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The Method of Socratic Proofs
for Normal Modal Propositional Logics

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INTRODUCTION

The method of Socratic proofs is a formal explication of the idea of solving problems by pure questioning. The method, invented by A. Wiśniewski, has been first described in [Wiśniewski:2004] for the case of Classical Propositional Calculus. Roughly speaking, a Socratic proof is a finite sequence of questions with the following properties:

(1) the questions – expressions of a certain formal language – represent simple yes-no questions of a natural language;

(2) the first question of the sequence concerns a certain logical problem, *e.g.*, whether a formula is a thesis of a given logic or not;

(3) each consecutive question results from the previous one by a certain rule of inference;

(4) the last question of this sequence concerns another problem of a similar nature, it is, however, evident that the answer to this question is affirmative;

(5) the structure of the rules that allow to infer a question from a question guarantee that if the answer to the last question is affirmative, then so is the answer to the first question.

It may be argued that a sequence of questions of this kind represents a complex reasoning in which a problem (expressed by the first question) is solved by reformulating and simplifying the consecutive questions. No declarative premise is used in such a reasoning. What is more, the inferential relations between questions occurring in this process may be analysed within the framework of **IEL** – *Inferential Erotetic Logic*.[1]

At the same time the method of Socratic proofs may be viewed as a general methodology of formalizing logics in a sequent calculus style. Socratic proofs are constructed in guidance with the rules of the so-called erotetic calculi.[2] An erotetic calculus for a given logic constitutes a proof-method for this logic.

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[1] A formal account of inferential relations between questions is the main topic of the book: [Wiśniewski:1995]. A concise account of the fundamental notions of **IEL** may be found in [Wiśniewski:2001].
[2] The term 'erotetic' comes from gr. '*erotema*' – question.

More specifically, an erotetic calculus may be viewed as a calculus for deriving sequents with rules of a Gentzen-style sequent calculus "turned upside-down" to the effect that a process of constructing a Socratic proof reflects the "root-first" proof-search procedure known from the Gentzen-style sequent calculi. In other words, all the rules of erotetic calculi are of eliminative character (there are no introduction rules) – in this sense an application of a rule of an erotetic calculus to a question results in a simplification of the question. What is more, the rules of an erotetic calculus are *invertible*, which means, *i.a.*, that their "inverses" may be used as rules of a Gentzen-style sequent calculus. In terms of questions, this means that the answer to a question to which an erotetic rule is applied is affirmative *if and only if* the answer to a question obtained under such a transformation is affirmative.

Except for the Classical Propositional Calculus, the method has been adjusted to classical first-order logic and to some paraconsistent propositional logics.[3] Erotetic calculus for a logic characterizes the logic in terms of a calculus of sequents with invertible elimination rules. Since there are no introduction rules, the process of constructing a Socratic transformation (a sequence of questions obtained by successive applications of the rules of an erotetic calculus) may be quite easily converted into an algorithmic procedure, thus constituting, in the case of decidable logics, a decision procedure.[4] Moreover, since the rules are invertible, Socratic proofs obtained by applications of the rules of an erotetic calculus may be transformed into proofs in a certain corresponding Genzten-style sequent calculus.[5]

However, the peculiarity of the method lies in its connection with the logic **IEL**. An erotetic calculus may be viewed as a variant of a sequent calculus but it is still a calculus of *questions* – questions are the "premises" and "conclusions" of erotetic rules. Hence the rules of erotetic calculi describe certain classes of inferences of questions from questions, and these inferences may be further analysed within the framework of **IEL**. In particular, the logic **IEL** develops certain notions to describe conditions of validity of such erotetic inferences. It may be shown that the syntactical relation of derivability of a question from a question encoded in the rules of erotetic calculi has a semantical counterpart in the relation of *pure erotetic implication*.[6] Thus the inferential steps of a Socratic proof are valid in the sense of **IEL**.

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[3] See: [Wiśniewski, Shangin:2006] and [Wiśniewski, Vanackere, Leszczyńska:2005].

[4] See: [Wiśniewski:2004] and [Wiśniewski, Vanackere, Leszczyńska:2005] where such procedures are sketched. The procedures are not given in exact terms in the papers, but they may be quite easily extracted from what is presented there. A theorem-prover for classical propositional logic and for paraconsistent propositional logic **CLuN**, based on these articles, is available on: http://logica.ugent.be/albrecht/socratic.html   The program has been written in Prolog language by Albrecht Heeffer.

[5] More details may be found in [Wiśniewski:2004, pp. 313-316] and in [Wiśniewski, Shangin:2006].

[6] In each of the papers: [Wiśniewski:2004, pp. 318-321], [Wiśniewski, Shangin:2006] and [Wiśniewski, Vanackere, Leszczyńska:2005, pp. 460-463] the reader may find a detailed account

The aim of this dissertation is to present the most familiar of normal modal propositional logics in the framework of erotetic calculi. The proof method we shall present is based on the analysis of the so-called "possible worlds" semantics. There is a wide family of semantically motivated proof-methods for modal logics. The family contains semantic tableaux of Kripke, semantic diagrams of Hughes and Cresswell, different variants of tableau methods (like Fitting's prefixed tableaux and Priest's Smullyan-type analytic tableaux) and, finally, the descriptions of modal logics in more general frameworks (like that of Labelled Deductive Systems).[7] What we mean by a "semantical motivation" of a proof method for modal logics is that the structure of a Kripke model is "mirrored" in the syntax of the proof method apparatus. This is obtained by using certain additional parameters that refer to possible worlds – usually these are single numerals and / or their sequences. The parameters may be operated on by the rules of a system, so the semantical information concerning the accessibility relation is available at the same level as the information concerning logical connectives. The same happens in the case of the method of Socratic proofs for modal logics, which is a semantically motivated method as well.[8]

Most of the semantically motivated proof methods for modal logics have a common feature. Namely, constructing a tableau (a tree, a diagram, *etc.*) for a formula to be proved consists in analysing the formula into its subformulas. Deductive systems having this property are called "analytic systems" (or systems with the "subformula property"). The derivation process in an analytic system may be viewed as a formalization of the procedure of a countermodel construction. The interpretation of a derivation process as an "indirect" reasoning is self-apparent when various tableaux methods are concerned – a proof of a formula *A* in a tableaux calculus is nothing but a permanently unsuccessful attempt to find a model satisfying '¬*A*'. Erotetic calculi for modal logics are also analytic systems. A Socratic proof in such a calculus may be interpreted as a systematic though unsuccessful countermodel construction. However, in contradistinction to the tableaux methods, an interpretation of the derivation process in an erotetic calculus as representing a *direct* reasoning is still more natural than as representing an indirect one.

Analytic, semantically motivated proof systems for modal logics have a common drawback as well. Namely, when transitive modal logics are concerned, infinite tableaux (trees, diagrams, *etc.*) occur to be possible, although the logics in question (like **S4**) have the finite model property. It is easy to observe that the infinity may be actually "reduced" to the so-called "loops". Usually, it is also

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of the erotetic calculus presented in the paper on the grounds of the logic **IEL**. The notion of erotetic implication is one of the central notions of **IEL**.

[7] Appropriate references may be found in Chapter IV where we have more to say about various proof-methods for modal logics.

[8] In [Goré:1999] the term "explicit system" is used for what we mean by a "semantically motivated system" (the point is that the accessibility relation is represented *explicitly* by some device).

quite easy to formulate very intuitive prescriptions of how to avoid loops and how to "interpret" loops in a derivation when the construction of a countermodel is concerned. Nevertheless, these observations and prescriptions are usually very difficult to formulate in a manner precise enough to receive a terminating decision procedure that could be implemented. One of our aims in the future is to formulate a solution to this problem within the framework of erotetic calculi.

The content of this work is the following. In Chapter I we present Wiśniewski's method of Socratic proofs for classical propositional logic in more detail. Two erotetic calculi are presented there: calculus **E*** described in [Wiśniewski:2004] and calculus **E**** which is a right-sided variant of **E***. We focus on the main ideas and omit most of the details (*e.g.* the proofs), since these may be found elsewhere. In Chapter II we present erotetic calculi for seven normal modal propositional logics (these are: **K**, **D**, **T**, **KB**, **K4**, **S4** and **S5**) which are, in a sense, built upon **E****. We present the proof of soundness of the calculi. In the last section of Chapter II we briefly analyse the calculi on the grounds of **IEL**. The analysis is sketchy, since the focus of this work is on the proof method, not on the logic of questions. In Chapter III a non-constructive proof of completeness is presented. The proof is based on the idea used in [Priest:2001]. We define a certain class of complete Socratic transformations and show how to construct a countermodel from a complete Socratic transformation which is not a Socratic proof. However, since Socratic transformations are not defined as trees, in order to adjust Priest's solution to our purposes we need some additional concepts and techniques. These we have found in [Wiśniewski, Shangin:2006] where the method of Socratic proofs for classical first-order logic is described. In the last chapter we briefly discuss the basic developments concerning proof methods for modal logics that may be compared with the method of Socratic proofs.

CHAPTER I: The Method of Socratic Proofs for Classical Propositional Calculus

We start with general remarks concerning notation used throughout this work.

I.1 Notation

We use the following set-theoretical notation:

$\in$ and $\notin$ for   membership and non-membership, respectively

$\varnothing$ the empty set

$\subseteq$ the inclusion of sets

$\subsetneqq$ the proper inclusion of sets

$\cup$ the union of sets

$\cap$ the intersection of sets

$X_1 \times \ldots \times X_n$ the Cartesian product of sets $X_1, \ldots, X_n$

A set of sets will be called a *family of sets*, or simply a *family*. *Relations* are defined as subsets of Cartesian products. If $R \subseteq X \times Y$, $x \in X$ and $y \in Y$, then we use the expression: $<x, y> \in R$ ($<x, y> \notin R$) to indicate that x stands (does not stand) in relation R to y.

By $f: X \mapsto Y$ we mean a *function defined on X with values in Y*. If function f is defined on Y, function g is defined on X such that $X \subseteq Y$, and $f(x) = g(x)$ for all $x \in X$, then we call f an *extension of g over Y* and we call g a *restriction of f to X*.

An *infinite sequence of the elements of a set* Σ is a function defined on the set of all positive integers with values in Σ. Infinite sequences will be represented by: $s = <s_1, s_2, \ldots>$, where s_i ($1 \leq i$) is the value of s at i. A function defined on a (finite) set of positive integers $\{1, \ldots, n\}$ with values in Σ will be called an *n-term sequence of the elements of* Σ. For technical reasons, we introduce the concept of empty sequence. By *finite sequences* we mean the empty sequence and *n*-term sequences, where *n* is a positive integer. *n*-term sequences will be represented by:

$s = \langle s_1, \ldots, s_n \rangle$, where s_i $(1 \leq i \leq n)$ is the value of s at i. s_i will be called the *i-th term of sequence s*.

If s and t are finite sequences, then by $s \,{}^\backprime\, t$ we refer to the concatenation of s and t. More precisely:

DEFINITION 1.1: By a *concatenation of an n-term sequence s*: $\{1, \ldots, n\} \mapsto \Sigma$ *and an m-term sequence t*: $\{1, \ldots, m\} \mapsto \Sigma^*$ we mean a function

$$s \,{}^\backprime\, t: \{1, \ldots, n + m\} \mapsto \Sigma \cup \Sigma^*$$

set by:

$$s \,{}^\backprime\, t(i) = \begin{cases} s(i) & \text{if } i \leq n \\ t(i\text{-}n) & \text{if } i > n \end{cases}$$

By a concatenation of an *n*-term sequence *s* and the empty sequence we mean the sequence *s*. Similarly, by a concatenation of the empty sequence and an *n*-term sequence *s* we mean the sequence *s*.

Thus the concatenation of an *n*-term sequence $s = \langle s_1, \ldots, s_n \rangle$ and an *m*-term sequence $t = \langle t_1, \ldots, t_m \rangle$ is an $(n+m)$-term sequence $s \,{}^\backprime\, t = \langle s_1, \ldots, s_n, t_1, \ldots, t_m \rangle$.

We use **CPC** for "Classical Propositional Calculus". The language of **CPC** will be designated by $\boldsymbol{L}$. The vocabulary of $\boldsymbol{L}$ contains propositional variables: p_1, p_2, …; logical constants: $\neg$, $\wedge$, $\vee$, $\rightarrow$; and parentheses: (,). We use *VAR* for the set of propositional variables.

Language $\boldsymbol{L}$ is the smallest set satisfying the following conditions:

(i) $VAR \subseteq \boldsymbol{L}$;

(ii)   if $A \in \boldsymbol{L}$, then '$\neg A$' $\in \boldsymbol{L}$;

(iii)   if $A, B \in \boldsymbol{L}$, then '$(A \wedge B)$', '$(A \vee B)$', '$(A \rightarrow B)$' $\in \boldsymbol{L}$.

If A is an element of $\boldsymbol{L}$, then A is called a *formula of $\boldsymbol{L}$*.

By a *literal* we mean a propositional variable, p_i, or a negation of a propositional variable, $\neg p_i$. For convenience, propositional variables will be referred to as p, q, r, …. When writing the formulas of $\boldsymbol{L}$, we adopt the usual conventions concerning omitting parentheses. The abbreviation *iff* stands for "if and only if".

By a **CPC**-*assignment* we mean any function defined over *VAR* with values in $\{\boldsymbol{0}, \boldsymbol{1}\}$. We introduce the following:

DEFINITION 1.2: If $V\#$ is a **CPC**-assignment, then an extension V of $V\#$ over $\boldsymbol{L}$ satisfying the following conditions:

1) $V(\neg A) = \boldsymbol{1}$ iff $V(A) = \boldsymbol{0}$;

2) $V(A \wedge B) = \mathbf{1}$ iff $V(A) = \mathbf{1}$ and $V(B) = \mathbf{1}$;

3) $V(A \vee B) = \mathbf{1}$ iff $V(A) = \mathbf{1}$ or $V(B) = \mathbf{1}$;

4) $V(A \rightarrow B) = \mathbf{1}$ iff $V(A) = \mathbf{0}$, or $V(B) = \mathbf{1}$;

is a **CPC**-*valuation*.

We say that a formula, A, of **L** is **CPC**-*valid* iff for each **CPC**-valuation V, $V(A) = \mathbf{1}$. A set of formulas, Σ, **CPC**-*entails* a formula A iff A is true under any **CPC**-valuation under which all the elements of Σ are true.

In general, sequents will be represented by '$S \vdash T$'. Depending on the calculus, T may be restricted to be a one-term sequence or S may be supposed to be empty (*cf.* Section I.2.1, Section I.3.1 and Section II.1 of Chapter II). For convenience, we will write: $A_1, \ldots, A_n \vdash B_1, \ldots, B_m$ instead of: $<A_1, \ldots, A_n> \vdash <B_1, \ldots, B_m>$.

We will be using the turnstile sign '$\vdash$' as a symbol of *object-level languages*, in a way the sequent arrow '$\Rightarrow$' is commonly used. This does not lead to any confusion as we do not use the turnstile sign as a meta-level expression.

I.2 Erotetic Calculus E*

We begin with a presentation of erotetic calculus **E*** for **CPC**. We summarize the description contained in [Wiśniewski:2004].

Till the end of this chapter letters $A, B, C, \ldots$ will stand for formulas of **L**, and letters S, T, U will represent finite (possibly empty) sequences of formulas of **L**.

I.2.1 Language L*

Erotetic calculus **E*** pertaining to **CPC** is worded in language **L***. This language is built upon language **L**, with certain expressions that belong to the metalanguage of **L** incorporated into **L***. The vocabulary of **L*** comprises the vocabulary of **L** and the signs: ?, $\vdash$, &, ng.

The well-formed formulas (wffs for short) of language **L*** belong to one of the two disjoint categories: declarative well-formed formulas (d-wffs) and erotetic well-formed formulas (questions). The wffs of **L*** are defined as follows.

An *atomic d-wff of* **L*** is an expression of the form:

(1.1) $S \vdash A$

where S is a finite (possibly empty) sequence of formulas of **L** and A is a single formula of **L**. Hence atomic d-wffs of **L*** are single-conclusioned sequents. The notion of a d-wff of **L*** is defined by:

DEFINITION 1.3:

 (i) each atomic d-wff of **L*** is a d-wff of **L***;

 (ii) if *X* is a d-wff of **L***, then ng(*X*) is a d-wff of **L***;

 (iii) if *X* and *Y* are d-wffs of **L***, then (*X*) & (*Y*) is a d-wff of **L***;

 (iv) nothing else is a d-wff of **L***.

  *Questions of **L*** are expressions of the form:

  (1.2) ? (Φ)

where Φ is a finite and non-empty sequence of atomic d-wffs (sequents) of **L***. If Φ = $\langle \varphi_1, \ldots, \varphi_n \rangle$, then we will write:

  (1.3) ? ($\varphi_1, \ldots, \varphi_n$)

instead of ? ($\langle \varphi_1, \ldots, \varphi_n \rangle$). Sequents $\varphi_1, \ldots, \varphi_n$ will be called *constituents of question* (1.3). A general reading of question (1.3) is the following:

  (1.4) Is it the case that: φ_1 and … and φ_n?

Questions of language **L*** are simple yes-no questions. The answers to question (1.3) are d-wffs of language **L***. Namely, the *affirmative answer to question* (1.3) is a d-wff of the form:

  (1.5) (…((φ_1) & (φ_2)) & …) & (φ_n)

and the *negative answer to question* (1.3) is a d-wff of the form:

  (1.6) ng((…((φ_1) & (φ_2)) & …) & (φ_n))

  The wffs of language **L*** may be said to represent certain expressions that normally belong to the metalanguage of **L**. Thus, *e.g.*, an atomic d-wff '*S* ⊢ *A*' of **L*** may be interpreted as representing an assertion: "formula *A* is **CPC**-derivable from the set of terms of *S*". Thus the sequents of language **L*** are given *operational* interpretation.[9] Consequently, questions of **L*** represent natural-language questions concerning **CPC**-derivability. This intuitive interpretation of the wffs of **L*** may be also formulated in terms of **CPC**-entailment.

  Let us emphasize that language **L*** is still an *object-level* language. Languages of other erotetic calculi developed so far have the same construction – meta-level expressions pertaining to the underlying logic are incorporated into the language of an erotetic calculus. We follow the same pattern in Section II.1, where we describe language **M***.

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[9] That is, an interpretation in terms of derivability. This is opposed to *denotational* interpretation of multi-conclusioned sequents, that is, interpretation in terms of entailment. *Cf.* [Negri, von Plato:2001, p.47].

In what follows we will use Greek lower-case letters φ, ψ as metalinguistic variables for sequents (atomic d-wffs) of L^*. Greek upper-case letters Φ, Ψ will represent sequences of sequents of L^*. For questions of L^* we use: Q, Q_1, Q^*.

I.2.2 The Rules of E*

Throughout this work we use two concatenation-signs. The symbol: $'$ is used for the concatenation of sequences of formulas of L, whereas the semicolon ';' is used for the concatenation of sequences of sequents of L^*. Thus we refer to the concatenation of S and T as '$S' T$', but the concatenation of sequences Φ and Ψ, where Φ and Ψ are sequences of sequents of L^*, is referred to as '$\Phi; \Psi$'. The distinction allows for a clear and precise presentation of the rules. Moreover, as we shall see in Section I.3.2, it may be argued that the two concatenation-signs correspond to different connectives on the meta-level.

We shall write $S' T' U$ for the concatenation of sequences $S' T$ and U. By $S' A$ we denote the concatenation of sequence S and a one-term sequence $<A>$. An analogous convention pertains to: $A' S$ or $S' A' T$. The concatenation-sign ; will be used in a similar manner.

The rules of **E*** are rules which transform questions into questions. A transformation starts with a question asking whether the logical relation of **CPC**-derivability holds in a particular case and it proceeds by simplifying the initial question. For example, a question whether a conjunction $A \wedge B$ is **CPC**-derivable from a set of formulas Σ is expressed by: ? $(S \vdash A \wedge B)$, where S is a sequence whose terms are all the elements of Σ. This may be simplified by asking whether A is **CPC**-derivable from this set and whether B is **CPC**-derivable from the same set. Hence question ? $(S \vdash A \wedge B)$ may be transformed into question ? $(S \vdash A; S \vdash B)$. A rule guiding this transition has the following form:

R$_\wedge$: $\qquad$? $(\Phi; S \vdash A \wedge B; \Psi)$
$$\overline{\qquad\qquad\qquad\qquad\qquad}$$
$\qquad$? $(\Phi; S \vdash A; S \vdash B; \Psi)$

Let us also observe that the answer to the question expressed by ? $(S \vdash A \wedge B)$ is affirmative *iff* both: the answer to the question expressed by ? $(S \vdash A)$ is affirmative and the answer to the question expressed by ? $(S \vdash B)$ is affirmative.

Similarly, if we ask whether C is derivable from the set: $\{A \wedge B\}$, we may ask as well whether C is derivable from the set: $\{A, B\}$. This transition is justified by the rule:

L$_\wedge$: $\qquad$? $(\Phi; S' A \wedge B' T \vdash C; \Psi)$
$$\overline{\qquad\qquad\qquad\qquad\qquad}$$
$\qquad$? $(\Phi; S' A' B' T \vdash C; \Psi)$

The rules of erotetic calculi are all of eliminative character. An application of such a rule results in, by and large, an elimination of a connective occurring in a formula of the antecedent or of the consequent of a sequent. However, since in the formulation of the erotetic rules the α-, β-notation is used, we may say that these rules rather simplify formulas of conjunctive (α-formulas) and disjunctive (β-formulas) type into their, appropriately defined, components.[10] The definition of components α_1, α_2 and β_1, β_2 of α- and β-formulas is given by the table below.[11]

α	α_1	α_2		β	β_1	β_2	β_1*
$A \wedge B$	A	B		$\neg(A \wedge B)$	$\neg A$	$\neg B$	A
$\neg(A \vee B)$	$\neg A$	$\neg B$		$A \vee B$	A	B	$\neg A$
$\neg(A \rightarrow B)$	A	$\neg B$		$A \rightarrow B$	$\neg A$	B	A

Table 1

β_1* is called the *complement* of β_1. This additional assignment is due to the fact that sequents of **L*** are single-conclusioned. When a β-formula is decomposed right of the turnstile, one of its components must be transferred from the right to the left side of the turnstile and, simultaneously, it must be changed into its complement. *E.g.* the schema of the rule of elimination of formula of the form $A \vee B$ from the right side of the turnstile is the following:

R$_\vee$: $\dfrac{?\,(\Phi;\ S \vdash A \vee B;\ \Psi)}{?\,(\Phi;\ S\,'\,\neg A \vdash B;\ \Psi)}$

If this is necessary, formulas of the form $\neg\neg A$ may be treated as α-formulas, as β-formulas or as formulas of both types. For our purposes, it is more convenient not to apply the distinction to formulas of such a form. However, we may extend the notion of a component of a formula in the following way: if B is a formula of the form $\neg\neg A$, then A is the *component of B*.

We present the rules of calculus **E*** in the α-, β-notation. The rules in the standard notation are presented in Appendix 1.

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[10] In [Konikowska:2002], a paper presenting a Rasiowa-Sikorski style system, the rules of the system are called by the author "decomposition" rules. We find this term more accurate than the one "elimination rules", since when the α-, β-notation is used, an application of a rule *does not* amount, strictly speaking, to an elimination of a single connective. Actually, the term comes from [Rasiowa, Sikorski:1963], where a method of diagrams of formulas is presented. "This method consists in the decomposition of formulas into parts of which it is built." ([Rasiowa, Sikorski:1963, p. 264]).

[11] The notation comes from [Smullyan:1968] and has been extensively used by many authors (for example by Fitting in [Fitting:1983]).

$\mathbf{L}_\alpha$: $$\dfrac{?\,(\Phi;\, S\,'\,\alpha\,'\,T \vdash C;\, \Psi)}{?\,(\Phi;\, S\,'\,\alpha_1\,'\,\alpha_2\,'\,T \vdash C;\, \Psi)}$$ $\mathbf{R}_\alpha$: $$\dfrac{?\,(\Phi;\, S \vdash \alpha;\, \Psi)}{?\,(\Phi;\, S \vdash \alpha_1;\, S \vdash \alpha_2;\, \Psi)}$$

$\mathbf{L}_\beta$: $$\dfrac{?\,(\Phi;\, S\,'\,\beta\,'T \vdash C;\, \Psi)}{?\,(\Phi;\, S\,'\,\beta_1\,'\,T \vdash C;\, S\,'\,\beta_2\,'\,T \vdash C;\, \Psi)}$$ $\mathbf{R}_\beta$: $$\dfrac{?\,(\Phi;\, S \vdash \beta;\, \Psi)}{?\,(\Phi;\, S\,'\,\beta_1* \vdash \beta_2;\, \Psi)}$$

$\mathbf{L}_{\neg\neg}$: $$\dfrac{?\,(\Phi;\, S\,'\,\neg\neg A\,'T \vdash C;\, \Psi)}{?\,(\Phi;\, S\,'A\,'T \vdash C;\, \Psi)}$$ $\mathbf{R}_{\neg\neg}$: $$\dfrac{?\,(\Phi;\, S \vdash \neg\neg A;\, \Psi)}{?\,(\Phi;\, S \vdash A;\, \Psi)}$$

The set of the rules of **E*** will be denoted by $\boldsymbol{R}(\mathbf{E^*})$.

The rules of calculus **E*** transform questions into questions. When a rule is applied to a question, one of its constituents is simplified by decomposing a formula that occurs in this constituent. What we mean by "decomposition" here is that the formula in the constituent is replaced by its component(s). This may be accompanied with transferring one of the components from the right to the left side of the turnstile, or by adding another sequent to the constituent(s) of the question. The notion of a *component* is given by Table 1 and the convention concerning formulas of the form '$\neg\neg A$' according to which A is a component of '$\neg\neg A$'.

Though the rules of **E*** operate on *questions*, **E*** is still a calculus for proving *sequents*. A sequent is proved, if under the "decomposition process" one reaches only sequents of some basic form or forms. More formally:

DEFINITION 1.4: A finite sequence $\langle Q_1, \ldots, Q_n\rangle$ of questions of $\boldsymbol{L^*}$ is a *Socratic proof of sequent* $S \vdash A$ in calculus **E*** iff the following conditions hold:
   (i) $Q_1 = ?\,(S \vdash A)$;
   (ii) Q_i, where $i = 2, \ldots, n$, results from Q_{i-1} by applying a rule $\mathbf{r} \in \boldsymbol{R}(\mathbf{E^*})$;
   (iii)   for each constituent φ of Q_n:
            (a) φ is of the form $T\,'\,B\,'\,U \vdash B$, or
            (b) φ is of the form $T\,'\,B\,'\,U\,'\,\neg B\,'\,W \vdash C$, or
            (c) φ is of the form $T\,'\,\neg B\,'\,U\,'\,B\,'\,W \vdash C$.[12]

Below we present an example of a Socratic proof in **E***. On the margin we indicate the rule applied to the question that occurs in the same line. (Let us observe that there is no need to number the lines, since each question results from the previous one.) We refer to the rules in the standard notation.

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[12] *Cf.*: [Wiśniewski:2004, p. 305].

EXAMPLE 1.1: A Socratic proof of sequent $\vdash ((p \to q) \land (q \to r)) \to (p \to r)$ in calculus **E***:

$?\,(\vdash ((p \to q) \land (q \to r)) \to (p \to r)\,)$ **R$_\to$**

$?\,((p \to q) \land (q \to r) \vdash p \to r\,)$ **L$_\land$**

$?\,(p \to q, q \to r \vdash p \to r\,)$ **R$_\to$**

$?\,(p \to q, q \to r, p \vdash r\,)$ **L$_\to$**

$?\,(\neg p, q \to r, p \vdash r\,;\; q, q \to r, p \vdash r\,)$ **L$_\to$**

$?\,(\neg p, q \to r, p \vdash r\,;\; q, \neg q, p \vdash r\,;\; q, r, p \vdash r\,)$

The constituents of the last question of the above Socratic proof are of the forms (respectively): (c), (b) and (a) specified in Definition 1.4. If we recall the interpretation of sequents of **L*** as expressions concerning the relation of **CPC**-derivability, we may say that sequents of any of the specified forms express certain basic properties of this relation. *E.g.*, the first constituent: '$\neg p, q \to r, p \vdash r$' expresses the fact that from $\neg p$ and p anything, r in particular, is **CPC**-derivable, whereas the third constituent: '$q, r, p \vdash r$' expresses the fact that a formula, r, is derivable from itself. The last question of the above Socratic proof may be read as follows: "is it the case that: $\neg p, q \to r, p \vdash r$ and $q, \neg q, p \vdash r$ and $q, r, p \vdash r$?". The answer to this question must be affirmative. Invertibility of the rules of **E***, a property which we shall discuss in the next section, warrants that the answer to the initial question of the above Socratic proof is also affirmative.

I.2.3 Semantical Invertibility of Rules

Let us define the following auxiliary notion: a sequent of **L***, $S \vdash A$, is **CPC**-*valid* iff A is **CPC**-entailed by the set of terms of S.[13] Every sequent which is of one of the forms (a), (b) or (c) of Definition 1.4 is easily seen to be **CPC**-valid. It is also easy to check that:

THEOREM 1.1: If question $Q_1 = ?(\Phi_1)$ results from question $Q = ?(\Phi)$ by a rule of **E***, then *each* term of Φ is **CPC**-valid iff *each* term of Φ_1 is **CPC**-valid.[14]

That is, the rules of **E*** preserve *joint* **CPC**-validity of the constituents of questions in both directions: from top to bottom and from bottom to top. From this

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[13] In [Wiśniewski:2004] the notion of an *atomic d-wff of **L*** being OK with respect to **CPC**-entailment* is introduced in this context. However, an analogous semantical property of sequents has been defined in [Wiśniewski, Shangin:2006] and in [Wiśniewski, Vanackere, Leszczyńska:2005] as "validity". We apply the latter convention.

[14] *Cf.*: [Wiśniewski:2004, p. 306 and p. 311].

result soundness of the calculus **E*** with respect to **CPC**-entailment (and hence also to **CPC**-derivability) follows by induction.

The property attributed to the rules of **E*** in Theorem 1.1 is called their *semantical invertibility*, as it is defined in semantical terms.[15] Proving completeness of a calculus with semantically invertible rules occurs to be a (relatively) simple task, provided a decidable logic is concerned. In the paper [Wiśniewski:2004] a decision procedure is sketched and then it is shown how to obtain a "countermodel" (a falsifying **CPC**-valuation) if a Socratic proof has not been found. The same line of reasoning has been used in the completeness proof of erotetic calculi for paraconsistent logics **CLuN** and **CLuNs**.[16]

Semantical invertibility of the rules of **E*** warrants that if a Socratic proof of a sequent of the form '$S \vdash A$' is obtained, it may be viewed as a positive solution to the problem: "is formula A **CPC**-derivable from the set of terms of S?". If an attempt to find a Socratic proof starts with a question of the form ? ($S \vdash A$) and ends with a question which has at least one constituent that is *not* of the required form ((a), (b) or (c)) and, moreover, no rule of **E*** may be applied to this question, then it may be easily shown that A is not **CPC**-derivable from the set of terms of S. Hence calculus **E*** constitutes a method of solving logical problems expressed by the questions of language **L***. Each such problem may be solved by a finite number of applications of the rules of **E***.

Invertible sequent calculi are perfect tools for proof-searching, as the "root-first" procedure, transforming a sequent to be derived into "axioms" (valid sequents) and / or unanalysable invalid sequents may be used almost mechanically. In [Wiśniewski:2004] it is shown how a Socratic proof in **E*** may be converted into a proof in a (Gentzen-style) sequent calculus **G***. The axioms of calculus **G*** are of the form:

(**Ax$_1$**): $T ' B ' U \vdash B$
(**Ax$_2$**): $T ' B ' U ' \neg B ' W \vdash C$
(**Ax$_3$**): $T ' \neg B ' U ' B ' W \vdash C$ [17]

The calculus **G*** is single-conclusioned, invertible and has no primary structural rules.[18]

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[15] Invertibility of rules may be defined purely syntactically as well. *E.g.* in [Negri, von Plato: 2001, p.19] by *invertibility of a sequent calculus* the following is meant: "From the derivability of a sequent of any of the forms given in the conclusions of the logical rules, the derivability of its premisses follows." Invertible rules may be also called "equivalent" rules (*e.g.* in [Konikowska:2002]) or symmetric rules (*cf.* for example [Sundholm:2001], where Kleene's sequent calculus for classical first-order logic with invertible rules is called a "symmetric calculus").

[16] For the details of the proofs we refer to [Wiśniewski:2004] and [Wiśniewski, Vanackere, Leszczyńska:2005].

[17] *Cf.* [Wiśniewski:2004, pp. 313-316]. It is also shown how the axiomatic basis of **G*** may be simplified.

[18] In [Negri, von Plato: 2001, pp. 48-60, 114-121] an invertible sequent calculus for **CPC** (called **G3cp**) as well as single-conclusioned sequent calculus for **CPC** (**G3ip**+**Gem-at**) may be found but

I.3 A Right-Sided Approach: Calculus E**

We present the right-sided version of erotetic calculus for **CPC** which we call calculus **E****.[19] The rules of **E**** are discussed in Section I.3.1, together with the soundness and completeness results. We omit all the proofs in this section, for these are either a reformulation of what has been presented in [Wiśniewski:2004] or a particular case of what is proved for the more general case of modal logics in the later parts of this work. Section I.3.2 is devoted to a certain consequence of the invertibility of the rules of **E****.

I.3.1 Language *L*** and the Rules of E**

The calculus is worded in language *L*** which has the same vocabulary as language *L**. By a *sequent of L*** we mean an expression of the form:

(1.7) $\vdash S$

where S is a finite and non-empty sequence of formulas of *L*. The sequents of *L*** are *right-sided* only. We may also think of the sequents of *L*** as of both-sided sequents with the empty sequence left of the turnstile. Right-sided sequents are intuitively interpreted in terms of entailment, but since the left side of (1.7) is empty, the interpretation "narrows down" to **CPC**-validity.[20] Hence what is expressed by a sequent of the form (1.7) is that under each **CPC**-valuation one of the terms of S is true. This amounts to **CPC**-validity of the disjunction of the terms of S. If a sequent, $\vdash S$, has this property, that is, if for each **CPC**-valuation V at least one term of S is true under V, then we will say that sequent $\vdash S$ is ***CPC-valid***. If a sequent is not **CPC**-valid, then we will say that it is ***CPC-invalid***.

As in the case of *L**, wffs of *L*** are: declarative wffs and questions. *Atomic d-wffs of L*** are sequents of *L***, *compound d-wffs of L*** are built from atomic d-wffs by means of & and / or ng in the same way the compound d-wffs of *L** are built (*cf.* Definition 1.3). Questions of *L*** are expressions of the form:

(1.3) $? (\varphi_1, \ldots, \varphi_n)$

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none of the calculi has both these features. **G3cp** is a multi-conclusioned – or "multisuccedent" in the authors terminology – sequent calculus (more than one formula is allowed in a succedent of a sequent). Thus sequents of **G3cp** are given denotational interpretation. Calculus **G3ip**+**Gem-at** is built upon intuitionistic sequent calculus **G3ip** by the addition of an invertible rule corresponding to the law of excluded middle. Sequents are single-conclusioned ("singlesuccedent") but the rules for $\vee$-introduction on the right side and $\rightarrow$-introduction on the left side are not invertible.

[19] The idea to present erotetic calculi in a right-sided format comes from Andrzej Wiśniewski. The very construction of the calculus **E**** has been also discussed during seminars with Andrzej Wiśniewski.

[20] In the general case, a sequent of the form: $S \vdash T$ is interpreted in the following way: the conjunction of the terms of S entails the disjunction of the terms of T. (This is the *denotational* interpretation of sequents. *Cf.* footnote 9.)

where $\varphi_1, \ldots, \varphi_n$ are sequents of language $\mathbf{\textit{L}}^{**}$. We apply all the notational conventions introduced in Sections I.2.1 and I.2.2 (including the convention concerning the concatenation-signs ' '' ' and ';').

We present the rules of $\mathbf{E}^{**}$ in the α-, β-notation again. The rules in the standard notation are presented in Appendix 2.

$$\mathbf{R_\alpha}: \quad \frac{? (\Phi; \vdash S\,'\,\alpha\,'\,T; \Psi)}{? (\Phi; \vdash S\,'\,\alpha_1\,'\,T; \vdash S\,'\,\alpha_2\,'\,T; \Psi)} \qquad\qquad \mathbf{R_\beta}: \quad \frac{? (\Phi; \vdash S\,'\,\beta\,'\,T; \Psi)}{? (\Phi; \vdash S\,'\,\beta_1\,'\,\beta_2\,'\,T; \Psi)}$$

$$\mathbf{R_{\neg\neg}}: \quad \frac{? (\Phi; \vdash S\,'\,\neg\neg A\,'\,T; \Psi)}{? (\Phi; \vdash S\,'\,A\,'\,T; \Psi)}$$

As in the case of calculus $\mathbf{E}^*$, rules of $\mathbf{E}^{**}$ transform questions into questions. A rule is always applied with respect to one constituent of a question and an application of the rule results in decomposing a formula occurring in this constituent into its component(s). What we now mean by "decomposition" is that the relevant formula of the constituent of the "question-premise" is replaced by its component(s); this may be accompanied with an introduction of another constituent to the "question-conclusion".

For convenience, we introduce the notion of a Socratic transformation.[21]

DEFINITION 1.5: A finite sequence $<Q_1, \ldots, Q_n>$ of questions of $\mathbf{\textit{L}}^{**}$ is a *Socratic transformation of a question Q via the rules of* $\mathbf{E}^{**}$ iff $Q_1 = Q$, and Q_{i+1} results from Q_i $(1 \le i < n)$ by an application of a rule of $\mathbf{E}^{**}$.

The notion of a Socratic proof may now be defined as follows:

DEFINITION 1.6: Let $\vdash A$ be a sequent of $\mathbf{\textit{L}}^{**}$. A *Socratic proof of* $\vdash A$ in $\mathbf{E}^{**}$ is a finite Socratic transformation s of the question $? (\vdash A)$ via the rules of $\mathbf{E}^{**}$ such that for each constituent φ of the last question of s:
  (a) φ is of the form $\vdash T\,'\,B\,'\,U\,'\,\neg B\,'\,W$, or
  (b) φ is of the form $\vdash T\,'\,\neg B\,'\,U\,'\,B\,'\,W$.

We present two examples of Socratic transformations via the rules of $\mathbf{E}^{**}$.

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[21] The notion has been introduced in [Wiśniewski, Vanackere, Leszczyńska:2005] and has been used also in [Wiśniewski, Shangin:2006] and [Leszczyńska:2004].

EXAMPLE 1.2: A Socratic transformation of question
$? (\vdash (p \to q) \to ((p \land \neg q) \to r))$

$? (\vdash (p \to q) \to ((p \land \neg q) \to r))$ **R$_\to$**

$? (\vdash \neg(p \to q), (p \land \neg q) \to r)$ **R$_\to$**

$? (\vdash \neg(p \to q), \neg(p \land \neg q), r)$ **R$_{\neg\land}$**

$? (\vdash \neg(p \to q), \neg p, \neg\neg q, r)$ **R$_{\neg\neg}$**

$? (\vdash \neg(p \to q), \neg p, q, r)$ **R$_{\neg\to}$**

$? (\vdash p, \neg p, q, r \; ; \; \vdash \neg q, \neg p, q, r)$

EXAMPLE 1.3: A Socratic transformation of question
$? (\vdash (\neg p \to (p \land q)) \to (p \land q))$

$? (\vdash (\neg p \to (p \land q)) \to (p \land q))$ **R$_\to$**

$? (\vdash \neg(\neg p \to (p \land q)), p \land q)$ **R$_{\neg\to}$**

$? (\vdash \neg p, p \land q \; ; \; \vdash \neg(p \land q), p \land q)$ **R$_\land$**

$? (\vdash \neg p, p \; ; \; \vdash \neg p, q \; ; \; \vdash \neg(p \land q), p \land q)$

The first Socratic transformation is a Socratic proof of the sequent $\vdash (p \to q) \to$ $((p \land \neg q) \to r)$ in calculus **E****. The second Socratic transformation is not a Socratic proof of the corresponding sequent, as the second constituent of the last question of this transformation does not have a required form. What is more, since only literals occur in this sequent, it is easy to find a **CPC**-valuation that "invalidates" the sequent. This observation may be generalized:

COROLLARY 1.1: Let $\varphi = \vdash S$ be a sequent of **L****.
    (i)   If φ is of the form (a) $\vdash T \, ' \, B \, ' \, U \, ' \, \neg B \, ' \, W$, or of the form (b) $\vdash T \, ' \, \neg B \, '$ $U \, ' \, B \, ' \, W$, then φ is **CPC**-valid.
    (ii)   If φ is of neither of the forms: (a), (b), and each term of S is a literal, then φ is **CPC**-invalid.

    The following lemma is also easily seen to be true.

LEMMA 1.1 (*semantical invertibility of the rules of* **E****): If question $Q_1 = ? (\Phi_1)$ results from question $Q = ? (\Phi)$ by a rule of **E****, then each term of Φ is **CPC**-valid iff each term of Φ_1 is **CPC**-valid.

The proof is by cases, analogously to the proof of Theorem 1.1.[22] From Corollary 1.1 (i) and Lemma 1.1, soundness of calculus **E**** follows by induction.

For the completeness result it suffices to observe that the process of constructing a Socratic transformation *via* the rules of **E**** must be finite. We sketch the reasons for this claim. Successive applications of the rules of **E**** result in the decomposition of the formulas occurring in the constituents of questions. This means, by and large, that after each application of a rule an occurrence of a binary connective or an occurrence of the double negation is eliminated. Therefore after each application of a rule the degree of complexity of a formula in some constituent decreases.[23] However, the number of occurrences of logical connectives in formulas of a constituent of a question is finite and so is the number of possible applications of rules of **E**** in a Socratic transformation. Completeness follows from this result, Corollary 1.1 (ii) and Lemma 1.1.

I.3.2 Semantical Invertibility – Semantical Duality

As we have noted above, the interpretation of sequents of **L**** is denotational – a general reading of a right-sided sequent, $\vdash S$, is that each **CPC**-valuation makes true at least one of the terms of S. The rules of **E**** may be also interpreted in such semantical terms. Let us consider rule $\mathbf{R_\wedge}$ as an example. The rule has the following form:

$\mathbf{R_\wedge}$:
$$\frac{?\,(\Phi;\ \vdash S\,'\,A \wedge B\,'\,T;\ \Psi)}{?\,(\Phi;\ \vdash S\,'\,A\,'\,T;\ \vdash S\,'\,B\,'\,T;\ \Psi)}$$

Let V stand for an arbitrary **CPC**-valuation. We may observe that if there is a formula in sequence $S\,'\,A \wedge B\,'\,T$ that is true under V, then in sequence $S\,'\,A\,'\,T$ there is a formula true under V and in sequence $S\,'\,B\,'\,T$ there is a formula true under V. On the other hand, if in both sequences $S\,'\,A\,'\,T$ and $S\,'\,B\,'\,T$, there is a formula true under V, then there is a term of $S\,'\,A \wedge B\,'\,T$ true under V. In other words, rule $\mathbf{R_\wedge}$ expresses the necessary *and* sufficient conditions for a sequent of the form $\vdash S\,'\,A \wedge B\,'\,T$ to be **CPC**-valid. Under this reading of the rule the comma separating formulas in sequents corresponds to meta-disjunction. The concatenation-sign: ' for sequences of formulas has a disjunctive reading as well, whereas the second concatenation-sign ';' corresponds to meta-conjunction.

This, however, is not the only way in which we can interpret rule $\mathbf{R_\wedge}$. If one thinks of formula $A \wedge B$ as false under a certain **CPC**-valuation, one may observe the following: all the formulas occurring in sequence $S\,'\,A \wedge B\,'\,T$ are

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[22] *Cf.* [Wiśniewski:2004].

[23] In Section III.4 of Chapter III we introduce the notion of degree of complexity of a formula which we find most adequate in a context like the one above. We have omitted the details here, since they will be repeated for the more general case of modal logics in Chapter III.

false under a certain **CPC**-valuation *iff* each term of sequence $S\,{'}\,A\,{'}\,T$ is false under this **CPC**-valuation *or* each term of sequence $S\,{'}\,B\,{'}\,T$ is false under this **CPC**-valuation. Obviously, under this reading of rule $\mathbf{R}_\wedge$ the interpretation of the concatenation-signs switches: the sign ''' (as well as the comma) corresponds to meta-conjunction, whereas the semicolon ';' corresponds to meta-disjunction.

The duality of the interpretation of the rules of **E**** is due to their invertibility. To sum up, invertible rules may be interpreted:

A   as stating the *necessary and sufficient* conditions for a sequent to be ***CPC**-valid*;

or

B   as stating the *necessary and sufficient* conditions for a sequent to be ***CPC**-invalid*.

A and B may be called, respectively, the *direct* and the *indirect* reading of the method of Socratic proofs. The observations presented here pertain to calculus **E*** as well. However, certain consequences of this duality of interpretation, such as the disjunctive nature of ''' under interpretation A, are blurred by the fact that sequents of *L** are single-conclusioned.

Rules of erotetic calculi for logics richer than **CPC** are also semantically invertible. Let us present the following rules of the right-sided calculus $\mathbf{E}^{\mathbf{RPQ}}$ pertaining to classical first-order logic:[24]

$\mathbf{R}_\forall$: $$\frac{?\,(\Phi;\ \vdash S\,{'}\,\forall x_i\,A\,{'}\,T;\ \Psi)}{?\,(\Phi;\ \vdash S\,{'}\,A(x_i\,/\,\tau)\,{'}\,T;\ \Psi)}$$ where x_i is free in A and τ is an individual parameter that does not occur in the terms of $S\,{'}\,\forall x_i\,A\,{'}\,T$

$\mathbf{R}_\exists$: $$\frac{?\,(\Phi;\ \vdash S\,{'}\,\exists x_i\,A\,{'}\,T;\ \Psi)}{?\,(\Phi;\ \vdash S\,{'}\,\exists x_i\,A\,{'}\,A(x_i\,/\,\tau)\,{'}\,T;\ \Psi)}$$ where x_i is free in A and τ is an arbitrary individual parameter

Again, it is easy to observe that the rules $\mathbf{R}_\forall$ and $\mathbf{R}_\exists$ may be viewed as stating the necessary and sufficient conditions for a relevant sequent to be valid, or as stating the necessary and sufficient conditions for a relevant sequent to be invalid. (Of course, the notion of validity of a sequent must be now relativized to the semantics of first-order logic.) In the case of rule $\mathbf{R}_\exists$, formula '$\exists x_i\,A$' must be rewritten in the relevant constituent of the "question-conclusion" in order to warrant invertibility of the rule. As we shall see in Section II.2 of Chapter II, the situation is similar in the case of erotetic rules for modal operators.

Let us go back to calculus **E****. If we think of interpretation A, the method of Socratic proofs for **CPC** in the right-sided version may be quite naturally

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[24] *Cf.*: [Wiśniewski:2005]. Calculus $\mathbf{E}^{\mathbf{RPQ}}$ is a right-sided version of calculus $\mathbf{E}^{\mathbf{PQ}}$ presented in [Wiśniewski, Shangin:2006]. The rules $\mathbf{R}_\forall$ and $\mathbf{R}_\exists$ of calculus $\mathbf{E}^{\mathbf{PQ}}$ are also semantically invertible.

viewed as a formalization of the procedure of deriving a conjunctive normal form of a given formula.[25] For example, the last term of the Socratic transformation of question:

$$(1.8) \quad ? \, (\vdash (p \to q) \to ((p \land \neg q) \to r) \,)$$

which we presented above (Example 1.2) had the form:

$$(1.9) \quad ? \, (\vdash p, \neg p, q, r \, ; \, \vdash \neg q, \neg p, q, r \,)$$

Reading the comma as the disjunction and the semicolon as the conjunction we obtain:

$$(1.10) \quad (((p \lor \neg p) \lor q) \lor r) \land (((\neg q \lor \neg p) \lor q) \lor r)$$

that is, a conjunctive normal form of the formula concerned in question (1.8).

However, following interpretation B it is also possible, though less natural, to read a Socratic transformation of a question $? \, (\vdash A)$ as a derivation of a disjunctive normal form of formula '$\neg A$'. What is needed in order to make this reading sensible is to make the following change in the last question of such a transformation: the propositional variables must be changed into their negations and the negations of propositional variables must be changed into their arguments. *E.g.* the formula:

$$(1.11) \quad (((\neg p \land p) \land \neg q) \land \neg r) \lor (((q \land p) \land \neg q) \land \neg r)$$

is a disjunctive normal form of the formula: $\neg((p \to q) \to ((p \land \neg q) \to r))$. As a matter of fact, the duality of the interpretations of the method of Socratic proofs may now be seen to be a consequence of the dual nature of the conjunction and the disjunction connectives.

I.4 The Method of Socratic Proofs and Inferential Erotetic Logic

The rules of erotetic calculi formalize certain inferential relations between questions. These relations may be analysed further on the grounds of the logic **IEL**. In particular, it may be shown that the inference that takes place when a rule of an erotetic calculus is applied to a question is, in a sense, a valid erotetic inference. Below we discuss very briefly the main results of such an analysis of calculus **E*** on the grounds of **IEL**. Basically, we summarize, as before, the developments that may be found in [Wiśniewski:2004]. Again, our aim is to present the main idea and thus we do not go into details.

The analysis proceeds by developing a semantics for the declarative part of **L***, which is then used to define some semantical concepts pertaining to questions of **L***. The semantics for language **L*** is based on the notion of an *admissible*

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[25] From this point of view calculus **E**** is an "erotetic formulation" of the method of diagrams of formulas presented in [Rasiowa, Sikorski: 1963, pp. 264-269].

partition of a language.[26] This is defined as follows. Let $\mathbf{D}_{L*}$ be the set of all d-wffs of L^*. A *partition* P of $\mathbf{D}_{L*}$ is an ordered pair $<T_P, U_P>$ such that: (a) $T_P \cap U_P = \varnothing$; and (b) $T_P \cup U_P = \mathbf{D}_{L*}$. Intuitively, the set T_P consists of "truths" and the set U_P – of "untruths" of partition P. A partition is *admissible* provided it satisfies certain conditions which are relativized to the semantics of the underlying language L. Now the following notion may be defined: a d-wff X of L^* *entails* a d-wff Y of L^* iff for each admissible partition $P = <T_P, U_P>$ of language L^*, if $X \in T_P$, then $Y \in T_P$.

The central notion pertaining to the erotetic part of L^* is that of positive equipollence of questions. A question, Q, is *positively equipollent to* a question, Q_1, iff the affirmative answers to Q and Q_1 entail each other and the negative answers to Q and Q_1 entail each other. It may be proved that if question Q_1 results from question Q by a rule of $\mathbf{E}^*$, then question Q_1 is positively equipollent to question Q. Moreover, the relation of positive equipollence of questions is a special case of the relation of pure erotetic implication of a question by a question. Hence if question Q_1 results from question Q by a rule of $\mathbf{E}^*$, then question Q_1 is erotetically implied by question Q on the basis of the empty set of declarative premises.

This "erotetic" analysis of the inferential rules of $\mathbf{E}^*$ may be extended so that it will cover calculus $\mathbf{E}^{**}$ as well. We will not do it here, however. In Section II.4 of Chapter II, we present the basic definitions needed for such an analysis of erotetic calculi for modal logics. Since these are based on the calculus $\mathbf{E}^{**}$, suitable definitions pertaining to $\mathbf{E}^{**}$ may be extracted from what is presented there.

Finally, let us emphasize that the very construction of the rules of an erotetic calculus is significant from the point of view of **IEL**, since the construction must warrant that the inferential relation between a "question-premise" and a "question-conclusion" satisfies the conditions of its validity as defined on the grounds of **IEL**. Semantical invertibility of the rules seems to be essential for these conditions to hold.

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[26] *Cf.* [Wiśniewski:2004, p. 319]. Further references may be found in this paper.

CHAPTER II: Erotetic Calculi for Normal Modal Propositional Logics[27]

In this chapter we adjust the method of Socratic proofs to modal logics: **K**, **D**, **T**, **KB**, **K4**, **S4** and **S5**. Logic **K** is the weakest normal modal logic semantically characterized by the class of all Kripke frames. For the sake of brevity, we have chosen only six of the well known proper extensions of **K**. The method, however, may be easily adjusted to any of the so-called basic normal modal logics.[28]

Before we continue, let us recall some widely known facts concerning modal logics. In each normal modal logic $p \models \Box p$ holds, where the sign '$\models$' refers to derivability or global entailment,[29] although in most of them it is not the case that $\models p \rightarrow \Box p$.[30] It is easy to observe that if we allowed the rules of a modal erotetic calculus to "switch" formulas from the right to the left of the turnstile (as it happens in the case of calculus **E***), it would be difficult to maintain the interpretation of '$\vdash$' as representing derivability, or otherwise the rules would not be invertible. Hence we allow the rules of the calculi to analyse formulas on the right of the turnstile only, that is, we choose the right-sided calculus **E**** as a "basis" for a modal erotetic calculus. Consequently, the turnstile symbol will be given the denotational interpretation. The symbol may be taken to express *L*-

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[27] An early version of this chapter has been published as [Leszczyńska:2004]. There is, however, an essential difference between the two versions, which concerns the account of transitive logics: **K4**, **S4** and **S5**.

[28] By basic normal modal logics we mean **K** and all of its proper extensions characterized by any combination of the following properties imposed on the accessibility relation: extendability, reflexivity, transitivity, symmetry and euclideaness. There are exactly fifteen different basic modal logics: **K**, **D**, **T**, **KB**, **DB**, **B**, **K4**, **D4**, **S4**, **KB4**, **S5**, **K5**, **D5**, **K45**, **D45**. (The notation we have used here comes from: [Goré:1999].)

[29] The notions of local and global entailment are defined for **K** as follows: a formula *A* is *locally entailed* by a set of formulas *X* iff for each Kripke model <*W, R, V*> and for every $w \in W$, if every element of *X* is true in *w*, then *A* is true in *w*. A formula *A* is *globally entailed* by a set of formulas *X* iff for each Kripke model, if every element of *X* is true in a model, then *A* is true in the model. Both notions are relativized to a logic *L* by imposing suitable conditions on *R*. The syntactical relation of derivability of a formula *A* from axioms of a modal logic *L* and a set of formulas *X* is complete with respect to the relation of global *L*-entailment of *A* from *X*.

[30] This simple example illustrates the fact that the Deduction Theorem for modal logics does not usually hold in the standard form. See, *e.g.*, [Fitting:1983, pp. 77-81] or [Perzanowski:1973].

validity of a formula (where *L* stands for a modal logic), that is *global L-entailment* of a formula from the empty set. Such a construction of an erotetic calculus may be extended further in order to include global *L*-entailment from non-empty sets, and this is what we partially aim at. We will not do it here, however.[31]

Our plan is the following. In Section II.1 we characterize language *M** of modal erotetic calculi. The calculi are presented in Section II.2, and in Section II.3 we address the problem of their soundness. We prove completeness of the calculi in Chapter III.

II.1 Language *M**

The vocabulary of the language of modal propositional logics contains: the elements of *VAR*; logical constants: ¬, ∧, ∨, →, □ (necessity operator), ◊ (possibility operator); and parentheses: (,). The language will be designated by *M*. More precisely, *language M* is the smallest set satisfying the following conditions:

(i) $VAR \subseteq M$;

(ii)   if $A \in M$, then '¬A', '□A', '◊A' $\in M$;

(iii)   if $A, B \in M$, then '$(A \wedge B)$', '$(A \vee B)$', '$(A \rightarrow B)$' $\in M$.

The elements of *M* are called *formulas of M*.

Similarly as before, we use *p*, *q*, *r*, … for propositional variables, and *A*, *B*, *C*, … as metavariables for formulas of *M*. When writing the formulas of *M*, we apply the usual conventions concerning omitting parentheses.

Following Fitting ([Fitting:1983]), we extend the α-, β-notation with the assignments presented in Table 2:

ν	ν_0	π	π_0
$\Box A$	A	$\Diamond A$	A
$\neg \Diamond A$	$\neg A$	$\neg \Box A$	$\neg A$

Table 2

Formula ν_0 (π_0) is called the *component of* a ν-formula (π-formula).

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[31] Of course, another solution is possible here: the turnstile symbol could be taken to represent the relation of local entailment, another sign would be needed then to represent the relation of global entailment, were the construction extended. (Tableau system with such a construction may be found in [Fitting: 1983, pp. 70-74].) We do not follow this line. However, modal erotetic calculi with both-sided sequents, based on the construction of erotetic calculus **E**$^{\textbf{PQ}}$ pertaining to first-order logic, may be quite easily obtained from what is presented in [Wiśniewski, Shangin:2006] and here.

As previously, language **M** will be extended to a language of modal erotetic calculi.

II.1.1 Syntax of *M**

The vocabulary of **M*** includes the vocabulary of **M**, the signs: ⊢, ?, &, ng, and the numerals 1, 2, … . Sequences of numerals will be called *indices*.

The role of the indices is twofold. First, we use numerals as "indicators" of possible worlds of a Kripke frame. Second, the order in which the numerals occur in an index gives us a partial description of the accessibility relation in the frame. Indices will be subscribed to formulas of **M** and they will carry semantical information concerning the particular formulas. Thus we will use indexed formulas as expressions of **M***. More specifically, if A is a formula of **M** and $<i_1, \ldots, i_n>$ is a finite, non-empty sequence of numerals (that is, an index), then an expression of the form:

(2.1) $(A)^{i_1, \ldots, i_n}$

is an *indexed formula* of **M***. According to the above definition, indices do not occur inside indexed formulas of **M***. Moreover, indexed formulas do not occur within the scopes of propositional connectives and modalities.

Till the end of this chapter we use S, T, … for finite sequences of indexed formulas. For convenience, we adopt the following convention. In a metalinguistic expression of the form:

(2.2) $(A)^{\phi(i_n)}$

symbol $\phi(i_n)$ represents a finite sequence of numerals which has numeral i_n as its last term. Thus the expression $(A)^{\phi(i_n)}$ represents any indexed formula of the form: $(A)^{i_1, \ldots, i_n}$, where $n \geq 1$. For example, indexed formulas: $(p)^3$, $(p \rightarrow q)^{1,2,3}$ and $(\Box p)^{1,3}$ are represented by the expression: $(A)^{\phi(3)}$. When referring to possibly different sequences ending with the same numeral we use two expressions with different Greek letters: $\phi(i)$ (or $\phi(i_n)$) and $\gamma(i)$ ($\gamma(i_n)$).

Language **M*** has declarative and erotetic well-formed formulas. *Atomic d-wffs of M** are right-sided sequents, that is, expressions of the form:

(2.3)   ⊢ S

where S is a finite and non-empty sequence of indexed formulas. As before, we will write:

(2.4)   ⊢ $(A_1)^{\phi(i_1)}, \ldots, (A_n)^{\phi(i_n)}$

instead of: ⊢ $<(A_1)^{\phi(i_1)}, \ldots, (A_n)^{\phi(i_n)}>$. Among sequents of **M*** we distinguish the class of *atomic sequents* which are of the form:

(2.5)   ⊢ $(A)^1$

*Compound d-wffs of **M***** are built up from atomic d-wffs of **M*** by means of &
and / or <u>ng</u> in a way described previously (*cf.* Definition 1.3, Section I.2.1).
Erotetic wff-s, that is, *questions of **M***, are of the form:

(2.6)   ? (Φ)

where Φ stands for a finite and non-empty sequence of sequents of **M***. We write:

(2.7)   ? (φ_1, …, φ_n)

instead of ? ($<\varphi_1$, …, $\varphi_n>$). Sequents φ_1, …, φ_n are called the *constituents of
question* (2.7). For simplicity, we use notions of a one-sequent question and a
many-sequent question. A *one-sequent question of **M*** is a question that has
exactly one constituent, and a *many-sequent question of **M*** is a question that has
more than one constituent.

Following the conventions introduced previously (Section I.2.1), we use
Greek lower-case letters φ, ψ as metalinguistic variables for sequents of **M***,
Greek upper-case letters Φ, Ψ for finite (possibly empty) sequences of sequents of
M*, and Q, Q^*, Q_1 for questions of **M***.

II.1.2 A Bit of Semantics

The intended reading of the turnstile symbol '├' as a sign of the object-
level language **M*** is more complex than in the case of languages **L*** and **L****.
Consequently, we will interpret sequents of **M*** in a way which requires some
explanation. Hence we shall explicate below our understanding of the turnstile '├'
by defining the notion of an interpretation of a sequent in a frame. Then we will
introduce the notion of validity of a sequent of **M***, which we are going to use
when we proceed to semantical invertibility of the rules and to the proof of
soundness.

In the case of language **M** we make use of standard notions of Kripke's
semantics. By a *frame* we mean an ordered pair $<W, R>$, where W is a non-empty
set and R is a binary relation in W. The elements of W are called *possible worlds*
and relation R is called *accessibility relation*. If $<W, R>$ is a frame, then by a $<W,
R>$-*assignment* we mean a function defined over $VAR \times W$ with values in $\{0, 1\}$.
We define the following:

DEFINITION 2.1: If $V\#$ is a $<W, R>$-assignment, then an extension of $V\#$ over
$M \times W$ that satisfies, for every $w \in W$, the following conditions:

1) $V(\neg A, w) = 1$ iff $V(A, w) = 0$;

2) $V(\alpha, w) = 1$ iff $V(\alpha_1, w) = 1$ and $V(\alpha_2, w) = 1$;

3) $V(\beta, w) = 1$ iff $V(\beta_1, w) = 1$ or $V(\beta_2, w) = 1$;

4) $V(\nu, w) = 1$ iff for each $w^* \in W$ such that $<w, w^*> \in R$, $V(\nu_0, w) = 1$;

5) $V(\pi, w) = \mathbf{1}$ iff there is $w^* \in W$ such that $<w, w^*> \in R$ and $V(\pi_0, w) = \mathbf{1}$

is a *valuation on a frame* $<W, R>$. *A model* $<W, R, V>$ is a frame $<W, R>$ together with a valuation V on it.

The letter L will stay for any of: **K**, **D**, **T**, **KB**, **K4**, **S4**, **S5**. By 'L-properties' we mean the properties of the accessibility relation that are characteristic to a given logic L. These are listed below.

Logic L:	L-properties:
K	no properties
D	extendability
T	reflexivity
KB	symmetry
K4	transitivity
S4	transitivity and reflexivity
S5	transitivity, reflexivity and symmetry

In what follows we may sometimes use the expression *extendable frame* (*transitive frame, reflexive frame, symmetric frame*) in order to refer to a frame $<W, R>$ with R extendable (transitive, reflexive, symmetric).

We shall say that a formula A is *true in a world w of a model* $<W, R, V>$ (or that it *holds in w*) iff $V(A, w) = \mathbf{1}$. A formula A is *valid in a model* $<W, R, V>$ iff for every $w \in W$, $V(A, w) = \mathbf{1}$. A formula A is **K**-*valid* iff A is valid in every model. The notions of **D**-, **T**-, **KB**-, **K4**-, **S4**-, **S5**-validity of a formula of M are defined as usual. Generally, a formula A of M is L-*valid* iff A is valid in every model $<W, R, V>$ in which R has the L-properties.

Let φ be a sequent of M^* of the form: $\vdash (A_1)^{\phi(i_1)}, \ldots, (A_n)^{\phi(i_n)}$. The sets $\mathbf{I}_W\{\varphi\}$ and $\mathbf{I}_R[\varphi]$ are defined as follows:

- $\mathbf{I}_W\{\varphi\} = \{j: j \text{ is a term of some } \phi(i_k), \text{ where } 1 \leq k \leq n\}$

- $\mathbf{I}_R[\varphi] = \{<j, j'>: j \text{ immediately precedes } j' \text{ in some } \phi(i_k), \text{ where } 1 \leq k \leq n\}$

Thus, if $\varphi = \vdash S$ is a sequent of M^*, then $\mathbf{I}_W\{\varphi\}$ is the set of all the numerals that occur in indices of terms of S, and $\mathbf{I}_R[\varphi]$ is the set of all the ordered pairs $<j, j'>$ such that j immediately precedes j' in an index of a term of S. The idea is simple. For a sequent φ and a frame $<W, R>$ we are going to map the set $\mathbf{I}_W\{\varphi\}$ into W, and, analogously, the set $\mathbf{I}_R[\varphi]$ – into R. We shall call such a mapping an interpretation of sequent φ in frame $<W, R>$. More formally:

DEFINITION 2.2: Let φ be a sequent of M^*. By an *interpretation of* sequent φ *in* a frame $<W, R>$ we mean a function $f: \mathbf{I}_W\{\varphi\} \mapsto W$ satisfying the following condition:

(*) if $<i, j> \in \mathbf{I}_R[\varphi]$, then $<f(i), f(j)> \in R$

We say that a sequent φ is *interpretable in* a frame $<W, R>$ iff there exists an interpretation of φ in $<W, R>$.

For clarity, let us note the following propositions about sequents and their interpretations.

PROPOSITION 2.1: It may happen that a sequent is not interpretable in a frame. *E.g.*, if a sequent φ is such that the set $\mathbf{I}_R[\varphi]$ is non-empty and a frame $<W, R>$ is such that R is empty, then, in view of condition (*) of Definition 2.2, there is no interpretation function of φ in $<W, R>$.

PROPOSITION 2.2: For every sequent φ there exists a frame $<W, R>$ such that φ is interpretable in $<W, R>$.

As an illustration we give a "recipe" for constructing, for a given sequent φ: (*i*) a certain frame $<W, R>$, and (*ii*) an interpretation of φ in $<W, R>$. Namely, we put: (*i*) $W = \mathbf{I}_W\{\varphi\}$ and $R = \mathbf{I}_R[\varphi]$, and (*ii*) the identity function f: $\mathbf{I}_W\{\varphi\} \mapsto W$ as the interpretation of φ in $<W, R>$.

A frame constructed for a sequent φ according to (*i*) will be called a *canonical frame for* φ, and an interpretation of φ in its canonical frame, constructed according to (*ii*), will be called the *canonical interpretation of φ in its canonical frame*.

PROPOSITION 2.3: Every sequent φ such that $\mathbf{I}_R[\varphi]$ is empty is interpretable in every frame $<W, R>$, as condition (*) of Definition 2.2 is vacuously satisfied. Indeed, in such a case *any* function f: $\mathbf{I}_W\{\varphi\} \mapsto W$ is an interpretation of φ in $<W, R>$.

The notion of validity of a sequent is relativized both to a frame and to an interpretation of the sequent in this frame. We shall start with a more elementary notion of satisfaction of a sequent in a model, which is also relativized to an interpretation of a sequent. Namely:

DEFINITION 2.3: Let $<W, R>$ be a frame and let V be a valuation on $<W, R>$. A sequent $\varphi = \vdash (A_1)^{\phi(i_1)}, \ldots, (A_n)^{\phi(i_n)}$ is *satisfied in a model $<W, R, V>$ under an interpretation f of φ in frame $<W, R>$* iff for some k ($1 \leq k \leq n$): $V(A_k, f(i_k)) = \mathbf{1}$.

Now we may say that a sequent, φ, is *valid in a frame $<W, R>$ under an interpretation f of φ in $<W, R>$* iff for every valuation V on frame $<W, R>$, the sequent φ is satisfied in a model $<W, R, V>$ under interpretation f of φ in frame $<W, R>$. And we will say that a sequent, φ, is *valid in a frame $<W, R>$* iff φ is valid in $<W, R>$ under every interpretation f of φ in $<W, R>$.

Let us observe that, according to the above definitions, sequent φ is not valid in a frame $<W, R>$ iff there exists an interpretation f of φ in $<W, R>$ such that φ is not valid in $<W, R>$ under f. Therefore we have:

COROLLARY 2.1: If a sequent φ is not interpretable in a frame $<W, R>$, then φ is valid in $<W, R>$.

We will also say that a sequent, φ, is **K**-*valid* iff φ is valid in every frame. The notion of **K**-validity of a sequent of **M*** may be easily adjusted to any *L*. Namely, we say that a sequent φ is *L-valid* iff φ is valid in every frame $<W, R>$ such that R has the *L*-properties.

The following corollary immediately follows from the above definitions:

COROLLARY 2.2: A sequent φ is not *L*-valid iff for some model $<W, R, V>$, where R has the *L*-properties, and for some interpretation f of φ in frame $<W, R>$, the sequent φ is not satisfied in the model $<W, R, V>$ under f.

The notion of validity defined for sequents of **M*** generalizes the notion of validity of formulas of the underlying modal language **M**. In the sequel we will be using the two notions of *L*-validity (*i.e. L*-validity of a formula of **M** and *L*-validity of a d-wff of **M***), but context should prevent any ambiguities. Now we shall prove:

THEOREM 2.1: An atomic sequent $\vdash (A)^1$ is *L*-valid iff the formula A of language **M** is *L*-valid.

PROOF: Let us observe, first, that the set $\mathbf{I}_R[(A)^1]$ is empty, and thus the sequent $\vdash (A)^1$ is interpretable in every frame (*cf.* Proposition 2.3). We show that the lack of *L*-validity of sequent $\vdash (A)^1$ is tantamount to the lack of *L*-validity of formula A.

If $\vdash (A)^1$ is not *L*-valid, then, by Corollary 2.2, for some frame $<W, R>$ (with R having the *L*-properties) and for some interpretation f of sequent $\vdash (A)^1$ in $<W, R>$, there is a valuation V on $<W, R>$ such that sequent $\vdash (A)^1$ is not satisfied in model $<W, R, V>$ under f. But then $V(A, f(1)) = \mathbf{0}$, and therefore A is not *L*-valid.

On the other hand, if A is not *L*-valid, then for some frame $<W, R>$ (with the *L*-properties imposed on R) and some valuation V on it there is $w \in W$ such that $V(A, w) = \mathbf{0}$. Observe that the function $f: \mathbf{I}_W\{\vdash (A)^1\} \mapsto W$ set by: $f(1) = w$ is an interpretation of $\vdash (A)^1$ in $<W, R>$ (*cf.* Proposition 2.3) and thus, obviously, sequent $\vdash (A)^1$ is not valid in $<W, R>$ under f, hence it is not *L*-valid.

∎

II.1.3 Questions of *M**: An Intuitive Account of Socratic Transformations

Theorem 2.1 justifies us in interpreting one-sequent questions of the form $?\,(\vdash (A)^1)$ as questions about *L*-validity of the formula A.[32] Therefore, for example, the following natural-language question:

(1)   Is it the case that axiom K: $\Box(p \to q) \to (\Box p \to \Box q)$ is valid in every Kripke model?

may be expressed in language *M** by:

(2) $?\,(\vdash (\Box(p \to q) \to (\Box p \to \Box q))^1)$

Numeral 1 in the upper index of the formula may be regarded as representing an arbitrary world of an arbitrary Kripke model. Question (2) asks whether axiom K is true in the world represented by 1. Our aim is to find an answer to this question by transforming (2) and to find rules governing such transformations. If we consider classical connectives, suitably modified rules of **E**** should be at place. This suggests that we may transform question (2) into:

(3) $?\,(\vdash (\neg\Box(p \to q))^1, (\neg\Box p)^1, (\Box q)^1)$

Just as in the case of the right-sided sequents of **E****, the comma separating the wffs in the constituent of question (3) may be interpreted as a meta-level disjunction of these wffs.[33]

Let us emphasize that the questions are now interpreted in view of the direct reading A (*cf.* Section I.3.2). Interpretation A may be also called the *generalizing* interpretation of the method of Socratic proofs, since under this interpretation numeral 1 occurring in questions (2) and (3) is thought of as representing an *arbitrary* world of an *arbitrary* Kripke model.

Let us also observe that if we assume that the formula $\Box q$ is true in a world (an arbitrary one) represented by 1, then the propositional variable q is true in any world accessible from the world denoted by 1, *if there is any such world*. This suggest the following tentative rule: an indexed ν-formula, $(\nu)^{\phi(i)}$, may be replaced by the component, ν_0, of the formula ν with an index extended with a *new* numeral, so that this new numeral represents an arbitrary accessible world. Under this generalized understanding of numerals we may transform question (3) into:

(4) $?\,(\vdash (\neg\Box(p \to q))^1, (\neg\Box p)^1, (q)^{1,2})$

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[32] Obviously, if a formula is *L*-valid, then it is *L*-entailed by the empty set in both, local and global, meanings. This is still coherent with our intended interpretation of the turnstile symbol.

[33] This should not be confused with a disjunction of formulas of language *M*. In contradistinction to what we may say about the sequents of language *L***, *L*-validity of a sequent of the form: $\vdash (A_1)^{\phi(i_1)}, \ldots, (A_n)^{\phi(i_n)}$ *does not* amount to *L*-validity of the disjunction: $A_1 \vee \ldots \vee A_n$. For example, sequent $\vdash (\Diamond p)^1, (\neg p)^{1,2}$ is **K**-valid but formula '$\Diamond p \vee \neg p$' is not. On the other hand, the formula '$p \vee \neg p$' of language *M* is obviously *L*-valid, but the sequent $\vdash (p)^{\phi(i)}, (\neg p)^{\phi(j)}$ is *L*-valid iff $i = j$.

The constituents of questions (3) and (4) differ with respect to frames in which they are interpretable. This is a consequence of adding a new numeral to an index of formula 'q' – the constituent of question (3) is interpretable in every frame (*cf.* Proposition 2.3), but the constituent of question (4) is not (*cf.* Proposition 2.1). However, if we consider an arbitrary frame $<W, R>$ such that the constituent of question (3) is valid in $<W, R>$, then either the constituent of question (4) is not interpretable in $<W, R>$, and hence it is valid in $<W, R>$ by Corollary 2.1, or an arbitrary interpretation of this sequent in this frame assigns to numeral 2 a world accessible from a world assigned to numeral 1, and this fact guarantees that q is true in the former, whenever $\square q$ is true in the latter. On the other hand, if the constituent of question (4) is **K**-valid, then for each frame such that *there is* a world accessible from a world assigned to 1, if q is true in an *arbitrary* such world (represented by 2), then $\square q$ must be true in a world assigned to 1. Moreover, if in a frame there is no world accessible from *an arbitrary world* that may be assigned to 1 (that is, if the constituent of question (4) is not interpretable in this frame), then the formula $\square q$ is, trivially, true in every model based on such a frame.

More problematic are the π-formulas, as under the generalized meaning of numerals (interpretation A) we may not eliminate a π-formula leaving its component π_0 in *some* world. For this reason, as could actually be expected, wffs of the form $(\pi)^{\phi(i)}$ are not eliminated in the course of a transformation, but their components may be, roughly speaking, introduced to any world which is accessible form the one assigned to numeral i. Hence question (4) may be transformed into:

(5) $? \, (\vdash (\neg\square(p \to q))^1, (\neg(p \to q))^2, (\neg\square p)^1, (q)^{1,2} \,)$

K-validity of the constituent of question (4) guarantees, quite trivially, **K**-validity of the constituent of question (5). The reason why the relation between **K**-validity of the constituents is preserved in the other direction is the following: if we consider an arbitrary model $<W, R, V>$, where $V(\neg(p \to q), f(2)) = \mathbf{1}$ for some interpretation f, then by the definition of an interpretation of a sequent in a frame, world $f(2)$ is accessible from $f(1)$, hence $V(\neg\square(p \to q), f(1)) = \mathbf{1}$, which suffices to guarantee that if the constituent of question (5) is **K**-valid, then the constituent of question (4) is **K**-valid as well. It should be emphasized here that the preservation of **K**-validity is guaranteed by the accessibility of $f(2)$ from $f(1)$, which suggests that replacing a wff $(\pi)^{\phi(i)}$ by a pair: $(\pi)^{\phi(i)}$ and $(\pi_0)^j$ is "safe" as long as the numerals: i, j occur one after another in an index of some formula of a considered sequent.

When we repeat the same step with respect to the wff $(\neg\square p)^1$, we get:

(6) $? \, (\vdash (\neg\square(p \to q))^1, (\neg(p \to q))^2, (\neg\square p)^1, (\neg p)^2, (q)^{1,2} \,)$

and an application of a rule analogous to $\mathbf{R}_{\to}$ of $\mathbf{E^{**}}$ results in:

(7) $? \, (\vdash (\neg\square(p \to q))^1, (p)^2, (\neg\square p)^1, (\neg p)^2, (q)^{1,2};$

$\qquad\qquad \vdash (\neg\square(p \to q))^1, (\neg q)^2, (\neg\square p)^1, (\neg p)^2, (q)^{1,2} \,)$

Obviously, for each frame and for an arbitrary interpretation f of the constituents of question (7) in this frame, it must be the case that one of: p and $\neg p$ (q and $\neg q$) is true in $f(2)$. Hence the answer to the question expressed by (7) is affirmative. It is clear that the transitions from a question to a question described so far were invertible. Therefore we may say that the answer to question (2) is affirmative.

As we have pointed out in Section I.3.2, a Socratic transformation performed within the framework of an erotetic calculus may be interpreted as representing both a direct and an indirect reasoning. The indirect reading (interpretation B) of the transformation of question (2) may seem more intuitive. In this case question (2) is assumed to represent the following natural-language question:

(8)   Is it the case that axiom K is false in some world of a Kripke model?

Consequently, the interpretation of numeral 1 changes into an "individualized" one. (For this reason we may call interpretation B the *individualizing* interpretation of the method of Socratic proofs.) By interpreting comma separating wffs in a sequent as representing conjunction of (false) wffs we arrive at question (3). Numeral 2 in question (4) does not represent an *arbitrary* world accessible from the one assigned to 1, but a *particular* accessible world in which the argument of the necessity operator is false. As to the wffs of the form $(\pi)^{\phi(i)}$, the component π_0 of π must be false in each world accessible from the one assigned to i, which justifies the transformation of question (4) into question (5). The constituents of question (7) express, in a sense, the *absurdum* we arrive at when assuming that axiom K is not **K**-valid.

What we have established so far for the case of logic **K** may be quite easily extended to any L other than **K**. For example, question about **K4**-validity of axiom **4**: $\Box p \to \Box\Box p$ may be represented in **M*** by:

(9) $? (\vdash (\Box p \to \Box\Box p)^1)$

And in a few steps this may be transformed into:

(10) $? (\vdash (\neg\Box p)^1, (p)^{1,2,3})$

Numeral 2 represents here an arbitrary world accessible from a world assigned to numeral 1, and numeral 3 represents an arbitrary world accessible from a world represented by 2. If we assume that the frames we consider are transitive, any world represented by 3 is accessible from a world assigned to 1, hence question (10) may be transformed into:

(11) $? (\vdash (\neg\Box p)^1, (\neg p)^3, (p)^{1,2,3})$

Let us also observe that if the initial question concerned **K**-validity of axiom **4**, the transformation would stop at:

(12) $? (\vdash (\Diamond\neg p)^1, (\neg p)^2, (p)^{1,2,3})$

without success but with enough information to construct a countermodel. Actually, in this case it suffices to consider the canonical frame for the constituent of question (12), namely, the frame with $W = \{1, 2, 3\}$ and $R = \{<1, 2>, <2, 3>\}$.[34] Obviously, we switch to the individualized understanding (interpretation B) of the numerals again. A valuation on our frame falsifying the formula assigns $\mathbf{0}$ to p in world 3 and $\mathbf{0}$ to $\neg p$ in world 2. It follows that $V(\Box p \to \Box \Box p, 1) = \mathbf{0}$.

Before we present the rules of the modal calculi of questions, let us summarize the results of this section. Theorem 2.1, as we have observed, shows that a question of the form $? (\vdash (A)^1)$ may be interpreted as a question about L-validity of the formula A. The choice of L determines, to some extent, the moves that are permitted in a transformation of such a question. During a transformation one arrives at questions with non-atomic sequents as constituents. Actually, there is no straightforward correspondence between L-validity of a non-atomic sequent and L-validity of formulas of M that occur in the sequent (*cf.* examples in footnote 33). However, we may say that a non-atomic, one-sequent question asks about L-validity of its constituent, and that a many-sequent question asks about *joint L*-validity of all of its constituents.

II.2 The Rules of Calculi $\mathbf{E}^L$. Socratic Transformations via the Rules of $\mathbf{E}^L$

II.2.1 Calculus $\mathbf{E}^K$

We present the rules of calculus $\mathbf{E}^K$ pertaining to logic $\mathbf{K}$. As the construction is based on $\mathbf{E^{**}}$, the rules $\mathbf{R_\alpha}$, $\mathbf{R_\beta}$, $\mathbf{R_{\neg\neg}}$ of $\mathbf{E}^K$ differ from those of $\mathbf{E^{**}}$ only in that *indexed* formulas are subjects of the decomposition procedure.

Let us recall that Φ and Ψ stand for finite (possibly empty) sequences of sequents, and letters S and T represent finite (possibly empty) sequences of indexed formulas. As previously, we use two concatenation-signs: the sign ' is used as the concatenation-sign for sequences of indexed formulas, and the semicolon ';' is used as the concatenation-sign for sequences of sequents. Here are the rules of $\mathbf{E}^K$ in the α-, β-notation:

$$\mathbf{R_\alpha}: \quad \frac{? (\Phi; \vdash S\, {}' (\alpha)^{\phi(i)}\, {}' T; \Psi)}{? (\Phi; \vdash S\, {}' (\alpha_1)^{\phi(i)}\, {}' T; \vdash S\, {}' (\alpha_2)^{\phi(i)}\, {}' T; \Psi)} \qquad \mathbf{R_\beta}: \quad \frac{? (\Phi; \vdash S\, {}' (\beta)^{\phi(i)}\, {}' T; \Psi)}{? (\Phi; \vdash S\, {}' (\beta_1)^{\phi(i)}\, {}' (\beta_2)^{\phi(i)}\, {}' T; \Psi)}$$

$$\mathbf{R_{\neg\neg}}: \quad \frac{? (\Phi; \vdash S\, {}' (\neg\neg A)^{\phi(i)}\, {}' T; \Psi)}{? (\Phi; \vdash S\, {}' (A)^{\phi(i)}\, {}' T; \Psi)}$$

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[34] The canonical interpretation of the constituent in this frame is set by $f(i) = i$, where $i = 1, 2, 3$. (*Cf.* Proposition 3.2.)

Just as in the case of **E****, an application of any of the rules: $\mathbf{R_\alpha}$, $\mathbf{R_\beta}$ or $\mathbf{R_{\neg\neg}}$, results in the decomposition of a formula. Indices of formulas are *not operated on* in case of these rules. Any modification of indices during the transformation is due to an application of rule $\mathbf{R_\nu}$ or of rule $\mathbf{R_\pi}$. The schemas of these rules are the following:

$$\mathbf{R_\nu}: \quad \frac{?\,(\Phi;\ \vdash S\,'\,(\nu)^{\phi(i)}\,'\,T;\ \Psi)}{?\,(\Phi;\ \vdash S\,'\,(\nu_0)^{\phi(i),j}\,'\,T;\ \Psi)} \qquad\qquad \mathbf{R_\pi}: \quad \frac{?\,(\Phi;\ \vdash S\,'\,(\pi)^{\phi(i)}\,'\,T;\ \Psi)}{?\,(\Phi;\ \vdash S\,'\,(\pi)^{\phi(i)}\,'\,(\pi_0)^j\,'\,T;\ \Psi)}$$

Rules $\mathbf{R_\nu}$ and $\mathbf{R_\pi}$ may be applied provided that the numerals i, j satisfy certain conditions, which have been already discussed in the previous section. Namely, rule $\mathbf{R_\nu}$ may be applied provided that numeral j is new with respect to sequent $\vdash S\,'\,(\nu)^{\phi(i)}\,'\,T$. That is:

- The proviso of applicability of rule $\mathbf{R_\nu}$: $j \notin \mathbf{I}_W\{\vdash S\,'\,(\nu)^{\phi(i)}\,'\,T\}$

For convenience, we assume that $j = \max(\mathbf{I}_W\{\vdash S\,'\,(\nu)^{\phi(i)}\,'\,T\}) + 1$.

Rule $\mathbf{R_\pi}$ may be applied provided that pair $\langle i, j\rangle$ already occurs in an index of a wff in sequent $\vdash S\,'\,(\pi)^{\phi(i)}\,'\,T$. That is:

- The proviso of applicability of rule $\mathbf{R_\pi}$: $\langle i, j\rangle \in \mathbf{I}_R[\vdash S\,'\,(\pi)^{\phi(i)}\,'\,T]$

In Appendix 3 we present the modal rules $\mathbf{R_\nu}$ and $\mathbf{R_\pi}$ without the ν-, π-notation.

We repeat the transformation of question $?\,(\vdash (\Box(p \to q) \to (\Box p \to \Box q))^1\,)$. Every question (except for the first one) of the sequence of questions presented below has been obtained from the previous one by an application of a rule of $\mathbf{E^K}$; on the margin we indicate which rule has been applied to a question. For transparency, we highlight the indexed formula acted upon.

1. $?\,(\vdash (\Box(p \to q) \to (\Box p \to \Box q))^1\,)$ $\mathbf{R_\beta}$

2. $?\,(\vdash (\neg\Box(p \to q))^1, (\Box p \to \Box q)^1\,)$ $\mathbf{R_\beta}$

3. $?\,(\vdash (\neg\Box(p \to q))^1, (\neg\Box p)^1, (\Box q)^1\,)$ $\mathbf{R_\nu}$

4. $?\,(\vdash (\neg\Box(p \to q))^1, (\neg\Box p)^1, (q)^{1,2}\,)$ $\mathbf{R_\pi}$

5. $?\,(\vdash (\neg\Box(p \to q))^1, (\neg(p \to q))^2, (\neg\Box p)^1, (q)^{1,2}\,)$ $\mathbf{R_\pi}$

6. $?\,(\vdash (\neg\Box(p \to q))^1, (\neg(p \to q))^2, (\neg\Box p)^1, (\neg p)^2, (q)^{1,2}\,)$ $\mathbf{R_\alpha}$

7. $?\,(\vdash (\neg\Box(p \to q))^1, (p)^2, (\neg\Box p)^1, (\neg p)^2, (q)^{1,2};$

                $\vdash (\neg\Box(p \to q))^1, (\neg q)^2, (\neg\Box p)^1, (\neg p)^2, (q)^{1,2})$

We introduce the key notions of this section:

DEFINITION 2.4: A *Socratic transformation of a question Q via the rules of* $\mathbf{E^K}$ is a sequence $s = Q_1, Q_2, \ldots$ of questions such that: $Q_1 = Q$, and for each $n > 1$, question Q_n results from question Q_{n-1} by an application of one of the rules of $\mathbf{E^K}$.

DEFINITION 2.5. Let $\vdash (A)^1$ be an atomic sequent of $\boldsymbol{M^*}$. A *Socratic proof of* $\vdash (A)^1$ in $\mathbf{E^K}$ is a finite Socratic transformation s of the question $?(\vdash (A)^1)$ via the rules of $\mathbf{E^K}$ such that for each constituent φ of the last question of s the following holds:

    (a) φ is of the form $\vdash S \mathbin{'} (B)^{\phi(i)} \mathbin{'} T \mathbin{'} (\neg B)^{\gamma(i)} \mathbin{'} U$, or
    (b) φ is of the form $\vdash S \mathbin{'} (\neg B)^{\gamma(i)} \mathbin{'} T \mathbin{'} (B)^{\phi(i)} \mathbin{'} U$.

    Let us remind that the symbols $\phi(i)$ and $\gamma(i)$ in (a) or (b) may not stand for occurrences of *the same* index. However, sequences $\phi(i)$ and $\gamma(i)$ end with the same numeral, and this is the crucial point as far as validity of a sequent is concerned, because for an indexed formula $(A)^{i_1, \ldots, i_n}$ it is only the numeral i_n that "indicates" the world in which the value of A is relevant. The following lemma holds:

LEMMA 2.1: If φ is a sequent of the form: $\vdash S \mathbin{'} (B)^{\phi(i)} \mathbin{'} T \mathbin{'} (\neg B)^{\gamma(i)} \mathbin{'} U$, or of the form: $\vdash S \mathbin{'} (\neg B)^{\phi(i)} \mathbin{'} T \mathbin{'} (B)^{\gamma(i)} \mathbin{'} U$, then φ is *L*-valid.

PROOF: Let $<W, R>$ stand for an arbitrary frame. Suppose that φ is of one of the forms specified above. If φ is not interpretable in frame $<W, R>$ then, by Corollary 2.1, φ is valid in $<W, R>$. Suppose that there exists an interpretation f of φ in $<W, R>$, and let V be a valuation on $<W, R>$. Obviously, either $V(B, f(i)) = \mathbf{1}$ or $V(\neg B, f(i)) = \mathbf{1}$. Hence sequent φ is valid in every frame, independently of the properties of R.

■

    Thus a Socratic proof is a Socratic transformation ending with a question whose constituents are *L*-valid. According to Definition 2.5, each Socratic proof has as its first term a one-sequent question whose only constituent is an atomic sequent, that is the first question of a Socratic proof is a question about *L*-validity of a formula of language $\boldsymbol{M}$. As we will show in Section II.3, the existence of such a Socratic proof amounts to the affirmative answer to the question.
    It is worth emphasizing that a "Socratic proof procedure" remains a direct procedure. It *does not* start with the negation of an initial assumption and, hence, it is not an indirect proof method. However, according to the discussion in Sections I.3.2 and II.1.3., a Socratic proof may still be interpreted as an unsuccessful attempt to find a countermodel. On the other hand, if a Socratic transformation stops without a success, then any of the constituents of its last question that is not of one of the forms (a), (b), specified in Definition 2.5, may be used to construct a countermodel. It is probably more convenient, when using the

method, to keep in mind its indirect reading. This reading has even some more merits, as it simplifies somewhat discussion *about* the method. Actually, in the proofs of semantical invertibility of rules (Section II.3) we implicitly refer to the indirect interpretation.

II.2.2 Calculi $\mathbf{E}^L$

In this section L will vary through the following proper extensions of $\mathbf{K}$: $\mathbf{D}$, $\mathbf{T}$, $\mathbf{KB}$, $\mathbf{K4}$, $\mathbf{S4}$, $\mathbf{S5}$. The rules of a calculus $\mathbf{E}^L$ are rules $\mathbf{R}_\alpha$, $\mathbf{R}_\beta$, $\mathbf{R}_{\neg\neg}$, $\mathbf{R}_\nu$ of $\mathbf{E}^K$ and rule $\mathbf{R}_\pi$ with a proviso of its applicability varying from a calculus to a calculus. The form of the proviso reflects the properties of the accessibility relation specific to L.

For each L we give the proviso on rule $\mathbf{R}_\pi$ and an example of a Socratic proof in $\mathbf{E}^L$. We use '$\mathbf{P}^L$' for proviso of applicability of rule $\mathbf{R}_\pi$ in calculus $\mathbf{E}^L$. We shall start with logic $\mathbf{T}$ and will return to the somewhat peculiar case of the extendable logic $\mathbf{D}$ at the end of this section. We repeat the schema of rule $\mathbf{R}_\pi$:

$$\frac{?\,(\Phi;\ \vdash S\,'\,(\pi)^{\phi(i)}\,'\,T;\,\Psi)}{?\,(\Phi;\ \vdash S\,'\,(\pi)^{\phi(i)}\,'\,(\pi_0)^{j}\,'\,T;\,\Psi)}$$

and we add the relevant proviso for calculus $\mathbf{E}^T$, which is:

$\mathbf{P}^T$: $<i,j> \in \mathbf{I}_R[\vdash S\,'\,(\pi)^{\phi(i)}\,'\,T]$ or

$\quad\quad\quad j = i$

In the case of calculus $\mathbf{E}^T$ rule $\mathbf{R}_\pi$ may be applied provided that numeral i immediately precedes numeral j in an index of some formula in the relevant sequent or numeral j is i. Obviously, in the case of a single application of rule $\mathbf{R}_\pi$ one part of the proviso is required to hold. The second part of the proviso corresponds to reflexivity of the accessibility relation. Sequent $\vdash (\Box p \to p)^1$ is provable in $\mathbf{E}^T$:

$(1.)\quad ?\,(\vdash (\Box p \to p)^1)$ $\mathbf{R}_\beta$

$(2.)\quad ?\,(\vdash (\neg\Box p)^1,\,(p)^1)$ $\mathbf{R}_\pi$

$(3.)\quad ?\,(\vdash (\neg\Box p)^1,\,(\neg p)^1,\,(p)^1)$

Question $(3.)$ has been obtained from the previous one by $\mathbf{R}_\pi$ on the second part of $\mathbf{P}^T$.

$\mathbf{P}^{KB}$: $<i,j> \in \mathbf{I}_R[\vdash S\,'\,(\pi)^{\phi(i)}\,'\,T]$ or

$\quad\quad\quad\quad <j,i> \in \mathbf{I}_R[\vdash S\,'\,(\pi)^{\phi(i)}\,'\,T]$

where the second part of $\mathbf{P^{KB}}$ corresponds to symmetry. Here is a Socratic proof of sequent $\vdash (p \rightarrow \Box \Diamond p)^1$ in $\mathbf{E^{KB}}$:

$(1.)$ $? (\vdash (p \rightarrow \Box \Diamond p)^1)$ $\mathbf{R_\beta}$

$(2.)$ $? (\vdash (\neg p)^1, (\Box \Diamond p)^1)$ $\mathbf{R_\nu}$

$(3.)$ $? (\vdash (\neg p)^1, (\Diamond p)^{1,2})$ $\mathbf{R_\pi}$

$(4.)$ $? (\vdash (\neg p)^1, (\Diamond p)^{1,2}, (p)^1)$

The last question results from the previous one on the second part of the proviso.

In the case of transitive logics we make use of the following notions:

DEFINITION 2.6: Let R be a binary relation in a set W.[35]
  (1) By an *R-chain* we mean a finite, at least two-term sequence $<w_1, \ldots, w_n>$ of elements of W such that for each k ($1 \leq k < n$): either $<w_k, w_{k+1}> \in R$ or $<w_{k+1}, w_k> \in R$.
  (2) By a *directed R-chain* we mean an *R-chain* $<w_1, \ldots, w_n>$ such that $<w_k, w_{k+1}> \in R$ for each k ($1 \leq k < n$).

$\mathbf{P^{K4}}$:   there is a directed $\mathbf{I}_R[\vdash S \; ' \; (\pi)^{\phi(i)} \; ' \; T]$-chain whose first term is i and whose last term is j

The proviso corresponds to transitivity of the accessibility relation. Let us observe that if $<i, j> \in \mathbf{I}_R[\vdash S \; ' \; (\pi)^{\phi(i)} \; ' \; T]$, then sequence $<i, j>$ is a directed $\mathbf{I}_R[\vdash S \; ' \; (\pi)^{\phi(i)} \; ' \; T]$-chain. Therefore $\mathbf{P^{K4}}$ includes the proviso specific to $\mathbf{K}$.

Formula '$\Box p \rightarrow \Box \Box p$' has been discussed above. A Socratic proof of sequent $\vdash (\Box p \rightarrow \Box \Box p)^1$ in calculus $\mathbf{E^{K4}}$ is the following:

$(1.)$ $? (\vdash (\Box p \rightarrow \Box \Box p)^1)$ $\mathbf{R_\beta}$

$(2.)$ $? (\vdash (\neg \Box p)^1, (\Box \Box p)^1)$ $\mathbf{R_\nu}$

$(3.)$ $? (\vdash (\neg \Box p)^1, (\Box p)^{1,2})$ $\mathbf{R_\nu}$

$(4.)$ $? (\vdash (\neg \Box p)^1, (p)^{1,2,3})$ $\mathbf{R_\pi}$

$(5.)$ $? (\vdash (\neg \Box p)^1, (\neg p)^3, (p)^{1,2,3})$

Question $(5.)$ results from question $(4.)$ by $\mathbf{R_\pi}$ (sequence $<1, 2, 3>$ is the required directed chain).

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[35] Let us emphasize, for clarity, that the metavariables W and R used in Definition 2.6 refer to an arbitrary set and an arbitrary binary relation in this set, respectively. W and R may, but *do not have to* refer to the elements of a frame $<W, R>$.

P^{S4}:   there is a directed $\mathbf{I}_R[\vdash S \,{}' (\pi)^{\phi(i)} \,{}' T]$-chain whose first term is i and whose last term is j or

$$j = i$$

Socratic proofs of sequents $\vdash (\square p \rightarrow \square\square p)^1$ and $\vdash (\square p \rightarrow p)^1$ presented above are available in **E^{S4}** as well. We present another example. Formula '$\square\Diamond\Diamond p \rightarrow \Diamond p$' is a thesis of **S4**.

(1.)	? ($\vdash (\square\Diamond\Diamond p \rightarrow \Diamond p)^1$)	$\mathbf{R}_\beta$
(2.)	? ($\vdash (\neg\square\Diamond\Diamond p)^1, (\Diamond p)^1$)	$\mathbf{R}_\pi$
(3.)	? ($\vdash (\neg\square\Diamond\Diamond p)^1, (\neg\Diamond\Diamond p)^1, (\Diamond p)^1$)	$\mathbf{R}_v$
(4.)	? ($\vdash (\neg\square\Diamond\Diamond p)^1, (\neg\Diamond p)^{1,2}, (\Diamond p)^1$)	$\mathbf{R}_v$
(5.)	? ($\vdash (\neg\square\Diamond\Diamond p)^1, (\neg p)^{1,2,3}, (\Diamond p)^1$)	$\mathbf{R}_\pi$
(6.)	? ($\vdash (\neg\square\Diamond\Diamond p)^1, (\neg p)^{1,2,3}, (\Diamond p)^1, (p)^3$)	

Question (*3.*) results from the previous one on the second part of the proviso, and question (*6.*) results from question (*5.*) on the first part of the proviso.

Let us observe that question (*5.*) of the above example may be also transformed into question:

(6'.)	? ($\vdash (\neg\square\Diamond\Diamond p)^1, (\neg\Diamond\Diamond p)^3, (\neg p)^{1,2,3}, (\Diamond p)^1$)	

by the same rule applied with respect to the wff $(\neg\square\Diamond\Diamond p)^1$. An application of rule **R$_v$** will lead us to:

(7.)	? ($\vdash (\neg\square\Diamond\Diamond p)^1, (\neg\Diamond p)^{3,4}, (\neg p)^{1,2,3}, (\Diamond p)^1$)	

and another application of rule **R$_\pi$** results in:

(8.)	? ($\vdash (\neg\square\Diamond\Diamond p)^1, (\neg\Diamond\Diamond p)^4, (\neg\Diamond p)^{3,4}, (\neg p)^{1,2,3}, (\Diamond p)^1$)	

It is easy to observe that the transformation may be continued *ad infinitum*. (This is not surprising, as we deal with a transitive logic.)

P^{S5}:   there is an $\mathbf{I}_R[\vdash S \,{}' (\pi)^{\phi(i)} \,{}' T]$-chain whose first term is i and whose last term is j or
$$j = i$$

Let us observe that if $<i, j> \in \mathbf{I}_R[\vdash S \,{}' (\pi)^{\phi(i)} \,{}' T]$ or $<j, i> \in \mathbf{I}_R[\vdash S \,{}' (\pi)^{\phi(i)} \,{}' T]$, then $<i, j>$ is an $\mathbf{I}_R[\vdash S \,{}' (\pi)^{\phi(i)} \,{}' T]$-chain, therefore **P^{S5}** includes both parts of the proviso **P^{KB}**. Moreover, since a directed $\mathbf{I}_R[\vdash S \,{}' (\pi)^{\phi(i)} \,{}' T]$-chain is an $\mathbf{I}_R[\vdash S \,{}' (\pi)^{\phi(i)} \,{}' T]$-chain, **P^{S5}** includes **P^{K4}** as a special case. Therefore the first part of **P^{S5}** combines conditions corresponding to transitivity and symmetry. The second part of the proviso corresponds to reflexivity.

Sequents $\vdash (\Box p \to \Box\Box p)^1$, $\vdash (\Box p \to p)^1$ and $\vdash (p \to \Box\Diamond p)^1$ are provable in $\mathbf{E}^{\mathbf{S5}}$. Another example is a Socratic proof of sequent $\vdash (p \to \Box\Box\Diamond p)^1$:

(1.)	$? (\vdash (p \to \Box\Box\Diamond p)^1)$	$\mathbf{R_\beta}$
(2.)	$? (\vdash (\neg p)^1, (\Box\Box\Diamond p)^1)$	$\mathbf{R_v}$
(3.)	$? (\vdash (\neg p)^1, (\Box\Diamond p)^{1,2})$	$\mathbf{R_v}$
(4.)	$? (\vdash (\neg p)^1, (\Diamond p)^{1,2,3})$	$\mathbf{R_\pi}$
(5.)	$? (\vdash (\neg p)^1, (\Diamond p)^{1,2,3}, (p)^1)$	

Question (5.) results from the previous one by $\mathbf{R_\pi}$, on the first part of $\mathbf{P^{S5}}$. The proviso is satisfied as the sequence $<3, 2, 1>$ is an $\mathbf{I}_R[\vdash (\neg p)^1, (\Diamond p)^{1,2,3}]$-chain.

Now we turn to logic $\mathbf{D}$. The peculiarity of this logic is due to the fact that the property of extendability of the accessibility relation requires another rule; the rule will be called $\mathbf{R_{\pi D}}$. Rule $\mathbf{R_\pi}$ is also present in $\mathbf{E^D}$, and it may be applied in a Socratic transformation via the rules of $\mathbf{E^D}$ provided that $<i, j> \in \mathbf{I}_R[\vdash S \,'\, (\pi)^{\phi(i)} \,'\, T]$ (that is, just as in the case of $\mathbf{E^K}$). Moreover, the rule:

$$\mathbf{R_{\pi D}}: \quad \frac{? (\Phi; \vdash S \,'\, (\pi)^{\phi(i)} \,'\, T; \Psi)}{? (\Phi; \vdash S \,'\, (\pi)^{\phi(i)} \,'\, (\pi_0)^{i,j} \,'\, T; \Psi)}$$

may be applied provided that $j \notin \mathbf{I}_W\{ \vdash S \,'\, (\pi)^{\phi(i)} \,'\, T\}$. The difference between the rules $\mathbf{R_\pi}$ and $\mathbf{R_{\pi D}}$ is subtle. In the case of rule $\mathbf{R_\pi}$, the component π_0 of a π-formula occurs in the conclusion with a one-term index $<j>$. In the case of rule $\mathbf{R_{\pi D}}$, the component π_0 must be introduced with a two-term index $<i, j>$. Extending the index is necessary in order to keep the information concerning accessibility relation "up to date".

In the case of $\mathbf{R_{\pi D}}$, just as in the case of rule $\mathbf{R_v}$, we assume that $j = \max(\mathbf{I}_W\{ \vdash S \,'\, (\pi)^{\phi(i)} \,'\, T\}) + 1$. We present a Socratic proof of sequent $\vdash (\Box p \to \Diamond p)^1$ in calculus $\mathbf{E^D}$:

(1.)	$? (\vdash (\Box p \to \Diamond p)^1)$	$\mathbf{R_\beta}$
(2.)	$? (\vdash (\neg\Box p)^1, (\Diamond p)^1)$	$\mathbf{R_{\pi D}}$
(3.)	$? (\vdash (\neg\Box p)^1, (\neg p)^{1,2}, (\Diamond p)^1)$	$\mathbf{R_\pi}$
(4.)	$? (\vdash (\neg\Box p)^1, (\neg p)^{1,2}, (\Diamond p)^1, (p)^2)$	

Let us observe that the seven modal logics discussed above are characterized by the corresponding erotetic calculi in a modular way. The characteristics of each of these logics is given by the proviso $\mathbf{P^L}$ of applicability of

rule $\mathbf{R_\pi}$ and by rule $\mathbf{R_{\pi D}}$ – if this rule is present in a calculus. There is a clear correspondence between the rules $\mathbf{R_\pi}$ and $\mathbf{R_{\pi D}}$ and the provisos of their applicability on the one hand, and the semantical properties characterizing the modal logics on the other. Our erotetic modular account of modal logics may be also extended in order to cover all of the 15 basic modal logics (cf. footnote 28). All the details may be found in Appendix 3.

Finally, let us present the following definitions of the generalized notions of a Socratic transformation of a question and of a Socratic proof of a sequent:

DEFINITION 2.7: A *Socratic transformation of a question Q via the rules of* $\mathbf{E}^L$ is a sequence $s = Q_1, Q_2, \ldots$ of questions such that $Q_1 = Q$ and for each $n > 1$, question Q_n results from question Q_{n-1} by an application of one of the rules of $\mathbf{E}^L$.

DEFINITION 2.8: Let $\vdash (A)^1$ be an atomic sequent of *M**. A *Socratic proof of* $\vdash (A)^1$ in $\mathbf{E}^L$ is a finite Socratic transformation *s* of the question $?(\vdash (A)^1)$ via the rules of $\mathbf{E}^L$ such that for each constituent φ of the last question of *s* the following holds:
  (a) φ is of the form $\vdash S \,{}^{\backprime} (B)^{\phi(i)} \,{}^{\backprime} T \,{}^{\backprime} (\neg B)^{\gamma(i)} \,{}^{\backprime} U$, or
  (b) φ is of the form $\vdash S \,{}^{\backprime} (\neg B)^{\gamma(i)} \,{}^{\backprime} T \,{}^{\backprime} (B)^{\phi(i)} \,{}^{\backprime} U$.

II.3 Semantical Invertibility of the Rules of $\mathbf{E}^L$. Soundness

In this section *L* stands, again, for any of: **K**, **D**, **T**, **KB**, **K4**, **S4**, **S5**. First, we prove that the rules of $\mathbf{E}^L$ are semantically invertible.

LEMMA 2.2: If question $Q_1 = ?(\Phi_1)$ results from question $Q = ?(\Phi)$ by one of the rules: $\mathbf{R_\alpha}$, $\mathbf{R_\beta}$, $\mathbf{R_{\neg\neg}}$ of $\mathbf{E}^L$, then each term of Φ is *L*-valid iff each term of Φ_1 is *L*-valid.

PROOF: We consider rule $\mathbf{R_\alpha}$ only, since the reasoning is similar in the case of the remaining rules.

If question $Q_1 = ?(\Phi_1)$ results from question $Q = ?(\Phi)$ by rule $\mathbf{R_\alpha}$, then the sequences Φ and Φ_1 are of the forms: $\Psi; \varphi; \Psi_1$ and $\Psi; \psi; \psi_1; \Psi_1$, where:

$$\varphi = \vdash S \,{}^{\backprime} (\alpha)^{\phi(i)} \,{}^{\backprime} T$$

$$\psi = \vdash S \,{}^{\backprime} (\alpha_1)^{\phi(i)} \,{}^{\backprime} T$$

$$\psi_1 = \vdash S \,{}^{\backprime} (\alpha_2)^{\phi(i)} \,{}^{\backprime} T$$

The terms of Ψ and Ψ_1 (if there are any) remain unchanged, therefore it suffices to show that the lack of *L*-validity of sequent φ is tantamount to the lack of *L*-validity of at least one of the sequents: ψ, ψ_1.

Observe that when rule $\mathbf{R_\alpha}$ is applied, no operation is performed on the indices of formulas of sequent φ, which means that the sets $\mathbf{I}_W\{\varphi\}$, $\mathbf{I}_W\{\psi\}$ and

$\mathbf{I}_W\{\psi_1\}$ are equal, and the same holds with respect to the sets $\mathbf{I}_R[\phi]$, $\mathbf{I}_R[\psi]$ and $\mathbf{I}_R[\psi_1]$. Thus an interpretation of any of the sequents: ϕ, ψ or ψ_1 in a frame $<W, R>$ is also an interpretation of any of the other two sequents in the same frame. Hence, in what follows, we are allowed to consider *one* interpretation of sequents ϕ, ψ and ψ_1 in a specified frame.

Suppose that sequent $\phi = \;\vdash S \;' (\alpha)^{\phi(i)} \;' T$ is not L-valid. Then, by Corollary 2.2, for some frame $<W, R>$ (with R having the L-properties), for an interpretation f of sequent ϕ in frame $<W, R>$, and for some valuation V on $<W, R>$, the sequent ϕ is not satisfied in model $<W, R, V>$ under f. Hence $V(A, f(j)) = \mathbf{0}$ for each term $(A)^{\phi(j)}$ of S and T, and also $V(\alpha, f(i)) = \mathbf{0}$. Therefore $V(\alpha_1, f(i)) = \mathbf{0}$ or $V(\alpha_2, f(i)) = \mathbf{0}$. If the first possibility holds, then sequent ψ is not satisfied in model $<W, R, V>$ under interpretation f of ψ in $<W, R>$. But then, by Corollary 2.2 again, ψ is not L-valid. If the second possibility holds then, by the same reasoning, ψ_1 is not L-valid. Hence, if sequent ϕ is not L-valid, then at least one of the sequents: ψ or ψ_1, is not L-valid.

Similar argument establishes that if one of the sequents ψ or ψ_1 is not L-valid, then the sequent ϕ is not L-valid.

For the other rules the reasoning goes analogously (the details of the proof concerning interpretation functions remain unchanged).

∎

Now we shall prove that the modal rules of $\mathbf{E}^L$ warrant transmission of joint L-validity of sequents in both directions.

LEMMA 2.3: If question $Q_1 = ?(\Phi_1)$ results from question $Q = ?(\Phi)$ by rule $\mathbf{R_v}$ of $\mathbf{E}^L$, then each term of Φ is L-valid iff each term of Φ_1 is L-valid.

PROOF: If question $Q_1 = ?(\Phi_1)$ results from question $Q = ?(\Phi)$ by rule $\mathbf{R_v}$ of $\mathbf{E}^L$, then the sequences Φ and Φ_1 are of the forms, respectively: $\Psi; \phi; \Psi_1$ and $\Psi; \psi; \Psi_1$, where:

$$\phi = \;\vdash S \;' (\nu)^{\phi(i)} \;' T$$

$$\psi = \;\vdash S \;' (\nu_0)^{\phi(i),j} \;' T$$

We show that non-L-validity of sequent ψ entails non-L-validity of sequent ϕ and *vice versa*. From the proviso on rule $\mathbf{R_v}$ we have: $j \notin \mathbf{I}_W\{\phi\}$.

If sequent ψ is not L-valid, then, by Corollary 2.2, there are: a frame $<W, R>$, where R has the L-properties, an interpretation f of sequent ψ in $<W, R>$, and a valuation V on $<W, R>$ such that ψ is not satisfied under the interpretation f in the model $<W, R, V>$. In particular, $V(\nu_0, f(j)) = \mathbf{0}$. We shall construct an interpretation g of sequent ϕ in $<W, R>$ such that ϕ is not satisfied in model $<W, R, V>$ under g.

First, observe that the set of numerals $\mathbf{I}_W\{\varphi\}$ is a (proper) subset of the set $\mathbf{I}_W\{\psi\}$, and the same holds with respect to sets $\mathbf{I}_R[\varphi]$ and $\mathbf{I}_R[\psi]$. Let g be a restriction of f to $\mathbf{I}_W\{\varphi\}$. It follows easily that g is an interpretation of φ in $<W, R>$. Indeed, if the set $\mathbf{I}_R[\varphi]$ is empty, then, trivially, g is an interpretation of φ in $<W, R>$ (*cf.* Proposition 2.3). Otherwise, the following holds. If $<k, k'> \in \mathbf{I}_R[\varphi]$, then $<k, k'> \in \mathbf{I}_R[\psi]$, and $g(k) = f(k)$ as well as $g(k') = f(k')$. Since f is an interpretation of ψ in $<W, R>$, it must be the case that $<f(k), f(k')> \in R$, and thus $<g(k), g(k')> \in R$, which establishes that if $<k, k'> \in \mathbf{I}_R[\varphi]$, then $<g(k), g(k')> \in R$.

Second, since numeral i immediately precedes numeral j in the index of the wff $(\nu_0)^{\phi(i),j}$, the ordered pair $<i, j>$ is an element of the set $\mathbf{I}_R[\psi]$ (that is, of the set $\mathbf{I}_R[\vdash S \,'\, (\nu_0)^{\phi(i),j} \,'\, T]$). As f is an interpretation of sequent ψ in $<W, R>$, the world $f(j)$ must be accessible from world $f(i)$. Hence, in view of $V(\nu_0, f(j)) = \mathbf{0}$, it follows that $V(\nu, f(i)) = \mathbf{0}$. Since $i \in \mathbf{I}_W\{\varphi\}$ (that is, i belongs to the domain of g), $g(i) = f(i)$ and hence $V(\nu, g(i)) = \mathbf{0}$. Moreover, for every term $(B)^{\phi(k)}$ of S and T: $V(B, g(k)) = V(B, f(k)) = \mathbf{0}$. Therefore sequent $\varphi = \vdash S \,'\, (\nu)^{\phi(i)} \,'\, T$ is not satisfied in model $<W, R, V>$ under g.

For the other direction we consider, as above, a frame $<W, R>$, an interpretation f of sequent $\varphi = \vdash S \,'\, (\nu)^{\phi(i)} \,'\, T$ in $<W, R>$, and a valuation V on $<W, R>$ such that φ is not satisfied in the model $<W, R, V>$. Since $V(\nu, f(i)) = \mathbf{0}$, there must be a world w in W accessible from $f(i)$ and such that $V(\nu_0, w) = \mathbf{0}$.

Let g be an extension of f over $\mathbf{I}_W\{\psi\}$ such that $g(j) = w$ (recall that j is not an element of $\mathbf{I}_W\{\varphi\}$). In order to establish that g is an interpretation of ψ in $<W, R>$ it is sufficient to observe that for $<i, j> \in \mathbf{I}_R[\psi]$ we have $<f(i), w> \in R$ and $f(i) = g(i)$. For every term $(B)^{\phi(k)}$ of S and T: $V(B, g(k)) = V(B, f(k)) = \mathbf{0}$. Since also $V(\nu_0, g(j)) = \mathbf{0}$, sequent $\psi = \vdash S \,'\, (\nu_0)^{\phi(i),j} \,'\, T$ is not valid in $<W, R>$ under g. ∎

LEMMA 2.4: If question $Q_1 = ?(\Phi_1)$ results from question $Q = ?(\Phi)$ by rule $\mathbf{R}_\pi$ or by rule $\mathbf{R}_{\pi\mathbf{D}}$ of $\mathbf{E}^L$, then each term of Φ is L-valid iff each term of Φ_1 is L-valid.

PROOF: We consider rule $\mathbf{R}_\pi$ first. If question $Q_1 = ?(\Phi_1)$ results from question $Q = ?(\Phi)$ by rule $\mathbf{R}_\pi$ of $\mathbf{E}^L$, then $\Phi = \Psi; \varphi; \Psi_1$, and $\Phi_1 = \Psi; \psi; \Psi_1$, where:

$$\varphi = \vdash S \,'\, (\pi)^{\phi(i)} \,'\, T$$

$$\psi = \vdash S \,'\, (\pi)^{\phi(i)} \,'\, (\pi_0)^j \,'\, T$$

Again, we make use of Corollary 2.2 and prove that there is a model in which sequent φ is not satisfied (under some interpretation) iff there is a model in which sequent ψ is not satisfied (under some interpretation). Let us observe that, as in the case of rules $\mathbf{R}_\alpha$, $\mathbf{R}_\beta$, $\mathbf{R}_{\neg\neg}$, $\mathbf{I}_W\{\varphi\} = \mathbf{I}_W\{\psi\}$ and $\mathbf{I}_R[\varphi] = \mathbf{I}_R[\psi]$, hence each interpretation of one of the sequents in an arbitrary frame is also an interpretation of the other sequent in the same frame.

The reasoning depends on the proviso of applicability of the rule satisfied in a particular case. Suppose that:

(i) $<i, j> \in \mathbf{I}_R[\varphi]$

Assume that for sequent $\psi = \vdash S \,'\, (\pi)^{\phi(i)} \,'\, (\pi_0)^j \,'\, T$ there exist: a frame $<W, R>$, an interpretation f of sequent ψ in this frame, and a valuation V on $<W, R>$ such that: first, for each term $(B)^{\phi(n)}$ of S or T, $V(B, f(n)) = \mathbf{0}$, and, second, $V(\pi, f(i)) = \mathbf{0}$. It follows that the sequent $\varphi = \vdash S \,'\, (\pi)^{\phi(i)} \,'\, T$ is not satisfied in the model $<W, R, V>$ under f.

Suppose that sequent $\varphi = \vdash S \,'\, (\pi)^{\phi(i)} \,'\, T$ is not satisfied in a model $<W, R, V>$ under a certain interpretation f of φ in $<W, R>$. Then for each term $(B)^{\phi(n)}$ of S or T: $V(B, f(n)) = \mathbf{0}$, and also $V(\pi, f(i)) = \mathbf{0}$. By assumption, $<i, j> \in \mathbf{I}_R[\varphi]$ and hence $<f(i), f(j)> \in R$. Thus $V(\pi_0, f(j)) = \mathbf{0}$. Hence sequent ψ is not satisfied in the considered model under f.

(ii) $j = i$

Hence L must be one of: $\mathbf{T}$, $\mathbf{S4}$, $\mathbf{S5}$. For the first implication we apply the argument presented in (i) (although R is reflexive, but this part of the reasoning does not depend on the properties of R). For the second one let us assume that for sequent $\varphi = \vdash S \,'\, (\pi)^{\phi(i)} \,'\, T$ there are: a reflexive frame $<W, R>$, an interpretation f of φ in $<W, R>$, and a valuation V on $<W, R>$ such that sequent φ is not satisfied in $<W, R, V>$ under f. Thus, in particular, $V(\pi, f(i)) = \mathbf{0}$. Since R is reflexive and $j = i$, $<f(i), f(j)> \in R$, hence $V(\pi_0, f(j)) = \mathbf{0}$. Therefore sequent ψ is not satisfied in $<W, R, V>$ under f.

(iii) $<j, i> \in \mathbf{I}_R[\varphi]$

In this case $L = \mathbf{KB}$ or $L = \mathbf{S5}$. For the first implication the reasoning goes on as previously. For the other direction suppose that there are: a symmetric frame $<W, R>$, an interpretation f of sequent φ in $<W, R>$, and a valuation V on $<W, R>$ such that, first, $V(B, f(n)) = \mathbf{0}$ for each term $(B)^{\phi(n)}$ of S and T, and second, $V(\pi, f(i)) = \mathbf{0}$. By assumption, $<j, i> \in \mathbf{I}_R[\varphi]$. Hence, and by symmetry of R, $<f(i), f(j)> \in R$. Therefore formula π_0 is false in $f(j)$ and sequent ψ is not satisfied in $<W, R, V>$ under f.

(iv)   there is a directed $\mathbf{I}_R[\varphi]$-chain $<i_1, \ldots, i_n>$ where $i_1 = i$ and $i_n = j$

Rule $\mathbf{R}_\pi$ may be applied on this proviso only if L is one of: $\mathbf{K4}$, $\mathbf{S4}$, $\mathbf{S5}$. For the first implication the reasoning is as previously. For the second one let us assume that sequent φ is not satisfied in a transitive model $<W, R, V>$ under a certain interpretation f of φ in $<W, R>$. Thus, in particular, $V(\pi, f(i)) = \mathbf{0}$. By assumption, for each k ($1 \leq k < n$): $<i_k, i_{k+1}> \in \mathbf{I}_R[\varphi]$, hence also $<f(i_k), f(i_{k+1})> \in R$. Since R is transitive, $<f(i), f(j)> \in R$. Thus $V(\pi_0, f(j)) = \mathbf{0}$ and therefore sequent ψ is not satisfied in $<W, R, V>$ under f.

(v)   there is an $\mathbf{I}_R[\varphi]$-chain $<i_1, \ldots, i_n>$ where $i_1 = i$ and $i_n = j$

L is **S5**. We consider the second implication only. Let *f* stand for an interpretation of sequent φ in a frame $<W, R>$, where *R* is both symmetric and transitive (reflexivity will play no part in this reasoning). We need to establish, first, that $<f(i), f(j)> \in R$. For this reason we prove by induction that $<f(i_1), f(i_k)> \in R$ for each *k*: $1 < k \leq n$.

By the definition of a chain, either $<i_1, i_2> \in \mathbf{I_R}[\varphi]$ or $<i_2, i_1> \in \mathbf{I_R}[\varphi]$. In the first case $<f(i_1), f(i_2)> \in R$ by assumption (*f* is an interpretation of φ in $<W, R>$). In the second case $<f(i_1), f(i_2)> \in R$ by symmetry of *R*. This finishes the initial step. Suppose that the thesis holds for *k* ($1 < k < n$), that is, $<f(i_1), f(i_k)> \in R$. We also have $<i_k, i_{k+1}> \in \mathbf{I_R}[\varphi]$ or $<i_{k+1}, i_k> \in \mathbf{I_R}[\varphi]$. In the first case $<f(i_k), f(i_{k+1})> \in R$ and thus $<f(i_1), f(i_{k+1})> \in R$ by transitivity of *R*. In the second case $<f(i_{k+1}), f(i_k)> \in R$. Hence $<f(i_k), f(i_{k+1})> \in R$ by symmetry of *R* and $<f(i_1), f(i_{k+1})> \in R$ by its transitivity. It follows that $<f(i_1), f(i_n)> \in R$, that is, $<f(i), f(j)> \in R$.

Now, suppose that φ is not satisfied in a model $<W, R, V>$ (where *R* is symmetric and transitive) under a certain interpretation *f*. Then $V(\pi, f(i)) = \mathbf{0}$. But, as we have established, $<f(i), f(j)> \in R$. Therefore $V(\pi_0, f(j)) = \mathbf{0}$, that is, sequent ψ is not satisfied in this model under *f*.

This finishes the part of the proof concerning rule $\mathbf{R_\pi}$. If question $Q_1 = ? (\Phi; \psi; \Psi)$ results from question $Q = ? (\Phi; \varphi; \Psi)$ by rule $\mathbf{R_{\pi D}}$, then sequents φ and ψ are of the forms:

$$\varphi = \vdash S \, ' \, (\pi)^{\phi(i)} \, ' \, T$$
$$\psi = \vdash S \, ' \, (\pi)^{\phi(i)} \, ' \, (\pi_0)^{i,j} \, ' \, T$$

and $j \notin \mathbf{I_W}\{\varphi\}$. Assume that sequent ψ is not satisfied in a model $<W, R, V>$ under some interpretation *f* of ψ in $<W, R>$. Let *g* stand for a restriction of *f* to $\mathbf{I_W}\{\varphi\}$. The function *g* is an interpretation of φ in $<W, R>$ (by an argument analogous to that presented in the case of $\mathbf{R_v}$). For each term $(B)^{\phi(k)}$ of *S* or *T* we have $V(B, g(k)) = V(B, f(k)) = \mathbf{0}$, and also we have $V(\pi, g(j)) = V(\pi, f(j)) = \mathbf{0}$. Hence sequent $\varphi = \vdash S \, ' \, (\pi)^{\phi(i)} \, ' \, T$ is not satisfied in $<W, R, V>$ under *g*.

Suppose that sequent $\varphi = \vdash S \, ' \, (\pi)^{\phi(i)} \, ' \, T$ is not **D**-valid. Again, for a certain frame $<W, R>$, where *R* is extendable, for some interpretation *f* of φ in $<W, R>$, and for some *V* on $<W, R>$, the sequent φ is not satisfied in $<W, R, V>$ under *f*. In particular, $V(\pi, f(i)) = \mathbf{0}$. As *R* is extendable, there must be $w \in W$ such that $<f(i), w> \in R$. Since *w* is accessible from *f(i)*, $V(\pi_0, w) = \mathbf{0}$. Let *g* be an extension of *f* over $\mathbf{I_W}\{\psi\}$ such that $g(j) = w$. It follows that sequent ψ is not satisfied in $<W, R, V>$ under *g*.

∎

From Lemmas 2.2, 2.3 and 2.4 we get:

THEOREM 2.2 (*semantical invertibility of the rules of* $\mathbf{E}^L$): If question $Q_1 = ?(\Phi_1)$ results from question $Q = ?(\Phi)$ by a rule of $\mathbf{E}^L$, then each term of Φ is *L*-valid iff each term of Φ_1 is *L*-valid.

And finally:

THEOREM 2.3 (*soundness*): If there exists a Socratic proof of a sequent $\vdash (A)^1$ in $\mathbf{E}^L$, then the formula *A* is *L*-valid.

PROOF: Let $s = \langle Q_1, \ldots, Q_n \rangle$ be a Socratic proof of sequent $\vdash (A)^1$ in calculus $\mathbf{E}^L$. By Lemma 2.1, each constituent of the last question Q_n of s is *L*-valid. By Theorem 2.2, if question $Q_{i+1} = ? (\Phi_{i+1})$ results from question $Q_i = ? (\Phi_i)$ (where $1 \leq i < n$) and each term of Φ_{i+1} is *L*-valid, then each term of Φ_i is *L*-valid. Hence (by induction), sequent $\vdash (A)^1$ is *L*-valid. Furthermore, by Theorem 2.1, the formula *A* of language *M* is *L*-valid.

∎

In view of Theorem 2.3 the existence of a Socratic proof of a sequent $\vdash (A)^1$ in $\mathbf{E}^L$ amounts to the affirmative answer to a question about *L*-validity of the formula *A*.

II.4 Modal Erotetic Calculi and Inferential Erotetic Logic

Language *M** is another example of a formalized language enriched with questions. In this section we analyse the erotetic part of *M** and supply it with a semantics. Similarly as in the case of language *L**, the aim of the semantical analysis presented in this section is an explication of the relation between questions that result one from another in a Socratic transformation. The analysis is performed within the framework of **IEL**.

Let us recall that the declarative part of *M** comprises sequents (atomic d-wffs) and compound d-wffs built up from sequents by means of & and / or ng. Within the class of sequents of *M** we have distinguished the class of atomic sequents of this language, that is, expressions of the form $\vdash (A)^1$. By Theorem 2.1 (Section II.1.2), a d-wff of *M** of such a form may be interpreted as an expression representing statement: "The formula *A* of language *M* is *L*-valid." Hence a one-sequent question of the form:

(2.8) $? (\vdash (A)^1)$

is a question about *L*-validity of the formula *A*. Let us emphasize that the statements concerning *L*-validity are made in an object-level language and the questions, as well, are posed at an object-level.

As to non-atomic sequents, a one-sequent question whose constituent is a non-atomic sequent may be interpreted, as we have established, as a question about *L*-validity of this sequent. The general reading of a question of the form:

(2.7)   ? (φ_1, …, φ_n)

is the following: "Is it the case that: φ_1 and … and φ_n?" Hence a many-sequent question of **M*** asks about joint *L*-validity of its constituents.

We define the notions of an affirmative and a negative answer to a question similarly as in the case of language **L***. Namely, if *Q* is a question of **M*** of the form (2.7), then the following d-wff:

(1.5)   (…((φ_1) & (φ_2)) & …) & (φ_n)

is the *affirmative answer* to *Q*, and a d-wff of the form:

(1.6)   ng((…((φ_1) & (φ_2)) & …) & (φ_n))

is the *negative answer* to *Q*. Hence questions of **M*** are simple yes-no questions.

We use letters *X, Y* for d-wffs of **M***. Let us denote by $\mathbf{D}_{M*}$ the set of all the d-wffs of **M***. The notion of a *partition of* $\mathbf{D}_{M*}$ (*partition of language M**) is defined as previously (*cf.* Section I.4 of Chapter I). The notion of an admissible partition of **M*** must be adjusted to the underlying semantics of **M**. Since we deal with more than one logic expressed in one language **M**, the notion must be relativized to *L*. *E.g.*, a **K**-admissible partition is defined by the following:

DEFINITION 2.9: A partition $\boldsymbol{P} = <\boldsymbol{T_P}, \boldsymbol{U_P}>$ of language **M*** is **K**-*admissible* iff the following conditions are satisfied:

(i)   ' $\vdash S$ ' $(\beta)^{\phi(i)}$ ' T ' $\in \boldsymbol{T_P}$ iff ' $\vdash S$ ' $(\beta_1)^{\phi(i)}$ ' $(\beta_2)^{\phi(i)}$ ' T ' $\in \boldsymbol{T_P}$;

(ii)   ' $\vdash S$ ' $(\alpha)^{\phi(i)}$ ' T ' $\in \boldsymbol{T_P}$ iff ' $\vdash S$ ' $(\alpha_1)^{\phi(i)}$ ' T ' $\in \boldsymbol{T_P}$ and ' $\vdash S$ ' $(\alpha_2)^{\phi(i)}$ ' T ' $\in \boldsymbol{T_P}$;

(iii)   ' $\vdash S$ ' $(\neg\neg A)^{\phi(i)}$ ' T ' $\in \boldsymbol{T_P}$ iff ' $\vdash S$ ' $(A)^{\phi(i)}$ ' T ' $\in \boldsymbol{T_P}$;

(iv)   ' $\vdash S$ ' $(\nu)^{\phi(i)}$ ' T ' $\in \boldsymbol{T_P}$ iff for $j = \max(\mathbf{I}_W\{\vdash S$ ' $(\nu)^{\phi(i)}$ ' $T\}) + 1$, ' $\vdash S$ ' $(\nu_0)^{\phi(i),j}$ ' T ' $\in \boldsymbol{T_P}$;

(v.i)   If ' $\vdash S$ ' $(\pi)^{\phi(i)}$ ' T ' $\in \boldsymbol{T_P}$, then for each numeral j such that $<i, j> \in \mathbf{I}_R[\vdash S$ ' $(\pi)^{\phi(i)}$ ' $T]$: ' $\vdash S$ ' $(\pi)^{\phi(i)}$ ' $(\pi_0)^{j}$ ' T ' $\in \boldsymbol{T_P}$;

(v.ii) If there is a numeral j such that $<i, j> \in \mathbf{I}_R[\vdash S$ ' $(\pi)^{\phi(i)}$ ' $T]$ and such that ' $\vdash S$ ' $(\pi)^{\phi(i)}$ ' $(\pi_0)^{j}$ ' T ' $\in \boldsymbol{T_P}$, then ' $\vdash S$ ' $(\pi)^{\phi(i)}$ ' T ' $\in \boldsymbol{T_P}$;

(vi) $X \in \boldsymbol{T_P}$ iff 'ng(X)' $\notin \boldsymbol{T_P}$;

(vii)   '(X) & (Y)' $\in \boldsymbol{T_P}$ iff $X \in \boldsymbol{T_P}$ and $Y \in \boldsymbol{T_P}$.

For simplicity, we have assumed that when rule $\mathbf{R_\nu}$ is applied, then the new numeral j introduced into the resulting sequent is obtained by adding 1 to the greatest numeral occurring in the "old" sequent. Hence the form of clause (iv) of the above definition.

In the definitions of L-admissible partitions for the extensions of **K**, we will make use of the clauses listed below.

(D) $'\vdash S\,'(\pi)^{\phi(i)}\,'\,T'\,\in\,\boldsymbol{T_P}$ iff for $j\,=\,\max(\mathbf{I}_W\{\vdash S\,'(\pi)^{\phi(i)}\,'\,T\})\,+\,1$, $'\vdash S\,'(\pi)^{\phi(i)}\,'(\pi_0)^{i,j}\,'\,T'\,\in\,\boldsymbol{T_P}$

(T) $'\vdash S\,'(\pi)^{\phi(i)}\,'\,T'\,\in\,\boldsymbol{T_P}$ iff $'\vdash S\,'(\pi)^{\phi(i)}\,'(\pi_0)^{i}\,'\,T'\,\in\,\boldsymbol{T_P}$

(KB.1)   If $'\vdash S\,'(\pi)^{\phi(i)}\,'\,T'\,\in\,\boldsymbol{T_P}$, then for each numeral j such that $<j,\,i>\,\in\,\mathbf{I}_R[\vdash S\,'(\pi)^{\phi(i)}\,'\,T]$: $'\vdash S\,'(\pi)^{\phi(i)}\,'(\pi)^{j}\,'\,T'\,\in\,\boldsymbol{T_P}$.

(KB.2)   If there is a numeral j such that $<j,\,i>\,\in\,\mathbf{I}_R[\vdash S\,'(\pi)^{\phi(i)}\,'\,T]$ and such that $'\vdash S\,'(\pi)^{\phi(i)}\,'(\pi)^{j}\,'\,T'\,\in\,\boldsymbol{T_P}$, then $'\vdash S\,'(\pi)^{\phi(i)}\,'\,T'\,\in\,\boldsymbol{T_P}$.

(K4.1)   If $'\vdash S\,'(\pi)^{\phi(i)}\,'\,T'\,\in\,\boldsymbol{T_P}$, then for each numeral j such that there is a directed $\mathbf{I}_R[\vdash S\,'(\pi)^{\phi(i)}\,'\,T]$-chain whose first term is i and whose last term is j: $'\vdash S\,'(\pi)^{\phi(i)}\,'(\pi_0)^{j}\,'\,T'\,\in\,\boldsymbol{T_P}$.

(K4.2)   If there is a numeral j such that there is a directed $\mathbf{I}_R[\vdash S\,'(\pi)^{\phi(i)}\,'\,T]$-chain whose first term is i and whose last term is j and such that $'\vdash S\,'(\pi)^{\phi(i)}\,'(\pi_0)^{j}\,'\,T'\,\in\,\boldsymbol{T_P}$, then $'\vdash S\,'(\pi)^{\phi(i)}\,'\,T'\,\in\,\boldsymbol{T_P}$.

(KB4.1)   If $'\vdash S\,'(\pi)^{\phi(i)}\,'\,T'\,\in\,\boldsymbol{T_P}$, then for each numeral j such that there is an $\mathbf{I}_R[\vdash S\,'(\pi)^{\phi(i)}\,'\,T]$-chain whose first term is i and whose last term is j: $'\vdash S\,'(\pi)^{\phi(i)}\,'(\pi_0)^{j}\,'\,T'\,\in\,\boldsymbol{T_P}$.

(KB4.2)   If there is a numeral j such that there is an $\mathbf{I}_R[\vdash S\,'(\pi)^{\phi(i)}\,'\,T]$-chain whose first term is i and whose last term is j and such that $'\vdash S\,'(\pi)^{\phi(i)}\,'(\pi_0)^{j}\,'\,T'\,\in\,\boldsymbol{T_P}$, then $'\vdash S\,'(\pi)^{\phi(i)}\,'\,T'\,\in\,\boldsymbol{T_P}$.

DEFINITION 2.10: Let $\boldsymbol{P}\,=\,<\boldsymbol{T_P},\,\boldsymbol{U_P}>$ stand for a partition of language $\boldsymbol{M^*}$. We say that $\boldsymbol{P}$ is L-*admissible* iff $\boldsymbol{P}$ is **K**-admissible and the following holds:

- if $L\,=\,\mathbf{D}$ (**T**), then $\boldsymbol{P}$ satisfies also clause (D) (clause (T));

- if $L\,=\,\mathbf{KB}$ (**K4**), then $\boldsymbol{P}$ satisfies also clauses (KB.1) and (KB.2) (clauses (K4.1) and (K4.2));

- if $L\,=\,\mathbf{S4}$, then $\boldsymbol{P}$ satisfies clauses (T), (K4.1) and (K4.2);

- if $L\,=\,\mathbf{S5}$, then $\boldsymbol{P}$ satisfies clauses (T), (KB4.1) and (KB4.2).

By means of the notion of an L-admissible partition we may define the notion of entailment in $\boldsymbol{M^*}$. Again, the notion must be relativized to L. Namely, we say that a d-wff X of $\boldsymbol{M^*}$ L-*entails* a d-wff Y of $\boldsymbol{M^*}$ iff for each L-admissible partition $\boldsymbol{P}\,=\,<\boldsymbol{T_P},\,\boldsymbol{U_P}>$ of language $\boldsymbol{M^*}$, if $X\,\in\,\boldsymbol{T_P}$, then $Y\,\in\,\boldsymbol{T_P}$.

As in the case of $\boldsymbol{L^*}$, the central notion pertaining to questions is that of positive equipollence of questions. This must be relativized to L. Namely, we will say that question Q is *positively equipollent to* question Q^* *under the L-admissible*

partitions of M* iff the affirmative answers to Q and Q^* L-entail each other and the negative answers to Q and Q^* L-entail each other. The following theorem holds:

THEOREM 2.4: If question Q^* results from question Q by a rule of calculus $\mathbf{E}^L$, then Q^* is positively equipollent to Q under L-admissible partitions of *M**.

PROOF: The proof is by cases. As an example we analyse the cases of the modal rules $\mathbf{R_v}$ and $\mathbf{R_\pi}$.

      If question Q^* results from question Q by rule $\mathbf{R_v}$ of $\mathbf{E}^L$, then Q is of the form:

     (a) $? (\varphi_1; \dots; \varphi_{n-1}; \varphi; \varphi_{n+1}; \dots; \varphi_{n+m})$

where φ is a sequent of the form $\vdash S \, ' \, (v)^{\phi(i)} \, ' \, T$; whereas Q^* is of the form:

     (b) $? (\varphi_1; \dots; \varphi_{n-1}; \psi; \varphi_{n+1}; \dots; \varphi_{n+m})$

where sequent ψ has the form $\vdash S \, ' \, (v_0)^{\phi(i),j} \, ' \, T$. The affirmative answer to question Q is of the following form (for transparency, we omit the parentheses around sequents):

     (c) $(\dots((((\dots(\varphi_1 \, \& \, \varphi_2) \, \& \, \dots) \, \& \, \varphi_{n-1}) \, \& \, \varphi) \, \& \, \varphi_{n+1}) \, \& \, \dots) \, \& \, \varphi_{n+m}$

and the affirmative answer to question Q^* is of the form:

     (d) $(\dots((((\dots(\varphi_1 \, \& \, \varphi_2) \, \& \, \dots) \, \& \, \varphi_{n-1}) \, \& \, \psi) \, \& \, \varphi_{n+1}) \, \& \, \dots) \, \& \, \varphi_{n+m}$

Let $\mathbf{P} = \langle T_\mathbf{P}, U_\mathbf{P} \rangle$ stand for an arbitrary L-admissible partition of language *M** and suppose that d-wff (c) is an element of $T_\mathbf{P}$. By Definition 2.10, $\mathbf{P}$ is a $\mathbf{K}$-admissible partition. Hence, and by clause (vii) of Definition 2.9, each of the sequents: $\varphi_1, \dots, \varphi_{n-1}, \varphi, \varphi_{n+1}, \dots, \varphi_{n+m}$ is an element of $T_\mathbf{P}$, in particular, $\varphi \in T_\mathbf{P}$. By the proviso of applicability of rule $\mathbf{R_v}$, $j = \max(\mathbf{I}_W\{\vdash S \, ' \, (v)^{\phi(i)} \, ' \, T\}) + 1$. Thus, by clause (iv) of Definition 2.9, also $\psi \in T_\mathbf{P}$. Since each of the sequents: $\varphi_1, \dots, \varphi_{n-1}, \psi, \varphi_{n+1}, \dots, \varphi_{n+m}$ is an element of $T_\mathbf{P}$, the d-wff (d) belongs to $T_\mathbf{P}$ by clause (vii) of Definition 2.9.

     For the converse direction suppose that $\psi \in T_\mathbf{P}$. Since $j = \max(\mathbf{I}_W\{\vdash S \, ' \, (v)^{\phi(i)} \, ' \, T\}) + 1$, sequent φ belongs to $T_\mathbf{P}$ by clause (iv) of Definition 2.9. Hence it follows easily that if d-wff (d) is an element of $T_\mathbf{P}$, then d-wff (c) is an element of $T_\mathbf{P}$. Therefore the affirmative answers to Q and Q^* L-entail each other.

     As to the negative answers we make use of clause (vi) of Definition 2.9. Namely, if the negative answer to question Q is an element of $T_\mathbf{P}$, where $\langle T_\mathbf{P}, U_\mathbf{P} \rangle$ is an arbitrary L-admissible partition of *M**, then, by clause (vi) of Definition 2.9, the d-wff (c) is not an element of $T_\mathbf{P}$. Hence, by clause (vii) of Definition 2.9, at least one of the sequents: $\varphi_1, \dots, \varphi_{n-1}, \varphi, \varphi_{n+1}, \dots, \varphi_{n+m}$ does not belong to $T_\mathbf{P}$. If this pertains to one of the sequents: $\varphi_1, \dots, \varphi_{n-1}, \varphi_{n+1}, \dots, \varphi_{n+m}$, then, by clauses (vii) and (vi) of Definition 2.9, the negative answer to question Q^* belongs to $T_\mathbf{P}$.

If $\varphi \notin \textbf{\textit{T}}_\textbf{P}$, then, by clause (iv) of Definition 2.9, $\psi \notin \textbf{\textit{T}}_\textbf{P}$. Thus in this case the negative answer to Q^* is an element of $\textbf{\textit{T}}_\textbf{P}$ by clauses (vii) and (vi) of Definition 2.9. For the converse direction the reasoning is analogous.

Suppose that question Q^* results from question Q by rule $\textbf{R}_\pi$ of $\textbf{E}^L$. In this case questions Q and Q^* are of the forms (a) and (b) indicated above, but φ is a sequent of the form: $\vdash S \mathbin{'} (\pi)^{\phi(i)} \mathbin{'} T$, and ψ is a sequent of the form: $\vdash S \mathbin{'} (\pi)^{\phi(i)} \mathbin{'} (\pi_0)^j \mathbin{'} T$. The affirmative answers to questions Q and Q^* are of the forms (c) and (d), respectively. Obviously, the reasoning depends on the logic L. We analyse the case of logic $\textbf{K}$ only, since the reasoning is similar in the remaining cases.

Let $\textbf{\textit{P}} = <\textbf{\textit{T}}_\textbf{P}, \textbf{\textit{U}}_\textbf{P}>$ stand for an arbitrary $\textbf{K}$-admissible partition of language M^* and suppose that d-wff (c) is an element of $\textbf{\textit{T}}_\textbf{P}$. As in the case of rule $\textbf{R}_\textbf{v}$, we arrive at the conclusion that $\varphi \in \textbf{\textit{T}}_\textbf{P}$. By the proviso $\textbf{P}^\textbf{K}$ of applicability of rule $\textbf{R}_\pi$ in $\textbf{E}^\textbf{K}$, $<i, j> \in \textbf{I}_R[\varphi]$. Thus, by clause (v.i) of Definition 2.9, also $\psi \in \textbf{\textit{T}}_\textbf{P}$. By clause (vii) of Definition 2.9, the d-wff (d) belongs to $\textbf{\textit{T}}_\textbf{P}$ (as previously).

Suppose that $\psi \in \textbf{\textit{T}}_\textbf{P}$. Again, by the proviso of applicability of $\textbf{R}_\pi$ in $\textbf{E}^\textbf{K}$, $<i, j> \in \textbf{I}_R[\varphi]$. Now we make use of clause (v.ii) of Definition 2.9. Since there is a numeral j such that $<i, j> \in \textbf{I}_R[\varphi]$ and such that '$\vdash S \mathbin{'} (\pi)^{\phi(i)} \mathbin{'} (\pi_0)^j \mathbin{'} T$' $\in \textbf{\textit{T}}_\textbf{P}$, it follows, by clause (v.ii), that $\varphi \in \textbf{\textit{T}}_\textbf{P}$. Therefore the affirmative answers to questions Q and Q^* $\textbf{K}$-entail each other, as required. We proceed to the negative answers.

As above, we assume that the negative answer to question Q is an element of $\textbf{\textit{T}}_\textbf{P}$ (where $<\textbf{\textit{T}}_\textbf{P}, \textbf{\textit{U}}_\textbf{P}>$ is a $\textbf{K}$-admissible partition of M^*), we make use of clauses (vi) and (vii) of Definition 2.9, and we consider the case when $\varphi \notin \textbf{\textit{T}}_\textbf{P}$. Again, by the proviso of applicability of rule $\textbf{R}_\pi$ in $\textbf{E}^\textbf{K}$, $<i, j> \in \textbf{I}_R[\varphi]$. Now we make use of the transposition of clause (v.ii) of Definition 2.9. Since $<i, j> \in \textbf{I}_R[\varphi]$, it follows that $\psi \notin \textbf{\textit{T}}_\textbf{P}$. For the converse direction the reasoning is analogous, but we make use of the transposition of clause (v.i) of Definition 2.9. The negative answers to questions Q and Q^* also $\textbf{K}$-entail each other, as required.

Finally, since the affirmative answers to questions Q and Q^* $\textbf{K}$-entail each other, and the negative answers to questions Q and Q^* $\textbf{K}$-entail each other, question Q^* is positively equipollent to question Q under $\textbf{K}$-admissible partitions of M^*.

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Each step of a Socratic transformation via the rules of $\textbf{E}^L$ may be considered as an example of an erotetic inference, that is, an inference with questions playing the roles of a premise and a conclusion. It follows from Theorem 2.4 that such questions are positively equipollent under L-admissible partitions of M^*. Since positive equipollence of questions is a special case of pure erotetic implication, we may say that the relation of pure erotetic implication holds between questions Q and Q^* whenever Q^* results from Q by a rule of $\textbf{E}^L$. Therefore each transition from a question to a question governed by a rule of $\textbf{E}^L$ is a valid erotetic inference in the sense of $\textbf{IEL}$.

CHAPTER III: Completeness of Modal Erotetic Calculi

In this chapter we prove completeness of the modal erotetic calculi presented in the previous part of this work. We follow Priest's idea of proving completeness by constructing a countermodel from a complete branch of an "unsuccessful" tableau (*i.e.*, tableau which is not a proof of a relevant formula).

We proceed as follows. We assume that there is no Socratic proof of a sequent $\vdash (A)^1$ in $\mathbf{E}^L$. We consider a certain Socratic transformation of question $? (\vdash (A)^1)$ via the rules of $\mathbf{E}^L$, which we call complete, and we prove that there is a path of this transformation (a path of a Socratic transformation is, in a sense, a counterpart of a branch of a tableau) that may be used in the construction of a countermodel for formula A.

However, since Socratic transformations are not defined as trees, we apply here certain non-standard methods. First of all, we follow the idea of "permanently unsuccessful sequent" that has been developed by Wiśniewski and Shangin, and used in the completeness proof of the erotetic calculus $\mathbf{E}^{PQ}$ pertaining to classical first-order logic.

We apply all the notational conventions introduced in Chapter I and Chapter II concerning the use of various types of metavariables.

III.1 Paths of Socratic Transformations

Let $Q = ? (\Phi)$ be a question of $\boldsymbol{M^*}$. By a *k-th constituent of* question Q we mean the *k*-th term of sequence Φ. We introduce the notion of a path of a Socratic transformation.[36]

DEFINITION 3.1: Let *s* be a Socratic transformation of a question of the form $? (\vdash (A)^1)$ via the rules of $\mathbf{E}^L$. By a *path of Socratic transformation s* we mean any sequence of sequents $\mathbf{p} = <\mathbf{p}_1, \mathbf{p}_2, \dots>$ such that:

(1) $\mathbf{p}_1 = \vdash (A)^1$;

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[36] The notion of a path of a Socratic transformation has been introduced in [Wiśniewski, Shangin:2006].

(2)   if question Q_{i+1} results from question Q_i ($i \geq 1$) by one of the rules: $\mathbf{R_\beta}$, $\mathbf{R_{\neg\neg}}$, $\mathbf{R_v}$, $\mathbf{R_\pi}$, $\mathbf{R_{\pi D}}$, and i-th term $\mathbf{p}_i$ of path $\mathbf{p}$ is k-th constituent of question Q_i, then $(i+1)$-st term $\mathbf{p}_{i+1}$ of $\mathbf{p}$ is k-th constituent of question Q_{i+1};

(3)   if question Q_{i+1} results from question Q_i ($i \geq 1$) by rule $\mathbf{R_\alpha}$, that is, the questions Q_i, Q_{i+1} are of the forms:

$$Q_i = ? \, (\Phi; \varphi; \Psi)$$

$$Q_{i+1} = ? \, (\Phi; \psi; \psi^*; \Psi)$$

then:

(3.1)   if i-th term $\mathbf{p}_i$ of path $\mathbf{p}$ is φ, then either $\mathbf{p}_{i+1}$ is ψ or $\mathbf{p}_{i+1}$ is ψ^*;

(3.2)   if i-th term $\mathbf{p}_i$ of path $\mathbf{p}$ is k-th term of sequence Φ (of sequence Ψ), then $(i+1)$-st term $\mathbf{p}_{i+1}$ of path $\mathbf{p}$ is k-th term of sequence Φ (of sequence Ψ).

Let us now present an example.

EXAMPLE 3.1: The following is a Socratic transformation of question $? \, (\vdash (\Box p \to (q \land (\Box p \land q)))^1)$ via the rules of $\mathbf{E^K}$:

1. $? \, (\vdash (\Box p \to (q \land (\Box p \land q)))^1)$ $\mathbf{R_\beta}$
2. $? \, (\vdash (\neg\Box p)^1, (q \land (\Box p \land q))^1)$ $\mathbf{R_\alpha}$
3. $? \, (\vdash (\neg\Box p)^1, (q)^1 \, ; \, \vdash (\neg\Box p)^1, (\Box p \land q))^1)$ $\mathbf{R_\alpha}$
4. $? \, (\vdash (\neg\Box p)^1, (q)^1 \, ; \, \vdash (\neg\Box p)^1, (\Box p)^1 \, ; \, \vdash (\neg\Box p)^1, (q)^1)$ $\mathbf{R_v}$
5. $? \, (\vdash (\neg\Box p)^1, (q)^1 \, ; \, \vdash (\neg\Box p)^1, (p)^{1,2} \, ; \, \vdash (\neg\Box p)^1, (q)^1)$ $\mathbf{R_\pi}$
6. $? \, (\vdash (\neg\Box p)^1, (q)^1 \, ; \, \vdash (\neg\Box p)^1, (\neg p)^2, (p)^{1,2} \, ; \, \vdash (\neg\Box p)^1, (q)^1)$

The transformation has three paths. For transparency, we write their terms vertically:

Path 1	Path 2	Path 3
$\vdash (\Box p \to (q \land (\Box p \land q)))^1$	$\vdash (\Box p \to (q \land (\Box p \land q)))^1$	$\vdash (\Box p \to (q \land (\Box p \land q)))^1$
$\vdash (\neg\Box p)^1, (q \land (\Box p \land q))^1$	$\vdash (\neg\Box p)^1, (q \land (\Box p \land q))^1$	$\vdash (\neg\Box p)^1, (q \land (\Box p \land q))^1$
$\vdash (\neg\Box p)^1, (q)^1$	$\vdash (\neg\Box p)^1, (\Box p \land q)^1$	$\vdash (\neg\Box p)^1, (\Box p \land q)^1$
$\vdash (\neg\Box p)^1, (q)^1$	$\vdash (\neg\Box p)^1, (\Box p)^1$	$\vdash (\neg\Box p)^1, (q)^1$
$\vdash (\neg\Box p)^1, (q)^1$	$\vdash (\neg\Box p)^1, (p)^{1,2}$	$\vdash (\neg\Box p)^1, (q)^1$
$\vdash (\neg\Box p)^1, (q)^1$	$\vdash (\neg\Box p)^1, (\neg p)^2, (p)^{1,2}$	$\vdash (\neg\Box p)^1, (q)^1$

Let us observe that we have defined the notion of a path only for Socratic transformations of one-sequent questions of the form $? \, (\vdash (A)^1)$. Let $s = \langle Q_1, Q_2,$

…> be a Socratic transformation of a question of such a form. Let Q_n be n-th ($1 \leq n$) question of this transformation and let φ_k be k-th ($1 \leq k$) constituent of question Q_n (both n and k are arbitrary). It is clear that the following two propositions are true:

PROPOSITION 3.1: There is a path of s whose n-th term is the k-th constituent φ_k of question Q_n.

PROPOSITION 3.2: If **p** and **p*** are paths of s such that the n-th term of **p** is the k-th constituent φ_k of Q_n, and the n-th term of **p*** is the k-th constituent φ_k of Q_n, then the paths **p** and **p*** do not differ with respect to the terms preceding φ_k.

Both propositions may be easily proved by induction with respect to n. Proposition 3.1 states that each sequent that occurs in a Socratic transformation of a question of the required form is a term of at least one path of this transformation. Hence sequents of a Socratic transformation may be ordered in two ways: an occurrence of a sequent is, both, a term of a path and a constituent of a question. We may also observe that k-th term of a path of a Socratic transformation must be a constituent of the k-th question of this transformation.

Let Q and Q^* represent questions of the forms, respectively, ? (Φ; φ; Ψ) and ? (Φ; Φ^*; Ψ), where Φ^* is a one-term or a two-term sequence of sequents. If question Q^* results from question Q by a rule **r** of $\mathbf{E}^L$, then we say that rule **r** has been applied to question Q *with respect to sequent* φ. Let φ^* be a term of sequence Φ^*. If sequent φ is of the form $\vdash S \, ' \, (A)^{\phi(i)} \, ' \, T$ and sequent φ^* is of the form $\vdash S \, ' \, U \, ' \, T$ (where U is a one-term or a two-term sequence of indexed formulas), then we say that rule **r** has been applied to question Q with respect to sequent φ and *with regard to indexed formula* $(A)^{\phi(i)}$. If **r** is $\mathbf{R_\pi}$, $(A)^{\phi(i)}$ is an indexed π-formula and U is of the form: $(A)^{\phi(i)} \, ' \, (\pi_0)^j$ (where π_0 is the component of A), then we will be more specific and we will say that rule $\mathbf{R_\pi}$ has been applied to question Q with respect to sequent φ and with regard to indexed formula $(A)^{\phi(i)}$ *and the pair* $<i, j>$. We will use the notion of *applicability* of a rule with respect to a sequent and with regard to a wff (and a pair) in a similar way.

To be honest, what we have said in the previous paragraph is not sufficiently precise. What we really mean by saying that a rule has been applied with respect to a *sequent* is that the rule has been applied with respect to a particular *occurrence of this sequent*. Similarly, when we say that a rule has been applied with respect to a sequent and with regard to a *wff*, then we have in mind a particular *occurrence of this wff*. However, it is more convenient to speak about sequents or wffs instead of their occurrences, and in most contexts the risk of a misunderstanding is quite unlikely. If such a misunderstanding is possible, we carefully distinguish between a sequent (wff) and its occurrence.

III.2 Permanently Unsuccessful Sequents

If there is a Socratic proof of a sequent in $\mathbf{E}^L$, then we will say that this sequent is *provable* in $\mathbf{E}^L$. Let us introduce the following notions:

DEFINITION 3.2: A sequent φ is *successful* iff φ is of one of the forms specified in Definition 2.10 of a Socratic proof:

(a) $\vdash S \, ' \, (B)^{\phi(i)} \, ' \, T \, ' \, (\neg B)^{\gamma(i)} \, ' \, U$, or

(b) $\vdash S \, ' \, (\neg B)^{\gamma(i)} \, ' \, T \, ' \, (B)^{\phi(i)} \, ' \, U$

A sequent is *unsuccessful* iff it is not successful.[37]

DEFINITION 3.3: Let $s = \langle Q_1, \ldots, Q_n \rangle$ (where $n \geq 1$) be a finite Socratic transformation of a question Q via the rules of $\mathbf{E}^L$. By an *extension of s* we mean any finite Socratic transformation $s^* = \langle Q^*_1, \ldots, Q^*_m \rangle$ of question Q^* via the rules of $\mathbf{E}^L$ such that $m > n$, and $Q_k = Q^*_k$ for $k = 1, \ldots, n$.

Obviously, if s^* is an extension of a Socratic transformation s of a question Q via the rules of $\mathbf{E}^L$, then s^* is also a Socratic transformation of question Q via the rules of $\mathbf{E}^L$.

We introduce the notion of a permanently unsuccessful sequent.[38] Roughly speaking, a constituent of a question of a Socratic transformation is permanently unsuccessful in this question if it is unsuccessful and, moreover, one can not obtain a Socratic transformation in which this sequent is "replaced" by a sequence of successful sequents. More formally:

DEFINITION 3.4: Let s be a Socratic transformation of a question Q via the rules of $\mathbf{E}^L$. Let $Q_n = ? (\Phi; \varphi; \Psi)$, where Φ and Ψ may be empty, be n-th ($1 \leq n$) question of s (n is arbitrary). Let $s\#$ stand for the Socratic transformation $\langle Q_1, \ldots, Q_n \rangle$. We say that sequent φ is *permanently unsuccessful in Q_n of s* iff φ is unsuccessful and there is no extension $s^* = \langle Q_1, \ldots, Q_m \rangle$ of $s\#$ such that Q_m is of the form: $? (\Phi; \Gamma; \Psi)$ where Γ is a sequence of successful sequents.

The following lemma states that if sequent $\vdash (A)^1$ is not provable in $\mathbf{E}^L$, then each Socratic transformation (either finite or infinite) of question $? (\vdash (A)^1)$ via the rules of $\mathbf{E}^L$ contains a path whose each term is a sequent permanently unsuccessful in a relevant question of the transformation. (Let us remind that n-th term of a path of a Socratic transformation is a constituent of the n-th question of this transformation.)

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[37] The notion of a successful / unsuccessful sequent comes from [Wiśniewski, Shangin:2006].
[38] We have borrowed this idea from [Wiśniewski, Shangin:2006].

LEMMA 3.1: Let s be a Socratic transformation of question ? ($\vdash (A)^1$) via the rules of $\mathbf{E}^L$. If the sequent $\vdash (A)^1$ is not provable in $\mathbf{E}^L$, then there is a path $\mathbf{p}$ of s such that for each term φ of $\mathbf{p}$: if φ is n-th term of path $\mathbf{p}$, then φ is a sequent permanently unsuccessful in the n-th question of s.

PROOF: Let s be an arbitrary Socratic transformation of question ? ($\vdash (A)^1$) via the rules of $\mathbf{E}^L$. We indicate the first term of a path with the desired property, and show how to determine $(k+1)$-st term of the path if the k-th term is already determined. We do it, however, in a non-effective way.

Let us start with the following observation. If sequent $\vdash (A)^1$ is not provable in $\mathbf{E}^L$, then this sequent is permanently unsuccessful in the first question ? ($\vdash (A)^1$) of transformation s. For suppose it is not permanently unsuccessful in this question. Obviously, sequent $\vdash (A)^1$ is not successful, so by the definition of a permanently unsuccessful sequent, there is an extension $s^* = <Q, ..., ? (\Gamma)>$ of the one-term Socratic transformation $<? (\vdash (A)^1)>$ such that each term of Γ is a successful sequent. But s^* is a Socratic proof of sequent $\vdash (A)^1$, contrary to assumption. Therefore the first term of each path of s is a sequent permanently unsuccessful in the first question of s. If s is a one-term sequence $<? (\vdash (A)^1)>$, then the only path of s has the desired property.

Suppose that s has more than one question. The second question of s is either of the form ? (φ) or of the form ? (φ; φ^*). If the second question of s is of the form ? (φ), then the sequent φ is permanently unsuccessful in the second question of s, for if it was not, then the sequent $\vdash (A)^1$ would not have been permanently unsuccessful in the first question of s. Similarly, if the second question of s is of the form ? (φ; φ^*), then at least one of the sequents: φ and φ^* is permanently unsuccessful in this question. For suppose that it is not the case. Let $s\#$ stand for the two-term Socratic transformation: $<? (\vdash (A)^1), ? (\varphi; \varphi^*)>$. By the definition of a permanently unsuccessful sequent, the following holds:

(1) if φ is not permanently unsuccessful in the second question of s, then there is an extension $s^* = <Q_1, Q_2, ..., Q_n>$ of $s\#$ such that Q_n is of the form: ? (Γ; φ^*), where Γ is a sequence of successful sequents; and

(2) if φ^* is not permanently unsuccessful in the second question of s, then there is an extension $s^{**} = <Q_1, Q_2, ..., Q_m>$ of $s\#$ such that Q_m is of the form: ? (φ; Γ^*), where Γ^* is a sequence of successful sequents.

Now we may "glue" the two transformations s^* and s^{**} together in order to obtain a Socratic proof of sequent $\vdash (A)^1$. We take the Socratic transformation $s^* = <Q_1, Q_2, ..., Q_n>$ as the starting point. Question Q_n is of the form: ? (Γ; φ^*), where Γ is a sequence of successful sequents. Let us observe that the questions $Q_3, ..., Q_m$ of Socratic transformation s^{**} are of the forms: ? (φ; Φ_1), ..., ? (φ; Φ_{m-2}), where Φ_{m-2} is Γ^*. Now we extend Socratic transformation s^* in the following way. We take ? (Γ; Φ_1) as the $(n+1)$-st question of the extension of s^*. Obviously, if question ? (φ; Φ_1) results from question ? (φ; φ^*) by a rule of $\mathbf{E}^L$,

then question ? (Γ; Φ_1) results from question ? (Γ; φ*) by the same rule of $\mathbf{E}^L$. We take question ? (Γ; Φ_2) as the (n+2)-nd question of the extension of s*, then ? (Γ; Φ_3) as the (n+3)-rd question, *etc*.; we proceed until we arrive at the (n+m-2)-nd question, which will be of the form ? (Γ; Φ_{m-2}), where Φ_{m-2} is Γ*. Each term of this sequence results from the previous one by a rule of $\mathbf{E}^L$. Observe that each constituent of question ? (Γ; Γ*) is a successful sequent. It follows, by Definition 3.4, that sequent $\vdash$ (A)^1 is not permanently unsuccessful in question ? ($\vdash$ (A)^1) of s#. (Moreover, it follows that this sequent is provable in $\mathbf{E}^L$.) We arrive at a contradiction. Hence at least one of the sequents: φ and φ* is permanently unsuccessful in the second question of Socratic transformation s.

We go back to the path **p** with permanently unsuccessful sequents. If the second question of Socratic transformation s is of the form ? (φ), then sequent φ is the second term of path **p**. If the second question of this transformation is of the form ? (φ; φ*), then the second term of path **p** is a sequent which is permanently unsuccessful in this question. We do not know which of the sequents satisfies this condition but we know that at least one does. If both sequents are permanently unsuccessful in this question, then we choose the leftmost one, that is, sequent φ.

We proceed in the way described above. Generally, if k-th term (for an arbitrary $k \geq 1$) of path **p** is determined, then the (k+1)-st term of **p** is determined in the following way (let us remind once again that k-th term of a path is a constituent of the k-th question):

Given that question Q_k is of the form: ? (Φ; φ; Ψ), where φ is the k-th term of path **p** (sequences Φ and Ψ may be empty), we have the following possibilities:

(1) question Q_{k+1} is of the form ? (Φ*; φ; Ψ), that is, Q_{k+1} results from Q_k by a rule of $\mathbf{E}^L$ applied with respect to a term of sequence Φ;

(2) question Q_{k+1} is of the form ? (Φ; φ; Ψ*), that is, Q_{k+1} results from Q_k by a rule of $\mathbf{E}^L$ applied with respect to a term of sequence Ψ;

(3) question Q_{k+1} is of the form ? (Φ; ψ; Ψ), that is, Q_{k+1} results from Q_k by a rule of $\mathbf{E}^L$ other than $\mathbf{R}_\alpha$ applied with respect to sequent φ;

(4) question Q_{k+1} is of the form ? (Φ; ψ; ψ*; Ψ), that is, Q_{k+1} results from Q_k by rule $\mathbf{R}_\alpha$ of $\mathbf{E}^L$ applied with respect to sequent φ.

If (1) or (2) holds, then the (k+1)-st term of path **p** is sequent φ. This sequent is permanently unsuccessful in question Q_{k+1} of s, for if it was not, then the sequent φ would not have been permanently unsuccessful in question Q_k.

If (3) holds, then sequent ψ is the (k+1)-st term of path **p**. Again, and for similar reasons as above, sequent ψ is permanently unsuccessful in question Q_{k+1}.

If (4) holds, then at least one of the sequents: ψ and ψ* is permanently unsuccessful in question Q_{k+1}. This may be established by a reasoning analogous to the one already presented above. (The only difference lies in the fact that sequences Φ and / or Ψ can be non-empty; their terms are not acted upon,

however.) If sequent ψ is permanently unsuccessful in question Q_{k+1}, then it is the $(k+1)$-st term of path **p**. Otherwise sequent ψ^* is the $(k+1)$-st term of path **p**.

This finishes the proof.

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III.3 Complete Socratic Transformations

We present a certain procedure of constructing a Socratic transformation of a question of the form $?\,(\vdash (A)^1)$ via the rules of $\mathbf{E}^L$. The term "procedure" is slightly misleading here since it does not have to terminate. Our aim, however, is to produce a Socratic transformation in which, roughly speaking, every rule that may be applied with regard to some wff is sooner or later applied with regard to this wff. Such transformations need not be finite. A Socratic transformation constructed according to this procedure will be called *complete*. We present the procedure and then we prove that each complete (*i.e.*, constructed according to this procedure) Socratic transformation has the desired property.

We explain the general idea first. At any stage, if there is no rule applicable to the last question of what we have constructed, then we have a complete Socratic transformation of the initial question. At the beginning, we apply a rule to question $?\,(\vdash (A)^1)$, if any rule is applicable. Then we act upon the leftmost occurrence of a wff such that a rule of $\mathbf{E}^L$ is applicable with regard to this wff. We apply the rule and we proceed with the wff which occurred next to the occurrence we have acted upon. Again, we apply a rule, if any is applicable with regard to a given wff, and proceed with the next wff. We do so until there is no "next" occurrence of a wff (that is, we have just acted upon the rightmost occurrence of a wff in the last question). Then we go back to the leftmost occurrence of a wff in the last question obtained so far.

Obviously, our explanation is very rough. Let us observe that some of the occurrences of wffs are simply rewritten from a question to a question, but some of them disappear under the decomposition process and some may be "doubled" by an application of rule $\mathbf{R}_\alpha$. In order to keep track of what occurrence of a wff has been already used and which occurrence is "the next one" to act upon, we mark the occurrences (*e.g.* by underlining it or by using *, *etc.*) in a way described below.

PROCEDURE: a complete Socratic transformation of a question $?\,(\vdash (A)^1)$ via the rules of $\mathbf{E}^L$

We start with the question $?\,(\vdash (A)^1)$. If there is no rule of $\mathbf{E}^L$ applicable to this question, then the complete Socratic transformation is ready. Otherwise we apply the rule applicable to this question (the rule is uniquely determined by the shape of formula A) and we mark each occurrence of a wff in the resulting question. Then we proceed according to the following principles:

(1) At any stage of the procedure, if each occurrence of a wff in the last question of the Socratic transformation obtained so far is marked, then unmark each of them.

(2) At any stage of the procedure, if no occurrence of a wff in the last question of the Socratic transformation obtained so far is marked, and no rule of $\mathbf{E}^L$ is applicable to this question, then stop. The transformation is finished.

(3) If neither of **(1)** and **(2)** applies, then choose the leftmost unmarked occurrence of a wff in the last question of the Socratic transformation obtained so far and:

(3.1) if there is no rule applicable with regard to this wff, then mark the wff.

(3.2) if there is a rule **r** applicable with regard to this wff, then this rule is uniquely determined by the shape of the wff and, possibly, by indices of the wffs that occur in the same constituent; in this case there are three possibilities:

(3.2.1) **r** is neither $\mathbf{R}_{\pi D}$ nor $\mathbf{R}_{\pi}$. Apply the rule. In the resulting question there exist(s) one or two "new" constituents. The decomposed indexed formula from the "old" constituent has been replaced by its indexed component(s) in the "new" constituent(s). Mark this (these) indexed component(s). Mark also each wff that has been marked in the previous question. (If rule $\mathbf{R}_{\alpha}$ was applied, then each wff that has been marked in the "old" constituent should be marked in the two "new" constituents.)

(3.2.2) **r** is $\mathbf{R}_{\pi D}$. Apply the rule. The rule has been applied with regard to an indexed π-formula, which has been "rewritten" in the "new" constituent of the resulting question. Mark this indexed π-formula, mark also its indexed component π_0 introduced by this application of the rule. Mark each wff that has been marked in the previous question.

(3.2.3) **r** is $\mathbf{R}_{\pi}$. This is the most complex case. Since the rule is applicable with regard to the chosen wff, this wff is of the form $(\pi)^{\phi(i)}$ and, moreover, there is a numeral that satisfies the relevant proviso of applicability of this rule in $\mathbf{E}^L$. It may happen that there is more than one such numeral. Let $j_1, \ldots, j_k$ be all the numerals that satisfy the proviso of applicability of $\mathbf{R}_{\pi}$ in $\mathbf{E}^L$ (these numerals must occur in the sequent in which $(\pi)^{\phi(i)}$ occurs, hence their number is finite). Apply rule $\mathbf{R}_{\pi}$ with regard to wff $(\pi)^{\phi(i)}$ and the pair $<i, j_1>$. The wff $(\pi)^{\phi(i)}$ has been "rewritten" in the "new" constituent of the resulting question, together with its indexed component $(\pi_0)^{j_1}$. Obviously, rule $\mathbf{R}_{\pi}$ is still applicable with respect to this new constituent and with regard to wff $(\pi)^{\phi(i)}$ and each of the pairs: $<i, j_2>, \ldots, <i, j_k>$. Apply the rule again, this time with regard to pair $<i, j_2>$. Repeat this step another (k-2) times. Mark the wff $(\pi)^{\phi(i)}$ in the "new" constituent of the last question obtained so far, mark also each of: $(\pi_0)^{j_1}, (\pi_0)^{j_2}, \ldots, (\pi_0)^{j_k}$ in this

constituent. Finally, mark each wff that has been marked in the question that we had started with.

The following corollary states that complete Socratic transformations have the desired properties (that is, that complete Socratic transformations are complete in the intuitive sense of the word). Actually, Corollary 3.1 follows immediately from the description of the procedure. Let us remind once again that n-th term of a path of a Socratic transformation is a constituent of the n-th question of this transformation.

COROLLARY 3.1: Let $s = <Q_1, Q_2, ...>$ be a complete Socratic transformation of a question of the form $? (\vdash (A)^1)$ via the rules of $\mathbf{E}^L$. Let $\mathbf{p}$ be a path of s and let φ be n-th ($1 \leq n$) term of path $\mathbf{p}$. The following holds:

(a) If a rule $\mathbf{r}$ of $\mathbf{E}^L$, other than $\mathbf{R_\pi}$, is applicable to question Q_n with respect to its constituent φ and with regard to a wff $(B)^{\phi(i)}$, then there is a question Q_l ($n < l$) of s such that Q_l results from Q_{l-1} by rule $\mathbf{r}$ applied with respect to the constituent of Q_{l-1} which is a term of $\mathbf{p}$ and with regard to wff $(B)^{\phi(i)}$.

(b) If rule $\mathbf{R_\pi}$ of $\mathbf{E}^L$ is applicable to question Q_n with respect to its constituent φ and with regard to a wff $(\pi)^{\phi(i)}$ and a pair $<i, j>$, then there is a question Q_l ($n < l$) of s such that Q_l results from Q_{l-1} by rule $\mathbf{R_\pi}$ applied with respect to the constituent of Q_{l-1} which is a term of $\mathbf{p}$ and with regard to wff $(\pi)^{\phi(i)}$ and the pair $<i, j>$.

PROOF: Before we start, let us introduce, for convenience, the following notion. We say that a wff occurring in a constituent of a question of a Socratic transformation via the rules of $\mathbf{E}^L$ is *active* iff there is a rule of $\mathbf{E}^L$ which is applicable to this question with respect to this constituent and with regard to this wff.

Let us start with the first part, (a), of Corollary 3.1. Suppose that $Q_n = ? (\psi_1; ...; \psi_{k-1}; \varphi; \psi_{k+1}; ...; \psi_m)$ is n-th question of a complete Socratic transformation, where φ is the n-th term of path $\mathbf{p}$ of this transformation. Suppose that a rule $\mathbf{r}$ of $\mathbf{E}^L$ (other than $\mathbf{R_\pi}$) is applicable to this question with respect to its constituent φ and with regard to a wff $(B)^{\phi(i)}$. Question Q_n has been obtained on a certain stage of the procedure described above. Let us consider the next stage of this procedure. By assumption, there is a rule $\mathbf{r}$ applicable to question Q_n, hence we have the following possibilities:

(1) Each wff that occurs in the constituents of question Q_n is already marked. In this situation all the wffs become unmarked (principle **(1)**), and then we proceed to the leftmost wff that occurs in Q_n (principle **(3)**).

(2) There are unmarked occurrences of wffs in Q_n but no such wff is active in a constituent of Q_n. In this case all these occurrences will be marked at the

next stage of the procedure (*cf.* **(3.1)**) and we will go back to the situation described in (1).

(3) There is an unmarked occurrence of a wff and this wff is active in a certain constituent of Q_n.

So we have to consider cases (1) and (3). In any of these cases we start with the leftmost unmarked wff in Q_n (*cf.* principle **(3)** of the procedure), we mark the consecutive wffs that are not active (if there are any) and we reach the leftmost unmarked occurrence of an active wff. Now we act upon this wff (since all the wffs that occur on the left from this wff are marked). We have the following possibilities:

(4) The wff we act upon on the current stage occurs in one of the constituents: $\psi_1, \ldots, \psi_{k-1}$ of question Q_n, or it occurs in constituent φ on the left of wff $(B)^{\phi(i)}$.

(5) The wff we act upon on the current stage is $(B)^{\phi(i)}$ in constituent φ of Q_n.

(6) The wff we act upon on the current stage occurs in constituent φ on the right of wff $(B)^{\phi(i)}$ or it occurs in one of the constituents: $\psi_{k+1}, \ldots, \psi_m$ of question Q_n.

If (5) holds, then, simply, we apply rule **r** and we obtain the required question Q_l. Moreover, it is easy to observe that if (4) or (6) holds, then, in a finite number of steps, the procedure will "reach" wff $(B)^{\phi(i)}$ in sequent φ.

For suppose that (4) holds. Let ψ be the constituent in which the active wff occurs. We proceed according to principle **(3.2)**. We apply an appropriate rule and we obtain question Q_{n+1} in which all wffs in the constituents that occur left from ψ are marked (for they were marked in Q_n), and, in the constituent that "replaced" constituent ψ, the wff that has been active is replaced by its indexed component(s). These indexed component(s) are marked, however, and also each wff that occurs in this "new" constituent left of the indexed component(s) is marked (for it was marked in Q_n). So in the "new" constituent there is less unmarked wffs than there were in constituent ψ. If rule $\mathbf{R_\alpha}$ has been applied, then we have two "new" constituents, but in both of them there are less unmarked wffs than there were in ψ. Let us also note that if rule $\mathbf{R_\pi}$ has been applied (that is, we have proceeded according to **(3.2.3)**), then we must repeat this step an appropriate number of times, until we make use of each pair of numerals that may be used. Nevertheless, also in this case we arrive at a question that has a "new" constituent with less unmarked wffs than there were in sequent ψ. We proceed to another stage of the procedure, apply a rule, and we continue until the leftmost unmarked occurrence of an active wff occurs in sequent φ. (Obviously, this sequent is a term of path **p**.) If the active wff is $(B)^{\phi(i)}$, then we apply rule **r**. In this case we have obtained the required question Q_l. If this wff is not $(B)^{\phi(i)}$, then it is a wff occurring left of $(B)^{\phi(i)}$ in sequent φ. In this case we apply a suitable rule and

obtain a question in which sequent φ is "replaced" by a "new" constituent (or by two "new" constituents). This "new" constituent, however, is a term of path $\mathbf{p}$, and wff $(B)^{\phi(i)}$ occurs in this constituent (if there are two "new" constituents, then only one of them is a term of $\mathbf{p}$, but wff $(B)^{\phi(i)}$ occurs in both of them). Moreover, the "new" constituent(s) has (have) less unmarked wffs than φ have had. Again, we continue by applying the rules, until the leftmost unmarked occurrence of an active wff is the wff $(B)^{\phi(i)}$ in a term of path $\mathbf{p}$. We apply rule $\mathbf{r}$ and we obtain the required question Q_l.

If (6) holds, then we first apply the rules with regard to active wffs that occur in constituents: ψ_{k+1}, …, ψ_m of question Q_n, until each wff in the last question obtained thus far is marked. Then we proceed according to principle **(1)** and then we use principle **(3)** again. The situation we now have is analogous to that described in (4).

As to the second part, (b), of Corollary 3.1, the reasoning is similar with the following two exceptions. First, when we finally "reach" the active occurrence of wff $(\pi)^{\phi(i)}$ in a term of path $\mathbf{p}$, then it can happen that we have to apply rule $\mathbf{R}_\pi$ more than once, thus producing questions Q^*_1, Q^*_2, …, Q^*_h. Each of these questions results from the previous question of s by rule $\mathbf{R}_\pi$ applied with respect to a term of path $\mathbf{p}$ and with regard to wff $(\pi)^{\phi(i)}$ and a pair of numerals. The number of such pairs is, however, finite, and the pair $<i, j>$ is, by assumption, one of them. Therefore one of the questions Q^*_1, Q^*_2, …, Q^*_h is the question Q_l of s, the existence of which we were to prove.

When applicability of rule $\mathbf{R}_\pi$ is concerned, there is still one more situation that we have not described yet (this is the second exception). Namely, it may happen that question Q_n results from the previous question by rule $\mathbf{R}_\pi$ applied with respect to a term of path $\mathbf{p}$ and with regard to wff $(\pi)^{\phi(i)}$ and pair $<i, j>$.[39] In this case the rule is still applicable to question Q_n with respect to its constituent φ and with regard to $(\pi)^{\phi(i)}$ and $<i, j>$,[40] but on the next stage of the procedure rule $\mathbf{R}_\pi$ is applied with regard to wff $(\pi)^{\phi(i)}$ and another pair of numerals that satisfies the relevant proviso (if there is any such pair). Obviously, after finite number of steps we will proceed to the next active wff, and thus we will have a situation analogous to that described in (6).

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[39] The situation will be similar if question Q_{n-1} has been obtained from the previous one by rule $\mathbf{R}_\pi$ applied with respect to a term of path $\mathbf{p}$ and with regard to wff $(\pi)^{\phi(i)}$ and pair $<i, j>$; also the situation will be similar if question Q_{n-2} has been obtained in the same way; *etc*. The point is that the pair $<i, j>$ may be omitted on the current stage of the procedure, because it has already been used in the current application of principle **(3.2.3)**.

[40] This is the reason why complete Socratic transformations via the rules of $\mathbf{E}^L$ may be infinite, even if $L = \mathbf{K}$.

III.4 Countermodels

In the proof of the completeness theorem we will make use of the following notion of *degree of complexity of a formula A* (in symbols: $deg(A)$):

DEFINITION 3.5: Let A be a formula of M,

(i) if A is a literal, then $deg(A) = 0$;

(ii) if A is of the form $\neg\neg B$, then $deg(A) = deg(B) + 1$;

(iii) if A is a β-formula, then $deg(A) = deg(\beta_1) + deg(\beta_2) + 1$;

(iv) if A is an α-formula, then $deg(A) = deg(\alpha_1) + deg(\alpha_2) + 1$;

(v) if A is a ν-formula, then $deg(A) = deg(\nu_0) + 1$;

(vi) if A is a π-formula, then $deg(A) = deg(\pi_0) + 1$.

The degree of complexity of a formula is usually defined as a number of occurrences of the logical constants in this formula. However, when the α, β, $\neg\neg$, ν, π-notation is used, it is more convenient to define the degree of complexity of a formula as we did it above.[41] The notion reflects the structure of the rules of $\mathbf{E}^L$ and it may be viewed as a measure of the number of possible applications of the rules with regard to a formula A, its components, their components, *etc*. (This is not strictly correct due to clause (vi) concerning π-formulas.)

Let us finally prove:

THEOREM 3.1 (completeness): If A is an L-valid formula of M, then sequent $\vdash (A)^1$ is provable in $\mathbf{E}^L$.

PROOF: We proceed by contraposition. We assume that a sequent $\vdash (A)^1$ is not provable in $\mathbf{E}^L$ and we construct a countermodel for the formula A.

Suppose that there is no Socratic proof of sequent $\vdash (A)^1$. Let us consider the complete Socratic transformation s of question $? (\vdash (A)^1)$ via the rules of $\mathbf{E}^L$. By Lemma 3.1, there is a path $\mathbf{p}$ of s whose each term is a sequent permanently unsuccessful in the relevant question. By Definition 3.4 of a permanently unsuccessful sequent, each term of $\mathbf{p}$ is an unsuccessful sequent. Therefore, in particular, the following holds:

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[41] What a definition of degree of complexity should warrant is that degree of a formula is higher than degree of its component(s). The standard definition of degree of complexity of a formula does not warrant that $deg(\beta) > deg(\beta_1)$. If a β-formula is of the form $B \to C$ and C is a propositional variable, then, under the standard definition, $deg(\beta) = deg(B \to C) = deg(B) + 0 + 1 = deg(\neg B) = deg(\beta_1)$. This exception unnecessarily complicates proofs by induction. Another possibility is to define the notion of degree of complexity of a formula as a measure of the number of occurrences of the *arguments* of logical constants in this formula. As a matter of fact, this idea, adjusted to our notation, is expressed in Definition 3.5.

(a) No term of **p** contains occurrences of both: $(p_k)^{\phi(i)}$ and $(\neg p_k)^{\gamma(i)}$, where p_k is an arbitrary propositional variable and i is an arbitrary numeral.

We define a frame $<W, R>$ as follows. W is the set of all the numerals that occur in indices of wffs of the terms of **p**. Let R_0 be the set of all the ordered pairs $<i, j>$ such that i immediately precedes j in an index of a wff occurring in a term of **p**. (In other words, $<i, j> \in R_0$ iff there is a term φ of **p** such that $<i, j> \in \mathbf{I}_R[\varphi]$.) The definition of R depends on L:

- if $L = \mathbf{K}$, then $R = R_0$;

- if $L = \mathbf{D}$, then $<i, j> \in R$ iff (1) $<i, j> \in R_0$, or (2) $i = j$ provided that there is no numeral k such that $<i, k> \in R_0$;

- if $L = \mathbf{T}$, then $<i, j> \in R$ iff (1) $<i, j> \in R_0$, or (2) $i = j$;

- if $L = \mathbf{KB}$, then $<i, j> \in R$ iff (1) $<i, j> \in R_0$, or (2) $<j, i> \in R_0$;

- if $L = \mathbf{K4}$, then $<i, j> \in R$ iff there is a directed R_0-chain whose first term is i and whose last term is j;

- if $L = \mathbf{S4}$, then $<i, j> \in R$ iff (1) there is a directed R_0-chain whose first term is i and whose last term is j, or (2) $i = j$;

- if $L = \mathbf{S5}$, then $<i, j> \in R$ iff (1) there is an R_0-chain whose first term is i and whose last term is j, or (2) $i = j$.

Let us observe that:

(b) For each L, if $<i, j> \in R_0$, then $<i, j> \in R$.

In the case of $L = \mathbf{K}$, $\mathbf{D}$, $\mathbf{T}$, $\mathbf{KB}$ this is obvious. For the other logics it is sufficient to observe that if $<i, j> \in R_0$, then $<i, j>$ is a directed R_0-chain (and hence also an R_0-chain, *cf.* Definition 2.6 from Chapter II).

Now we prove that:

(c) R has the L-properties.

Let $L = \mathbf{D}$ and let i be an arbitrary element of W. We show that there is a numeral "R-accessible" from i. There are two possibilities. Either there is a numeral k such that $<i, k> \in R_0$ or there is no such numeral. If the first possibility holds, then, by condition (1) imposed on R, $<i, k> \in R$. If the second possibility takes place, then, by condition (2) imposed on R, $<i, i> \in R$. In both cases there is a numeral "R-accessible" from i, hence R is extendable.

Let $L = \mathbf{T}$ and let i be an arbitrary element of W. By condition (2) imposed on R, $<i, i> \in R$, which proves that R is reflexive.

Let $L = \mathbf{KB}$ and let $<i, j>$ be an arbitrary element of R. There are two possibilities: either $<i, j> \in R_0$ or $<j, i> \in R_0$. In the first case $<j, i> \in R$ by condition (2) imposed on R. If the second possibility holds, then $<j, i> \in R$ by condition (1). It follows that R is symmetric.

Let $L = \mathbf{K4}$ and suppose that $<i, j> \in R$ and $<j, k> \in R$. By the definition of R for $\mathbf{K4}$, if $<i, j> \in R$, then there is a directed R_0-chain $<i_1, \ldots, i_n>$, where $i_1 = i$ and $i_n = j$. Since $<j, k> \in R$, there is also a directed R_0-chain $<j_1, \ldots, j_m>$ such that $j_1 = j$ and $j_m = k$. Let us observe that $i_n = j_1$, and for this reason $<i_n, j_2> \in R_0$. But if this is the case, then the sequence $<i_1, \ldots, i_n, j_2, \ldots, j_m>$ is also a directed R_0-chain. Since the first term of this directed R_0-chain is i and the last one is k, $<i, k> \in R$ by our definition of R. It follows that R is transitive.

Let $L = \mathbf{S4}$. Condition (1) imposed on R guarantees that R is transitive (the reasoning is exactly as for $\mathbf{K4}$), and condition (2) guarantees that it is reflexive (the reasoning is as for $\mathbf{T}$).

Let $L = \mathbf{S5}$. Condition (2) guarantees that R is reflexive (as above), whereas condition (1) warrants both transitivity and symmetry. We start with transitivity (the reasoning is a slight modification of that for $\mathbf{K4}$). Suppose that $<i, j> \in R$ and $<j, k> \in R$. Then, by condition (1) imposed on R, there is an R_0-chain $<i_1, \ldots, i_n>$, where $i_1 = i$ and $i_n = j$, and there is also an R_0-chain $<j_1, \ldots, j_m>$ such that $j_1 = j$ and $j_m = k$ (these chains need not be directed). Again, $i_n = j_1$, and for this reason either $<i_n, j_2> \in R_0$ or $<j_2, i_n> \in R_0$ (by the definition of a chain). In both cases the sequence: $<i_1, \ldots, i_n, j_2, \ldots, j_m>$ is also an R_0-chain. Since $i_1 = i$ and $j_m = k$, $<i, k> \in R$ by condition (1). Therefore R is transitive. We proceed to symmetry. Suppose that $<i, j> \in R$. Then, by condition (1) again, there is an R_0-chain $<i_1, \ldots, i_n>$, with $i_1 = i$ and $i_n = j$. Let us observe that, by the definition of a chain, each pair $<i_k, i_{k+1}>$ (for $k \geq 1$) satisfies at least one of the conditions: $<i_k, i_{k+1}> \in R_0$ or $<i_{k+1}, i_k> \in R_0$. It makes no difference which of them is satisfied (that is, so to say, the "direction" makes no difference). For example, it may be easily checked that if $<i, j, k>$ is an R_0-chain, then $<k, j, i>$ is an R_0-chain as well. For this reason the sequence $<i_n, \ldots, i_1>$ which has as its l-th term the $i_{(n-l)+1}$ term of sequence $<i_1, \ldots, i_n>$, is also an R_0-chain. This proves that R is symmetric.

Next, we define the following $<W, R>$-assignment $V\#: VAR \times W \mapsto \{\mathbf{0}, \mathbf{1}\}$

(d) $V\#(p_k, i) = \mathbf{0}$ iff wff $(p_k)^{\phi(i)}$ occurs in a term of path $\mathbf{p}$

We extend assignment $V\#$ to a valuation V on frame $<W, R>$, and we prove the following:

(e) For each wff $(B)^{\phi(i)}$ that occurs in a term of path $\mathbf{p}$, $V(B, i) = \mathbf{0}$.

The proof is by induction on the degree of B.

Suppose that wff $(B)^{\phi(i)}$ occurs in a term φ of path $\mathbf{p}$ and that $deg(B) = 0$. Then B is a literal. If B is a propositional variable, then B is assigned value $\mathbf{0}$ (by (d)). Hence in this case $V(B, i) = \mathbf{0}$. Suppose that B is a negation of a propositional variable, $\neg p_k$. Then, by (a), $(p_k)^{\gamma(i)}$ does not occur in term φ of $\mathbf{p}$. What is more, it follows that $(p_k)^{\gamma(i)}$ does not occur in *any* term of path $\mathbf{p}$. For suppose that it does occur in a term of path $\mathbf{p}$ that precedes term φ. Then, since indexed literals are not eliminated in the course of a Socratic transformation, $(p_k)^{\gamma(i)}$ occurs in term φ as well, which contradicts (a). And if $(p_k)^{\gamma(i)}$ occurs in a term of $\mathbf{p}$ that succeeds φ,

then, for the same reason, $(\neg p_k)^{\phi(i)}$ also occurs in this term, contrary to (a). Since wff $(p_k)^{\gamma(i)}$ does not occur in any term of $\mathbf{p}$, we have, by (d), $V\#(p_k, i) = \mathbf{1}$. Therefore $V(\neg p_k, i) = \mathbf{0}$. This finishes the initial step.

Suppose that (e) holds for each formula of degree lower than k ($k > 0$), and let $deg(B) = k$. The reasoning depends on the shape of formula B.

Suppose that B is a β-formula and that wff $(B)^{\phi(i)}$ occurs in some term of path $\mathbf{p}$. Let φ be such a term of $\mathbf{p}$ and suppose that φ is n-th ($n \geq 1$) term of $\mathbf{p}$. Sequent φ is a constituent of the n-th question of s. Let us observe that rule $\mathbf{R_\beta}$ is applicable to the n-th question of s with respect to sequent φ and with regard to wff $(B)^{\phi(i)}$. By Corollary 3.1, there is a question Q_l ($n < l$) of s such that Q_l results from Q_{l-1} by rule $\mathbf{R_\beta}$ applied with respect to the constituent of Q_{l-1} which is a term of $\mathbf{p}$ and with regard to wff $(B)^{\phi(i)}$. Therefore question Q_l is of the form:

$$Q_l = ? \; (\Phi; \; \vdash S \, ' \, (\beta_1)^{\phi(i)} \, ' \, (\beta_2)^{\phi(i)} \, ' \, T; \Psi)$$

where sequent $\vdash S \, ' \, (\beta_1)^{\phi(i)} \, ' \, (\beta_2)^{\phi(i)} \, ' \, T$ is a term of path $\mathbf{p}$, and β_1 and β_2 are the components of formula B. It follows that wffs $(\beta_1)^{\phi(i)}$ and $(\beta_2)^{\phi(i)}$ occur in a term of path $\mathbf{p}$. Moreover, $deg(\beta_1) < k$ and $deg(\beta_2) < k$. Therefore, by assumption, $V(\beta_1, i) = \mathbf{0}$ and $V(\beta_2, i) = \mathbf{0}$. Hence also $V(\beta, i) = \mathbf{0}$.

Suppose that B is an α-formula and that $(B)^{\phi(i)}$ occurs in a term of path $\mathbf{p}$. Let φ be such a term of $\mathbf{p}$ and suppose that φ is n-th ($n \geq 1$) term of $\mathbf{p}$. Sequent φ is a constituent of the n-th question of s. Again, rule $\mathbf{R_\alpha}$ is applicable to the n-th question of s with respect to its constituent φ and with regard to wff $(B)^{\phi(i)}$. By Corollary 3.1, there is a question Q_l ($n < l$) of s such that Q_l results from Q_{l-1} by rule $\mathbf{R_\alpha}$ applied with respect to the constituent of Q_{l-1} which is a term of $\mathbf{p}$ and with regard to wff $(B)^{\phi(i)}$. Therefore question Q_l is of the form:

$$Q_l = ? \; (\Phi; \; \vdash S \, ' \, (\alpha_1)^{\phi(i)} \, ' \, T; \; \vdash S \, ' \, (\alpha_2)^{\phi(i)} \, ' \, T; \Psi)$$

where α_1 and α_2 are the components of formula B. What is more, one of the sequents: $\vdash S \, ' \, (\alpha_1)^{\phi(i)} \, ' \, T$ and $\vdash S \, ' \, (\alpha_2)^{\phi(i)} \, ' \, T$ is a term of path $\mathbf{p}$. Therefore wff $(\alpha_1)^{\phi(i)}$ or wff $(\alpha_2)^{\phi(i)}$ occurs in a term of path $\mathbf{p}$. If $(\alpha_1)^{\phi(i)}$ occurs in a term of path $\mathbf{p}$, then, since $deg(\alpha_1) < k$, $V(\alpha_1, i) = \mathbf{0}$ by assumption, and hence also $V(\alpha, i) = \mathbf{0}$. Similarly, in the second case we also have $V(\alpha, i) = \mathbf{0}$.

Suppose that B is of the form $\neg\neg C$ and that $(B)^{\phi(i)}$ occurs in a term φ of path $\mathbf{p}$. The reasoning is analogous to these already presented. Let φ be n-th ($n \geq 1$) term of $\mathbf{p}$. Rule $\mathbf{R_{\neg\neg}}$ is applicable to the n-th question of s with respect to φ and with regard to $(B)^{\phi(i)}$, therefore, by Corollary 3.1, there is a question Q_l ($n < l$) of s such that Q_l is of the form:

$$Q_l = ? \; (\Phi; \; \vdash S \, ' \, (C)^{\phi(i)} \, ' \, T; \Psi)$$

where sequent $\vdash S \, ' \, (C)^{\phi(i)} \, ' \, T$ is a term of path $\mathbf{p}$, and C is the component of formula B. Since $(C)^{\phi(i)}$ occurs in a term of path $\mathbf{p}$ and $deg(C) < k$, $V(C, i) = \mathbf{0}$, and therefore $V(B, i) = \mathbf{0}$.

Suppose that B is ν-formula. If $(B)^{\phi(i)}$ occurs in n-th ($n \geq 1$) term φ of path $\mathbf{p}$, then rule $\mathbf{R_\nu}$ is applicable to the n-th question of s with respect to φ and with regard to $(B)^{\phi(i)}$. By Corollary 3.1, there is a question Q_l ($n < l$) of s such that Q_l is of the form:

$$Q_l = ?\,(\Phi;\ \vdash S\,'\,(\nu_0)^{\phi(i),\,j}\,'\,T;\ \Psi)$$

where sequent $\vdash S\,'\,(\nu_0)^{\phi(i),\,j}\,'\,T$ is a term of path $\mathbf{p}$, and ν_0 is the component of formula B. Therefore wff $(\nu_0)^{\phi(i),\,j}$ occurs in a term of path $\mathbf{p}$. Let us observe that $<i, j> \in \mathbf{I}_R[\vdash S\,'\,(\nu_0)^{\phi(i),\,j}\,'\,T]$, and for this reason $<i, j> \in R_0$ (by the definition of R_0). But, by (b), $<i, j> \in R$. Since $(\nu_0)^{\phi(i),\,j}$ occurs in a term of path $\mathbf{p}$ and $deg(\nu_0) < k$, $V(\nu_0, j) = \mathbf{0}$. Hence, and by the fact that $<i, j> \in R$, also $V(B, i) = \mathbf{0}$.

Suppose that B is a π-formula and that $(B)^{\phi(i)}$ occurs in a term φ of path $\mathbf{p}$. Let π_0 be the component of B. We need to prove that $V(\pi_0, j) = \mathbf{0}$ for each j such that $<i, j> \in R$. The reasoning depends on the relation R, so we will have to consider each L separately.

Let $L = \mathbf{K}$. It may happen that there is no j such that $<i, j> \in R$. In this case, trivially, $V(\pi, i) = \mathbf{0}$, that is, $V(B, i) = \mathbf{0}$. Suppose that there is at least one such numeral, and let j be an arbitrary numeral for which $<i, j> \in R$ holds. Let us remind that for $L = \mathbf{K}$, $R = R_0$, therefore if $<i, j> \in R$, then $<i, j> \in R_0$. By the definition of R_0, there is a term ψ of path $\mathbf{p}$ such that $<i, j> \in \mathbf{I}_R[\psi]$. By assumption, wff $(B)^{\phi(i)}$ occurs in a term φ of path $\mathbf{p}$. Let us observe that indexed π-formulas are never eliminated in the course of a Socratic transformation, and, similarly, numerals do not disappear from indices of wffs. Therefore there is a term χ of $\mathbf{p}$ such that $(B)^{\phi(i)}$ occurs in this term and the pair $<i, j>$ occurs in an index of a wff in this term. (More specifically, if term ψ precedes term φ on path $\mathbf{p}$, then χ is φ, and if term φ precedes term ψ on path $\mathbf{p}$, then χ is ψ. If it happens that $\psi = \varphi$, then this is χ.) Sequent χ is a constituent of some question Q of s. Moreover, rule $\mathbf{R_\pi}$ is applicable to question Q with respect to sequent χ and with regard to wff $(B)^{\phi(i)}$ and pair $<i, j>$. Therefore, by Corollary 3.1, there is a question Q^* such that: Q^* succeeds question Q in s, and question Q^* results from the previous question of s by rule $\mathbf{R_\pi}$ applied with respect to a constituent which is a term of $\mathbf{p}$ and with regard to $(B)^{\phi(i)}$ and pair $<i, j>$. Therefore question Q^* is of the form:

$$Q^* = ?\,(\Phi;\ \vdash S\,'\,(B)^{\phi(i)}\,'\,(\pi_0)^j\,'\,T;\ \Psi)$$

where sequent $\vdash S\,'\,(B)^{\phi(i)}\,'\,(\pi_0)^j\,'\,T$ is a term of path $\mathbf{p}$, and π_0 is the component of formula B. Hence it follows that wff $(\pi_0)^j$ occurs in a term of path $\mathbf{p}$. Since $deg(\pi_0) < k$, $V(\pi_0, j) = \mathbf{0}$ by assumption. But j was an arbitrary numeral such that $<i, j> \in R$. Therefore $V(\pi, i) = \mathbf{0}$.

Suppose that $L = \mathbf{D}$. Let j be an arbitrary numeral such that $<i, j> \in R$. (By (c), R is extendable, hence such a numeral exists.) By the conditions imposed on R, either $<i, j> \in R_0$, or there is no numeral k such that $<i, k> \in R_0$ and thus $i = j$.

We prove that in our case the second possibility can not take place, that is, $<i, j>$ must be in R_0. Then the reasoning is exactly as for **K**.

Let us observe that since wff $(B)^{\phi(i)}$ occurs in a term of path **p**, rule $\mathbf{R_{\pi D}}$ is applicable with respect to this term, in a certain question Q of s, and with regard to wff $(B)^{\phi(i)}$. Therefore, by Corollary 3.1, there is a question Q^*, that succeeds question Q in s, and such that question Q^* results from the previous question of s by rule $\mathbf{R_{\pi D}}$ applied with respect to a constituent which is a term of **p** and with regard to wff $(B)^{\phi(i)}$. Therefore question Q^* is of the form:

$$Q^* = ? \ (\Phi; \ \vdash S \ ' \ (B)^{\phi(i)} \ ' \ (\pi_0)^{i,h} \ ' \ T; \ \Psi)$$

where sequent $\vdash S \ ' \ (B)^{\phi(i)} \ ' \ (\pi_0)^{i,h} \ ' \ T$ is a term of path **p**, h is the "new" numeral introduced by $\mathbf{R_{\pi D}}$, and π_0 is the component of formula B. Observe that $<i, h> \in \mathbf{I}_R[\vdash S \ ' \ (B)^{\phi(i)} \ ' \ (\pi_0)^{i,h} \ ' \ T]$, and hence $<i, h> \in R_0$, as required.

For the other logics, we reason as follows. If it happens that there is no j such that $<i, j> \in R$ (this is possible, if L is **KB** or **K4**), then, trivially, $V(B, i) = \mathbf{0}$ (as in the case of **K**). So we assume that there is such a numeral. Let j be an arbitrary numeral such that $<i, j> \in R$. We need to prove that there is a term of path **p** such that rule $\mathbf{R_\pi}$ is applicable with respect to this term (in a relevant question) and with regard to wff $(B)^{\phi(i)}$ and pair $<i, j>$. If this is the case, then we use Corollary 3.1 and we reason exactly as in the case of **K**.

Let $L = \mathbf{T}$. In this case $<i, j> \in R$ iff $<i, j> \in R_0$, or $i = j$. Suppose that $i = j$. Wff $(B)^{\phi(i)}$ occurs in a term φ of path **p**. Since $i = j$, pair $<i, j>$ satisfies the proviso $\mathbf{P^T}$ of applicability of rule $\mathbf{R_\pi}$ in $\mathbf{E^T}$. Therefore, in this case, rule $\mathbf{R_\pi}$ is applicable with respect to sequent φ (in a relevant question) and with regard to wff $(B)^{\phi(i)}$ and pair $<i, i>$. If $<i, j> \in R_0$, then the reasoning is as for **K**.

Let $L = \mathbf{KB}$. In this case $<i, j> \in R$ iff $<i, j> \in R_0$, or $<j, i> \in R_0$. Suppose that $<j, i> \in R_0$. As in the case of **K**, we arrive at a conclusion that there is a term χ of path **p** such that wff $(B)^{\phi(i)}$ occurs in χ and $<j, i> \in \mathbf{I}_R[\chi]$. Sequent χ is a constituent of some question Q of s. Since $<j, i> \in \mathbf{I}_R[\chi]$, the pair $<i, j>$ satisfies the proviso $\mathbf{P^{KB}}$ of applicability of rule $\mathbf{R_\pi}$ in $\mathbf{E^{KB}}$. Therefore rule $\mathbf{R_\pi}$ is applicable to question Q with respect to sequent χ and with regard to wff $(B)^{\phi(i)}$ and pair $<i, j>$ (as required). If $<i, j> \in R_0$, then the reasoning is exactly as for **K**.

Let $L = \mathbf{K4}$. In this case $<i, j> \in R$ iff there is a directed R_0-chain $<i_1, \ldots, i_n>$ such that $i_1 = i$ and $i_n = j$. By the definition of a directed chain, $<i_l, i_{l+1}> \in R_0$ for each $l \geq 1$. By the definition of R_0, each such pair $<i_l, i_{l+1}>$ occurs in some term of path **p**. Let us observe, once again, that pairs of numerals do not disappear from a path in the course of a Socratic transformation. Since the number of pairs $<i_l, i_{l+1}>$ in the directed R_0-chain is finite, there is a term φ of path **p** such that $<i_l, i_{l+1}> \in \mathbf{I}_R[\varphi]$ for each $l \geq 1$. Wff $(B)^{\phi(i)}$ occurs in a term ψ of path **p**. As in the previous cases, we arrive at a conclusion that there is a term χ such that $(B)^{\phi(i)}$ occurs in χ and $<i_l, i_{l+1}> \in \mathbf{I}_R[\chi]$ for each $l \geq 1$. Observe that the pairs $<i_l, i_{l+1}>$ form a directed

$\mathbf{I}_R[\chi]$-chain, and therefore the pair of numerals $<i, j>$ satisfies the proviso $\mathbf{P}^{\mathbf{K4}}$ of applicability of rule $\mathbf{R}_\pi$ in $\mathbf{E}^{\mathbf{K4}}$ (as required).

Let $L = \mathbf{S4}$. In this case $<i, j> \in R$ iff (1) there is a directed R_0-chain whose first term is i and whose last term is j, or (2) $i = j$. If (1) holds, then the reasoning is as for $\mathbf{K4}$. If (2) holds, then the reasoning is as for $\mathbf{T}$.

Let $L = \mathbf{S5}$. In this case $<i, j> \in R$ iff (1) there is an R_0-chain whose first term is i and whose last term is j, or (2) $i = j$. In the second case the reasoning is as for $\mathbf{T}$, and if (1) holds, then we reason as in the case of $\mathbf{K4}$ (the chain need not be directed, but this has no effect on the reasoning).

We have proved that (e) is true. Let us remind that sequent $\vdash (A)^1$ is the first term of each path of s. In particular, $\vdash (A)^1$ is the first term of path $\mathbf{p}$. Therefore, by (e), $V(A, 1) = \mathbf{0}$. By (c), R has the L-properties, thus A is not L-valid. This finishes the proof.

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CHAPTER IV: Related Work

In this chapter we present a brief overview of the basic developments in the field of deductive systems for modal logics. We pay more attention to proof-theoretical traditions that are close to our approach and we make some comparisons. In our overview we will use the turnstile symbol '⊢' in the standard manner. We end this chapter with a summary of the work that we plan in the future.

IV.1 Sequent Calculi

Sequent calculi were proposed by Gentzen (*cf.* [Gentzen:1935]) and they constitute one of the most important frameworks to study logics. The first Gentzen-style sequent systems for modal logics (**S4** and **S5**) appeared in the fifties of the previous century (*cf.* [Onishi, Matsumoto:1957], [Onishi, Matsumoto:1959]). Recently, there are many formulations of modal logics as sequent systems.[42] For instance, logic **K** may be formalized as such a system by adding the following rule to a sequent system for **CPC**:[43]

$$(\rightarrow \Box)_1 \qquad \frac{\Delta \vdash A}{\Box \Delta \vdash \Box A}$$

where Δ is a finite (possibly empty) set of formulas of the language of modal logic, A is a single formula of this language and $\Box \Delta = \{\Box B : B \in \Delta\}$. (For simplicity, only the necessity operator is considered as primitive in this account.) Logic **K4** may be presented by adding the following rule:

$$(\rightarrow \Box)_2 \qquad \frac{\Delta, \Box \Delta \vdash A}{\Box \Delta \vdash \Box A}$$

to a sequent system for **CPC**.

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[42] An overview of standard and non-standard sequent systems for modal logics may be found in [Wansing:2002].
[43] We follow [Wansing:2002] in this presentation. (The names of the rules come from this paper; we modify the way of presenting rules, however.)

The basic problem with sequent systems for modal logics is that they are neither uniform nor modular. For instance, in an axiomatic account each of the basic modal logics is formalized by extending the axiomatic system for **K** with axioms (axiom schemas) that correspond to the properties characterizing a given logic. In the case of sequent systems, even if a good formalization of a modal logic is known, one usually has to start from "scratch" when formalizing another modal logic. The problem is that the rules of an ordinary sequent system fail to capture single properties of the accessibility relation in a way that the axioms do. For instance, rule $(\rightarrow\Box)_2$ displayed above corresponds not just to the transitivity axiom, but also to the normality axiom, which is captured by rule $(\rightarrow\Box)_1$ as well. Modularity is a common requirement nowadays (especially when automated deduction is concerned, since modularity supports implementation), thus a good deductive system should have the nice "combinational" properties that the axiomatic systems for modal logics have, and which the Gentzen systems for these logics lack.

This is probably one of the reasons why no unifying framework for modal logics in the style of standard sequent calculus is known.[44] An interesting proposal of a solution to this problem may be found in the paper by Avron: [Avron:1996]. In Avron's account the differences between modal logics are reflected on the level of structural rules acting on hypersequents. This approach, however, considerably differs from ours.

There are also approaches that combine sequent calculi for modal logics with the framework of Labelled Deductive Systems (*cf.* [Basin, Matthews, Viganò:1997b], [Mateus, Sernadas, Sernadas, Viganò:2004], [Mateus, Rasga, Sernadas:2005], [Governatori, Rotolo:2001]), and thus obtain uniform and modular account of many modal propositional logics. We will refer to these developments in Section IV.5.

IV.2 Tableau Systems

The breakthrough in the proof-theory of modal logics comes with the possible worlds semantics. The intuitions behind Kripke semantics are exceptionally clear and simple, and so the tradition of constructing proof methods for modal logics through an analysis of this semantics is already quite long. The very semantical approach to proof-theoretical issues has been initiated by Beth (*cf.* [Beth:1955]). Beth's tableau method is essentially a method of a countermodel construction. The idea of relational semantics derives from works of Kanger, Hintikka and, finally, Kripke ([Kripke:1959]), who presented Beth-style tableaux for modal logics together with their semantical characteristics in [Kripke:1963].

Kripke follows Beth's idea of dividing a tableau into the left and the right column for formulas that are assumed to be true or false, respectively. However, Kripke uses a "web" of tableaux rather than one tableau – an auxiliary tableau is

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[44] For this opinion *cf.* [Wansing:2002] and [Avron:1996].

added for each possible world and the tableaux are interrelated by an auxiliary accessibility relation. The method of semantic diagrams of Hughes and Cresswell ([Hughes, Cresswell:1968]) is a reformulation of Kripke's method, with the difference that Hughes and Cresswell use 1 and 0 to distinguish between formulas that are true or false in a given world.

In the previous parts of this work we have already pointed to the connection between modal erotetic calculi and tableau systems. As we have observed in Section II.1.3 of Chapter II, a Socratic transformation may be viewed as a systematic countermodel construction (when the indirect interpretation B is concerned), just as it happens in the case of tableaux. Let us also observe that the operations performed on formulas of right-sided sequents correspond to operations performed on formulas occurring in the right column of a Beth's (Kripke's) tableau. However, in the case of right-sided sequents of language *M** indices are used to control the information flow between possible worlds, hence there is no need for Kripke's auxiliary tableaux.

The connections between sequent-type formalizations of logics and tableau-type formalizations of logics are, obviously, very close anyway. If a sequent system and a tableau system are two notational variants of the same proof procedure, then the method of Socratic proofs may be viewed as a "compromise between them". It is, in a sense, a sequent-type formalization of the tableau method.

Today, tableaux are usually defined as trees, with formulas occurring in their nodes. Up to now, many variants of modal tableau systems have been formulated. We will make two useful distinctions here. First, there are *Smullyan-type* formulations of tableau systems, where single formulas occur in the nodes of a tableau; and there are *Hintikka-type* formulations of tableau systems, where finite sets of formulas occur in the nodes of a tableau. The first type of formulation of tableaux comes from Smullyan ([Smullyan:1968]),[45] and is probably more popular than the second one (it may seem more intuitive, and thus more often used in teaching). Modal tableaux of this type were presented by Fitting ([Fitting:1983]) and Priest ([Priest:2001]). Tableau systems of the second type had been proposed by Hintikka ([Hintikka:1955]), and were presented for modal logics by Rautenberg ([Rautenberg:1983]). The correspondence between sequent systems and tableau systems is more transparent when the second formulation is used.

The second distinction is that between *explicit* and *implicit* tableau systems for modal logics (*cf.* [Goré:1999]). In the case of explicit tableau systems, the accessibility relation is represented in the deductive mechanism explicitly by some device (*e.g.* by the structure of labels or as it is done in Kripke's

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[45] Actually, similar results had been achieved by Zbigniew Lis ([Lis:1960]), who presented an elegant variant of analytic tableaux with symbols '+' and '-' playing the roles analogous to that of truth-signs used by Smullyan. Unfortunately, the paper by Lis appeared in Polish only and has not become widely known. However, in a historical introduction to tableau methods in [Fitting:1999] a summary of the paper of Lis is contained.

formulation). The implicit tableau systems are based on "pure" modal language. Since there is no device to represent the accessibility relation, its properties must be built into the rules. We will not discuss the implicit formulations of modal tableaux, since these are distinct from our work.[46] We now refer to explicit systems of Fitting and Priest.

The work of Fitting and Priest belongs, actually, to the tradition of "labelled deductive systems", as they are called nowadays. Prefixed tableau systems were presented in [Fitting:1983, Chapter VIII]. In Fitting's approach each formula occurs with a prefix (a finite sequence of positive integers). The prefix is thought of as a name of a possible world, and the accessibility relation is encoded in the structure of the prefixes. *E.g.,* the rules handling with the modal operators are of the forms:

ν-rule $\dfrac{\sigma\ \nu}{\sigma'\ \nu}$ **π-rule** $\dfrac{\sigma\ \pi}{\sigma'\ \pi}$

where σ and σ' are prefixes, and, in the case of the ν-rule, σ' is a prefix that has been used on the branch and that is accessible from prefix σ, whereas in the case of the π-rule, prefix σ' is new to the branch. The relation of accessibility between prefixes is defined separately for each modal logic and it reflects the semantic properties of the accessibility relation in Kripke models. (For some modal logics considered by Fitting, certain additional side conditions for the rules must be added, but we omit these details in order to simplify this presentation.) Fitting's account of modal logics is modular, in that the differences between various logics are reflected only in the side conditions for the modal rules, and the correspondence between the side conditions and the semantical properties is very clear.

Interestingly, Fitting's prefixed tableaux may be reformulated to the effect that the side conditions are simplified. In [Goré:1999] the author presents labelled tableau calculi for a wide class of modal logics, that are based on Fitting's work. In a labelled tableau calculus a modal logic is characterized by a set of rules manipulating both formulas and prefixes (now called labels), without the side conditions concerning accessibility relation between prefixes. The price is that more rules are needed. For example, logic **S4** requires the following modal rules:

(lK) $\dfrac{\sigma :: \Box P}{\sigma.n :: P}$ (lT) $\dfrac{\sigma :: \Box P}{\sigma :: P}$ ($l4$) $\dfrac{\sigma :: \Box P}{\sigma.n :: \Box P}$ ($l\pi$) $\dfrac{\sigma :: \neg\Box P}{\sigma.n :: \neg P}$

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[46] Implicit Smullyan-type tableau systems are presented in [Fitting:1983]. In [Goré:1999] the author presents a multitude of Hintikka-type implicit tableau systems for modal and temporal logics.

where σ and σ.*n* are prefixes, that is, finite sequences of positive integers (as previously), *n* is a positive integer, and σ.*n* is the concatenation of prefix σ and the one-term prefix <*n*>. In the case of rules (*lK*) and (*l4*), prefix σ.*n* must already exist on the branch. In the case of rule (*lπ*), prefix σ.*n* must be new to the branch.

Fitting's formulation of modal tableaux, with only one schema rule for ν-formulas and the side conditions for its application varying for each modal logic, may seem more intuitive and is more convenient if the method is actually used by a human. The second formulation, however, is obviously better for computer applications.

The way Fitting's rules handle with prefixed modal formulas is analogous to our treatment of indexed formulas. (With the obvious exception that the ν-formulas and the π-formulas switch their roles in our calculi. However, if we think of the formulas occurring in the constituents of questions of a Socratic transformation as of formulas occurring in the right column of Beth's tableau, then the duality disappears.) This analogy becomes even more transparent when we come to Priest's tableaux. As a matter of fact, the inspiration to use indices in the way we did had came from Priest's work.

In Priest's account, in a node of a tableau there is either an expression of the form: *A*, *i*, where *A* is a formula and *i* is a natural number (the name of a possible world in which *A* is true), or an expression of the form *i r j* (such expressions define the accessibility relation in a Kripke model). There are rules for dealing with the modalities and separate rules, corresponding to the properties of the accessibility relation, for introducing expressions of the form *i r j*. The rule for the necessity operator allows to introduce an expression of the form *A*, *j* on a branch, if expressions: □*A*, *i* and *i r j* already occur on the branch. In the case of the possibility operator, if an expression ◊*A*, *i* occurs on a branch, then one is allowed to extend the branch with two nodes: the first is *i r j*, where *j* is new on the branch, and the second is *A*, *j*. The rule corresponding to, *e.g.*, the property of transitivity allows to introduce an expression *i r j* on a branch if for some natural number *k*, the expressions *i r k* and *k r j* occur on the branch. No side conditions are needed, since the properties of the accessibility relation are encoded in the rules for introducing expressions of the form *i r j*. The tableau systems for consecutive modal logics differ only with respect to these "accessibility rules". Again, the account is modular.

In our approach we have brought the expressions *i r j* into indices of indexed formulas of language *M**, and we have used side conditions, as in Fitting's formulation. However, it is clear that the way modalities are dealt with in the three systems is analogous. For example, in a modal erotetic calculus, if we have a π-formula in a world named *i* (and a π-formula behaves like the ν-formulas do in Fitting's and Priest's calculi), then its component π_0 may be introduced into a world named *j*, provided that *j* is accessible from *i*. Whether the last proviso holds depends on the structure of the indices, just as in Fitting's case it depends on

the structure of prefixes, and in Priest's case on the presence of the expression *i r j* on a branch.

However, there is also an obvious difference between the two mentioned explicit tableau formulations of modal logics and modal erotetic calculi. The rules of our calculi do not operate on single indexed formulas, but on constituents of questions. Therefore, as far as modal erotetic calculi may be viewed as a sequent-style formulation of the tableau method, it is a Hintikka-type formulation. We suppose that this is an advantage of erotetic modal calculi over tableau systems, when the issues of automated deduction and implementation are concerned. When building a tableau, the whole tree must be searched for decomposable formulas which may occur high above the leaves of the tree constructed so far. In a Socratic transformation the rules always act upon the last question obtained so far, thus only the last question must be searched for the necessary information.

Let us observe that the tableau systems that we have briefly discussed above (that of Kripke, Hughes and Cresswell, Fitting, Priest) belong to the family of analytic, semantically motivated proof methods for modal logics, as we called them in the introduction to this work. We also mentioned there that such systems have a common drawback. The problem is that when transitive modal logics are concerned, the procedures of applying rules of these systems may fail to give a solution in a finite number of steps. In [Hughes, Cresswell:1968] the method of semantic diagrams is claimed to constitute a decision procedure for the modal logics considered there, though the solution of the problem of "loops" is described only in an intuitive manner. A systematic proof procedure in a prefixed tableau system is given in [Fitting:1983, Chapter VIII], but the procedure also permits infinite tableaux (obviously, formulating a terminating decision procedure for transitive modal propositional logics was not the aim of Fitting's book). The problem of loops is described there quite precisely, this is not, however, what we could call an algorithmic terminating procedure.[47] Such a procedure is available in the framework of erotetic modal calculi, but we leave its presentation for future work.

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[47] A very elegant solution of this problem may be found in [Rautenberg:1983]. (As we have already mentioned, Rautenberg presents implicit Hintikka-type tableau systems for modal logics.) A tableau is defined as a finite tree constructed by the rules of a system and "such that if a node *E* bears a set *Y* and *Y* appeared already on the branch to *E* then *E* is an end node" of the tableau. (*Cf.* [Rautenberg:1983, p. 407].) This warrants that when a loop occurs on a branch, the branch is not extended any more.

An interesting example of a terminating and *loop-free* procedure for logic **S4** may be found in the paper [Matsumoto:2003], where the author presents a Hintikka-type tableau system. Actually, the system does not fit the distinction explicit / implicit systems, since there is no device to represent the accessibility relation, but there is a device, called "history", to keep track of the operations performed so far in a tableau. The rules manipulate both: a set of formulas and a history. No loops occur at all in this system.

IV.3 Rasiowa-Sikorski Deduction Systems

So far we have focussed on the well-known tableau systems for modal logics, a method which is dual with respect to our erotetic modal calculi, in that a tableau starts with formula '$\neg A$' and proceeds by constructing a model for this formula, whereas the first question of a Socratic transformation concerns validity of formula A simply.[48] Now we are going to present a deductive system which is also dual with respect to tableau systems, in the way erotetic modal calculi are. Moreover, the construction of this system is guided by a general methodology which resembles the "Socratic paradigm" in many respects. It is a Rasiowa-Sikorski style deduction system developed by Konikowska. Our presentation is based on [Konikowska:2002] (*cf.* also [Konikowska:1999]).

A Rasiowa-Sikorski system (R-S system for short) is a sequence-type formalization of logics, developed for **CPC** and for first-order logic by Rasiowa and Sikorski (*cf.* [Rasiowa, Sikorski:1963, pp. 264-269, 299-306]). An R-S deduction system consists of *decomposition rules* and *fundamental sequences*. The decomposition rules act upon finite sequences of formulas in quite the same way as the erotetic rules act upon sequents, that is, they break down a complex formula that occurs in a sequence into its component parts. The rules are used to construct a *decomposition tree* of a formula (or of a sequence of formulas), which is a tree with finite sequences of formulas in its nodes.

Semantically, the comma separating formulas in a sequence corresponds to meta-disjunction, and the branching in a decomposition tree corresponds to meta-conjunction. Again, the situation is analogous in the case of erotetic calculi with right-sided sequents. Moreover, the decomposition rules are semantically invertible. A formula is provable in an R-S system if it has a finite decomposition tree with a fundamental sequence in each of its leaves.[49]

The fundamental sequences (also called the "axioms" of the R-S system) play the same role that we have prescribed to sequents of the "basic" forms specified in the definitions of Socratic proof – fundamental sequences are sequences whose validity is warranted by some simple semantical facts.

In the paper [Konikowska:2002] the author shows examples of applications of R-S methodology to various brands of computer science logics. There is no separate presentation of modal logics, but the author investigates, *in. a.*, three-valued temporal logic, and a suitable "subsystem" for modal logic **S4** may be easily extracted from this presentation. This is what we shall do. We

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[48] But let us remind again that if we concern the indirect interpretation B of our method and keep in mind the "left column – right column" division, then the duality disappears, since we perform the operations on formula A (and its components, their components, *etc.*) *as if it was false*, that is, as if '$\neg A$' was true.

[49] Strictly speaking, the decomposition rules are defined in a way which warrants that the decomposition tree of a given formula is *unique*. For simplicity, we have omitted this aspect of the R-S systems.

simplify Konikowska's presentation and we show the "purely modal" part of her system.

First, Konikowska uses truth-signs (in order to deal with three logical values): **t** (the "truth operator") and **n** (the "non-truth operator").[50] Second, in order to capture modal operators Konikowska introduces *state variables* representing possible worlds and the constant $\prec$ representing the accessibility relation. Formulas of the deduction language are: *state formulas* of the form $x.\alpha$, where x is a state variable and α is a formula of the modal language; and *accessibility formulas* of the form $x \prec y$, where x and y are state variables. *Signed formulas* are of one of the forms:

- **t**$(x.\alpha)$ (intuitively, such a formula claims that α holds in x)

- **n**$(x.\alpha)$ (α does not hold in x)

- **t**$(x \prec y)$ (the accessibility relation holds between x and y)

- **n**$(x \prec y)$ (the accessibility relation does not hold between x and y)

The deduction system developed by Konikowska for the three-valued temporal logic L_T is called DR_T. The rules of DR_T operate on sequences of signed formulas. There are eight specific modal rules in this system. Below we present two of them (these two are representative for the remaining six)[51]:

rule (t□):   **rule (t◊):**

$$\frac{\Omega', \mathbf{t}(x.\Box\alpha), \Omega''}{\Omega', \mathbf{n}(x \prec z), \mathbf{t}(z.\alpha), \Omega''} \qquad \frac{\Omega', \mathbf{t}(x.\Diamond\alpha), \Omega''}{\Omega', \mathbf{t}(x \prec y), \Omega'', \mathbf{t}(x.\Diamond\alpha) \ | \ \Omega', \mathbf{t}(y.\alpha), \Omega'', \mathbf{t}(x.\Diamond\alpha)}$$

The following side conditions are imposed on the state variables z and y: z (rule (t□)) must not occur in sequence $\Omega', \mathbf{t}(x.\Box\alpha), \Omega''$ above the inference line; whereas y (rule (t¬□)) is arbitrary. The line | in rule (t◊) indicates branching.

The rules are invertible (*cf.* [Konikowska:2002, pp. 355-357]) and they considerably resemble the modal rules of erotetic calculi $\mathbf{E}^L$, with the following two exceptions. First, the information "x sees y" ("x does not see y") is represented separately by use of the accessibility formulas, which causes branching in rule (t◊) (and in the three analogous rules). Second, after the (t◊) rule is applied, the active formula $\mathbf{t}(x.\Diamond\alpha)$ is rewritten as the last term of the resulting

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[50] In Konikowska's paper there are upper-case letters **T** and **N** used as the truth-signs. We have switched them to lower-case letters, since we use **T** for the reflexive modal logic. The truth-sign **t** corresponds to the designated value in the three-valued temporal logic investigated by Konikowska, whereas the truth-sign **n** corresponds to the other two values. However, in order to obtain an R-S system for a two-valued logic, it is sufficient to interpret "non-truth" **n** as "false".

[51] The remaining six rules are: (t¬◊), (n¬□), (n◊) – they are analogous to rule (t□), and (t¬□), (n□), (n¬◊) – analogous to rule (t◊).

sequences. This is a part of the "control mechanism" in R-S systems, due to which the simple "take the leftmost" procedure may be applied without causing trivial loops.

Interestingly, the properties of reflexivity and transitivity are dealt with by different means in this system. As to reflexivity, the side condition on state variables for rule $(t\lozenge)$ permits the situation when $x = y$. Moreover, a sequence of signed formulas that has as its term an expression of the form $\mathbf{t}(x \rightslice x)$ is defined as fundamental, and the decomposition rules are not applied to fundamental sequences in R-S systems. Therefore the reflexivity property is actually built into the rules of this system and into the "terminating conditions". Transitivity is encoded in a separate rule, which is:

rule $(tr \rightslice)$:

$$\frac{\Omega', \mathbf{n}(x \rightslice y), \mathbf{n}(y \rightslice z), \Omega''}{\Omega', \mathbf{n}(x \rightslice y), \mathbf{n}(y \rightslice z), \mathbf{n}(x \rightslice z), \Omega''}$$

Here the notation is somewhat simplified, since the formulas in sequence $\Omega', \mathbf{n}(x \rightslice y), \mathbf{n}(y \rightslice z), \Omega''$ are allowed to appear in any order and can be separated by arbitrary sequences of formulas. Rule $(tr \rightslice)$ is also invertible.

The advantage of this account over ours is that except from the information "x sees y", also the information "x *does not* see y" may be represented in the deductive language, and thus it is possible to encode properties such as weak antisymmetry or irreflexivity in the rules. (We think that this is not possible in our approach. If this is true, then the erotetic framework which we have presented in Chapter II is probably not capable of capturing temporal logics, unless modified in some way – *e.g.* by extending the syntax of *M** to the effect that the information "i does not see j" may be represented by some of its expressions. Using additional terminating conditions also seems to be an interesting idea in this context.)

The disadvantage of this system, when compared with ours, is that adding separate accessibility formulas causes branching in rule $(t\lozenge)$, which potentially increases complexity of decomposition trees. On the other hand, one of the main aims of the author is to show the merits of the R-S methodology in the field of automated deduction. Our another hypothesis is that, as far as purely modal logics are concerned, the method is more efficient if the information concerning accessibility is treated in a manner analogous to the way we deal with indices.

IV.4 Natural Deduction

There are also examples of semantically motivated proof methods for modal logics in natural deduction (ND) style. The main reference here is

[Fitting:1983] again. Fitting follows the idea of Fitch and introduces "strict subordinate derivations" which are displayed in a "main" derivation in "strict boxes". A strict box represents a possible world and the rule of creation of a strict box guides us from a world to an accessible world. Fitting presents two basic types of modal ND systems: A-style ND systems and I-style ND systems. The difference between them lies in the way the strict boxes are interpreted. A strict box may be thought of as representing an *arbitrary* possible world – in this case we have the A-style ND system – or as representing a *particular* possible world – in this case we have the I-style ND system. Formally, the ND systems of the two types differ with respect to the rules of creating and closing a strict box.

Fitting's ND systems have been reformulated in the framework of Labelled Deductive Systems (LDS) by Russo ([Russo:1995]). We will refer to the paper by Russo after we sketch the main ideas of LDS.

IV.5 Labelled Deductive Systems

Before we proceed, let us observe that all the systems considered so far in this chapter have the following common features. There is a clear semantical motivation behind the construction of the rules (and, possibly, behind the terminating conditions), and in most cases some form of "labelling" is used in order to refer to possible worlds.

Labelling (also called prefixing or annotating) is well-known in proof theory for modal logics. Modal labelled deduction traces back at least to Fitting (also Kanger and Prior are sometimes mentioned in this context). Today, however, the paradigmatic approach is that of Dov Gabbay ([Gabbay:1996]), who has first proposed labelling as a *systematic* approach to deduction. This approach is very wide-spread today, so we will now focus on the basic developments in this field. However, before we start, let us make one observation. There are deductive systems with labels in which labelling is used in a simple and modest way – such as the explicit tableau systems – and there are labelled deductive systems in which the labelling mechanism is very rich and / or combined with other frameworks. Our work belongs to the first type of labelled systems and, in many respects, it is not comparable with the systems of the second type.

Gabbay's aim is to provide a general unifying framework in which many of the new logics found in various fields of computer science can be presented and investigated. Such a framework must be capable of formalizing the more traditional logics as well. What is needed for an investigation of various logics in one paradigm is a good presentation of differences between logics within this paradigm. The differences are, however, described on the meta-level, therefore the new unifying framework must be able to combine the meta-level features of logics with their object-level properties.

This is exactly the basic idea of LDS methodology. Generally, an LDS is a triple <**A**, **L**, **R**>, where **L** is a logical language (*e.g.* that of modal logics), **A** is the so-called algebra of labels (syntactically, this is usually a logical language

simply) and **R** – in the simplest case – is the set of rules of an LDS. The rules operate on declarative units, or complex structures composed of declarative units, where a declarative unit is a pair $<t: A>$. Here A is a formula of **L** and t is a term of **A** (a label). Intuitively, the label t represents some additional information concerning formula A, that is usually represented on the meta-level. Thus the meta-level features are reflected in the algebra of labels and the object-level features are reflected in the logic of formulas. In the LDS framework the traditional notion of consequence relation between formulas: $A_1, \ldots, A_n \vdash B$ is replaced by the notion of consequence between labelled formulas: $t_1: A_1, \ldots, t_n: A_n \vdash s: B$.

As we have mentioned, the idea of using labels in a deductive system is old. What is new in Gabbay's approach is that Gabbay uses an algebra of labels and considers the labelling as part of the logic. Gabbay's standpoint is, in a sense, very strong, since he claims that "the notion of a logic is an LDS. This is not the same as the occasional use of labelling with some specific purpose in mind."[52] The motivations behind Gabbay's work are application-oriented. He wants to have a notion of a logical system in which the computational aspects of a logic play an important role. For instance, when it comes to define an LDS in the context of Practical Reasoning Systems, the third element **R** of an LDS is equipped with: the deductive mechanism (rules, as above), the notion of a database, and algorithms for performing operations like abduction, explanation, updating, *etc.* That is, the notion of a logical system understood as an LDS becomes the notion of an *agent* in the AI sense. Moreover, in the LDS account a labelled system for a logic is usually a basis for its implementation. (*Cf.* for example [Basin, Matthews, Viganò:1997a] and [Basin, Matthews, Viganò:1998].)

Gabbay puts some effort to show that the framework of LDS is rich and flexible enough to formalize "traditional" classical and non-classical logics, and presents ND formulations of many such logics in the framework of LDS. In particular, this is successfully done for modal logics. The general LDS approach proposed by Gabbay is developed further and with more details by Russo (in the paper we have already mentioned), who presents Modal Labelled Deductive Systems (MLDS) for logics **K**, **T**, **K4**, **KB**, **S4**, **S5**, **D**, **D4** and **DB**. We now present some details of this approach.

As in Gabbay's account, the basic unit of information in an MLDS is a *declarative unit* of the form $t : A$. A modal labelled deductive language contains also *R*-literals of the forms: $R(t_1, t_2)$ and $\neg R(t_1, t_2)$. The inference rules of an MLDS operate on *configurations*. Roughly speaking, a configuration is a structure that contains a set of *R*-literals (the information about the accessibility relation) and a set of labelled formulas (the information about the truth-values of formulas in possible worlds). An MLDS consists of a modal labelled deductive language, a labelling algebra **A** and a set **R** of inference rules which generate one configuration from another. For example, the rule for □-Elimination allows to

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^{52} See [Gabbay:1996, footnote 5 on page 12].

"infer" from a configuration containing both $t_1 : \Box A$ and $R(t_1, t_2)$ a configuration that, *in addition*, contains $t_2 : A$. Thus the consequence relation in an MLDS is defined between configurations.

The algebra **A** consists of "axioms" – first-order formulas expressing semantical properties defining a given modal logic. The axioms may be used to "infer" new *R*-literals from the *R*-literals already present in a configuration. The representation of different modal logics affects only the labelling algebra, leaving the proof system unchanged. In order to obtain an MLDS for, *e.g.*, logic **S4**, one adds to the labelling algebra **A** the first-order formulas expressing reflexivity and transitivity of *R*. The resulting deductive framework is general, modular and uniform, at least for modal logics that are first-order definable.

The power of the labelling mechanism lies in the control over the information flow between possible worlds. The notion of a "strict subordinate derivation" is not needed any more in this ND-style formulation of modal logics. Moreover, in an MLDS the distinction between an arbitrary accessible world and a particular accessible world is introduced in the language of the labelling algebra *via* special unary symbols. Hence Fitting's distinction between an A-style and an I-style ND systems is not necessary here.

Russo's work is, in a way, representative for the LDS framework in the paradigmatic Gabbay style. There are, however, many other developments in this field. We briefly sketch some of them.

First, there is interesting work of Basin, Matthews and Viganò (*cf.* [Basin, Matthews, Viganò:1997a]). The authors present ND systems for modal propositional logics in the LDS style, but they combine the LDS framework with a logical framework (what the authors mean by a logical framework is a formal notation providing support for the uniform implementation of different logics). The implementation of their systems in such a framework is also studied in this paper. An interesting point is that the labelling algebras used by these authors are relational theories comprised of Horn clause axioms (which formalize the properties of the accessibility relation). Thus they obtain a modular account of all the modal logics that fall under the generalized Geach axiom schema. Their approach has been later generalized for other non-classical logics whose semantics may be presented in Kripke-style fashion, like relevance or intuitionistic logic (*cf.* [Basin, Matthews, Viganò:1998]). In the paper [Basin, Matthews, Viganò:1997b] the authors also present cut-free labelled sequent systems for modal logics that have been obtained by translation of their ND systems.

Second, there is also an approach in which labels are used not as "names of possible worlds", but as referring to truth-values. Such an account of modal logics may be found in [Mateus, Sernadas, Sernadas, Viganò:2004] (*cf.* also [Mateus, Rasga, Sernadas:2005]). The authors introduce algebras of truth values as labelling algebras (in the context of modal logics a truth-value is simply a set of possible worlds), and they present labelled deductive systems for modal logics in a sequent-calculus setting. The calculi have structural rules, order rules which

express basic properties of the ordering imposed on truth-values, and specific rules for the logical connectives. Even the very "specific rules" have to deal with both logical connectives and the ordering relation imposed on truth-values. The resulting approach is very complex, but, on the other hand, it is also very general, as the "truth-values labelling" approach may be extended, in a uniform way, to other types of logics, like intuitionistic, relevance or many-valued logics. (*Cf.* also the paper [Rasga, Sernadas, Sernadas, Viganò:2002] where the approach of using algebras of truth-values as labelling algebras is investigated for many different classes of logics in an ND setting.)

Third, there is the "internalized labelled deduction" approach, represented by Blackburn (*cf.* [Blackburn:2000]). Blackburn uses the *basic hybrid language*, that is, language in which labels (called *nominals* in his paper) are treated as atomic symbols, just like propositional variables.[53] Syntactically, it means that labels may be the arguments of logical connectives. The basic hybrid language is also equipped with a binary operator : which forms expressions of the form $i : \varphi$ (here i is a label and φ is a formula of the basic hybrid language). Such expressions are called the *satisfaction statements*, their intuitive interpretation is that formula φ is true in the unique world labelled by i. This intuition is expressed in the extended Kripke semantics in a very elegant way. What is striking in this approach, when compared to Gabbay-style paradigm of labelled deduction, is that the basic hybrid language *internalizes the labelling algebra*. No separate algebra of labels is needed, since the behaviour of labels is determined by the elegant extended Kripke semantics. The resulting labelled systems are simpler than those of Gabbay and the other authors which we have mentioned above, but, probably, they are not so general.

Let us conclude this subject. The framework of LDS certainly goes much further than "the occasional use of labelling with some specific purpose in mind." The advantage of this approach lies in the power of the separate labelling mechanism. The terms of a labelling algebra are used as labels, its predicates may be used to define properties of labels, the consequence relation *between labels* may be imposed on the algebra and used in the proof theory of a given logical system. The separation of the labelling mechanism from a language of a logic allows for a uniform account of many different logics in one framework. Even if we consider modal logics only, the clear advantage of LDSs over the more moderate formulations using labels (like ours) is the expressive power of the labelling language. It is usually a first-order language, whose well formed expression are, *in. a.*, $\neg R(t_1, t_2)$, like in Russo's systems. This is something our language $\textbf{\textit{M}}^*$ lacks. (Interesting notes on the expressive power of languages with labels may be found in the paper [Blackburn:2002].)

The obvious disadvantage of the LDS approach is its complexity. This is not a human-oriented framework. On the other hand, LDS is an approach suitable

[53] *Cf.* also [Blackburn, Seligman:1998] and [Blackburn, Seligman:1995].

for practically-oriented research, and this is not applications by a human, but computer applications the framework aims at.

FINAL REMARKS

Modal erotetic calculi presented in this dissertation constitute a proof method for normal modal logics which is grounded in the logic of questions **IEL**. The rules of modal erotetic calculi describe inferences whose premises and conclusions are questions of formal language *M**. These inferences may be further analysed within the framework of **IEL**. In Section II.4 we showed that the relation of positive equipollence of questions, and hence also that of pure erotetic implication of a question by a question, holds between a "premise" and a "conclusion" of such an erotetic inference. Thus the inferences are *valid*, in a well defined sense of the word.

At the same time, the method of Socratic proofs characterizes the modal logics considered in this work in terms of "inverted" sequent calculi with semantically invertible elimination rules. The construction of a Socratic transformation reflects the "root-first" proof procedure known from Gentzen-style sequent calculi, thus erotetic calculi may be used as convenient tools for proof-searching. Another interesting feature of invertible sequent calculi is their duality (*cf.* Section II.3). The method of Socratic proofs is a direct proof method, since it does not start with the negation of a formula to be proved. On the other hand, in the course of a Socratic transformation the operations are performed on formulas as if they were false, therefore the method may be also viewed as a method of a countermodel construction.

Moreover, the method is clearly motivated by the possible-worlds semantics for modal logics. As we observed in Chapter IV, the idea of relating the deductive apparatus of a proof method for a modal logic to its Kripke semantics has about 50 years, and is still vital. It is a powerful tool which makes the deductive steps intuitive, and clearly reflects the differences between various modal logics in the deductive apparatus, thus allowing for modularity. In Chapter II we presented such a modular account of modal propositional logics **K**, **D**, **T**, **KB**, **K4**, **S4** and **S5**, and we proved its completeness with respect to the underlying Kripke semantics in Chapter III. In Appendix 3 we also showed how our approach may be extended to other basic modal logics (that is, **B**, **KB4**, **K5**, **K45**, **DB**, **D4**, **D5**, **D45**).

Our method may be also adjusted to many interesting extensions of logic **S4** which we have not concerned in this work. Moreover, we think that the "modal Socratic paradigm" is fruitful enough to capture logics as rich as temporal ones

(probably, Gabbay's message to treat the labelling language "more seriously" is a good advice here). We leave it, however, for future work. We have also postponed for the future a formulation of a loop-free terminating decision procedure for transitive modal logics that may serve as a basis for implementation of modal erotetic calculi. Another aims for future research which we find interesting is extending modal erotetic calculi in order to capture the relation of global entailment, and converting modal erotetic calculi into Gentzen-style sequent calculi.

APPENDIX 1

The rules of calculus **E*** in the standard notation:

L$_\wedge$:
$$\frac{?\,(\Phi;\ S\,'\,A \wedge B\,'\,T \vdash C;\ \Psi)}{?\,(\Phi;\ S\,'\,A\,'\,B\,'\,T \vdash C;\ \Psi)}$$

R$_\wedge$:
$$\frac{?\,(\Phi;\ S \vdash A \wedge B;\ \Psi)}{?\,(\Phi;\ S \vdash A;\ S \vdash B;\ \Psi)}$$

L$_{\neg\wedge}$:
$$\frac{?\,(\Phi;\ S\,'\,\neg(A \wedge B)\,'\,T \vdash C;\ \Psi)}{?\,(\Phi;\ S\,'\,\neg A\,'\,T \vdash C;\ S\,'\,\neg B\,'\,T \vdash C;\ \Psi)}$$

R$_{\neg\wedge}$:
$$\frac{?\,(\Phi;\ S \vdash \neg(A \wedge B);\ \Psi)}{?\,(\Phi;\ S\,'\,A \vdash \neg B;\ \Psi)}$$

L$_\vee$:
$$\frac{?\,(\Phi;\ S\,'\,A \vee B\,'\,T \vdash C;\ \Psi)}{?\,(\Phi;\ S\,'\,A\,'\,T \vdash C;\ S\,'\,B\,'\,T \vdash C;\ \Psi)}$$

R$_\vee$:
$$\frac{?\,(\Phi;\ S \vdash A \vee B;\ \Psi)}{?\,(\Phi;\ S\,'\,\neg A \vdash B;\ \Psi)}$$

L$_{\neg\vee}$:
$$\frac{?\,(\Phi;\ S\,'\,\neg(A \vee B)\,'\,T \vdash C;\ \Psi)}{?\,(\Phi;\ S\,'\,\neg A\,'\,\neg B\,'\,T \vdash C;\ \Psi)}$$

R$_{\neg\vee}$:
$$\frac{?\,(\Phi;\ S \vdash \neg(A \vee B);\ \Psi)}{?\,(\Phi;\ S \vdash \neg A;\ S \vdash \neg B;\ \Psi)}$$

L$_\rightarrow$:
$$\frac{?\,(\Phi;\ S\,'\,A \rightarrow B\,'\,T \vdash C;\ \Psi)}{?\,(\Phi;\ S\,'\,\neg A\,'\,T \vdash C;\ S\,'\,B\,'\,T \vdash C;\ \Psi)}$$

R$_\rightarrow$:
$$\frac{?\,(\Phi;\ S \vdash A \rightarrow B;\ \Psi)}{?\,(\Phi;\ S\,'\,A \vdash B;\ \Psi)}$$

L$_{\neg\rightarrow}$:
$$\frac{?\,(\Phi;\ S\,'\,\neg(A \rightarrow B)\,'\,T \vdash C;\ \Psi)}{?\,(\Phi;\ S\,'\,A\,'\,\neg B\,'\,T \vdash C;\ \Psi)}$$

R$_{\neg\rightarrow}$:
$$\frac{?\,(\Phi;\ S \vdash \neg(A \rightarrow B);\ \Psi)}{?\,(\Phi;\ S \vdash A;\ S \vdash \neg B;\ \Psi)}$$

L$_{\neg\neg}$:
$$\frac{?\,(\Phi;\ S\,'\,\neg\neg A\,'\,T \vdash C;\ \Psi)}{?\,(\Phi;\ S\,'\,A\,'\,T \vdash C;\ \Psi)}$$

R$_{\neg\neg}$:
$$\frac{?\,(\Phi;\ S \vdash \neg\neg A;\ \Psi)}{?\,(\Phi;\ S \vdash A;\ \Psi)}$$

APPENDIX 2

The rules of calculus **E**** in the standard notation:

R$_\wedge$:
$$\frac{?\,(\Phi;\ \vdash S\,'\,A \wedge B\,'\,T;\ \Psi)}{?\,(\Phi;\ \vdash S\,'\,A\,'\,T;\ \vdash S\,'\,B\,'\,T;\ \Psi)}$$

R$_{\neg\wedge}$:
$$\frac{?\,(\Phi;\ \vdash S\,'\,\neg(A \wedge B)\,'\,T;\ \Psi)}{?\,(\Phi;\ \vdash S\,'\,\neg A\,'\,\neg B\,'\,T;\ \Psi)}$$

R$_{\neg\vee}$:
$$\frac{?\,(\Phi;\ \vdash S\,'\,\neg(A \vee B)\,'\,T;\ \Psi)}{?\,(\Phi;\ \vdash S\,'\,\neg A\,'\,T;\ \vdash S\,'\,\neg B\,'\,T;\ \Psi)}$$

R$_\vee$:
$$\frac{?\,(\Phi;\ \vdash S\,'\,A \vee B\,'\,T;\ \Psi)}{?\,(\Phi;\ \vdash S\,'\,A\,'\,B\,'\,T;\ \Psi)}$$

$\mathbf{R}_{\neg\rightarrow}$: $\quad\dfrac{?\,(\Phi;\; \vdash S\,'\neg(A \rightarrow B)\,'\,T;\,\Psi)}{?\,(\Phi;\; \vdash S\,'A\,'\,T;\; \vdash S\,'\neg B\,'\,T;\,\Psi)}$ $\qquad$ $\mathbf{R}_{\rightarrow}$: $\quad\dfrac{?\,(\Phi;\; \vdash S\,'A \rightarrow B\,'\,T;\,\Psi)}{?\,(\Phi;\; \vdash S\,'\neg A\,'\,B\,'\,T;\,\Psi)}$

$\mathbf{R}_{\neg\neg}$: $\quad\dfrac{?\,(\Phi;\; \vdash S\,'\neg\neg A\,'\,T;\,\Psi)}{?\,(\Phi;\; \vdash S\,'A\,'\,T;\,\Psi)}$

APPENDIX 3

The rules $\mathbf{R}_\nu$ and $\mathbf{R}_\pi$ in the standard notation:

$\mathbf{R}_\square$: $\quad\dfrac{?\,(\Phi;\; \vdash S\,'(\square A)^{\phi(i)}\,'\,T;\,\Psi)}{?\,(\Phi;\; \vdash S\,'(A)^{\phi(i),j}\,'\,T;\,\Psi)}$ $\qquad$ $\mathbf{R}_{\neg\square}$: $\quad\dfrac{?\,(\Phi;\; \vdash S\,'(\neg\square A)^{\phi(i)}\,'\,T;\,\Psi)}{?\,(\Phi;\; \vdash S\,'(\neg\square A)^{\phi(i)}\,'(\neg A)^{j}\,'\,T;\,\Psi)}$

$\mathbf{R}_{\neg\lozenge}$: $\quad\dfrac{?\,(\Phi;\; \vdash S\,'(\neg\lozenge A)^{\phi(i)}\,'\,T;\,\Psi)}{?\,(\Phi;\; \vdash S\,'(\neg A)^{\phi(i),j}\,'\,T;\,\Psi)}$ $\qquad$ $\mathbf{R}_\lozenge$: $\quad\dfrac{?\,(\Phi;\; \vdash S\,'(\lozenge A)^{\phi(i)}\,'\,T;\,\Psi)}{?\,(\Phi;\; \vdash S\,'(\lozenge A)^{\phi(i)}\,'(A)^{j}\,'\,T;\,\Psi)}$

proviso of applicability of rule $\mathbf{R}_\nu$ (rule $\mathbf{R}_\square$ and $\mathbf{R}_{\neg\lozenge}$) in each $\mathbf{E}^L$: $j \notin \mathbf{I}_W\{\vdash S\,'(\nu)^{\phi(i)}\,'\,T\}$

The inferential rules of $\mathbf{E}^L$ for $L = \mathbf{K}$, $\mathbf{T}$, $\mathbf{KB}$, $\mathbf{B}$, $\mathbf{K4}$, $\mathbf{S4}$, $\mathbf{KB4}$, $\mathbf{S5}$, $\mathbf{K5}$, $\mathbf{K45}$ are the following: $\mathbf{R}_\beta$, $\mathbf{R}_\alpha$, $\mathbf{R}_{\neg\neg}$, $\mathbf{R}_\nu$, $\mathbf{R}_\pi$. For extendable non-reflexive logics: $\mathbf{D}$, $\mathbf{DB}$, $\mathbf{D4}$, $\mathbf{D5}$, $\mathbf{D45}$ the rules of the corresponding calculus are: $\mathbf{R}_\beta$, $\mathbf{R}_\alpha$, $\mathbf{R}_{\neg\neg}$, $\mathbf{R}_\nu$, $\mathbf{R}_\pi$ and the following rule:

$\mathbf{R}_{\pi D}$: $\quad\dfrac{?\,(\Phi;\; \vdash S\,'(\pi)^{\phi(i)}\,'\,T;\,\Psi)}{?\,(\Phi;\; \vdash S\,'(\pi)^{\phi(i)}\,'(\pi_0)^{i,j}\,'\,T;\,\Psi)}$

The proviso of applicability of rule $\mathbf{R}_{\pi D}$ is the same for each $\mathbf{E}^L$ that has this rule: $j \notin \mathbf{I}_W\{\vdash S\,'(\pi)^{\phi(i)}\,'\,T\}$.

Here is the list of the provisos of applicability of rule $\mathbf{R}_\pi$ ($\mathbf{R}_{\neg\square}$ and $\mathbf{R}_\lozenge$) for the 15 basic modal propositional logics.

Calculus:	proviso:
$\mathbf{E^K}$, $\mathbf{E^D}$	$<i,j> \in \mathbf{I}_R[\vdash S\,'(\pi)^{\phi(i)}\,'\,T]$
$\mathbf{E^T}$	$<i,j> \in \mathbf{I}_R[\vdash S\,'(\pi)^{\phi(i)}\,'\,T]$ **or** $j = i$
$\mathbf{E^{KB}}$, $\mathbf{E^{DB}}$	$<i,j> \in \mathbf{I}_R[\vdash S\,'(\pi)^{\phi(i)}\,'\,T]$ **or** $<j,i> \in \mathbf{I}_R[\vdash S\,'(\pi)^{\phi(i)}\,'\,T]$
$\mathbf{E^B}$	$<i,j> \in \mathbf{I}_R[\vdash S\,'(\pi)^{\phi(i)}\,'\,T]$ **or** $<j,i> \in \mathbf{I}_R[\vdash S\,'(\pi)^{\phi(i)}\,'\,T]$ **or** $j = i$

$\mathbf{E}^{\mathbf{K4}}$, $\mathbf{E}^{\mathbf{D4}}$	there is a directed $\mathbf{I}_R[\vdash S \,'\, (\pi)^{\phi(i)} \,'\, T]$-chain $<i_1, \ldots, i_n>$ such that $i_1 = i$ and $i_n = j$
$\mathbf{E}^{\mathbf{S4}}$	there is a directed $\mathbf{I}_R[\vdash S \,'\, (\pi)^{\phi(i)} \,'\, T]$-chain $<i_1, \ldots, i_n>$ such that $i_1 = i$ and $i_n = j$ **or** $j = i$
$\mathbf{E}^{\mathbf{KB4}}$	there is an $\mathbf{I}_R[\vdash S \,'\, (\pi)^{\phi(i)} \,'\, T]$-chain $<i_1, \ldots, i_n>$ such that $i_1 = i$ and $i_n = j$
$\mathbf{E}^{\mathbf{S5}}$	there is an $\mathbf{I}_R[\vdash S \,'\, (\pi)^{\phi(i)} \,'\, T]$-chain $<i_1, \ldots, i_n>$ such that $i_1 = i$ and $i_n = j$ **or** $j = i$
$\mathbf{E}^{\mathbf{K5}}$, $\mathbf{E}^{\mathbf{D5}}$	there is k such that $<k, i>$, $<k, j> \in \mathbf{I}_R[\vdash S \,'\, (\pi)^{\phi(i)} \,'\, T]$
$\mathbf{E}^{\mathbf{K45}}$, $\mathbf{E}^{\mathbf{D45}}$	for certain k: there is a directed $\mathbf{I}_R[\vdash S \,'\, (\pi)^{\phi(i)} \,'\, T]$-chain $<i_1, \ldots, i_n>$ such that $i_1 = k$ and $i_n = i$ and there is a directed $\mathbf{I}_R[\vdash S \,'\, (\pi)^{\phi(i)} \,'\, T]$-chain $<j_1, \ldots, j_m>$ such that $j_1 = k$ and $j_m = j$

REFERENCES

[Avron:1996] A. Avron, 'The Method of Hypersequents in Proof Theory of Propositional Non-Classical Logics', in: W. Hodges, M. Hyland, Ch. Steinhorn, J. Truss (eds.), **Logic: From Foundations to Applications**, Oxford University Press, Oxford 1996, pp. 1-32

[Basin, Matthews, Viganò:1997a] D. Basin, S. Matthews, L. Viganò, 'Labelled Propositional Modal Logics: Theory and Practice', *Journal of Logic and Computation*, Vol. 7, No. 6, 1997, pp. 685-717

[Basin, Matthews, Viganò:1997b] D. Basin, S. Matthews, L. Viganò, 'A new method for bounding the complexity of modal logics', in: G. Gottlob, A. Leitsch, D. Mundici (eds.), **Computational Logic and Proof Theory**, Proceedings of the 5th Kurt Gödel Colloquium on Computational Logic and Proof Theory (KGC'97), number 1289 in Lecture Notes in Computer Science, Springer, Heidelberg 1997, pp. 89-102

[Basin, Matthews, Viganò:1998] D. Basin, S. Matthews, L. Viganò, 'Natural Deduction for Non-Classical Logics', *Studia Logica*, Vol. 60, No. 1, 1998, pp. 119-160

[Beth:1955] E. W. Beth, 'Semantic entailment and formal derivability', reprinted in: J. Hintikka (ed.), **The Philosophy of Mathematics**, Oxford University Press, London 1969, pp. 9-41

[Blackburn:2000] P. Blackburn, 'Internalizing Labelled Deduction', *Journal of Logic and Computation*, Vol. 10, No. 1, 2000, pp. 137-168

[Blackburn, Seligman:1995] P. Blackburn, J. Seligman, 'Hybrid Languages', *Journal of Logic, Language and Information*, Vol. 4, 1995, pp. 251-272

[Blackburn, Seligman:1998] P. Blackburn, J. Seligman, 'What are Hybrid Languages?', in: M. Kracht, M. de Rijke, H. Wansing, M. Zakharyaschev (eds.), **Advances in Modal Logic**, Vol. 1, CSLI Publications, Standford University 1998, pp. 41-62

[Fitting:1983] M. Fitting, **Proof Methods for Modal and Intuitionistic Logic**, Vol. 169 of *Synthese Library*, Reidel Publ. Comp., Dordrecht 1983

[Fitting:1999] M. Fitting, 'Introduction', in: M. D'Agostino, D. M. Gabbay, R. Hähnle, J. Posegga (eds.), **Handbook of Tableau Methods**, Kluwer Academic Publishers, Dordrecht 1999, pp. 1-43

[Gabbay:1996] D. M. Gabbay, **Labelled Deductive Systems**, Vol. 1, Clarendon Press, Oxford 1996

[Gentzen:1935] G. Gentzen, 'Untersuchungen über das logische Schließen', *Mathematische Zeitschrift*, Vol. 39, 1935, pp. 176-210; English translation in: 'Investigations into logical deduction', in: **The Collected Papers of Gerhard Gentzen**, M. E. Szabo (ed.), North-Holland Publishing Co., Amsterdam and London 1969, pp. 68-131

[Goré:1999] R. Goré, 'Tableau Methods for Modal and Temporal Logics', in: M. D'Agostino, D. M. Gabbay, R. Hähnle, J. Posegga (eds.), **Handbook of Tableau Methods**, Kluwer Academic Publishers, Dordrecht 1999, pp. 297-396

[Governatori, Rotolo:2001] G. Governatori, A. Rotolo, 'Labelled Modal Sequents', in: Roy Dyckhoff (ed.), Position Papers and Tutorials, Tableaux 2000, Scientific Report CS/00/0001, School of Computer Science, University of St. Andrews 2001, pp. 3-21

[Hintikka:1955] J. Hintikka, 'Form and content in quantification theory', *Acta Philosophica Fennica*, Vol. 8, 1955, pp. 8-55

[Hughes, Cresswell:1968] G. Hughes, M. Cresswell, **An Introduction to Modal Logic**, Methuen, London 1968

[Konikowska:1999] B. Konikowska, 'Rasiowa-Sikorski deduction system: a handy tool for computer science logic', Proceedings WADT'98, Springer Lecture Notes in Computer Science, Vol. 1589, Springer, Berlin 1999, pp. 183-197

[Konikowska:2002] B. Konikowska, 'Rasiowa-Sikorski deduction systems in computer science applications', *Theoretical Computer Science*, Vol. 286, 2002, pp. 323-366

[Kripke:1959] S. Kripke, 'A Completeness Theorem in Modal Logic', *Journal of Symbolic Logic*, Vol. 24, No. 1, 1959, pp. 1-14

[Kripke:1963] S. Kripke, 'Semantical Analysis of Modal Logic I, Normal Propositional Calculi', *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, Vol. 9, 1963, pp. 67-96

[Leszczyńska:2004] D. Leszczyńska, 'Socratic Proofs for Some Normal Modal Propositional Logics', *Logique et Analyse*, Vol. 47, No. 185-188, 2004, pp. 259-285

[Lis:1960] Z. Lis, 'Wynikanie semantyczne a wynikanie formalne', *Studia Logica*, Vol. 10, 1960, pp. 39-60

[Mateus, Rasga, Sernadas:2005] P. Mateus, J. Rasga, C. Sernadas, 'Modal Sequent Calculi Labelled with Truth Values: Cut Elimination', *Logic Journal of the Interest Group in Pure and Applied Logics*, Vol. 13, No. 2, 2005, pp. 173-199

[Mateus, Sernadas, Sernadas, Viganò:2004] P. Mateus, A. Sernadas, C. Sernadas, L. Viganò, 'Modal Sequent Calculi Labelled with Truth Values: Completeness, Duality and Analyticity', *Logic Journal of the Interest Group in Pure and Applied Logics*, Vol. 12, No. 3, 2004, pp. 227-274

[Matsumoto:2003] T. Matsumoto, 'A tableau system for modal logic S4 with an efficient proof-search procedure', conference paper, Proceeding of the 5th Japan Society for Software Science and Technology Workshop on Programming and Programming Languages 2003, pp. 75-86

[Negri, von Plato:2001] S. Negri, J. von Plato, **Structural Proof Theory**, Cambridge University Press, Cambridge 2001

[Onishi, Matsumoto:1957] M. Onishi, K. Matsumoto, 'Gentzen method in modal calculi I', *Osaka Mathematical Journal*, Vol. 9, 1957, pp. 113-130

[Onishi, Matsumoto:1959] M. Onishi, K. Matsumoto, 'Gentzen method in modal calculi II', *Osaka Mathematical Journal*, Vol. 11, 1959, pp. 115-120

[Perzanowski:1973] J. Perzanowski, 'The deduction theorems for the modal propositional calculi formalized after the manner of Lemmon. Part I', *Reports on Mathematical Logic*, Vol. 1, 1973, pp. 1-12

[Priest:2001] G. Priest, **An Introduction to Non-Classical Logic**, Cambridge University Press, Cambridge 2001

[Rasga, Sernadas, Sernadas, Viganò:2002] J. Rasga, A. Sernadas, C. Sernadas, L. Viganò, 'Labelled deduction over algebras of truth values', in: A. Armando (ed.), **Frontiers of Combining Systems 4**, Volume 2309 of Lecture Notes in Artificial Intelligence, Springer-Verlag, 2002, pp. 222-238

[Rasiowa, Sikorski:1963] H. Rasiowa, R. Sikorski, **The Mathematics of Metamathematics**, PAN, Warszawa 1963

[Rautenberg:1983] W. Rautenberg, 'Modal tableau calculi and interpolation', *Journal of Philosophical Logic*, Vol. 12, No. 4, 1983, pp. 403-423

[Russo:1995] A. Russo, 'Modal Labelled Deductive Systems', Technical Report 95/7, Department of Computing, Imperial College, London, UK 1995

[Smullyan:1968] R. M. Smullyan, **First-Order Logic**, Springer-Verlag, Berlin 1968

[Sundholm:2001] G. Sundholm, 'Systems of Deduction', in: D. M. Gabbay, F. Guenthner (eds.), **Handbook of Philosophical Logic**, 2nd Edition, Vol. 2, Kluwer Academic Publishers, Dordrecht, Boston, London 2001, pp. 1-52

[Wansing:2002] H. Wansing, 'Sequent systems for modal logics', in: D. M. Gabbay, F. Guenthner (eds.), **Handbook of Philosophical Logic**, 2nd Edition, Vol. 8, Kluwer Academic Publishers, Dordrecht, Boston, London 2002, pp. 61-145

[Wiśniewski:1995] A. Wiśniewski, **The Posing of Questions: Logical Foundations of Erotetic Inferences**, Kluwer Academic Publishers, Dordrecht, Boston, London 1995

[Wiśniewski:2001] A. Wiśniewski, 'Questions and Inferences', *Logique et Analyse*, Vol. 44, No. 173-175, 2001, pp. 5-43

[Wiśniewski:2004] A. Wiśniewski, 'Socratic Proofs', *Journal of Philosophical Logic*, Vol. 33, No. 3, 2004, pp. 299-326

[Wiśniewski:2005] A. Wiśniewski, 'A Right-Sided Socratic Calculus for Classical Logic. Research Report', Section of Logic and Cognitive Science, Institute of Psychology, Adam Mickiewicz University, Poznań

[Wiśniewski, Shangin:2006] A. Wiśniewski, V. Shangin, 'Socratic Proofs for Quantifiers', *Journal of Philosophical Logic*, Vol. 35, No. 2, 2006, pp. 147-178

[Wiśniewski, Vanackere, Leszczyńska:2005] A. Wiśniewski, G. Vanackere, D. Leszczyńska, 'Socratic Proofs and Paraconsistency: A Case Study', *Studia Logica*, Vol. 80, No. 2-3, 2005, pp. 433-468

LIST OF SYMBOLS

IEL	4	β	13
∈	8	α_1	13
∉	8	α_2	13
∅	8	β_1	13
⊆	8	β_2	13
⊊	8	β_1*	13
∪	8	**G***	16
∩	8	**E***	17-20, 83-84
$X_1 \times \ldots \times X_n$	8	$\mathbf{E^{RPQ}}$	21
↦	8	$\mathbf{E^{PQ}}$	21
'	9	$\mathbf{D}_{L*}$	23
CPC	9	*P*	23, 47
¬	9	$<T_P, U_P>$	23, 47
∧	9	□	25
∨	9	◊	25
→	9	*M*	25
(,)	9	ν	25
VAR	9	ν_0	25
V(A)	9	π	25
⊢	10	π_0	25
E*	10-15, 83	*M***	26
A, B, C	10	$\phi(i), \gamma(i)$	26
S, T, U	10	*V(A, w)*	27
L*	10	$\mathbf{I}_W\{\varphi\}$	28
φ, ψ	12, 27	$\mathbf{I}_R[\varphi]$	28
Φ, Ψ	12, 27	$\mathbf{E^K}$	34-37, 84
Q, Q_1, Q^*	12, 27	$\mathbf{E^L}$	37-41, 84-85
S ' T	12, 26	$\mathbf{D}_{M*}$	47
Φ; Ψ	12	**p**	51-52
α	13	*deg(A)*	62

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