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STABILITY OF THE VISCOELASTIC MISES TRUSS

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A nonlinear equilibrium equation of Mises truss was derived in the paper. It was differential equation with respect to time due to the fact that the material from which the rods were made was viscoelastic. The six parameter rheological model was used for this material. These parameters were identified in creep bending test. The identified parameters were used in numerical example inserted in the paper. In this example the critical times for various levels of load were calculated. The differential equation was solved by means of numerical procedure *NDSolve* from the *Mathematica*™ packet.

Keywords: viscoelasticity, truss stability, critical time, creep bending test.

1. INTRODUCTION

In the stability analysis of structural elements viscoelastic properties of the material are usually neglected. In the case of many materials it is correct because actually their properites do not depend on the time. Metals and their alloys do not exhibit rhelological properties in the room temperature. The creep in such conditions are not observed. Some materials encountered in the engineering practice creep and this fact has to be taken into account in the static analysis of structures and particularly in the stability analysis. Plastics are examples of such materials. The creep of structural elements made of such materials is so significant that it must be taken into account in structural calculations. It has great significance in stability analysis of shallow trusses, arcs and shells loaded laterally. Some examples of such structures are shown in Fig. 1. In this structures the critical load depends on the rise in the power of three. The creep causes drop of the rise and this leads to the significant lowering of the critical value of the load. The buckling is just the matter of time. It has to happen. If the load level

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was reduced in advance the problem of determination of the critical time appears. It is the time which elapses between the moment of loading and the moment of buckling.

The problem of stability of the Mises truss fabricated from the material manifesting rheological properties was solved in the paper analytically. It was shown how to obtain the critical time for the given load level. The whole procedure was illustarted on the example of the Mises truss fabricated from the polimethacrylate of methyl. The procedure of identification of material parameters was presented in the paper as well.

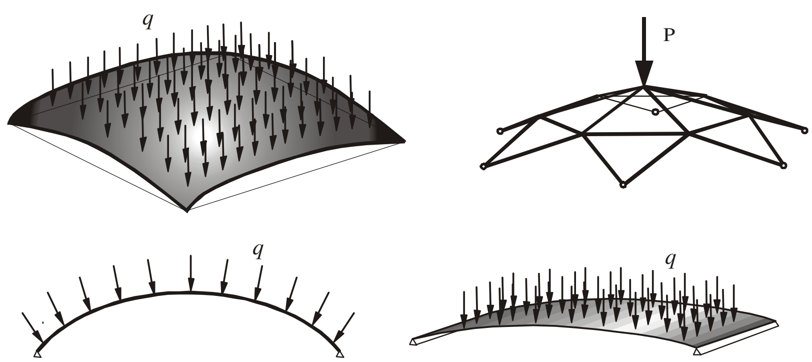


Fig. 1. Shallow space trusses, arches and shells

2. STABILITY OF MISES TRUSS MADE OF ELASTIC MATERIAL

Let us consider deformations of the two rods, plane truss known as the Mises truss. It was shown in Fig. 2. The shortening of rods as a result of the linear axial deformation may be determined from the relationship

$$\varepsilon = \frac{\Delta L}{L} = \frac{\dfrac{l}{\cos\varphi_o} - \dfrac{l}{\cos\varphi}}{\dfrac{l}{\cos\varphi_o}} = \frac{\sqrt{l^2 + H_o^2} - \sqrt{l^2 + H^2}}{\sqrt{l^2 + H_o^2}} = 1 - \sqrt{\frac{l^2 + H^2}{l^2 + H_o^2}} \, . \quad (1)$$

All notations are indicated in the figure.

If one assumes that the truss rise is small then the foregoing relationship adopts the simplified form

$$\varepsilon = \frac{H_0^2 - H^2}{2l^2} \quad . \tag{1a}$$

From the equilibrium condition written for the current configuration described by $H(t)$ the following relationship is obtained

$$T(H) = 2N \sin \varphi = 2N \frac{H}{\sqrt{l^2 + H^2}} \quad , \tag{2}$$

or

$$T(H) = 2N \frac{H}{l} \quad , \tag{2a}$$

when the small rise was assumed.

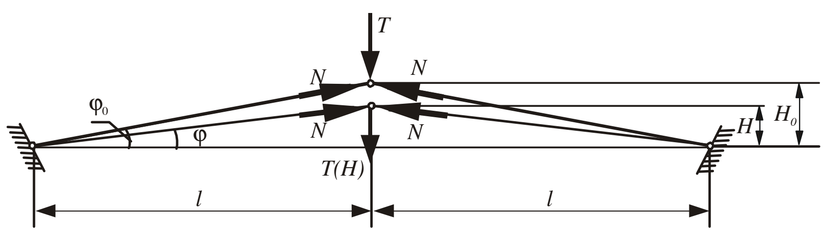


Fig. 2. The Mises truss

In these relations $T(H)$ means the value of external load for the current rise $H(t)$, and N is the axial force within the rod. The force N can be calculated on the basis of the constitutive law as follows

$$N = EA\varepsilon = EA\left(1 - \sqrt{\frac{l^2 + H^2}{l^2 + H_o^2}}\right). \tag{3}$$

From the equation (2) one obtains

$$T(H) = 2EA\left(1 - \sqrt{\frac{l^2 + H^2}{l^2 + H_o^2}}\right)\frac{H}{\sqrt{l^2 + H^2}} \quad . \tag{4}$$

This relationship was illustrated in the form of plot and shown in Fig. 3.

It is just the nonlinear equilibrium path for the whole range of the load T and the rise H. The critical load corresponding to the limit point can be easy obtained from the condition

$$\frac{dT}{dH} = 0 \;\Rightarrow\; H_{cr} = \sqrt{\left(H_o^2 l^4 + l^6\right)^{1/3} - l^2} \tag{5}$$

Substituting this result to the expression (4) one can obtain the final result for the critical value of load

$$T_{cr} = 2\,E\,A\sqrt{\left(H_o^2 l^4 + l^6\right)^{1/3} - l^2}\left(1 - \sqrt{\frac{\left(H_o^2 l^4 + l^6\right)^{1/3}}{H_o^2 + l^2}}\right)\frac{1}{\left(H_o^2 l^4 + l^6\right)^{1/6}}\,. \tag{6}$$

It is the particular value of the load for which the truss suddenly jumps adopting the inverted configuration. It is just the classical snap-through.

The formulae for $T(H)$, H_{cr}, T_{cr} are far simpler if one assumes during derivations that the rise is small. In this case

$$T(H) = H\,E\,A\,\frac{H_o^2 - H^2}{l^3}\,, \quad T_{cr} = \frac{2}{3\sqrt{3}}\frac{H_o^3}{l^3}E\,A\,, \quad H_{cr} = \frac{H_o}{\sqrt{3}}\,. \tag{7}$$

The solution in this form presents Kliusznikov [1].

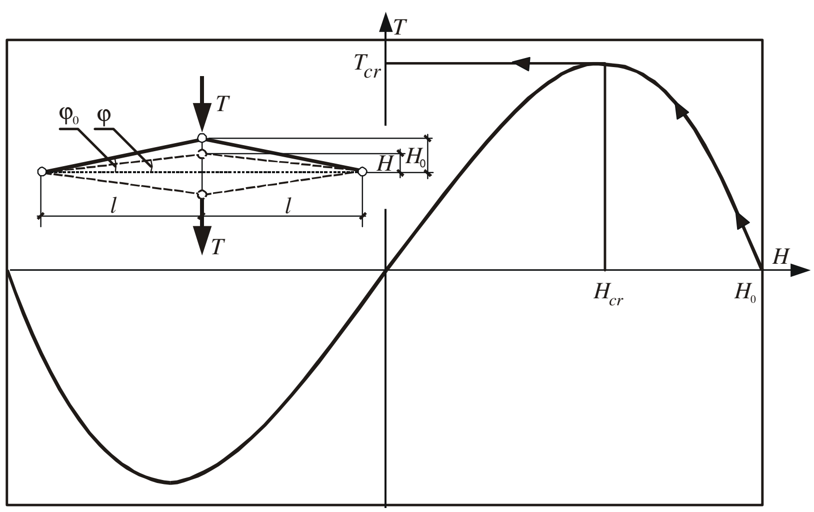


Fig. 3. The elastic solution of the Mises truss

3. THE MISES TRUSS FABRICATED FROM THE SIX PARAMETER RHEOLOGICAL MATERIAL

The constitutive law for a viscoelastic material can be written in integral or differential form (comp. Findley et al. [2]). The differential form will be used in this work. For the material model of which is shown in Fig. 4 the constitutive law can be derived without any particular difficulties. It adopts the following form

$$p_0\,\sigma + p_1\,\dot{\sigma} + p_2\,\ddot{\sigma} + p_3\,\dddot{\sigma} = q_1\,\dot{\varepsilon} + q_2\,\ddot{\varepsilon} + q_3\,\dddot{\varepsilon}\ , \tag{8}$$

where:

$$\begin{gathered} p_0 = E_1\,E_3\,E_4, \quad p_1 = E_3\,E_4\,\eta_2 + E_1\,E_3\,\eta_4 + E_1\,E_4\,\eta_3 + E_1\,E_4\,\eta_2 + E_1\,E_3\,\eta_2\,, \\ p_2 = E_3\,\eta_2\,\eta_4 + E_4\,\eta_2\,\eta_3 + E_1\,\eta_3\,\eta_4 + E_1\,\eta_2\,\eta_4 + E_1\,\eta_2\,\eta_3, \\ p_3 = \eta_2\,\eta_3\,\eta_4\,, \;\; q_1 = E_1\,E_3\,E_4\,\eta_2\,, \quad q_2 = E_1\,\eta_2\left(E_3\,\eta_4 + E_4\,\eta_3\right), \\ q_3 = E_1\,\eta_2\,\eta_3\,\eta_4\,. \end{gathered} \tag{9}$$

and E_i are Yung module of spring elements, η_k – coefficients of viscous damping of Newton elements. The notation $\dot{(\;)} = \dfrac{d(\,)}{dt}$ means differentiation with respect to time.

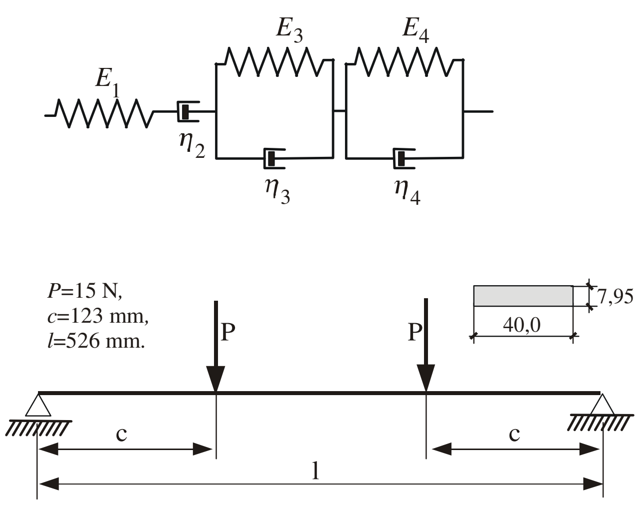


Fig. 4. The six parameter mechanical model of the material and the static scheme for the creep bending test

The equation (8) can be multiplied by the cross-sectional area A. Remembering that $N=\sigma A$ one obtains

$$p_0 N + p_1 \dot{N} + p_2 \ddot{N} + p_3 \dddot{N} = A\left(q_1 \dot{\varepsilon} + q_2 \ddot{\varepsilon} + q_3 \dddot{\varepsilon}\right). \tag{10}$$

The relation (1a) has been obtained from geometrical considerations and now from this relation one can calculate

$$\dot{\varepsilon} = -\frac{H\dot{H}}{l^2}, \quad \ddot{\varepsilon} = -\frac{1}{l^2}\left(\dot{H}\dot{H} + H\ddot{H}\right), \quad \dddot{\varepsilon} = -\frac{1}{l^2}\left(3\dot{H}\ddot{H} + H\dddot{H}\right). \tag{11}$$

From the relation (2a) one obtains

$$N = \frac{T_0 l}{2H}, \quad \dot{N} = -\frac{T_0 l}{2}\frac{\dot{H}}{H^2},$$
$$\ddot{N} = \frac{T_0 l}{2}\frac{2H\dot{H}^2 - \ddot{H}H^2}{H^4}, \quad \dddot{N} = \frac{T_0 l}{2}\frac{-6\dot{H}^3 H^4 + 6\ddot{H}\dot{H}H^5 - \dddot{H}H^6}{H^8}, \tag{12}$$

where: T_0 – the specified magnitude of the external load.

Substituting expressions (11) and (12) to the equation (10) one can obtain the final form of the equation

$$-q_1 H\dot{H} - q_2\left(\dot{H}^2 + H\ddot{H}\right) - q_3\left(3\dot{H}\ddot{H} + H\dddot{H}\right) =$$
$$a\left(p_0 \frac{1}{H} - p_1\frac{\dot{H}}{H^2} + p_2\frac{2H\dot{H}^2 - \ddot{H}H^2}{H^4} + p_3\frac{6\ddot{H}\dot{H}H^5 - 6\dot{H}^3H^4 - \dddot{H}H^6}{H^8}\right), \tag{13}$$

where:

$$a = \frac{\alpha}{3\sqrt{3}}H_0^3 E_1, \quad \alpha = \frac{T_0}{T_{cr}^D}, \quad T_{cr}^D = \frac{2}{3\sqrt{3}}\frac{H_0^3 E_1 A}{l^3}. \tag{14}$$

T_{cr}^D is the instantaneous critical load, the load which will cause buckling immediately after applying it to the truss. The specified value of the load T_0 must be smaller then T_{cr}^D from the obvious reason. It means that α coefficient must be smaller than 1.

The equation (13) must be supplemented by the following initial conditions

$$H(0) = H_p, \quad \dot{H}(0) = \dot{H}_p, \quad \ddot{H}(0) = \ddot{H}_p. \tag{15}$$

H_p is the instantaneous elastic deflection of the truss made of Hooke's material of Young modulus E_1. The value of H_p one obtains equating the force $T_0 = \alpha T_{cr}^D = \alpha \frac{2}{3\sqrt{3}} \frac{H_0^3 E_1 A}{l^3}$ to the force resulting from the relation (7) in which E should be replaced by E_1. One obtains

$$\alpha \frac{2}{3\sqrt{3}} \frac{H_0^3 E_1 A}{l^3} = H E_1 A \frac{H_o^2 - H^2}{l^3} \tag{16}$$

and finally the equation of third order on H_p for given value of α

$$\left(\frac{H}{H_0}\right)^3 - \frac{H}{H_0} + \frac{2\alpha}{3\sqrt{3}} = 0 . \tag{17}$$

As far as the remaining initial conditions are concerned, it was assumed that the load was applied very slowly (quasi-static way of loading), hence $\dot{H}(0) = 0, \quad \ddot{H}(0) = 0$.

The equation (13) is the nonlinear differential equation of the third order. It has been solved for given values of parameters by means of the procedure *NDSolve* from the *Mathematica*™ packet [3]. As the result of the solution the function $H(t)$ in the numerical form has been obtained. When the $H(t)$ attains the critical rise $\frac{H_0}{\sqrt{3}}$, the procedure of the numerical solution becomes singular. The lack of the solution in the vicinity of this particular point was the reason of this singularity. The next point of the solution occurs at the distance $2H_0$ in the inverted configuration. This particular value of the time for which the singularity occurs is the critical value of time which was looked for.

4. IDENTIFICATION OF RHEOLOGICAL PARAMETERS OF THE MATERIAL

The Mises truss was fabricated from the polymethacrylate of methyl. It was necessary to identify rheological parameters of this material. The six parameter model shown in Fig. 4 was chosen as the mechanical model of the material. The constitutive law of this model is described by the relation (8). In this equation Parameters p_i and q_i being the combination of Young module E_k and coefficients of viscosity η_k (compare relations (9)).

In order to determine numerical values of these parameters the creep bending test was performed. The specimen in the shape of single span beam is sown in Fig. 4. The beam was loaded by two concentrated forces 15 N each. The test duration was 24 hours. During the experiment, the beam central deflection was measured by means of inductive displacement transducer. Measurement were done automatically by the apparatus composed of the voltage signal amplifier, the analog-digital card, the computer and computer program written deliberately for this experiment. Results of the test were shown in form of the solid black curve shown in Fig. 5.

There exist relationship between visco-elastic deflection $w(x, t, \boldsymbol{K})$ and elastic deflection of the beam fabricated from the Hooke's material with Young modulus E. The relation was derived in [2] and takes the form

$$f(x,t,\boldsymbol{K}) = w(x,t,\boldsymbol{K}) = w^e(x)\, E_0\, J(t,\boldsymbol{K}) \,, \tag{18}$$

where:
$w^e(x)$ – the bending of the beam made of elastic material of Young modulus E_0, known for the given static scheme, $J(t,\boldsymbol{K})$ – the creep function of the viscoelastic material, $\boldsymbol{K}$ – the vector of material parameters E_i and η_i.

This relation follows directly from the so called Alfrey's analogy (cf. Findley et al. [2]).

For the static scheme shown in Fig. 4, the deflection of the middle point

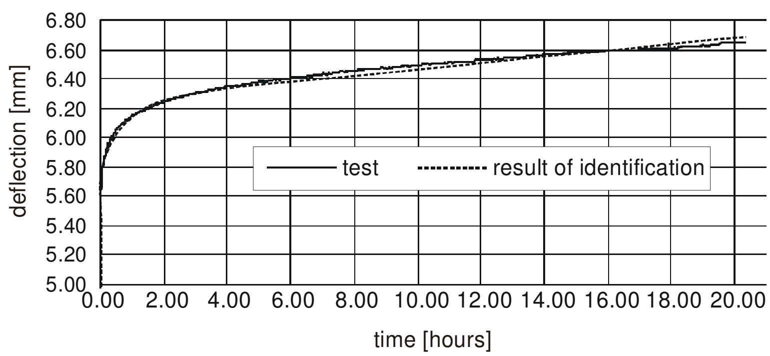


Fig. 5. Identification of rheological parameters of the material

of the elastic beam can be obtained from the formula

$$w^e = \frac{P\,a}{24\,E_0\,J}\left(3\,L^2 - 4\,a^2\right). \tag{19}$$

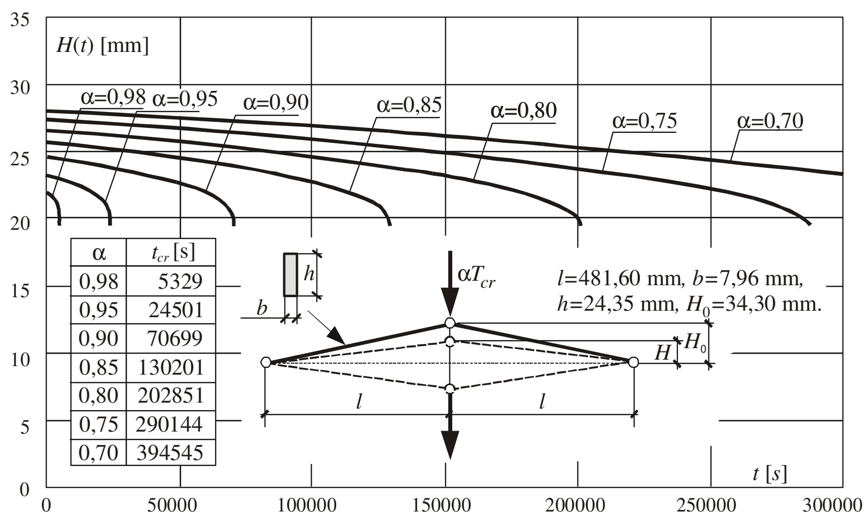
The creep function for the six parameter model can be obtained in standard way (cf. Findley et al. [2]) from the constitutive law (8). It takes the following form

$$J(t)=\frac{1}{E_1}+\frac{t}{\eta_2}+\frac{1}{E_3}\left[1-e^{\frac{-E_3 t}{\eta_3}}\right]+\frac{1}{E_3}\left[1-e^{\frac{-E_4 t}{\eta_4}}\right]. \tag{20}$$

All six parameters were determined by means of the numerical procedure purpose of which was minimization of discrepancies between measured and calculated deflections according to the relation (18). In this stage of the analysis procedures *NonlinearFit* and *NonlinearRegression* from the *Mathematica*™ packet were exploited. As the result the following numerical values for the six material parameters were obtained

$$E_1 = 3205{,}1 \text{ MPa}, \quad E_3 = 25010{,}0 \text{ MPa}, \quad E_4 = 36140{,}9 \text{ MPa},$$
$$\eta_2 = 3156963700 \text{ MPa/s}, \quad \eta_3 = 1521516 \text{ MPa/s}, \eta_4 = 111552552 \text{ MPa/s}.$$

The comparison between deflections measured in the creep bending test and the prediction following from the right hand side of the relation (18) with material parameters given above is shown in Fig. 5.



α	t_{cr} [s]
0,98	5329
0,95	24501
0,90	70699
0,85	130201
0,80	202851
0,75	290144
0,70	394545

Fig. 6. Stability of the viscoelstic Mises truss

5. NUMERICAL EXAMPLE

Knowing the actual parameters of the material and geometrical parameters of the truss shown in Fig. 6 the problem of the truss stability has been solved for various values of α parameter. As a result the family of curves was obtained. Some of them were shown in Fig. 6. Calculations were performed by means of *Mathematica*™ packet [3] (the procedure *NDSolve* were exploited). The procedure has became singular every time when $H(t)$ attained $H_{cr} = \frac{H_o}{\sqrt{3}} = 19{,}80\ \text{mm}$ indicating in this way the location of the critical time. The critical times for particular values of α parameter were presented in the table inserted in Fig. 6. According to the relation given above and for the particular data

$$T_0 = \alpha T_{cr}^D = \alpha \frac{2}{3\sqrt{3}} \frac{H_0^3 E_1 A}{l^3} = \alpha\ 86{,}39\ \text{N},$$

where α definies the actual level of loading.

6. FINAL REMARKS

The differential equation derived in the paper describes large deformations of the viscoelastic Mises truss and can be the basis for calculation of the critical time. The equation was very complicated and it was the reason that it was solved numerically by *NDSolve* procedure from the *Mathematica*™ packet. The family of solutions in the form of functions $H(t)$ for various values of α parameter was the basis for determination of the critical time. The knowledge of it has great significance from the practical point of view. The critical time is the time durnig which the structure is able to sustain savely the external load of given value. When this time elapses the structure snaps suddenly adopting the inverted configuration.

The analog procedure can be adopted for more complicated mechanical models of viscoelastic material. The identification of material parameters could be performed in similar way as it was done in this work.

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STATECZNOŚĆ LEPKOSPRĘŻYSTEJ KRATOWNICY MISESA

S t r e s z c z e n i e

W referacie przedstawiono analityczne rozwiązanie problemu stateczności kraty Misesa wykonanej z materiału liniowo sprężystego. Na wstępie przedstawiono rozwiązanie problemu stateczności tej kratownicy wykonanej z materiału liniowo sprężystego. Rozwiązanie to w sposób istotny wykorzystano w dalszym postępowaniu. Następnie rozważono problem stateczności tej kratownicy przy założeniu, że pręty wykonano z materiału opisanego sześcioparametrowym modelem reologicznym. Wyprowadzono nieliniowe równanie, którego rozwiązanie pozwala ustalić czas krytyczny, będący przedmiotem dociekań. Równanie to rozwiązywano numerycznie wykorzystując przy tym pakiet Mathematica□. Parametry reologiczne materiału, z którego wykonano pręty kratownicy Misesa (był to polimetakrylan metylu) zidentyfikowano w zgięciowej próbie pełzania. Szczegóły procedury identyfikacji parametrów materiałowych wraz z opisem stanowiska badawczego zamieszczono w pracy.