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THE PENALTY AND LAGRANGE MULTIPLIER METHODS IN THE FRICTIONAL 3D BEAM-TO-BEAM CONTACT PROBLEM

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This paper is concerned with the frictional contact between beams. The purpose is to compare the results obtained by the penalty and Lagrange multiplier methods. The advantages and disadvantages of both methods are generally known and widely discussed but not in the quantitative manner and not specifically for the problem of beam-to-beam contact. The paper sketches briefly both formulations pointing out the main differences and features of both methods. The section with examples presents several cases of analysed beam-to-beam contact scenarios. The accuracy of results is taken into account, also the comparison to the full 3D analysis performed using the program ABAQUS is made. The computation times and the length of the codes for both methods are compared as well. These criteria allow one to conclude that for the beam-to-beam contact problem the Lagrange multiplier method is more attractive than the penalty method.

Keywords: beam-to-beam contact, Coulomb friction, penalty parameter method, Lagrange multiplier method

1. INTRODUCTION

The paper presents a numerical comparison between two standard methods of contact analysis for a case of beam-to-beam contact. The penalty parameter method and the Lagrange multiplier method are successfully used in the numerical modeling of contact problems by the FEM, also for the beam-to-beam case [1, 2, 5]. Accompanied by an active set strategy [3] the methods are used to solve the problem of constrained minimization of a functional, usually a strain energy for a system of several bodies. The constraints represent the impenetra-

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bility condition and the limitations resulting from a friction law in an interface. Merits and drawbacks of both methods are theoretically well known and discussed extensively, e.g. in [3]. This paper will present the quantitative analysis of the results obtained using both methods in the beam-to-beam contact problem.

In Section 2 the basic assumptions are formulated. The normal part of contact that was discussed in more detail in [1] is briefly presented in Section 3. Section 4 contains a description of the model of Coulomb friction and a summary of sliding variables, which were given in more detail in [2]. In this Section also the way to incorporate the friction constraints into the strain energy using both methods is given. The consistent linearization of the residual and the FE treatment of the friction contributions are sketched briefly. The interested Reader will find all the details in [2]. Several numerical examples are presented in Section 5. They are aimed at the comparison of results obtained with both considered formulations. The reference to the full 3D analysis performed with the program ABAQUS and the comparison of code length and computation times are also included. The last section gives the concluding remarks.

2. MAIN ASSUMPTIONS

A set of beams with rectangular cross-sections is considered. Large displacements and small strains are assumed. The contact element is formulated on the base of a linear-elastic 3D 12-dof beam element [1, 2].

The point-wise contact between beams is considered. For the beams with rectangular cross-sections their edges are four possible lines of contact. When the large strains case is excluded it is only allowed that for a pair of beams no more than two edges per beam will be involved in contact. So there can be no more than four contact points for a pair of beams. The full description of the contact search and the penetration check routines was given in the paper [1]. There the orthogonality conditions are exploited (Fig. 1):

$$\mathbf{x}_{i,i}(r_i)\cdot\left[\mathbf{x}_2(r_2)-\mathbf{x}_1(r_1)\right]=0 \tag{2.1}$$

were $\mathbf{x}_i$ are the current position vectors of points on the edges and $(\ )_{,i}=\frac{\partial(\ )}{\partial r_i}$.

Eq.2.1 usually forms a set of non-linear equations which are solved by the Newton method to yield the current local co-ordinates of contact points: r_{c1} and r_{c2}.

The virtual work G for a pair of beams consisting of two contributions due to their deformation is in the case of contact between one pair of edges supplemented by one term due to the normal contact and two terms due to two independent friction forces acting along each of the contacting beam edges:

$$G = G_1 + G_2 + \sum_1^4 \left(G_N + G_{T1} + G_{T2}\right) = 0 \tag{2.2}$$

Linearisation of this set of non-linear equations is necessary to solve it by the Newton-Raphson method:

$$\Delta G = \Delta G_1 + \Delta G_2 + \sum_1^4 \Delta\left(G_N + G_{T1} + G_{T2}\right) = 0 \tag{2.3}$$

3. NORMAL CONTACT

The normal part of the contact contribution in the penalty method has the form:

$$\begin{aligned} G_N &= \varepsilon_N g_N \delta g_N \\ \Delta G_N &= \varepsilon_N \Delta g_N \delta g_N + \varepsilon_N g_N \Delta\delta g_N \end{aligned} \tag{3.1}$$

and in the Lagrange multiplier method:

$$\begin{aligned} G_N &= \lambda_N \delta g_N + \delta\lambda_N g_N \\ \Delta G_N &= \Delta\lambda_N \delta g_N + \delta\lambda_N \Delta g_N + \lambda_N \Delta\delta g_N \end{aligned} \tag{3.2}$$

where g_N is the normal gap (see Fig. 1), λ_N is the Lagrange multiplier for the normal contact, equivalent to the normal force in the contact point and g_N is the penalty parameter, equivalent to a stiffness of a fictitious spring in the contact point.

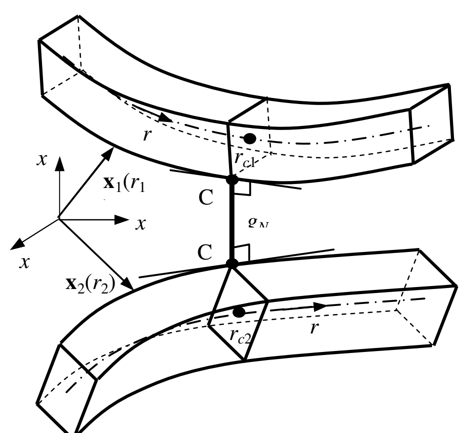

Fig. 1. A pair of contacting edges

Kinematic variables for the normal contact have the form:

$$
\begin{aligned}
\delta g_N &= \left(\delta \mathbf{u}_2 - \delta \mathbf{u}_1\right)^T \mathbf{n} \\
\Delta g_N &= \left(\Delta \mathbf{u}_2 - \Delta \mathbf{u}_1\right)^T \mathbf{n} \\
\Delta\delta g_N &= \Big[\Delta\delta \mathbf{u}_2 - \Delta\delta \mathbf{u}_1 + \left(\delta \mathbf{u}_2\right)_{,2} \Delta r_2 - \left(\delta \mathbf{u}_1\right)_{,1} \Delta r_1 + \\
&\quad + \left(\Delta \mathbf{u}_2\right)_{,2} \delta r_2 - \left(\Delta \mathbf{u}_1\right)_{,1} \delta r_1 \\
&\quad + \mathbf{x}_{2,22} \delta r_2 \Delta r_2 - \mathbf{x}_{1,11} \delta r_1 \Delta r_1 \Big] \mathbf{n} + \\
&\quad \left(\delta \mathbf{u}_2 + \mathbf{x}_{2,2} \Delta r_2 - \delta \mathbf{u}_1 + \mathbf{x}_{1,1} \Delta r_1\right) \cdot \\
&\quad \left(\mathbf{1} - \mathbf{n} \otimes \mathbf{n}\right)\left[\Delta\left(\mathbf{x}_{2,2} \times \mathbf{x}_{1,1}\right)\right] \frac{1}{\left\| \mathbf{x}_{2,2} \times \mathbf{x}_{1,1} \right\|}
\end{aligned}
\tag{3.3}
$$

where $\mathbf{u}_i$ is the displacement vector of the contact point and $\mathbf{n}$ is a unit normal vector defined as:

$$
\mathbf{n} \equiv \left(n_1, n_2, n_3\right)^T = \frac{\left(\mathbf{x}_2 - \mathbf{x}_1\right)}{\left\| \mathbf{x}_2 - \mathbf{x}_1 \right\|} \equiv \frac{\mathbf{x}_{2,2} \times \mathbf{x}_{1,1}}{\left\| \mathbf{x}_{2,2} \times \mathbf{x}_{1,1} \right\|} \tag{3.4}
$$

One may encounter the problems with the former form of the normal vector in the Lagrange multiplier method. This method leads to the exact fulfilment of the constraints and the singularity occurs. The latter form of normal vector gives a stable numerical behaviour. In the penalty method either of these definitions can be used. The same applies to the last component of $\Delta\delta g_N$ (Eq.3.3(3)), see [1], which is given here in the form which can be used in both methods.

The finite element discretisation of contact contribution bases on two two-node 12-dof beam elements within which the contact points lie. Let us denote nodal displacements of both elements as $\mathbf{u}_{M1}$ and $\mathbf{u}_{M2}$ which together represent 24 degrees of freedom of the beam-to-beam contact element in the penalty formulation:

$$
\mathbf{u}_M = \left[\mathbf{u}_{M2}{}^T, \mathbf{u}_{M1}{}^T\right]^T \tag{3.5}
$$

In the Lagrange formulation three extra unknowns must be included – the Lagrange multipliers: normal g_N and two tangential ones g_{T1} and g_{T2} yielding the vector of unknowns:

$$
\mathbf{u}_M = \left[\mathbf{u}_{M2}{}^T, \mathbf{u}_{M1}{}^T, \lambda_N, \lambda_{T1}, \lambda_{T2}\right]^T \tag{3.6}
$$

corresponding to the 27-degrees-of-freedom contact element.

The discretisation of kinematic variables yields the normal contact contributions to the tangent stiffness matrix and to the residual vectors, see [1, 2]. In the penalty method they can be expressed as follows:

$$\Delta G_N = \delta\mathbf{u}_M{}^T\left[\varepsilon_N g_N \mathbf{K}_1 + \varepsilon_N \mathbf{R}_1 \otimes \mathbf{R}_1\right] \tag{3.7}$$

$$G_N = \delta\mathbf{u}_M{}^T\left[\varepsilon_N g_N \mathbf{R}_1\right] \tag{3.8}$$

and in the Lagrange method:

$$\Delta G_N = \delta\mathbf{u}_M{}^T \begin{bmatrix} \lambda_N \mathbf{K}_1 & \mathbf{R}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{R}_1{}^T & 0 & 0 & 0 \\ \mathbf{0}^T & 0 & 0 & 0 \\ \mathbf{0}^T & 0 & 0 & 0 \end{bmatrix} \Delta\mathbf{u}_M \tag{3.9}$$

$$G_N = \delta\mathbf{u}_M{}^T \begin{bmatrix} \lambda_N \mathbf{R}_1 \\ g_N \\ 0 \\ 0 \end{bmatrix} \tag{3.10}$$

where $\mathbf{0}$ is a 24-element zero vector. The explicit forms of the (24×24)-submatrix $\mathbf{K}_1$ and 24-component vector $\mathbf{R}_1$ are given in [1].

4. FRICTION IN BEAM-TO-BEAM CONTACT

Friction between beams is modelled by the Coulomb law with a constant friction coefficient μ. The analogy to the rigid plasticity [4] is employed with the stick (elastic) and slip (plastic) cases distinguished. If the tangential force F_T has a value below its limit μF_N then beams are in stick. When the limit value is exceeded than the sliding starts and the value of the tangential force must be pulled back to its maximum possible level.

In the beam-to-beam contact the distinction between two independent friction forces for a pair of contacting edges must be considered. The two forces correspond to two possible relative movements along each of the edges and consequently two independent tangential gaps are introduced: g_{T1} and g_{T2}. Each gap is a sum of elastic and plastic part. The elastic part is subject to constraint:

$$g_{Ti}^{e} = 0 \tag{4.1}$$

while the plastic part must fulfil the non-associated flow rule:

$$\dot{g}_{Ti}^{p} = \dot{\gamma} \frac{\partial f_i}{\partial F_{Ti}} \tag{4.2}$$

The yield function for the Coulomb friction reads:

$$f_i = \left| F_{Ti} \right| - \mu F_N \le 0 \tag{4.3}$$

In the penalty method the contact forces are:

$$\begin{aligned} F_N &= \varepsilon_N g_N \\ F_{Ti} &= \varepsilon_T g_{Ti}^{e} \end{aligned} \tag{4.4}$$

where ε_T is the common penalty parameter for both friction forces. In the Lagrange multiplier method:

$$\begin{aligned} F_N &= \lambda_N \\ F_{Ti} &= \lambda_{Ti} \end{aligned} \tag{4.5}$$

where λ_{Ti} are two Lagrange multipliers for the friction part of contact.

If the condition (4.3) is fulfilled than the contact is of the stick type and the elastic gap is zero (Lagrange) or close to zero (penalty). Otherwise the slip case occurs, the tangent force is at its maximum possible level:

$$F_{Ti} = F_{T\max} = \mu F_N \tag{4.6}$$

and the edge of one beam slides along the edge of the second beam.

The tangential gap is updated [2] in the current configuration at every load, see Fig.2. In the Lagrange formulation it is exactly zero for the stick case and for the sliding its entire value represents the plastic gap. In the penalty method the elastic gap will never be zero, the bigger the penalty parameter ε_{Ti} the closer to zero it is.

In the following the subscript i denoting the number of the contacting beam (edge) is omitted to simplify the notation. The kinematic variables required to formulate the friction contributions to the virtual work may be written as:

$$\begin{aligned} g_T &= s \left\| \mathbf{x} - \mathbf{x}_p \right\| \\ \delta g_T &= \mathbf{t}^T \left(\mathbf{x}_{,i} \delta r + \delta \mathbf{u} - \delta \mathbf{u}_p \right) \\ \Delta \delta g_T &= \frac{1}{\left\| \mathbf{x}_{,i} \right\|} \left(\mathbf{x}_{,i} \delta r + \delta \mathbf{u} - \delta \mathbf{u}_p \right) \left(\mathbf{1} - \mathbf{t} \otimes \mathbf{t} \right) \left(\mathbf{x}_{,ii} \Delta r + \Delta \mathbf{u}_{,i} \right) + \\ &\quad + \mathbf{t} \left(\mathbf{x}_{,ii} \Delta r \delta r + \Delta \mathbf{u}_{,i} \delta r + \delta \mathbf{u}_{,i} \Delta r + \mathbf{x}_{,i} \Delta \delta r + \Delta \delta \mathbf{u} - \Delta \delta \mathbf{u}_p \right) \end{aligned} \tag{4.7}$$

The subscript p in Eq.4.7 corresponds to the values for the previous contact point $C_{i(n-1)}^{f}$ mapped onto the current beam configuration (see Fig. 2). The parameter s controls the sliding:

$$s = \mathrm{sign}(\lambda_T) \quad \text{or} \quad s = \mathrm{sign}(r - r_p) \tag{4.8}$$

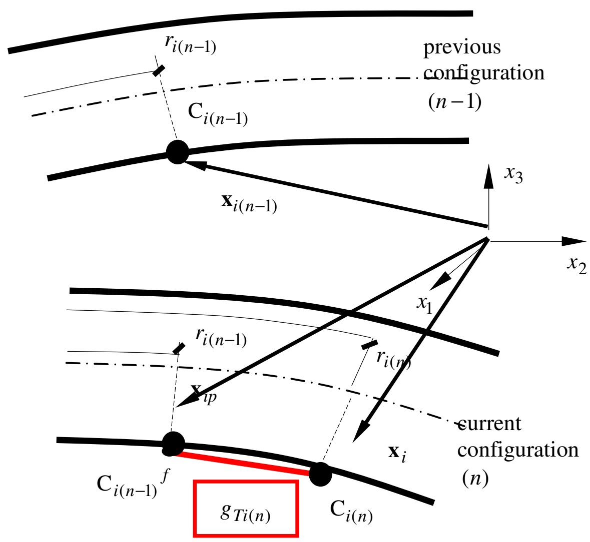

Fig. 2. A tangential gap increment

When the Lagrange multiplier method is used the similar problem as with normal gap may occur here. Due to an exact fulfilment of the friction constraints it is necessary to use the alternative form for the variable $\Delta\delta g_T$ (Eq.4.7(3)) and the following definition of the tangent vector $\mathbf{t}$:

$$\mathbf{t}_i = \frac{\mathbf{x}_{,i}}{\left\| \mathbf{x}_{,i} \right\|} \tag{4.8}$$

Now the friction contributions to the virtual work and its linearization can be written for both formulations. For the stick case and the penalty method they read:

$$\begin{aligned} G_T^{el} &= \varepsilon_T g_T \delta g_T \\ \Delta G_T^{el} &= \varepsilon_T \Delta g_T \delta g_T + \varepsilon_T g_T \Delta\delta g_T \end{aligned} \tag{4.10}$$

and for the Lagrange multiplier method:

$$\begin{aligned} G_T^{el} &= \lambda_T \delta g_T + \delta\lambda_T g_T \\ \Delta G_T^{el} &= \Delta\lambda_T \delta g_T + \delta\lambda_T \Delta g_T + \lambda_T \Delta\delta g_T \end{aligned} \tag{4.11}$$

while for the slip case and the penalty method:

$$\begin{aligned} G_T^{pl} &= \mu\varepsilon_N g_N \delta g_T^{el} \\ \Delta G_T^{pl} &= \mu\varepsilon_N \Delta g_N \delta g_T^{el} + \mu\varepsilon_N g_N \Delta\delta g_T^{el} \end{aligned} \tag{4.12}$$

and for the Lagrange multiplier method:

$$\begin{aligned} G_T^{pl} &= \mu\lambda_N \delta g_T^{el} \\ \Delta G_T^{pl} &= \mu\Delta\lambda_N \delta g_T^{el} + \mu\lambda_N \Delta\delta g_T^{el} \end{aligned} \tag{4.13}$$

The FE discretization of kinematic variables for friction given by Eqs.4.7 was presented in detail in [1] and [2] so just the general form of matrices and vectors to be added to the tangent stiffness matrix and the residual vector is given here for the consistence of the paper.

Tangent stiffness matrices for the stick case and the penalty method are:

$$\Delta G_{T1}^{el} = \delta\mathbf{u}_M{}^T \left[\varepsilon_T g_{T1} \mathbf{K}_2 + \varepsilon_T \mathbf{R}_2 \otimes \mathbf{R}_2\right] \tag{4.14}$$

$$\Delta G_{T2}^{el} = \delta\mathbf{u}_M{}^T \left[\varepsilon_T g_{T1} \mathbf{K}_3 + \varepsilon_T \mathbf{R}_3 \otimes \mathbf{R}_3\right] \tag{4.15}$$

and for the Lagrange multiplier method:

$$\Delta G_{T1}^{el} = \delta\mathbf{u}_M{}^T \begin{bmatrix} \lambda_{T1}\mathbf{K}_2 & \mathbf{0} & \mathbf{R}_2 & \mathbf{0} \\ \mathbf{0}^T & 0 & 0 & 0 \\ \mathbf{R}_2{}^T & 0 & 0 & 0 \\ \mathbf{0}^T & 0 & 0 & 0 \end{bmatrix} \Delta\mathbf{u}_M \tag{4.16}$$

$$\Delta G_{T2}^{el} = \delta\mathbf{u}_M{}^T \begin{bmatrix} \lambda_{T1}\mathbf{K}_3 & \mathbf{0} & \mathbf{0} & \mathbf{R}_3 \\ \mathbf{0}^T & 0 & 0 & 0 \\ \mathbf{0}^T & 0 & 0 & 0 \\ \mathbf{R}_3{}^T & 0 & 0 & 0 \end{bmatrix} \Delta\mathbf{u}_M \tag{4.17}$$

Residual vectors for the stick case and the penalty method are:

$$G_{T1}^{el} = \delta\mathbf{u}_M{}^T \left[\varepsilon_T g_T \mathbf{K}_2\right] \tag{4.18}$$

$$G_{T2}^{el} = \delta\mathbf{u}_M{}^T \left[\varepsilon_T g_T \mathbf{K}_3\right] \tag{4.19}$$

and for the Lagrange multiplier method:

$$G_{T1}^{el} = \delta\mathbf{u}_M{}^T \begin{bmatrix} \lambda_{T1}\mathbf{R}_2 \\ 0 \\ g_{T1} \\ 0 \end{bmatrix} \tag{4.20}$$

$$G_{T2}^{el} = \delta\mathbf{u}_M{}^T \begin{bmatrix} \lambda_{T2}\mathbf{R}_3 \\ 0 \\ 0 \\ g_{T2} \end{bmatrix} \tag{4.21}$$

Tangent stiffness matrices for the slip case and the penalty method are:

$$\Delta G_{T1}^{el} = \delta\mathbf{u}_M{}^T \left[\mu\varepsilon_N g_N \mathbf{K}_2 + \mu\varepsilon_N \mathbf{R}_1 \otimes \mathbf{R}_2\right] \tag{4.22}$$

$$\Delta G_{T2}^{el} = \delta\mathbf{u}_M{}^T \left[\mu\varepsilon_N g_N \mathbf{K}_3 + \mu\varepsilon_N \mathbf{R}_1 \otimes \mathbf{R}_3\right] \tag{4.23}$$

and for the Lagrange multiplier method:

$$\Delta G_{T1}^{pl} = \delta\mathbf{u}_M{}^T \begin{bmatrix} \mu\lambda_N\mathbf{K}_2 & \mu\mathbf{R}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^T & 0 & 0 & 0 \\ \mathbf{0}^T & -\mu & \operatorname{sign}(\lambda_{T1}) & 0 \\ \mathbf{0}^T & 0 & 0 & 0 \end{bmatrix} \Delta\mathbf{u}_M \tag{4.24}$$

$$\Delta G_{T2}^{pl} = \delta\mathbf{u}_M{}^T \begin{bmatrix} \lambda_{T1}\mathbf{K}_3 & \mu\mathbf{R}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^T & 0 & 0 & 0 \\ \mathbf{0}^T & 0 & 0 & 0 \\ \mathbf{0}^T & -\mu & 0 & \operatorname{sign}(\lambda_{T2}) \end{bmatrix} \Delta\mathbf{u}_M \tag{4.25}$$

Residual vectors for the slip case and the penalty method are:

$$G_{T1}^{pl} = \delta\mathbf{u}_M{}^T \left[\mu\varepsilon_N g_N \mathbf{K}_2\right] \tag{4.26}$$

$$G_{T2}^{el} = \delta\mathbf{u}_M{}^T \left[\mu\varepsilon_N g_N \mathbf{K}_3\right] \tag{4.27}$$

and for the Lagrange multiplier method:

$$G_{T1}^{pl} = \delta\mathbf{u}_M{}^T \begin{bmatrix} \mu\lambda_N\mathbf{R}_2 \\ 0 \\ |\lambda_{T1}| - \mu\lambda_N \\ 0 \end{bmatrix} \tag{4.28}$$

$$G_{T2}^{pl} = \delta \mathbf{u}_M{}^T \begin{bmatrix} \lambda_{T2}\mathbf{R}_3 \\ 0 \\ 0 \\ |\lambda_{T2}| - \mu\lambda_N \end{bmatrix} \tag{4.29}$$

The explicit forms of the (24×24)-submatrices $\mathbf{K}_2$ and $\mathbf{K}_3$ and 24-component vectors $\mathbf{R}_2$ and $\mathbf{R}_3$ can be taken from [1].

Note that the stiffness matrices and the residual vectors for the slip case in Lagrange formulation contain the terms, which lead to the fulfilment of the condition given in Eq.4.6.

5. NUMERICAL EXAMPLES

The performance of the beam-to-beam contact elements formulated on the base of the penalty method and the Lagrange multipliers method is shown for three numerical examples. The two methods are compared with respect to the accuracy of results and computation time. The first example includes also the comparison to the 3D analysis results obtained using the program ABAQUS. All the examples show also the sensitivity of results to the values of penalty parameters. The results are obtained using the self-written programs in FORTRAN. The codes were compiled using the Compaq compiler. The exe-file for the penalty formulation has 749 kB, the one for the Lagrange multiplier method – 777 kB. The difference results from the fact that the latter method requires more complicated treatment. More checks must be performed and more possible cases of values of normal and tangential gaps and their corresponding Lagrange multipliers must be considered. So the code is longer and one must expect longer computer times.

5.1. Example 1

Contact between two beams shown in Fig. 3 is analysed. This example was solved in [2] using the penalty method.

Beam 1 is the cantilever beam and Beam 2 has the constrained rotation around axis Y at its ends and the constraints in the central node (point B) due to the symmetry with respect to XZ-plane. The following data are used: dimensions of Beam 1: 10×10×100, dimensions of Beam 2: 5×5×100, Young's moduli for both beams: 30000, Poisson's ratios for both beams: 0.17, initial gap: 0.5, friction coefficient $\mu = 0.5$. Both beams are divided into 10 finite elements, displacements are applied using 50 increments, parameter $0.0 \le T \le 1.0$ is used to control the level of load.

Selected stages of the deformation process for the are shown in Fig. 4. These are the results for the Lagrange multiplier method. Tables 1 and 2 show the comparison of displacements of points A and B (see Fig. 3) obtained by the penalty method with various values of penalty parameters, by the Lagrange multiplier method and from the full 3D analysis. The latter was performed using the program ABAQUS. Beam 1 (master) was modelled by 2×2×20 and Beam 2 (slave) by 2×2×40 brick elements C3D8.

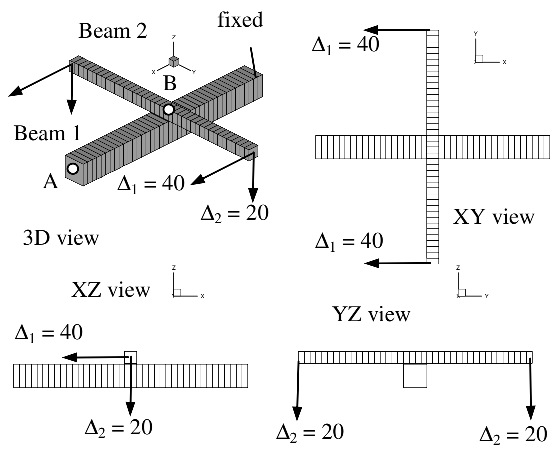

Fig. 3. Beams layout for the Example 1

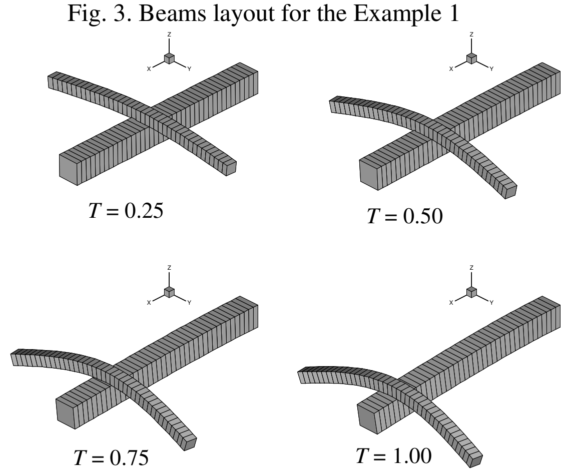

Fig. 4. Example 1 – four stages of deformation process

Table 1. Displacements of the point A

Formulation		δ_X	δ_Z
penalty			
ε_N	ε_T		
10000	400	.537	9.79
10000	800	.538	9.79
10000	1500	.539	9.80
20000	400	.539	9.81
20000	800	.539	9.82
20000	1500	.540	9.82
30000	400	.539	9.80
30000	800	.540	9.81
30000	1500	.541	9.82
Lagrange		.571	9.89
3D (ABAQUS)		.562	9.87

Table 2. Displacements of the point B

Formulation		δ_X	δ_Z
penalty			
ε_N	ε_T		
10000	400	8.02	35.6
10000	800	8.02	35.5
10000	1500	8.02	35.5
20000	400	8.01	35.6
20000	800	8.01	35.5
20000	1500	8.01	35.5
30000	400	8.00	35.6
30000	800	8.00	35.5
30000	1500	8.00	35.4
Lagrange		8.08	36.2
3D (ABAQUS)		8.27	37.2

The comparison of displacements at the points A and B shows clearly the better accuracy of the Lagrange multiplier method. For the penalty method one observes relatively small influence of the value of penalty parameter (in the considered range) on the values of displacements.

The computer time on PC, 900 MHz for the Lagrange method was 87 seconds, for the penalty method – 37 seconds. This confirms the earlier comment that the former method requires more complicated code and longer computation times.

5.2. Example 2

Two cantilever beams shown in Fig. 5 are considered.

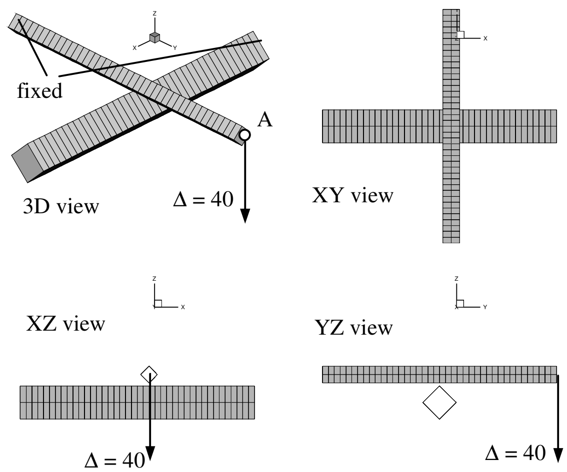

Fig. 5. Beams layout for the Example 2

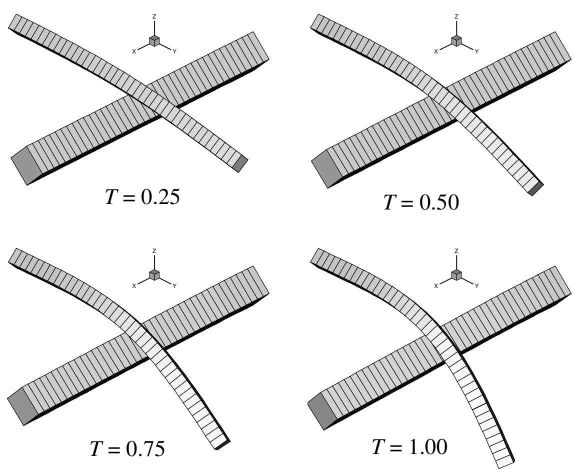

Fig. 6. Example 2 – four stages of deformation process

The data used in the example are: dimensions of Beam 1: 10×10×100, dimensions of Beam 2: 5×5×100, Young's moduli: Beam 1: 30000, Beam 2: 20000, Poisson's ratios: Beam 1: 0.17, Beam 2: 0.3, initial gap: 1.4, friction coefficient μ = 1.0. Both beams are divided into 10 finite elements, displacements are applied using 40 increments, parameter $0.0 \leq T \leq 1.0$ is used to control the level of load.

Fig. 6 presents some selected stages of the loading process obtained from the Lagrange formulation.

Graphs in Figs. 7, 8 and 9 show the influence of the penalty parameters on the percentage differences in the values of displacement δ_X at the point A, normal contact force F_N and tangential contact force F_{T1}, respectively, obtained by both considered methods.

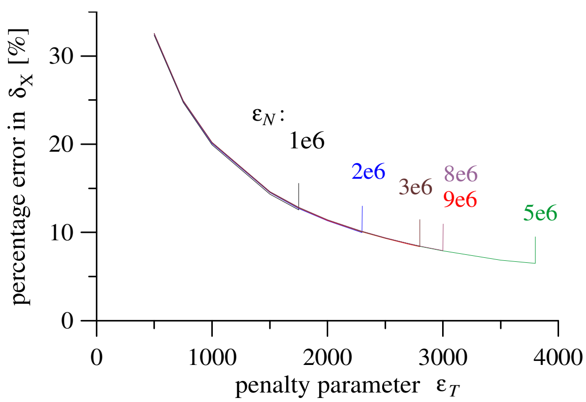

Fig. 7. Example 2 – displacements of the point A

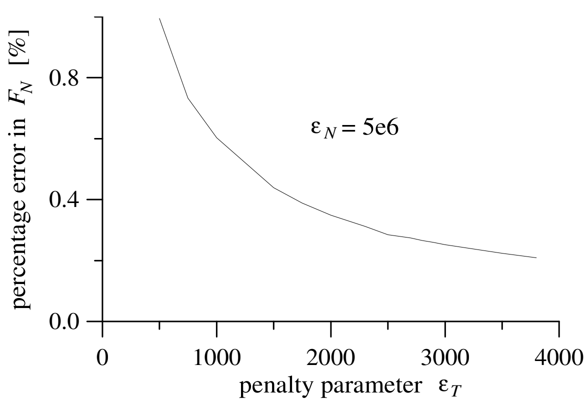

Fig. 8. Example 2 – normal contact force

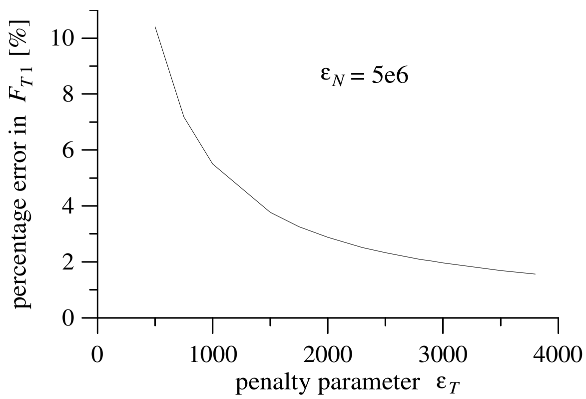

Fig. 9. Example 2 – tangential contact force

The results in Fig. 7 indicate that for the considered case the tangential penalty parameter ε_T has the clear influence on the accuracy. On the other hand the results are almost insensitive to the normal penalty parameter ε_N. The vertical lines in Fig. 7 point out the maximum values of ε_T for which the solution could still be achieved for a given normal penalty parameter ε_N. For the higher values the well known phenomenon occurred – the ill-conditioning of the equations and the convergence could not be achieved. It is also interesting to note that in this case the optimal value of the normal penalty parameter is not its highest possible level but ε_N = 5e6.

The normal contact force in this case can be calculated with a relatively high accuracy below 1% error even with low (not optimally chosen) penalty parameters. The friction force on the other hand is more sensitive to the choice of these parameters.

The computer time on PC, 900 MHz for the Lagrange method was 36 seconds, for the penalty method – 24 seconds.

5.3. Example 3

Another scenario of two cantilever beams getting in contact is considered, see Fig. 10.

The following data are used: dimensions of Beam 1: 10×10×100, dimensions of Beam 2: 5×5×100, Young's moduli: Beam 1: 30000, Beam 2: 20000, Poisson's ratios: Beam 1: 0.17, Beam 2: 0.3, Initial gap: 0.15, friction coefficient μ = 1.0. Both beams are divided into 10 finite elements, displacements are applied using 40 increments, parameter $0.0 \le T \le 1.0$ is used to control the level of load.

Fig. 11 presents some selected stages of the loading process for the Lagrange formulation.

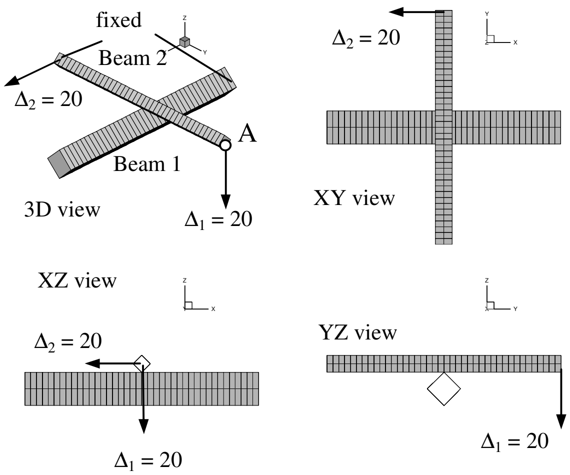

Fig. 10. Beams layout for the Example 3

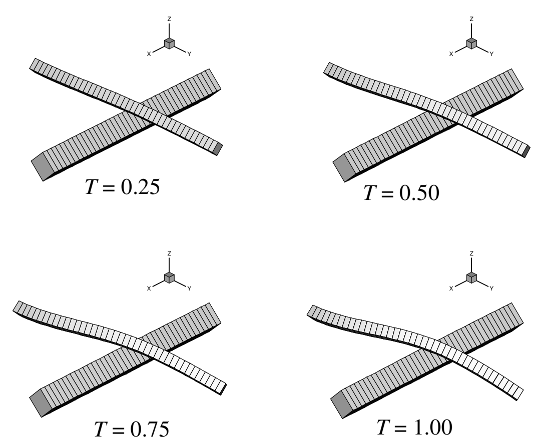

Fig. 11. Example 3 – four stages of deformation process

Graphs in Figs. 12, 13 and 14 show the influence of the penalty parameters on the percentage differences in the values of displacement δ_X at the point A, normal contact force F_N and tangential contact force F_{T1}, respectively, obtained by both considered methods.

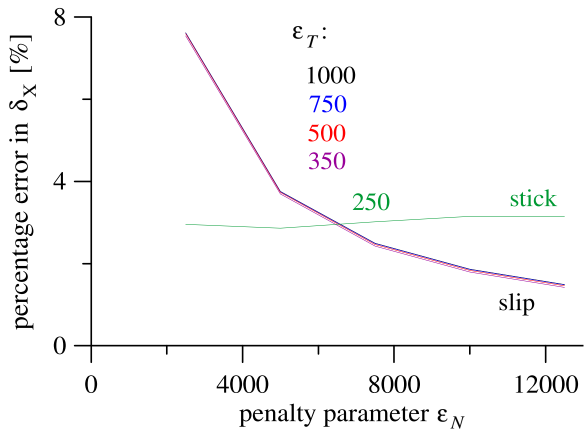

Fig. 12. Example 3 – displacements of the point A

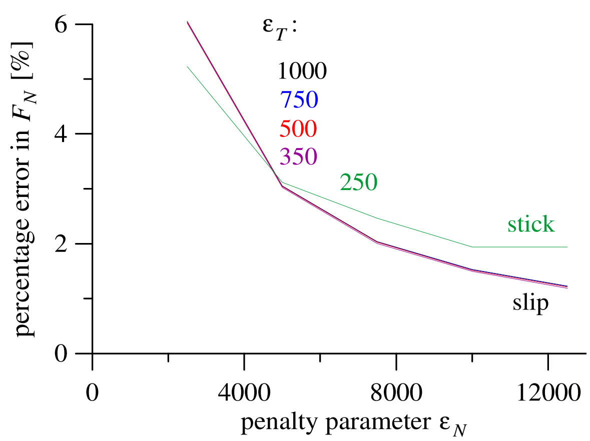

Fig. 13. Example 3 – normal contact force

In this case the results are more sensitive to the normal penalty parameter ε_N than to the tangential penalty parameter ε_T. This concerns equally displacements and forces. It is also interesting to note that there is a limiting value of ε_T between 350 and 250 for which the friction behaviour changes from slip to stick. Hence, the behaviour in the contact point may be modelled completely wrong if care is not taken when choosing the penalty parameters.

The computer time on PC, 900 MHz for the Lagrange method was 50 seconds, for the penalty method – 35 seconds.

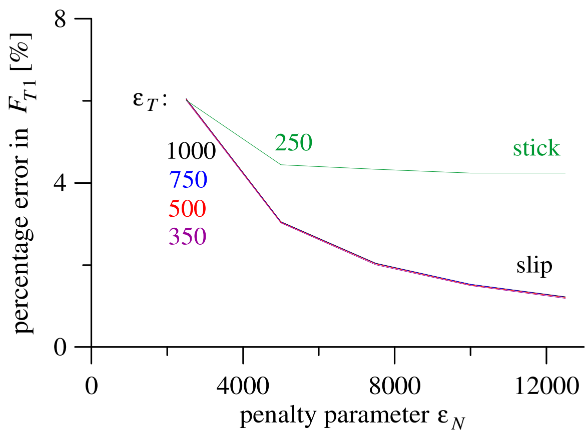

Fig. 14. Example 3 – tangential contact force

6. CONCLUDING REMARKS

In this paper results of the beam-to-beam contact analysis using the Lagrange multiplier and penalty methods are compared. It is well known that the Lagrange multiplier method introduces extra unknowns but for the case of beam-to-beam contact there are few of them. Although three multipliers per contact point are necessary but there are no more than four contact points for a pair of beams. So the increase of the problem dimension is not big. This method requires also some additional checks to be incorporated in the code. Some concern the fact that the limitations of the computer precision may result in the artificial very small separation of beams when the positive value of contacting force suggests that in fact there is still contact between beams. Generally one can say that the code in the Lagrange multiplier method is longer and more complicated.

But the method gives two main benefits: the exact fulfilment of constraints and no problem with the choice of penalty parameters. Hence the results are clearly closer to the reality. And they do not depend on the parameters which must be chosen specifically for the problem in hand. This is usually done by the 'trial-and-error' method and obliges to run the program several times unless some complicated and not fully reliable techniques are used to assess the penalty parameters on the basis of the stiffness of the beams. So in the author's opinion the Lagrange multiplier method is to be preferred in the case of the beam-to-beam contact.

Acknowledgement

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METODY WSPÓŁCZYNNIKA KARY I MNOŻNIKÓW LAGRANGE'A W ZAGADNIENIU KONTAKTU Z TARCIEM MIĘDZY BELKAMI

S t r e s z c z e n i e

Artykuł dotyczy kontaktu z tarciem między belkami. Jego celem jest porównanie wyników obliczeń uzyskanych metodami współczynnika kary i mnożników Lagrange'a. Zalety i wady obu tych metod są ogólnie znane i dyskutowane w literaturze, brakuje jednak ich porównania ilościowego dla przypadku kontaktu między belkami. W artykule krótko przedstawiono oba sformułowania i wskazano na podstawowe różnice między nimi. Przedstawiono wyniki obliczeń dla kilku przykładów kontaktujących się belek. Wzięto pod uwagę dokładność obliczeń, wyniki porównano z pełną analizą trójwymiarową wykonaną za pomocą programu ABAQUS. Porównano także długość kodu i czas obliczeń. Te kryteria pozwoliły stwierdzić, że w przypadku kontaktu między belkami metoda mnożników Lagrange'a jest bardziej atrakcyjna niż metoda współczynnika kary.