_2))=\Gamma(\theta_1-\theta_2)$, for θ_1, $\theta_2\in\mathbb{R}^d$, the space correlation Γ is positive-definite and $|\Gamma(\theta_1-\theta_2)|\le\Gamma(0)<\infty$.

Let now μ be the finite spectral measure of a homogeneous Gaussian random field Y. Then Γ is a positive definite continuous function. By Kolmogorov's existence theorem, there exists a family $\widetilde{Y}(\theta)$, $\theta\in\mathbb{R}^d$, such that:

$$\mathbb{E}\,(\widetilde{Y}(\theta_1)\,\widetilde{Y}(\theta_2))=\Gamma(\theta_1-\theta_2),\quad \theta_1,\theta_2\in\mathbb{R}^d.$$

From the continuity of Γ, it follows that the family $\widetilde{Y}$, is stochastically continuous and therefore has a measurable version. Since the laws of the random fields Y, $\widetilde{Y}$ coincide, the result follows. □

We finish the section recalling a continuity criterium which will be used in the proof of the continuity results, see ([2, Theorem 3.4.3]).

Proposition 4.2. *Let $Y(\theta)$, $\theta\in\mathbb{R}^d$, be a homogeneous, Gaussian random field with the spectral measure μ. If, for some $\varepsilon>0$,*

$$\int_{\mathbb{R}^d}(\ln(1+|\lambda|))^{1+\varepsilon}\mu(d\lambda)<\infty,$$

then Y has a version with almost surely continuous sample functions.

4.2. Regularity of solutions to stochastic Volterra equations

This section is concerned with the following stochastic Volterra equation

$$X(t,\theta) = X_0(\theta) + \int_0^t b(t-\tau) A X(\tau,\theta)\, d\tau + W(t,\theta), \tag{4.1}$$

where $t \in \mathbb{R}_+$, $\theta \in \mathbb{R}^d$, $X_0 \in S'(\mathbb{R}^d)$, $b \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ and W is a spatially homogeneous Wiener process which takes values in the space of real, tempered distributions $S'(\mathbb{R}^d)$. The class of operators A covered in the present chapter contains in particular the Laplace operator Δ and its fractional powers $-(-\Delta)^{\beta/2}$, $\beta \in (0,2]$.

The equation (4.1) is a generalization of stochastic heat and wave equations studied by many authors, see e.g. [20], [55], [53], [61]–[63], [67] and [77] and references therein. In the context of infinite particle systems stochastic heat equation of a similar type has been investigated by Bojdecki with Jakubowski [9]–[11] and by Dawson with Gorostiza in [25].

As we have already said, our aim is to obtain conditions under which solutions to the stochastic Volterra equation (4.1) are function-valued and even continuous with respect to the space variable. In the chapter we treat the case of general dimension and the correlated, spatially homogeneous noise W_Γ of the general form.

4.2.1. Stochastic integration. In this section we will integrate operator-valued functions $\mathcal{R}(t)$, $t \geq 0$, with respect to a Wiener process W. The operators $\mathcal{R}(t)$, $t \geq 0$, will be non-random and will act from some linear subspaces of $S'(\mathbb{R}^d)$ into $S'(\mathbb{R}^d)$. We shall assume that $W(t)$, $t \geq 0$, is a continuous process with independent increments taking values in $S'(\mathbb{R}^d)$. The process W is space homogeneous in the sense that, for each $t \geq 0$, random variables $W(t)$ are stationary, Gaussian, generalized random fields. We denote by Γ the covariance of $W(1)$ and the associated spectral measure by μ. To underline the fact that the probability law of W is determined by Γ we will write W_Γ. From now on we denote by q a scalar product on $S(\mathbb{R}^d)$ given by the formula

$$q\langle \phi, \psi \rangle = \langle \Gamma, \phi * \psi_{(s)} \rangle, \quad \phi, \psi \in S(\mathbb{R}^d).$$

Let us present three examples of spatially homogeneous Wiener processes.

Examples. (a) Important examples of random fields are provided by symmetric α-stable distributions $\Gamma(x) = e^{-|x|^\alpha}$, where $\alpha \in \,]0,2]$. For $\alpha = 1$ and $\alpha = 2$ the densities of the spectral measures are given by the formulas $c_1(1+|x|^2)^{-(d+1)/2}$ and $c_2 e^{-|x|^2}$, where c_1 and c_2 are appropriate constants.

(b) Let $q(\psi,\varphi) = \langle (-\Delta + m^2)^{-1}\psi, \varphi \rangle$, where Δ is the Laplace operator on $\mathbb{R}^d$ and m is a strictly positive constant. Then Γ is a continuous function on $\mathbb{R}^d \setminus \{0\}$ and $(d\mu/dx)(x) = (2\pi)^{-d/2}(|x|^2 + m^2)^{-1}$. The law of $W(1)$ is the so-called Euclidean free field.

(c) Let $q(\psi, \varphi) = \langle \psi, \varphi \rangle$. Then Γ is equal to the Dirac δ_0-function, its spectral density $d\mu/dx$ is the constant function $(2\pi)^{-d/2}$ and $\partial W/\partial t$ is a white noise on $L^2([0,\infty[\times \mathbb{R}^d)$. If $B(t,x)$, $t \geq 0$ and $x \in \mathbb{R}^d$, is a Brownian sheet on $[0,\infty[\times \mathbb{R}^d$, then W can be defined by the formula

$$W(t,x) = \frac{\partial^d B(t,x)}{\partial x_1 \dots \partial x_d}, \quad t \geq 0.$$

The crucial role in the theory of stochastic integration with respect to W_Γ is played by the Hilbert space $S'_q \subset S'(\mathbb{R}^d)$ called the *kernel* or the *reproducing kernel* of W_Γ. Namely the space S'_q consists of all distributions $\xi \in S'(\mathbb{R}^d)$ for which there exists a constant C such that

$$|\langle \xi, \psi \rangle| \leq C\sqrt{q(\psi,\psi)}, \quad \psi \in S(\mathbb{R}^d).$$

The norm in S'_q is given by the formula

$$|\xi|_{S'_q} = \sup_{\psi \in S} \frac{|\langle \xi, \psi \rangle|}{\sqrt{q(\psi,\psi)}}.$$

Let us assume that we require that the stochastic integral should take values in a Hilbert space H continuously imbedded into $S'(\mathbb{R}^d)$. Let $L_{HS}(S'_q, H)$ be the space of Hilbert–Schmidt operators from S'_q into H. Assume that $\mathcal{R}(t)$, $t \geq 0$ is measurable $L_{HS}(S'_q, H)$-valued function such that

$$\int_0^t \|\mathcal{R}(\sigma)\|^2_{L_{HS}(S'_q,H)}\, d\sigma < \infty, \quad \text{for all } t \geq 0.$$

Then the stochastic integral

$$\int_0^t \mathcal{R}(\sigma) dW_\Gamma(\sigma), \quad t \geq 0$$

can be defined in a standard way, see [44], [23] or [66]. The stochastic integral is an H-valued martingale for which

$$\mathbb{E}\left(\int_0^t \mathcal{R}(\sigma) dW_\Gamma(\sigma)\right) = 0, \quad t \geq 0$$

and

$$\mathbb{E}\left|\int_0^t \mathcal{R}(\sigma) dW_\Gamma(\sigma)\right|^2_H = \mathbb{E}\left(\int_0^t \|\mathcal{R}(\sigma)\|^2_{L_{HS}(S'_q,H)}\, d\sigma\right), \quad t \geq 0.$$

We will need a characterization of the space S'_q. In the proposition below, $L^2_{(s)}(\mathbb{R}^d, \mu)$ denotes the subspace of $L^2(\mathbb{R}^d, \mu; \mathbb{C})$ consisting of all functions u such that $u_{(s)}(\theta) = u(-\theta)$ for $\theta \in \mathbb{R}^d$.

Proposition 4.3 ([66, Proposition 1.2]). *A distribution ξ belongs to S'_q if and only if $\xi = \widehat{u\mu}$ for some $u \in L^2_{(s)}(\mathbb{R}^d, \mu)$. Moreover, if $\xi = \widehat{u\mu}$ and $\eta = \widehat{v\mu}$, then*

$$\langle \xi, \eta \rangle_{S'_q} = \langle u, v \rangle_{L^2_{(s)}(\mathbb{R}^d,\mu)}.$$

The operators $\mathcal{R}(t)$, $t \geq 0$, of convolution type are of special interest

$$\mathcal{R}(t)\xi = \mathbf{r}(t) * \xi, \quad t \geq 0,\ \xi \in S'(\mathbb{R}^d),$$

with $\mathbf{r}(t) \in S'(\mathbb{R}^d)$. The convolution operator is not, in general, defined for all $\xi \in S'(\mathbb{R}^d)$ and for the stochastic integration it is important to know under what conditions on $\mathbf{r}(\,\cdot\,)$ and ξ the convolution is well-defined. For many important cases the Fourier transform $\mathcal{F}\mathbf{r}(t)(\lambda)$, $t \geq 0$, $\lambda \in \mathbb{R}^d$, is continuous in both variables and, for any $T \geq 0$,

$$\sup_{t\in[0,T]} \sup_{\lambda\in\mathbb{R}^d} |\mathcal{F}\mathbf{r}(t)(\lambda)| = M_T < \infty. \tag{4.2}$$

If this is the case then the operators $\mathcal{R}(t)$ can be defined using Fourier transforms

$$\mathcal{R}(t)\xi = \mathcal{F}^{-1}(\mathcal{F}\mathbf{r}(t)\,\mathcal{F}\xi),$$

for all ξ such that $\mathcal{F}\xi$ has a representation as a function.

Now, we can characterize the stochastic convolution as follows.

Theorem 4.4. *Assume that the function $\mathcal{F}\mathbf{r}$ is continuous in both variables and satisfies condition* (4.2)*. Then the stochastic convolution*

$$\mathcal{R} * W_\Gamma(t) = \int_0^t \mathcal{R}(t-\sigma)\, dW_\Gamma(\sigma), \quad t \geq 0,$$

is a well-defined $S'(\mathbb{R}^d)$-valued stochastic process. For each $t \geq 0$, $\mathcal{R} * W_\Gamma(t)$ is a Gaussian, stationary, generalized random field with the spectral measure

$$\mu_t(d\lambda) = \left(\int_0^t |\mathcal{F}\mathbf{r}(\sigma)(\lambda)|^2\, d\sigma\right)\mu(d\lambda), \tag{4.3}$$

and with the covariance

$$\Gamma_t = \int_0^t \mathbf{r}(\sigma) * \Gamma * \mathbf{r}_{(s)}(\sigma)\, d\sigma. \tag{4.4}$$

Proof. Let p be an arbitrary continuous scalar product on $S(\mathbb{R}^d)$, such that the embedding $S'_q(\mathbb{R}^d) \subset S'_p(\mathbb{R}^d)$ is Hilbert–Schmidt; here $S'_p(\mathbb{R}^d)$ denotes a space of distributions on $S(\mathbb{R}^d)$ endowed with p. For more information on a family of Hilbert spaces of distributions we refer to [43]. Note that for $\xi \in S'_q(\mathbb{R}^d)$,

$$\xi = \mathcal{F}^{-1}(u\mu) \quad \text{with } u \in L^2_{(s)}(\mathbb{R}^d, \mu).$$

By (4.2), $\mathcal{F}(\mathbf{r}(t))\mathcal{F}(\xi)$ is a measure

$$\mathcal{F}(\mathbf{r}(t))(\lambda)u(\lambda)\mu(d\lambda),$$

belonging again to $S'_q(\mathbb{R}^d)$. Moreover, for $t \in [0,T]$,

$$||\mathcal{R}(t)||_{L(S'_q,S'_q)} \leq \sup_{t\in[0,T]} \sup_{\lambda\in\mathbb{R}^d} |\mathcal{F}\mathbf{r}(t)(\lambda)| = M_T < \infty.$$

Since the embedding $S'_q \subset S'_p$ is Hilbert–Schmidt, the stochastic integral, by the very definition, is an S'_p-valued random variable. Denote

$$Z_t = \mathcal{R} * W_\Gamma(t).$$

Then we may write

$$\begin{aligned}
&\mathbb{E}\left(\langle Z_t,\varphi\rangle\langle Z_t,\psi\rangle\right)\\
&\quad= \mathbb{E}\left(\left\langle \int_0^t \mathcal{R}(t-\sigma)\,dW_\Gamma(\sigma),\varphi\right\rangle\left\langle \int_0^t \mathcal{R}(t-u)\,dW_\Gamma(u),\psi\right\rangle\right)\\
&\quad= \mathbb{E}\left(\int_0^t \langle \mathbf{r}(t-\sigma)*\varphi, dW_\Gamma(\sigma)\rangle \int_0^t \langle \mathbf{r}(t-u)*\psi, dW_\Gamma(u)\rangle\right)\\
&\quad= \int_0^t \langle \Gamma, (\mathbf{r}(\sigma)*\varphi)*(\mathbf{r}(\sigma)*\psi)_{(s)}\rangle\,d\sigma
\end{aligned}$$

where φ, $\psi \in S(\mathbb{R}^d)$. This implies the formula (4.4) of the theorem, from which (4.3) easily folows. □

As an application define

$$v(\lambda) = \frac{1}{2}\langle Q\lambda,\lambda\rangle - \int_{R^d} (e^{i\langle\lambda,y\rangle}-1)\nu(dy) \tag{4.5}$$

In the formula (4.5), Q is a symmetric, non-negative definite matrix and ν is a symmetric measure concentrated on $R^d \setminus \{0\}$ such that

$$\int_{|y|\leq 1} |y|^2\nu(dy) < \infty, \qquad \int_{|y|>1} 1\nu(dy) < \infty. \tag{4.6}$$

This is the *Levy–Khinchin exponent* of an infinitely divisible symmetric law. From Theorem 4.4 we have the following proposition.

Proposition 4.5. *Assume that*

$$\mathcal{F}\mathbf{r}(t)(\lambda) = e^{-tv(\lambda)}, \quad t \geq 0$$

where v is the Levy–Khinchin exponent given by (4.5) *and* (4.6). *Then the conditions of Theorem* 4.4 *are satisfied.*

For more information on stochastic integral with values in the Schwartz space of tempered distributions $S'(\mathbb{R}^d)$ we refere to Itô ([43], [44]), Bojdecki with Jakubowski ([8]–[11]), Bojdecki with Gorostiza ([7]) and Peszat with Zabczyk ([66]–[67]).

4.2.2. Stochastic Volterra equation. We finally pass to the linear, stochastic, Volterra equation in $S'(\mathbb{R}^d)$

$$X(t) = X_0 + \int_0^t b(t-\tau)AX(\tau)\,d\tau + W_\Gamma(t), \tag{4.7}$$

where $X_0 \in S'(\mathbb{R}^d)$, A is an operator given in the Fourier transform form

$$\mathcal{F}(A\xi)(\lambda) = -v(\lambda)\,\mathcal{F}(\xi)(\lambda), \tag{4.8}$$

v is a locally integrable function and W_Γ is an $S'(\mathbb{R}^d)$-valued space homogeneous Wiener process.

Note that if $v(\lambda) = |\lambda|^2$, then $A = \Delta$ and if $v(\lambda) = |\lambda|^\alpha$, $\alpha \in (0,2)$, then $A = -(-\Delta)^{\alpha/2}$ is the fractional Laplacian.

We shall assume the following hypothesis:

(H) For any $\gamma \geq 0$, the unique solution $s(\,\cdot\,,\gamma)$ to the equation

$$\mathbf{s}(t) + \gamma \int_0^t b(t-\tau)\,\mathbf{s}(\tau)\,d\tau = 1, \quad t \geq 0 \tag{4.9}$$

fullfils the following condition:

$$\sup_{t\in[0,T]}\sup_{\gamma\geq 0}|\mathbf{s}(t,\gamma)| < \infty \quad \text{for any } T \geq 0.$$

Comment. Let us note that under assumption, that the function b is a locally integrable function, the solution $s(\,\cdot\,,\gamma)$ of the equation (4.9) is locally integrable function and measurable with respect to both variables $\gamma \geq 0$ and $t \geq 0$.

For some special cases the function $\mathbf{s}(t;\gamma)$ may be found explicitely. Namely, we have (see e.g. [70]):

$$\text{for } b(t) = 1, \quad \mathbf{s}(t;\gamma) = e^{-\gamma t}, \quad t \geq 0,\ \gamma \geq 0; \tag{4.10}$$

$$\text{for } b(t) = t, \quad \mathbf{s}(t;\gamma) = \cos(\sqrt{\gamma}t), \quad t \geq 0,\ \gamma \geq 0; \tag{4.11}$$

$$\text{for } b(t) = e^{-t}, \quad \mathbf{s}(t;\gamma) = (1+\gamma)^{-1}[1+\gamma e^{-(1+\gamma)t}], \quad t \geq 0,\ \gamma \geq 0. \tag{4.12}$$

We introduce now the so called *resolvent* family $\mathcal{R}(\,\cdot\,)$ determined by the operator A and the function v. Namely,

$$\mathcal{R}(t)\xi = \mathbf{r}(t) * \xi, \quad \xi \in S'(\mathbb{R}^d),$$

where,

$$\mathbf{r}(t) = \mathcal{F}^{-1}\mathbf{s}(t, v(\,\cdot\,)), \quad t \geq 0.$$

As in the deterministic case the solution to the stochastic Volterra equation (4.7) is of the form

$$X(t) = \mathcal{R}(t)X_0 + \int_0^t \mathcal{R}(t-\tau)\,dW_\Gamma(\tau), \quad t \geq 0. \tag{4.13}$$

By Theorem 4.4, we can formulate the following result.

Theorem 4.6 ([54, Theorem 1]). *Let W_Γ be a spatially homogeneous Wiener process and $\mathcal{R}(t)$, $t \geq 0$, the resolvent for the equation* (4.7). *If hypothesis* (H) *holds then the stochastic convolution*

$$\mathcal{R} * W_\Gamma(t) = \int_0^t \mathcal{R}(t-\sigma)\, dW_\Gamma(\sigma), \quad t \geq 0,$$

is a well-defined $S'(\mathbb{R}^d)$-valued process. For each $t \geq 0$ the random variable $\mathcal{R}*W_\Gamma(t)$ is generalized, stationary random field on $\mathbb{R}^d$ with the spectral measure

$$\mu_t(d\lambda) = \left[\int_0^t (\mathbf{s}(\sigma, v(\lambda)))^2\, d\sigma\right] \mu(d\lambda). \tag{4.14}$$

By Propositions 4.1 and 4.2, we can conclude the below result.

Theorem 4.7 ([54, Theorem 2]). *Assume that the hypothesis* (H) *holds. Then the process $\mathcal{R} * W_\Gamma(t)$ is function-valued for all $t \geq 0$ if and only if*

$$\int_{\mathbb{R}^d} \left(\int_0^t (\mathbf{s}(\sigma, v(\lambda)))^2\, d\sigma \right) \mu(d\lambda) < \infty, \quad t \geq 0.$$

If for some $\varepsilon > 0$ and all $t \geq 0$,

$$\int_0^t \int_{\mathbb{R}^d} (\ln(1+|\lambda|))^{1+\varepsilon} (\mathbf{s}(\sigma, v(\lambda)))^2\, d\sigma \mu(d\lambda) < \infty,$$

then, for each $t \geq 0$, $\mathcal{R} * W_\Gamma(t)$ is a sample continuous random field.

4.2.3. Continuity in terms of Γ. In this subsection we provide sufficient conditions for continuity of the solutions in terms of the covariance kernel Γ of the Wiener process W_Γ rather than in terms of the spectral measure as we have done up to now. Analogical conditions for existence of function-valued solutions can be derived in a similar way.

Theorem 4.8. *Assume that $d \geq 2$, that Γ is a non-negative measure and $A = -(-\Delta)^{\alpha/2}$, $\alpha \in\,]0,2]$. If for some $\delta > 0$,*

$$\int_{|\lambda|\leq 1} \frac{1}{|\lambda|^{d-\alpha+\delta}} \Gamma(d\lambda) < \infty, \qquad \int_{|\lambda|>1} \frac{1}{|\lambda|^{d+\alpha-\delta}} \Gamma(d\lambda) < \infty,$$

then for the cases (4.10)–(4.12) *the solution to the stochastic Volterra equation* (4.7) *has continuous version.*

The proof will be based on several lemmas. For any $\gamma \in\,]0,2]$ denote by p_t^γ the density of the γ-stable, rotationally invariant, density on the d-dimensional space. Thus,

$$e^{-t|\lambda|^\gamma} = \mathcal{F} p_t^\gamma(\lambda). \tag{4.15}$$

Lemma 4.9. *For arbitrary $t>0$ and arbitrary $x \in \mathbb{R}^d$,*

$$p_t^\gamma(x) = t^{-d/\gamma} p_1^\gamma(xt^{-1/\gamma}).$$

Proof. From (4.15) we have

$$I := e^{-|t^{1/\gamma}\lambda|^\gamma} = \int_{\mathbb{R}^d} e^{i\langle t^{1/\gamma}\lambda, x\rangle}\, p_1^\gamma(x)\,dx = \int_{\mathbb{R}^d} e^{i\langle \lambda, t^{1/\gamma}x\rangle}\, p_1^\gamma(x)\,dx.$$

Introducing a new variable $y = t^{1/\gamma}x$, one has

$$I := \int_{\mathbb{R}^d} e^{i\langle \lambda, y\rangle}\, p_1^\gamma(yt^{-1/\gamma})\,dy$$

and the result follows. □

Lemma 4.10. *There exists a constant $c>0$ such that for all $\gamma \le 2$,*

$$G_d^\gamma(x) \overset{\mathrm{df}}{=} \int_0^\infty e^{-t} p_t^\gamma(x)\,dt \le \frac{c}{|x|^{d+\gamma}}, \quad x \in \mathbb{R}^d.$$

Proof. It is well-known, see e.g. Gorostiza and Wakolbinger [37, p. 286], that for some constant $c_1 > 0$:

$$p_1^\gamma(x) \le \frac{c_1}{1+|x|^{d+\gamma}}, \quad x \in \mathbb{R}^d. \tag{4.16}$$

From Lemma 4.9 and the estimate (4.16) we obtain:

$$\begin{aligned} G_d^\gamma(x) &= \int_0^\infty e^{-t} t^{-d/\gamma} p_1^\gamma(x\,t^{-1/\gamma})\,dt \\ &\le \int_0^\infty e^{-t} t^{-d/\gamma} \frac{c_1}{1+|xt^{-1/\gamma}|^{d+\gamma}}\,dt \le \int_0^\infty e^{-t} t^{-d/\gamma} \frac{c_1 t^{(d+\gamma)/\gamma}}{t^{(d+\gamma)/\gamma}+|x|^{d+\gamma}}\,dt \\ &\le \int_0^\infty e^{-t} t \frac{c_1}{t^{(d+\gamma)/\gamma}+|x|^{d+\gamma}}\,dt \le \frac{c_1}{|x|^{d+\gamma}} \int_0^\infty e^{-t} t\,dt. \end{aligned}$$

□

Lemma 4.11. *If $\gamma < d$, $\gamma \in]0,2]$, then there exists a constant $c>0$ such that*

$$G_d^\gamma(x) \le \frac{c}{|x|^{d-\gamma}} \quad \textit{for } |x| < 1.$$

Proof. Since

$$G_d^\gamma(x) \le \int_0^\infty p_t^\gamma(x)\,dt,$$

the result follows from the well-known formula for Riesz γ-potential, see e.g. Landkof [56]. □

Conclusion. There exists a constant $c>0$ such that, if $\gamma < d$, $\gamma \le 2$, then:

$$G_d^\gamma(x) \le \frac{c}{|x|^{d-\gamma}} \quad \text{if } |x| \le 1, \qquad \text{and} \qquad G_d^\gamma(x) \le \frac{c}{|x|^{d+\gamma}} \quad \text{if } |x| \ge 1.$$

Proof of Theorem 4.8. Now, we pass to the proof of the theorem and restrict to the case of $b(t) = 1$, $t \geq 0$, that is, to stochastic heat equation. (Proof for the next two cases may be obtained in a similar way.) We have $\mathbf{s}(t;\gamma) = e^{-\gamma t}$, $t \geq 0$, $\gamma \geq 0$ and therefore

$$\mathbf{s}(t; v(\lambda)) = e^{-v(\lambda)t}, \quad \lambda \in \mathbb{R}^d,\ t \geq 0.$$

By Theorem 4.7, if for some $\varepsilon > 0$ and all $t > 0$,

$$\int_{\mathbb{R}^d} (\ln(1+|\lambda|))^{1+\varepsilon} \left[\int_0^t e^{-2v(\lambda)\sigma} d\sigma\right] \mu(d\lambda) < \infty, \tag{4.17}$$

then for all $t > 0$, the solution of the stochastic equation (in the general form), has a continuous version. Taking into account that a is a non-negative continuous function, one can replace (4.17) by

$$\int_{\mathbb{R}^d} (\ln(1+|\lambda|))^{1+\varepsilon} \frac{1}{1+v(\lambda)} \mu(d\lambda) < \infty. \tag{4.18}$$

Since we have assumed that $A = -(-\Delta)^{\alpha/2}$, then $v(\lambda) = |\lambda|^\alpha$, with $\alpha \in\ ]0, 2]$ and therefore (4.18) becomes

$$\int_{\mathbb{R}^d} (\ln(1+|\lambda|))^{1+\varepsilon} \frac{1}{1+|\lambda|^\alpha}\, \mu(d\lambda) < \infty. \tag{4.19}$$

However, the condition (4.19) holds for some $\varepsilon > 0$ if for some $\delta > 0$

$$\int_{\mathbb{R}^d} \frac{1}{1+|\lambda|^{\alpha-\delta}}\, \mu(d\lambda) < \infty.$$

In the same way as in the paper [53] by Karczewska and Zabczyk, for some constant $c > 0$:

$$\int_{\mathbb{R}^d} \frac{1}{1+|\lambda|^\gamma} \mu(d\lambda) = c \int_{\mathbb{R}^d} G_d^\gamma(x) \Gamma(dx),$$

where $\gamma := \alpha - \delta$. Taking into account Lemmas 4.10 and 4.11, the result follows. $\square$

4.2.4. Some special cases. In this subsection we illustrate the main results obtained considering several special cases.

Let us recall that the linear stochastic Volterra equation (4.7) considered in the chapter has the following form

$$X(t) = X_0 + \int_0^t b(t-\tau) A X(\tau)\, d\tau + W_\Gamma(t),$$

where $X_0 \in S'(\mathbb{R}^d)$, A is an operator given in the Fourier transform form

$$\mathcal{F}(A\xi)(\lambda) = -v(\lambda)\mathcal{F}(\xi)(\lambda), \quad \xi \in S'(\mathbb{R}^d),$$

v is a locally integrable function and W_Γ is an $S'(\mathbb{R}^d)$-valued space homogeneous Wiener process. This equation is determined by three objects: the spatial correlation Γ of the process W_Γ, the operator A and the function v or, equivalently, by the spectral measure μ, the function a and the function $\mathbf{s}$, respectively.

We apply our Theorems 4.7 and 4.8 to several special cases corresponding to particular choices of functions v, a and of the measure μ. We will assume, for instance, that $b(t) = 1$ or $b(t) = t$ or $b(t) = e^{-t}$, $t \geq 0$, that $v(\lambda) = |\lambda|^\alpha$, $\alpha \in]0,2]$, $\lambda \in \mathbb{R}^d$ and that the measure μ is either finite or $\mu(d\lambda) = (1/|\lambda|^\gamma)d\lambda$, $\gamma \in]0,d[$. Note that if $v(\lambda) = |\lambda|^2$, then $A = \Delta$ and if $v(\lambda) = |\lambda|^\alpha$, $\alpha \in]0,2[$, then $A = -(-\Delta)^{\alpha/2}$ is the fractional Laplacian. In all considered cases we assume that hypothesis (H), on the function v, holds.

Case 1. If (H) holds, the function a is given by (4.5) and (4.6) and the measure μ is finite then $\mathcal{R} * W_\Gamma$ is a function-valued process. To see this note that by (H) and Theorem 4.6, the measure μ_t given by (4.14) is finite. So, the result follows from Theorem 4.7.

Case 2. If (H) holds, the function a is given by (4.5) and (4.6) and μ is a measure such that for some $\varepsilon > 0$,

$$\int_{\mathbb{R}^d} (\ln(1+|\lambda|)^{1+\varepsilon} \mu(d\lambda) < \infty,$$

then for arbitrary $t > 0$, $\mathcal{R} * W_\Gamma(t)$ is a continuous random field. This follows immediately from Theorem 4.7.

Case 3. Assume that $b(t) = 1$ or $b(t) = t$ or $b(t) = e^{-t}$, $t \geq 0$, $A = \Delta$ (Laplace operator) and $\Gamma(x) = \Gamma_\beta(x) = 1/|x|^\beta$, $\beta \in [0,d[$. Then function $\mathbf{s}$ is given by formulas (4.10)–(4.12), respectively. Function $v(\lambda) = |\lambda|^2$, and the spectral measure μ_β corresponding to Γ_β is of the form $\mu_\beta(d\lambda) = c_\beta/|\lambda|^{d-\beta}$, with c_β a positive constant. To simplify notation we assume that $d \geq 2$. Then $\mathcal{R} * W_\Gamma$ is a function-valued process if an only if $\beta \in]0,2[$, see ([53]). Moreover, if $\beta \in]0,2[$ then for each $t > 0$, $\mathcal{R} * W_\Gamma(t)$, is a continuous random field. To prove this we use Theorem 4.8 and show that for some $\delta > 0$,

$$\int_{|x|<1} \frac{1}{|x|^{d-2+\delta}} \, \Gamma_\beta(x) \, dx < \infty \tag{4.20}$$

and

$$\int_{|x|\geq 1} \frac{1}{|x|^{d+2-\delta}} \, \Gamma_\beta(x) \, dx < \infty. \tag{4.21}$$

Condition (4.21) is always satisfied because (4.21) is equivalent to: $\beta > \delta - 2$. Condition (4.20) may be replaced by the following one:

$$\int_{|x|<1} \frac{1}{|x|^{d-2+\delta+\beta}} \, dx = c \int_0^1 \frac{1}{r^{d-2+\delta+\beta}} \, r^{d-1} \, dr = c \int_0^1 \frac{1}{r^{\beta-1+\delta}} \, dr < \infty,$$

equivalent to $\beta < 2 - \delta$, which holds for sufficiently small $\delta > 0$.

Case 4. Assume that $b(t) = 1$ and the operator A is given by the formula

$$\mathcal{F}(A\xi)(\lambda) = -v(\lambda)\,\mathcal{F}(\xi),$$

where

$$v(\lambda) = \langle Q\lambda, \lambda\rangle + \int_{\mathbb{R}^d} (1 - \cos\langle \lambda, x\rangle)\nu(dx)$$

and ν is a symmetric measure such that

$$\int_{\mathbb{R}^d} (|x|^2 \wedge 1)\,\nu(dx) < \infty.$$

Then the equation (4.7) has a function-valued solution if and only if

$$\int_{\mathbb{R}^d} \frac{1}{1 + v(\lambda)}\,\mu(d\lambda) < \infty.$$

Additionally, if $X_0 = 0$ and

$$\int_{\mathbb{R}^d} (\ln(1 + |\lambda|)^{1+\varepsilon}) \frac{1}{1 + v(\lambda)}\,\mu(d\lambda) < \infty,$$

then equation (4.7) has continuous version for each $t \geq 0$.

In this situation, $\mathbf{s}(\sigma, v(\lambda)) = e^{-\sigma v(\lambda)}$. By Theorem 4.7 the condition for function-valued solution of the equation (4.7) becomes:

$$\int_{\mathbb{R}^d} \left(\int_0^t (\mathbf{s}(\sigma, v(\lambda)))^2 d\sigma \right) \mu(d\lambda) = \int_{\mathbb{R}^d} \int_0^t e^{-2\sigma v(\lambda)} d\sigma\, \mu(d\lambda) < \infty,$$

and it is equivalent to

$$\int_{\mathbb{R}^d} \int_0^t \frac{1}{1 + v(\lambda)}\,\mu(d\lambda) < \infty.$$

4.3. Limit measure to stochastic Volterra equations

This section is a natural continuation of the previous one. Description of asymptotic properties of solutions to stochastic evolution equations in finite dimensional spaces and Hilbert spaces is well-known and has been collected in the monograph [24]. This problem has been studied for generalized Langevin equations in conuclear spaces also by Bojdecki and Jakubowski [11]. The question of existence of invariant and limit measures in the space of distributions seems to be particularly interesting. Especially for stochastic Volterra equations, because this class of equations is not well-investigated.

In the section we give necessary and sufficient conditions for the existence of a limit measure and describe all limit measures to the equation (4.1). Our results are in a sense analogous to those formulated for the finite-dimensional and Hilbert space cases obtained for stochastic evolution equations, see [24, Chapter 6].

Let us recall the stochastic Volterra equation (4.1) in the simpler form (4.7), that is,

$$X(t) = X_0 + \int_0^t b(t-\tau)AX(\tau)\,d\tau + W_\Gamma(t).$$

As previously, we study this equation in the space $S'(\mathbb{R}^d)$, where $X_0 \in S'(\mathbb{R}^d)$, A is an operator given in the Fourier transform form (4.8), i.e.

$$\mathcal{F}(A\xi)(\lambda) = -v(\lambda)\,\mathcal{F}(\xi)(\lambda),$$

where v is a locally integrable function and W_Γ is an $S'(\mathbb{R}^d)$-valued space homogeneous Wiener process.

4.3.1. The main results. In this subsection we formulate results providing the existence of a limit measure and the form of any limit measure for the stochastic Volterra equation (4.7) with the operator A given by (4.8). In our considerations we assume that the hypothesis (H) holds.

Let us recall the definition of weak convergence of probability measures defined on the space $S'(\mathbb{R}^d)$ of tempered distributions.

Definition 4.12. We say that a sequence $\{\gamma_t\}$, $t \geq 0$, of probability measures on $S'(\mathbb{R}^d)$ *converges weakly* to probability measure γ on $S'(\mathbb{R}^d)$ if for any function $f \in C_b(S')$

$$\lim_{t\to\infty} \int_{S'(\mathbb{R}^d)} f(x)\,\gamma_t(dx) = \int_{S'(\mathbb{R}^d)} f(x)\,\gamma(dx). \tag{4.22}$$

More general definition on weak convergence of probability measures defined on topological spaces may be found, e.g. in [6] or [45].

By ν_t we denote the law $\mathcal{L}(\widetilde{Z}(t)) = \mathcal{N}(0,\Gamma_t)$ of the process

$$\widetilde{Z}(t) := \int_0^t \mathcal{R}(t-\sigma)\,dW_\Gamma(\sigma), \quad t \geq 0. \tag{4.23}$$

Let us define

$$\mu_\infty(d\lambda) := \left[\int_0^\infty (\mathbf{s}(\sigma, a(\lambda)))^2\,d\sigma\right]\mu(d\lambda). \tag{4.24}$$

Convergence of measures in the distribution sense is a special kind of weak convergence of measures. This means that

$$\int_{\mathbb{R}^d} \varphi(\lambda)\,d\mu_t(\lambda) \xrightarrow{t\to\infty} \int_{\mathbb{R}^d} \varphi(\lambda)\,d\mu_\infty(\lambda) \tag{4.25}$$

for any test function $\varphi \in S(\mathbb{R}^d)$.

Now, we can formulate the following results.

Lemma 4.13. *Let μ_t and μ_∞ be measures defined by* (4.14) *and* (4.24)*, respectively. If μ_∞ is a slowly increasing measure, then the measures $\mu_t \to \mu_\infty$, as $t \to \infty$, in the distribution sense.*

Proof. First of all, let us notice that, by Theorem 4.6, μ_t, $t \geq 0$, are spectral measures of stationary generalized Gaussian random fields. Moreover, the measures μ_t, $t \geq 0$, are slowly increasing. Since the function $\mathbf{s}(\tau, a(\lambda))$, $\tau \geq 0$, $\lambda \in \mathbb{R}^d$ is bounded, then the integral $g_t(\lambda) = \int_0^t (\mathbf{s}(\tau, a(\lambda)))^2\, d\tau$, for $t < \infty$, is bounded, as well. In the proof we shall use the specific form of the measures μ_t, $t \geq 0$, defined by (4.14).

We assume that the measure μ_∞ is slowly increasing, that is, there exists $k > 0$:

$$\int_{\mathbb{R}^d} (1 + |\lambda|^2)^{-k}\, d\mu_\infty(\lambda) = \int_{\mathbb{R}^d} (1 + |\lambda|^2)^{-k} \left[\int_0^\infty (\mathbf{s}(t, a(\lambda)))^2\, d\tau \right] d\mu(\lambda) < \infty.$$

Hence, the function $g_\infty(\lambda) = \int_0^\infty (\mathbf{s}(\tau, a(\lambda)))^2\, d\tau < \infty$ for μ-almost every λ.

In our case, because of formulae (4.14) and (4.24), we have to prove the following convergence:

$$\lim_{t\to\infty} \int_{\mathbb{R}^d} \varphi(\lambda)\, g_t(\lambda)\, d\mu(\lambda) = \int_{\mathbb{R}^d} \varphi(\lambda)\, g_\infty(\lambda)\, d\mu(\lambda), \tag{4.26}$$

where $\varphi \in S(\mathbb{R}^d)$, and g_t and g_∞ are as above.

In other words, the convergence (4.25) of the measures μ_t, $t \geq 0$, to the measure μ_∞ in the distribution sense, in our case is equivalent to the weak convergence (4.26) of functions g_t, $t \geq 0$, to the function g_∞.

Let us recall that the function $\mathbf{s}$ determining the measures μ_t, $t \geq 0$, and μ_∞, satisfies the Volterra equation (4.9) (see hypothesis (H)):

$$\mathbf{s}(t) + \gamma \int_0^t b(t-\tau)\mathbf{s}(\tau)\, d\tau = 1.$$

Additionally, by Lemma 2.1 from [15], $\lim_{t\to\infty} \mathbf{s}(t) = 0$.

For any $\varphi \in S(\mathbb{R}^d)$ we have the following estimations

$$\begin{aligned} &\left| \int_{\mathbb{R}^d} \varphi(\lambda)\, g_t(\lambda)\, d\mu(\lambda) - \int_{\mathbb{R}^d} \varphi(\lambda)\, g_\infty(\lambda)\, d\mu(\lambda) \right| \\ &\quad \leq \int_{\mathbb{R}^d} |\varphi(\lambda)|\, |g_t(\lambda) - g_\infty(\lambda)|\, d\mu(\lambda) \\ &\quad = \int_{\mathbb{R}^d} |\varphi(\lambda)| \left| \int_0^t (\mathbf{s}(\tau, a(\lambda)))^2\, d\tau - \int_0^\infty (\mathbf{s}(\tau, a(\lambda)))^2\, d\tau \right| d\mu(\lambda) \\ &\quad \leq \int_{\mathbb{R}^d} |\varphi(\lambda)| \left(\int_t^\infty (\mathbf{s}(\tau, a(\lambda)))^2\, d\tau \right) d\mu(\lambda). \end{aligned} \tag{4.27}$$

The right hand side of (4.27) tends to zero because

$$h_\infty(\lambda) = \int_t^\infty (\mathbf{s}(\tau, a(\lambda)))^2 \, d\tau$$

tends to zero, as $t \to \infty$.

Hence, we have proved the convergence (4.26) which is equivalent to the convergence (4.25) of the measures μ_t, as $t \to \infty$, to the measure μ_∞ in the distribution sense. $\square$

Lemma 4.14. *Let Γ_t, Γ_∞ be covariance kernels of the stochastic convolution* (4.23) *for $t < \infty$ and $t = \infty$, respectively, and let μ_t, μ_∞ be defined by* (4.14) *and* (4.24)*. Assume that μ_∞ is a slowly increasing measure on $\mathbb{R}^d$. Then $\Gamma_t \to \Gamma_\infty$, as $t \to \infty$, in the distribution sense if and only if the measures $\mu_t \to \mu_\infty$, for $t \to \infty$, in the distribution sense.*

Proof. The sufficiency comes from the convergence of measures in the distribution sense which, in fact, is a type of weak convergence of measures. Actually, the convergence of μ_t, $t \geq 0$, to the measure μ_∞ in the distribution sense means that $\langle \mu_t, \varphi \rangle \xrightarrow{t\to\infty} \langle \mu_\infty, \varphi \rangle$ for any $\varphi \in S(\mathbb{R}^d)$. Particularly, because the Fourier transform acts from $S(\mathbb{R}^d)$ into $S(\mathbb{R}^d)$, we have $\langle \mu_t, \mathcal{F}(\varphi) \rangle \xrightarrow{t\to\infty} \langle \mu_\infty, \mathcal{F}(\varphi) \rangle$ for any $\varphi \in S(\mathbb{R}^d)$. This is equivalent to the convergence $\langle \mathcal{F}^{-1}(\mu_t), \varphi \rangle \xrightarrow{t\to\infty} \langle \mathcal{F}^{-1}(\mu_\infty), \varphi \rangle \rangle$, $\varphi \in S(\mathbb{R}^d)$.

This means the convergence of the Fourier inverse transforms of considered measures μ_t, as $t \to \infty$, to the inverse transform of the measure μ_∞ in the distribution sense.

Because the measures μ_t, $t \geq 0$, and μ_∞ are positive, symmetric and slowly increasing on $\mathbb{R}^d$, then their Fourier inverse transforms define, by Bochner–Schwartz theorem, covariance kernels $\Gamma_t = \mathcal{F}^{-1}(\mu_t)$, $t \geq 0$, and $\Gamma_\infty = \mathcal{F}^{-1}(\mu_\infty)$, respectively. Hence, $\Gamma_t \to \Gamma_\infty$ as $t \to \infty$, in the distribution sense.

The necessity is the version of Lévy–Cramér's theorem generalized for a sequence of slowly increasing measures $\{\mu_t\}$, $t \geq 0$, and their Fourier inverse transforms which are their characteristic functionals. $\square$

Now, we are able to formulate the main results of the section.

Theorem 4.15. *There exists the limit measure $\nu_\infty = \mathcal{N}(0, \Gamma_\infty)$, the weak limit of the measures $\nu_t = \mathcal{N}(0, \Gamma_t)$, as $t \to \infty$, if and only if the measure μ_∞ defined by* (4.24) *is slowly increasing.*

Theorem 4.16. *Assume that the measure μ_∞ defined by* (4.24) *is slowly increasing. Then any limit measure of the stochastic Volterra equation* (4.7) *is of the form*

$$m_\infty * \mathcal{N}(0, \Gamma_\infty), \tag{4.28}$$

where m_∞ is the limit measure for the deterministic version of the equation (4.7) *with condition* (4.8)*, and $\mathcal{N}(0,\Gamma_\infty)$ is the limit measure of the measures ν_t, as $t\to\infty$.*

We would like to emphasize that Theorems 4.15 and 4.16 have been formulated in the spirit analogous to well-known theorems giving invariant measures for linear evolution equations, see e.g. [24] or [11]. Such results first give conditions for the existence of invariant measure and next describe all invariant measures provided they exist. Our theorems extend, in some sense, Theorem 6.2.1 from [24]. Because we consider stochastic Volterra equations we can not study invariant measures but limit measures.

4.3.2. Proofs of theorems.

Proof of Theorem 4.15. ($\Rightarrow$) Let us notice that, by Theorem 4.6, the laws $\nu_t = \mathcal{N}(0,\Gamma_t)$, $t\geq 0$, are laws of Gaussian, stationary, generalized random fields with the spectral measures μ_t and the covariances Γ_t. The weak convergence (4.22) is equivalent to the convergence of the characteristic functionals corresponding to the measures ν_t, $t\geq 0$ and ν_∞, respectively. Particularly

$$\widehat{\nu}_t(\varphi) \xrightarrow{t\to\infty} \widehat{\nu}_\infty(\varphi) \quad \text{for any } \varphi\in S(\mathbb{R}^d).$$

We may use the specific form of the characteristic functionals of Gaussian fields. Namely, we have

$$\widehat{\nu}_t(\varphi) = \mathbb{E}\, e^{i\langle \widetilde{Z}(t),\varphi\rangle} = \exp\biggl(-\frac{1}{2}q_t(\varphi,\varphi)\biggr) = \exp\biggl(-\frac{1}{2}\langle \Gamma_t, \varphi * \varphi_{(s)}\rangle\biggr),$$

where $t\geq 0$, $\varphi\in S(\mathbb{R}^d)$ and $\widetilde{Z}(t)$ is the stochastic convolution given by (4.23). Analogously

$$\widehat{\nu}_\infty(\varphi) = \exp\biggl(-\frac{1}{2}\langle \Gamma_\infty, \varphi * \varphi_{(s)}\rangle\biggr), \quad \varphi\in S(\mathbb{R}^d).$$

Hence, we have the following convergence

$$\exp\biggl(-\frac{1}{2}\langle \Gamma_t, \varphi * \varphi_{(s)}\rangle\biggr) \xrightarrow{t\to\infty} \exp\biggl(-\frac{1}{2}\langle \Gamma_\infty, \varphi * \varphi_{(s)}\rangle\biggr)$$

for any $\varphi\in S(\mathbb{R}^d)$.

Because Γ_t, $t\geq 0$, are positive-definite generalized functions then Γ_∞ is a positive-definite generalized function, too. So, by Bochner–Schwartz theorem, there exists a slowly increasing measure μ_∞ such that $\Gamma_\infty = \mathcal{F}^{-1}(\mu_\infty)$.

($\Leftarrow$) Assume that the measure μ_∞, defined by the formula (4.24) is slowly increasing. Then, by Bochner–Schwartz theorem, there exists a positive-definite distribution Γ_∞ on S such that $\Gamma_\infty = \mathcal{F}^{-1}(\mu_\infty)$ and

$$\langle \Gamma_\infty, \varphi\rangle = \int_{\mathbb{R}^d} \varphi(x)\, d\mu_\infty(x).$$

Now, we have to show, that Γ_∞ is the limit, in the distribution sense, of the functionals Γ_t, $t \geq 0$. In order to do this, by Lemma 4.14, we have to prove the convergence of the spectral measures $\mu_t \to \mu_\infty$, as $t \to \infty$, in the distribution sense. But, by Lemma 4.13, the measures μ_t, $t \geq 0$, defined by (4.14), converge to the measure μ_∞ in the distribution sense. This fact implies, by Lemma 4.14, that $\Gamma_t \to \Gamma_\infty$, as $t \to \infty$, in the distribution sense. Then, the following convergence

$$\exp\Big(-\frac{1}{2}\langle \Gamma_t, \varphi * \varphi_{(s)} \rangle\Big) \xrightarrow{t\to\infty} \exp\Big(-\frac{1}{2}\langle \Gamma_\infty, \varphi * \varphi_{(s)} \rangle\Big)$$

holds for any $\varphi \in S(\mathbb{R}^d)$. This means the convergence of characteristic functionals of the measures $\nu_t = \mathcal{N}(0, \Gamma_t)$, $t \geq 0$, to the characteristic functional of the measure $\nu_\infty = \mathcal{N}(0, \Gamma_\infty)$. Hence, there exists the weak limit ν_∞ of the sequence ν_t, $t \geq 0$, and $\nu_\infty = \mathcal{N}(0, \Gamma_\infty)$. $\square$

Proof of Theorem 4.16. Consider a limit measure for the stochastic Volterra equation (4.7) with the condition (4.8). This means that we study a limit distribution of the solution given by (4.13) to the considered equation (4.7).

Let us introduce the following notation for distributions, when $0 \leq t < \infty$:

- $\eta_t = \mathcal{L}(X(t))$ means the distribution of the solution $X(t)$;
- $m_t = \mathcal{L}(\mathcal{R}(t)X_0)$ denotes the distribution of the part $\mathcal{R}(t)X_0$ of the solution $X(t)$;
- $\nu_t = \mathcal{L}(\widetilde{Z}(t)) = \mathcal{L}(\int_0^t \mathcal{R}(t-\tau)\, dW_\Gamma(\tau))$ is, as earlier, the distribution of the stochastic convolution $\widetilde{Z}(t)$, that is, $\nu_t = \mathcal{N}(0, \Gamma_t)$.

We assume that η_∞ is any limit measure of the stochastic Volterra equation (4.7) with the condition (4.8). This means that distributions η_t of the solution $X(t)$, as $t \to \infty$, converge weakly to η_∞.

We have to show the formula (4.28), that is the distribution η_∞ has the form $\eta_\infty = m_\infty * \mathcal{N}(0, \Gamma_\infty)$.

The distribution of the solution (4.13) can be written

$$\mathcal{L}(X(t)) = \mathcal{L}\Big(\mathcal{R}(t)X_0 + \int_0^t \mathcal{R}(t-\tau)\, dW_\Gamma(\tau) \Big)$$

for any $0 \leq t < \infty$.

Because the initial value X_0 is independent of the process $W_\Gamma(t)$, we have

$$\mathcal{L}(X(t)) = \mathcal{L}(\mathcal{R}(t)X_0) * \mathcal{L}(\widetilde{Z}(t))$$

or, using the above notation

$$\eta_t = m_t * \nu_t, \quad \text{for any } 0 \leq t < \infty.$$

This formula can be rewritten in terms of characteristic functionals of the above distributions:

$$\widehat{\eta}_t(\varphi) = \widehat{m}_t(\varphi)\,\widehat{\nu}_t(\varphi), \tag{4.29}$$

where $\varphi \in S(\mathbb{R}^d)$ and $0 \leq t < \infty$.

Then, letting in (4.29) t to tend to ∞, we have

$$\widehat{\eta}_\infty(\varphi) = \mathcal{C}(\varphi)\,\widehat{\nu}_\infty(\varphi), \quad \varphi \in S(\mathbb{R}^d),$$

where $\widehat{\eta}_\infty(\varphi)$ is the characteristic functional of the limit distribution η_∞, $\mathcal{C}(\varphi) = \lim_{t\to\infty} \widehat{m}_t(\varphi)$ and $\widehat{\nu}_\infty(\varphi)$ is the characteristic functional of the limit measure $\nu_\infty = \mathcal{N}(0,\Gamma_\infty)$; moreover, $\widehat{\nu}_\infty(\varphi) = \exp(-(1/2)\langle\Gamma_\infty, \varphi * \varphi_{(s)}\rangle)$.

Now, we have to prove that $\mathcal{C}(\varphi)$ is the characteristic functional of the weak limit measure m_∞ of the distributions $m_t = \mathcal{L}(\mathcal{R}(t)X_0)$.

In fact,

$$\mathcal{C}(\varphi) = \widehat{\eta}_\infty(\varphi)\exp\left(\frac{1}{2}\langle\Gamma_\infty, \varphi * \varphi_{(s)}\rangle\right),$$

where the right hand side of this formula, as the product of characteristic functionals, satisfies conditions of the generalized Bochner's theorem (see e.g. [44]). So, using the generalized Bochner's theorem once again, there exists a measure m_∞ in $S'(\mathbb{R}^d)$, such that $\mathcal{C}(\varphi) = \widehat{m}_\infty$, as required. Hence, we have obtained $\eta_\infty = m_\infty * \mathcal{N}(0,\Gamma_\infty)$. □

4.3.3. Some special case. Stochastic Volterra equations have been considered by several authors, see e.g. [14]–[17] and [54], and are studied in connection with problems arising in viscoelasticity. Particularly, in [15] the heat equation in materials with memory is treated. In that paper the authors consider an auxiliary equation of the form

$$z(t) + \int_0^t [\mu\, c(t-\tau) + \beta(t-\tau)]\, z(\tau)\, d\tau = 1, \tag{4.30}$$

$t \geq 0$, where μ is a positive constant and c, β are some functions specified below.

Let us notice that if in the Volterra equation (4.9) we take $b(\tau) = [\mu\, c(\tau) + \beta(\tau)]/\gamma$, we arrive at the equation (4.30). On the contrary, if we assume in the equation (4.30) that $\beta(\tau) = 0$, $\mu = \gamma$ and $b(\tau) = c(\tau)$, we obtain the equation (4.9).

Assume, as in [15], the following hypothesis:

(H1) (a) Function β is nonnegative nonincreasing and integrable on $\mathbb{R}^+$.
(b) The constants μ, c_0 are positive.
(c) There exists a function $\delta \in L^1(\mathbb{R}^+)$ such that:

$$c(t) := c_0 - \int_0^t \delta(\sigma)\, d\sigma \quad \text{and} \quad c_\infty := c_0 - \int_0^\infty |\delta(\sigma)|\, d\sigma > 0.$$

Proposition 4.17 ([15, Lemma 2.1]). *Let functions β, δ and c be as in hypothesis* (H1). *Then the solution to* (4.30) *satisfies:*

(a) $0 \le |z(t)| \le 1$, $t \ge 0$;
(b) $\int_0^\infty |z(t)|\, dt \le (\mu\, c_\infty)^{-1} < \infty$.

In the next result we will use the above assumption and Proposition 4.17 of Clément and Da Prato and follow the spirit of their argumantation.

Proposition 4.18. *Assume that the stochastic Volterra equation* (4.7) *has the kernel function v given in the form*

$$b(t) = c_0 - \int_0^t |\delta(\sigma)| d\sigma > 0, \quad c_0 > 0 \quad \textit{where } \delta \in L^1(\mathbb{R}^+),$$

and the operator A is given by (4.8). *In this case the limit measure μ_∞ given by the formula* (4.24) *is a slowly increasing measure.*

Proof. This proposition is the direct consequence of the definition (4.24) of the measure μ_∞ and Proposition 4.17. In fact, from Proposition 4.17 we have

$$\int_0^\infty |\mathbf{s}(\tau,\gamma)|\, d\tau \le (\gamma\, c_\infty)^{-1} , \tag{4.31}$$

where γ satisfies hypothesis (H1), so c_∞ is finite. Hence, the right hand side of (4.31) is finite for any finite γ. In our case, because A satisfies (4.8), $\gamma = a(\lambda)$.

From the definition (4.24) of the measure μ_∞ we have:

$$\int_{\mathbb{R}^d} (1+|\lambda|^2)^{-k}\, d\mu_\infty(\lambda) = \int_{\mathbb{R}^d} (1+|\lambda|^2)^{-k} \left[\int_0^\infty (\mathbf{s}(t,a(\lambda)))^2\, d\tau \right] d\mu(\lambda) \tag{4.32}$$

for $k > 0$. Let us notice that, by Proposition 4.17, $0 \le |\mathbf{s}(t,a(\lambda))| \le 1$ for $t \ge 0$. So, $(\mathbf{s}(t,a(\lambda)))^2 \le |\mathbf{s}(t,a(\lambda))|$. Therefore, because (4.31) holds and the measure μ is slowly increasing, the right hand side of (4.32) is finite. Hence, the measure μ_∞ is slowly increasing, too. □

4.4. Regularity of solutions to equations with infinite delay

4.4.1. Introduction and setting the problem. In this section we consider the following integro-differential stochastic equation with infinite delay

$$X(t,\theta) = \int_{-\infty}^t b(t-s)[\Delta X(s,\theta) + \dot{W}_\Gamma(s,\theta)]\, ds, \quad t \ge 0,\ \theta \in T^d, \tag{4.33}$$

where $b \in L^1(\mathbb{R}_+)$, Δ is the Laplace operator and T^d is the d-dimensional torus. In (4.33), W_Γ is a spatially homogeneous Wiener process with the space covariance Γ taking values in the space of tempered distributions $\mathcal{S}'(T^d)$ and $\dot{W}_\Gamma$ denotes its partial derivative with respect to the first argument in the sense of distributions. Such equation arises, in the deterministic case, in the study of heat flow in materials of fading memory type (see [13], [64]).

In this section we address the following question: under what conditions on the covariance Γ the process X takes values in a Sobolev space $H^\alpha(T^d)$, particularly in $L^2(T^d)$?

We remark that the knowledge of the regularity of solutions is important in the study of nonlinear stochastic equations (see e.g. [20] and [62]).

We study a particular case of weak solutions under the basis of an explicit representation of the solution to (4.33) (cf. Definition 4.20).

Observe that equation (4.33) can be viewed as the *limiting equation* for the stochastic Volterra equation

$$X(t,\theta) = \int_0^t b(t-s)[\Delta X(s,\theta) + \dot{W}_\Gamma(s,\theta)]ds, \quad t \geq 0,\ \theta \in T^d. \tag{4.34}$$

If b is sufficiently regular, we get, by differentiating (4.34) with respect to t,

$$\begin{aligned} \frac{\partial X}{\partial t}(t,\theta) &= b(0)[\Delta X(t,\theta) + \dot{W}_\Gamma(t,\theta)] \\ &\quad + \int_0^t b'(t-s)[\Delta X(s,\theta) + \dot{W}_\Gamma(s,\theta)]\, ds, \end{aligned} \tag{4.35}$$

where $t \geq 0$ and $\theta \in T^d$.

Taking in (4.35), $b(t) \equiv 1$ we obtain

$$\begin{cases} \dfrac{\partial X}{\partial t}(t,\theta) = \Delta X(t,\theta) + \dot{W}_\Gamma(t,\theta), & t > 0,\ \theta \in T^d, \\ X(0,\theta) = 0, & \theta \in T^d. \end{cases} \tag{4.36}$$

Similarly, taking $b(t) = t$ and differentiating (4.34) twice with respect to t we obtain

$$\begin{cases} \dfrac{\partial^2 X}{\partial t^2}(t,\theta) = \Delta X(t,\theta) + \dot{W}_\Gamma(t,\theta), & t > 0,\ \theta \in T^d, \\ X(0,\theta) = 0, & \theta \in T^d, \\ \dfrac{\partial X}{\partial t}(0,\theta) = 0, & \theta \in T^d. \end{cases} \tag{4.37}$$

It has been shown in [53, Theorem 5.1] (see also [55, Theorem 1]) that equations (4.36) and (4.37) on the d-dimensional torus T^d have an $H^{\alpha+1}(T^d)$-valued solutions if and only if the Fourier coefficients (γ_n) of the space covariance Γ of the process W_Γ satisfy

$$\sum_{n \in \mathbb{Z}^d} \gamma_n (1 + |n|^2)^\alpha < \infty. \tag{4.38}$$

Observe that for both, stochastic heat (4.36) and wave (4.37) equations, the conditions are exactly the same, despite of the different nature of the equations. On the other hand, the obtained characterization form a natural framework in which nonlinear heat and wave equations can be studied.

We will prove that condition (4.38) even characterizes $H^{\alpha+1}(T^d)$-valued solutions for the stochastic Volterra equation (4.33), provided certain conditions on the kernel b are satisfied. This is a strong contrast with the deterministic case, where regularity of (4.33) is dependent on the kernel b. The conditions that we impose on b are satisfied by a large class of functions. Moreover, the important example $b(t) = e^{-t}$ is shown to satisfy our assumptions.

We use, instead of resolvent families, a direct approach to the equation (4.33) finding an explicit expression for the solution in terms of the kernel b. This approach reduces the considered problem to questions in harmonic analysis and lead us with a complete answer.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a complete filtered probability space. By T^d we denote the d-dimensional torus which can be identified with the product $(-\pi,\pi]^d$. Let $D(T^d)$ and $D'(T^d)$ denote, respectively, the space of test functions on T^d and the space of distributions. By $\langle \xi, \phi\rangle$ we denote the value of a distribution ξ on a test function. We assume that W_Γ is a $D'(T^d)$-valued spatially homogeneous Wiener process with covariance Γ which is a positive-definite distribution.

As we have already written, any arbitrary spatially homogeneous Wiener process W_Γ is uniquely determined by its covariance Γ according to the formula

$$\mathbb{E}\,\langle W_\Gamma(t,\theta),\phi\rangle\,\langle W_\Gamma(\tau,\theta),\psi\rangle = \min(t,\tau)\,\langle \Gamma, \phi\star\psi_{(s)}\rangle, \tag{4.39}$$

where $\phi,\psi\in D(T^d)$ and $\psi_{(s)}(\eta) = \psi(-\eta)$, for $\eta\in T^d$. Because W_Γ is spatially homogeneous process, the distribution $\Gamma = \Gamma(\theta-\eta)$ for $\theta,\eta\in T^d$.

The space covariance Γ, like distribution in $D'(T^d)$, may be uniquely expanded (see e.g. [33] or [74]) into its Fourier series (with parameter $w = 1$ because the period is 2π)

$$\Gamma(\theta) = \sum_{n\in\mathbb{Z}^d} e^{i(n,\theta)}\gamma_n, \quad \theta\in T^d, \tag{4.40}$$

convergent in $D'(T^d)$. In (4.40), $(n,\theta) = \sum_{i=1}^d n_i\theta_i$ and $\mathbb{Z}^d$ denotes the product of integers.

The coefficients γ_n, in the Fourier series (4.40), satisfy:

(1) $\gamma_n = \gamma_{-n}$ for $n\in\mathbb{Z}^d$,
(2) the sequence (γ_n) is *slowly increasing*, that is

$$\sum_{n\in\mathbb{Z}^d} \frac{\gamma_n}{1+|n|^r} < \infty, \quad \text{for some } r>0. \tag{4.41}$$

Let us introduce, by induction, the following set of indexes. Denote $\mathbb{Z}_s^1 := \mathbb{N}$, the set of natural numbers and define $\mathbb{Z}_s^{d+1} := (\mathbb{Z}_s^1\times\mathbb{Z}^d)\cup\{(0,n) : n\in\mathbb{Z}_s^d\}$. Let us notice that $\mathbb{Z}^d = \mathbb{Z}_s^d\cup(-\mathbb{Z}_s^d)\cup\{0\}$. For instance, for $d=2$, $\mathbb{Z}_s^2 = \mathbb{N}\times\mathbb{Z}\cup\{(0,n) : n\in\mathbb{Z}\}$.

Now, the spatially homogeneous Wiener process W_Γ corresponding to the covariance Γ given by (4.40) may be represented in the form

$$W_\Gamma(t,\theta) = \sqrt{\gamma_0}\beta_0(t) + \sum_{n\in\mathbb{Z}_s^d} \sqrt{2\gamma_n}[\cos(n,\theta)\,\beta_n^1(t) + \sin(n,\theta)\,\beta_n^2(t)], \tag{4.42}$$

for $t \geq 0$ and $\theta \in T^d$. In (4.42) $\beta_0, \beta_n^1, \beta_n^2, n \in \mathbb{Z}_s^d$, are independent real Brownian motions and γ_0, γ_n are coefficients of the series (4.40). The series (4.42) is convergent in the sense of $D'(T^d)$.

Because any periodic distribution with positive period is a tempered distribution (see, e.g. [33]), we may restrict our considerations to the space $S'(T^d)$ of tempered distributions. By $S(T^d)$ we denote the space of infinitely differentiable rapidly decreasing functions on the torus T^d.

Let us denote by $H^\alpha = H^\alpha(T^d)$, $\alpha \in \mathbb{R}$, the real Sobolev space of order α on the torus T^d. The norms in such spaces may be expressed in terms of the Fourier coefficients (see [1])

$$\begin{aligned} ||\xi||_{H^\alpha} &= \left(\sum_{n\in\mathbb{Z}^d} (1+|n|^2)^\alpha |\xi_n|^2\right)^{1/2} \\ &= \left(|\xi_0|^2 + 2\sum_{n\in\mathbb{Z}_s^d} (1+|n|^2)^\alpha ((\xi_n^1)^2 + (\xi_n^2)^2)\right)^{1/2}, \end{aligned}$$

where $\xi_n = \xi_n^1 + i\xi_n^2, \xi_= \bar{\xi}_{-n}, n \in \mathbb{Z}^d$.

There is another possibility to define the Sobolev spaces (see, e.g. [71]). We say that a distribution $\xi \in S'(T^d)$ belongs to $H^\alpha, \alpha \in \mathbb{R}$, if its Fourier transform $\widehat{\xi}$ is a measurable function and

$$\int_{T^d} (1+|\lambda|^2)^\alpha\, |\widehat{\xi}(|\lambda|)|^2\, d\lambda < \infty.$$

4.4.2. Main results. If $b \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ and $\mu \in \mathbb{C}$, we shall denote by $r(t,\mu)$ the unique solution in $L^1_{\mathrm{loc}}(\mathbb{R}_+)$ to the linear Volterra equation

$$r(t,\mu) = b(t) + \mu \int_0^t b(t-s) r(s,\mu)\, ds, \quad t \geq 0. \tag{4.43}$$

In many cases the function $r(t,\mu)$ may be found explicitly. For instance:

$$\begin{aligned} b(t) &\equiv 1, & r(t,\mu) &= e^{\mu t}, & & \\ b(t) &= t, & r(t,\mu) &= \frac{\sinh\sqrt{\mu}\,t}{\sqrt{\mu}} & & \mu \neq 0, \\ b(t) &= e^{-t}, & r(t,\mu) &= e^{(1+\mu)t}, & & \\ b(t) &= te^{-t}, & r(t,\mu) &= e^{-t}\,\frac{\sinh\sqrt{\mu}\,t}{\sqrt{\mu}}, & & \mu \neq 0. \end{aligned}$$

For more examples, see monograph [70] by Prüss.

Let us denote by $\widehat{f}(k)$, $k \in \mathbb{Z}$, the k-th Fourier coefficient of an integrable function f:

$$\widehat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} e^{ikt} f(t)\, dt.$$

Given $b \in L^1(\mathbb{R}_+)$, we find that, for $F(t) := \int_{-\infty}^t b(t-s) f(s)\, ds$, we have

$$\widehat{F}(k) = \widetilde{b}(ik)\widehat{f}(k), \quad k \in \mathbb{Z}, \tag{4.44}$$

where $\widetilde{b}(\lambda) := \int_0^\infty e^{-\lambda t} b(t)\, dt$ denotes the Laplace transform of b.

In what follows we will assume that $\widetilde{b}(ik)$ exists for all $k \in \mathbb{Z}$ and suppose that $\lambda \to \widetilde{b}(\lambda)$ admits an analytical extension to a sector containing the imaginary axis, and still denote this extension by $\widetilde{b}$. We introduce the following definition.

Definition 4.19. We say that a kernel $b \in L^1(\mathbb{R}_+)$ is *admissible* for equation (4.33) if

$$\lim_{|n| \to \infty} |n|^2 \int_0^\infty [r(s, -|n|^2)]^2\, ds =: C_b$$

exists.

Examples. (a) In the case $b(t) = e^{-t}$ we obtain

$$|n|^2 \int_0^\infty [r(s, -|n|^2)]^2\, ds = \frac{-|n|^2}{2(1-|n|^2)}$$

and hence $C_b = 1/2$.

(b) In the case $b(t) = te^{-t}$ we obtain

$$|n|^2 \int_0^\infty [r(s, -|n|^2)]^2\, ds = \frac{|n|^2}{4 + 4|n|^2}$$

and hence $C_b = 1/4$.

Denote by $W_n(t,\theta) := \cos(n,\theta)\, \beta_n^1(t) + \sin(n,\theta)\, \beta_n^2(t)$, $n \in \mathbb{Z}_s^d$, that is the n-th element in the expansion (4.42).

Definition 4.20. By a solution $X(t,\theta)$ to the stochastic Volterra equation (4.33) we will understand the process of the form

$$X(t,\theta) = \sqrt{\gamma_0}\, \beta_0(t) + \sum_{n \in \mathbb{Z}_s^d} \sqrt{2\gamma_n} \int_{-\infty}^t r(t-s, -|n|^2)\, dW_n(s,\theta), \tag{4.45}$$

where the function r is as above, $t \geq 0$ and $\theta \in T^d$.

The process X given by (4.45) is a particular form of the weak solution to the equation (4.33) (cf. [53]) and takes values in the space $S'(T^d)$.

The following is our main result.

Theorem 4.21. *Assume $b \in L^1(\mathbb{R}_+)$ is admissible for* (4.33). *Then, the equation* (4.33) *has an $H^{\alpha+1}(T^d)$-valued solution if and only if the Fourier coefficients (γ_n) of the covariance Γ satisfy*

$$\sum_{n\in\mathbb{Z}^d} \gamma_n (1+|n|^2)^\alpha < \infty. \tag{4.46}$$

In particular, equation (4.33) *has an $L^2(T^d)$-valued solution if and only if*

$$\sum_{n\in\mathbb{Z}^d} \frac{\gamma_n}{1+|n|^2} < \infty.$$

Proof. We shall use the representation (4.42) for the Wiener process $W_\Gamma(t,\theta)$ with respect to the basis: 1, $\cos(n,\theta)$, $\sin(n,\theta)$, where $n \in \mathbb{Z}^d_s$ and $\theta \in T^d$. Equation (4.33) may be solved coordinatewise as follows.

Assume that

$$X(t,\theta) = \sum_{n\in\mathbb{Z}^d_s} [\cos(n,\theta)\, X^1_n(t) + \sin(n,\theta)\, X^2_n(t)] + X_0(t). \tag{4.47}$$

Introducing (4.47) into (4.33), we obtain

$$\begin{aligned}
\cos(n,\theta)X^1_n(t) &+ \sin(n,\theta)X^2_n(t)\\
&= -\,|n|^2 \int_{-\infty}^t b(t-s)[\cos(n,\theta)X^1_n(s) + \sin(n,\theta)X^2_n(s)]\, ds\\
&\quad + \sqrt{2\gamma_n}\int_{-\infty}^t b(t-s)[\cos(n,\theta)\,\beta^1_n(s) + \sin(n,\theta)\,\beta^2_n(s)]\, ds,
\end{aligned}$$

or, equivalently

$$\begin{aligned}
&[\cos(n,\theta),\sin(n,\theta)] \begin{bmatrix} X^1_n(t)\\ X^2_n(t)\end{bmatrix}\\
&\qquad = -\,|n|^2 \int_{-\infty}^t b(t-s)[\cos(n,\theta),\sin(n,\theta)] \begin{bmatrix} X^1_n(s)\\ X^2_n(s)\end{bmatrix} ds\\
&\qquad\quad + \sqrt{2\gamma_n}\int_{-\infty}^t b(t-s)[\cos(n,\theta),\sin(n,\theta)] \begin{bmatrix} d\beta^1_n(s)\\ d\beta^2_n(s)\end{bmatrix}.
\end{aligned}$$

Denoting

$$X_n(t) := \begin{bmatrix} X^1_n(t)\\ X^2_n(t)\end{bmatrix} \quad\text{and}\quad \beta_n(t) := \begin{bmatrix} \beta^1_n(t)\\ \beta^2_n(t)\end{bmatrix}$$

we arrive at the equation

$$X_n(t) = -|n|^2 \int_{-\infty}^t b(t-s) X_n(s)\, ds + \sqrt{2\gamma_n}\int_{-\infty}^t b(t-s)\, d\beta_n(s). \tag{4.48}$$

Taking Fourier transform in t, and making use of (4.43) with $\mu = -|n|^2$ and (4.44), we get the following solution to the equation (4.48):

$$X_n(t) = \int_{-\infty}^{t} r(t-s,-|n|^2)\sqrt{2\gamma_n}\,d\beta_n(s) = \int_0^\infty r(s,-|n|^2)\sqrt{2\gamma_n}\,d\beta_n(t-s).$$

Hence, we deduce the following explicit formula for the solution to the equation (4.33):

$$X(t,\theta) = \sqrt{\gamma_0}\beta_0(t) + \sum_{n\in\mathbb{Z}^d_s} \sqrt{2\gamma_n}\Big[\cos(n,\theta)\int_0^\infty r(s,-|n|^2)d\beta^1_n(t-s) + \sin(n,\theta)\int_0^\infty r(s,-|n|^2)\,d\beta^2_n(t-s)\Big]. \tag{4.49}$$

Since the series defining the process X converges in $S'(T^d)$, P-almost surely, it follows from the definition of the space H^α that $X(t) \in H^{\alpha+1}$, P-almost surely if and only if

$$\sum_{n\in\mathbb{Z}^d} (1+|n|^2)^{\alpha+1}\gamma_n\Big[\Big(\int_{-\infty}^t r(t-s,-|n|^2)\,d\beta^1_n(s)\Big)^2 + \Big(\int_{-\infty}^t r(t-s,-|n|^2)\,d\beta^2_n(s)\Big)^2\Big] < \infty. \tag{4.50}$$

Because the stochastic integrals in (4.50) are independent Gaussian random variables, we obtain that (4.50) holds P-almost surely if and only if

$$\sum_{n\in\mathbb{Z}^d} (1+|n|^2)^{\alpha+1}\gamma_n\mathbb{E}\Big[\Big(\int_{-\infty}^t r(t-s,-|n|^2)\,d\beta^1_n(s)\Big)^2 + \Big(\int_{-\infty}^t r(t-s,-|n|^2)\,d\beta^2_n(s)\Big)^2\Big] < \infty. \tag{4.51}$$

Or equivalently, using properties of stochastic integrals, if and only if

$$\sum_{n\in\mathbb{Z}^d} (1+|n|^2)^{\alpha+1}\gamma_n \int_0^\infty [r(s,-|n|^2)]^2\,ds < \infty. \tag{4.52}$$

Since b is admissible for the equation (4.33), we conclude that (4.52) holds if and only if

$$\sum_{n\in\mathbb{Z}^d} (1+|n|^2)^{\alpha+1}\frac{\gamma_n}{|n|^2} < \infty,$$

and the proof is completed. □

Concerning uniqueness, we have the following result.

Proposition 4.22. *Assume $b \in L^1(\mathbb{R}_+)$ is admissible for* (4.33) *and the following conditions hold:*

(a) $\sum_{n\in\mathbb{Z}^d} \gamma_n (1+|n|^2)^\alpha < \infty$,
(b) $\{1/\widetilde{b}(ik)\}_{k\in\mathbb{Z}} \subset \mathbb{C} \setminus \{-|n|^2 : n \in \mathbb{Z}^d\}$.

Then, (4.33) *has a unique $H^{\alpha+1}(T^d)$-valued solution.*

Proof. Let $X(t,\theta)$ be solution of

$$X(t,\theta) = \int_{-\infty}^{t} b(t-s)\Delta X(s,\theta)\, ds.$$

Taking Fourier transform in θ and denoting by $X_n(t)$ the n-th Fourier coefficient of $X(t,\theta)$ (t fixed), we obtain

$$X_n(t) = -|n|^2 \int_{-\infty}^{t} b(t-s) X_n(s)\, ds$$

for all $n \in \mathbb{Z}^d$. Taking now Fourier transform in t, we get that the Fourier coefficients of $X_n(t)$ (n fixed) satisfy

$$(1 + |n|^2 \widetilde{b}(ik))\widehat{X_n}(k) = 0$$

for all $k \in \mathbb{Z}$. According to (b) we obtain that $\widehat{X_n}(k) = 0$ for all $k \in \mathbb{Z}$ and all $n \in \mathbb{Z}^d$. Hence, the assertion follows by uniqueness of the Fourier transform. □

The following corollaries are an immediate consequence of Theorem 4.21. The arguments are the same as in [55]. We give here the proof for the sake of completeness.

Corollary 4.23. *Suppose $b \in L^1(\mathbb{R}_+)$ is admissible for* (4.33) *and assume $\Gamma \in L^2(T^d)$. Then the integro-differential stochastic equation* (4.33) *has a solution with values in $L^2(T^d)$ for $d = 1, 2, 3$.*

Proof. We have to check equation (4.46) with $\alpha = -1$. Note, that if $\Gamma \in L^2(T^d)$ then $\widehat{\Gamma} = (\gamma_n) \in l^2(\mathbb{Z}^d)$. Consequently

$$\sum_{n\in\mathbb{Z}^d} \frac{\gamma_n}{1+|n|^2} \le \Bigg(\sum_{n\in\mathbb{Z}^d} {\gamma_n}^2\Bigg)^{1/2} \Bigg(\sum_{n\in\mathbb{Z}^d} \frac{1}{(1+|n|^2)^2}\Bigg)^{1/2}.$$

But $\sum_{n\in\mathbb{Z}^d} {\gamma_n}^2 < \infty$ and

$$\sum_{n\in\mathbb{Z}^d} \frac{1}{(1+|n|^2)^q} < \infty \quad \text{if and only if} \quad 2q > d. \tag{4.53}$$

Hence, the result follows. □

Corollary 4.24. *Suppose $b \in L^1(\mathbb{R}_+)$ is admissible for* (4.33) *and assume $\widehat{\Gamma} \in l^p(\mathbb{Z}^d)$ for $1 < p \leq 2$. Then the integro-differential stochastic equation* (4.33) *has a solution with values in $L^2(T^d)$ for all $d < 2p/(p-1)$.*

Proof. Note that

$$\sum_{n\in\mathbb{Z}^d} \frac{\gamma_n}{1+|n|^2} \leq \Big(\sum_{n\in\mathbb{Z}^d} {\gamma_n}^p \Big)^{1/p} \Big(\sum_{n\in\mathbb{Z}^d} \frac{1}{(1+|n|^2)^q} \Big)^{1/q},$$

where $1/p + 1/q = 1$. Hence the result follows from (4.53) with $q = p/(p-1)$.□

For $\alpha = -1$, the condition (4.46) can be written as follows.

Theorem 4.25. *Let b be admissible for the equation* (4.33)*. Assume that the covariance Γ is not only a positive definite distribution but is also a non-negative measure. Then the equation* (4.33) *has $L^2(T^d)$-valued solution if and only if*

$$(\Gamma, G_d) < \infty, \tag{4.54}$$

where

$$G_d(x) = \sum_{n\in\mathbb{Z}^d} \int_0^\infty \frac{1}{\sqrt{(4\pi t)^d}}\, e^{-t}\, e^{-|x+2\pi n|^2/t}\, dt, \quad x \in T^d. \tag{4.55}$$

The proof of Theorem 4.25 is the same that for Theorem 2, part 2) in [55], so we omit it. For more details concerning the function G_d we refer to [55] and [56].

Additionally, from properties of function G_d defined by (4.55) and the condition (4.54) we obtain the following result (see [53, Theorem 6.1]).

Corollary 4.26. *Assume that Γ is a non-negative measure and b is admissible. Then equation* (4.33) *has function valued solutions:*

(a) *for all Γ if $d = 1$;*
(b) *for exactly those Γ for which $\int_{|\theta|\leq 1} \ln|\theta|\, \Gamma(d\theta) < \infty$ if $d = 2$;*
(c) *for exactly those Γ for which $\int_{|\theta|\leq 1} (1/|\theta|^{d-2})\, \Gamma(d\theta) < \infty$ if $d \geq 3$.*

In what follows, we will see that formula (4.49) also provides hölderianity of X with respect to t. In order to do that, we need assumptions very similar to those in [14].

(H2) Assume that there exist $\delta \in (0,1)$ and $C_\delta > 0$ such that, for all $s \in (-\infty, t)$ we have:
(a) $\int_s^t [r(t-\tau, -|n|^2)]^2\, d\tau \leq C_\delta |n|^{2(\delta-1)}\, |t-s|^\delta$;
(b) $\int_{-\infty}^s [r(t-\tau, -|n|^2) - r(s-\tau, -|n|^2)]^2\, d\tau \leq C_\delta |n|^{2(\delta-1)}\, |t-s|^\delta$.

Proposition 4.27. *Assume that* $\sum_{n\in\mathbb{Z}^d}\gamma_n/(1+|n|^2)<\infty$. *Under hypothesis* (H2)*, the trajectories of the solution* X *to the equation* (4.33) *are almost surely* η*-Hölder continuous with respect to* t*, for every* $\eta\in(0,\delta/2)$.

Proof. From the expansion (4.49) and properties of stochastic integral, we have

$$
\begin{aligned}
&\mathbb{E}\,||X(t,\theta)-X(s,\theta)||^2_{L^2}\\
&=\mathbb{E}\,\Bigg\|\sqrt{\gamma_0}(\beta_0(t)-\beta_0(s))+\sum_{n\in\mathbb{Z}^d_s}\sqrt{2\gamma_n}\Bigg[\cos(n,\theta)\Bigg(\int_{-\infty}^t r(t-\tau,-|n|^2)\,d\beta^1_n(\tau)\\
&\quad-\int_{-\infty}^s r(s-\tau,-|n|^2)\,d\beta^1_n(\tau)\Bigg)\\
&\quad+\sin(n,\theta)\Bigg(\int_{-\infty}^t r(t-\tau,-|n|^2)\,d\beta^2_n(\tau)-\int_{-\infty}^s r(s-\tau,-|n|^2)\,d\beta^2_n(\tau)\Bigg)\Bigg]\Bigg\|^2_{L^2}\\
&=(2\pi)^d\Bigg(\gamma_0|t-s|+\sum_{n\in\mathbb{Z}^d_s}2\gamma_n\Bigg[\int_{-\infty}^s[r(t-\tau,-|n|^2)-r(s-\tau,-|n|^2)]^2\,d\tau\\
&\quad+\int_s^t r^2(t-\tau,-|n|^2)\,d\tau\Bigg]\Bigg).
\end{aligned}
$$

According to assumptions (a) and (b) of the hypothesis (H2), we have

$$\mathbb{E}\,||X(t,\theta)-X(s,\theta)||^2_{L^2}\le C_\delta\sum_{n\in\mathbb{Z}^d_s}2\gamma_n\,|n|^{2(\delta-1)}|t-s|^\delta.$$

Because X is a Gaussian process, then for any $m\in N$, there exists a constant $C_m>0$ that

$$\mathbb{E}\,||X(t,\theta)-X(s,\theta)||^{2m}_{L^2}\le C_m\Bigg[C_\delta\sum_{n\in\mathbb{Z}^d_s}2\gamma_n\,|n|^{2(\delta-1)}\Bigg]^m|t-s|^{m\delta}.$$

Taking m such that $m\delta>1$ and using the Kolmogorov test, we see that the solution $X(t,\theta)$ is η-Hölder continuous, with respect to t, for $\eta=\delta/2-1/(2m)$.□

Example. Let us consider the particular case $b(t)=e^{-t}$, $t\ge 0$. Then, by previous considerations, $r(t,-|n|^2)=e^{(1-|n|^2)\,t}$. One can check that in this case the hypothesis (H2) is fulfilled.

Remark. We observe that the condition (a) in hypothesis (H2) is the same as

$$|n|^2\int_0^t[r(s,-|n|^2)]^2\,ds\le C_\delta|n|^{2\delta}|t|^\delta$$

and hence it is nearly equivalent to say that the function b is admissible.

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