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AN EXTENSION OF THE CAYLEY–HAMILTON THEOREM FOR NONLINEAR TIME–VARYING SYSTEMS

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The classical Cayley-Hamilton theorem is extended to nonlinear time-varying systems with square and rectangular system matrices. It is shown that in both cases system matrices satisfy many equations with coefficients being the coefficients of characteristic polynomials of suitable square matrices. The proposed theorems are illustrated with numerical examples.

Keywords: extension, Cayley-Hamilton theorem, nonlinear, time-varying system

1. Introduction

The classical Cayley-Hamilton theorem (Gantmacher, 1974; Kaczorek, 1988; Lancaster, 1969) says that every square matrix satisfies its own characteristic equation. Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ (the set of $n \times n$ complex matrices) and $p(s) = \det[I_n s - A] = \sum_{i=0}^{n} a^i s_i$, $(a_n = 1)$ be the characteristic polynomial of $\mathbf{A}$. Then $p(\mathbf{A}) = \sum_{i=0}^{n} a_i \mathbf{A}^i = 0_n$ (the $n \times n$ zero matrix). The Cayley Hamilton theorem was extended to rectangular matrices (Kaczorek, 1988; 1995c), block matrices (Kaczorek, 1995b; Victoria, 1982), pairs of commuting matrices (Chang and Chan, 1992; Lewis, 1982; 1986; Kaczorek, 1988), pairs of block matrices (Kaczorek, 1988; 1998) as well as standard and singular two-dimensional linear (2-D) systems (Kaczorek, 1992; 1995a; Smart and Barnett, 1989; Theodoru, 1989). The Cayley-Hamilton theorem and its generalizations were used in control systems, electrical circuits, systems with delays, singular systems, 2-D linear systems, etc., cf. (Busłowicz, 1981; 1982; Kaczorek, 1992; 1994; Lewis, 1982; Mcrtizios and Christodolous, 1986).

In (Kaczorek, 2005a), the Cayley-Hamilton theorem was extended to n-dimensional (n-D) real polynomial matrices. An extension of the Cayley-Hamilton theorem for discrete-time and continuous-time linear systems with delay was given in (Busłowicz and Kaczorek, 2004; Kaczorek, 2005b).

In this paper, the Cayley-Hamilton theorem will be extended to the case of nonlinear time-varying systems with square and rectangular system matrices. To the best of the author's knowledge, the extension of the Cayley-Hamilton theorem for nonlinear time-varying systems has not been considered yet.

2. Square System Matrices

Consider the nonlinear time-varying system

$$\dot{x}(t) = \mathbf{A}(x,t)x(t) + \mathbf{B}(x,t)u(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input vector and $\mathbf{A} = \mathbf{A}(x,t) \in \mathbb{R}^{n\times n}$, $\mathbf{B} = \mathbf{B}(x,t) \in \mathbb{R}^{n\times m}$. The well-known notion of the characteristic polynomial (equation) for linear systems can be extended for nonlinear systems of the form (1) as follows.

Definition 1. *The polynomial*

$$\begin{aligned} p(s) &= \det\Big[\mathbf{I}_n s - \mathbf{A}(x,t)\Big] \\ &= s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 \end{aligned} \tag{2}$$

with the coefficients $a_k = a_k(x,t)$, $k = 0,1,\dots,n-1$ depending on x and t is called the characteristic polynomial *of the system (1). The equation $p(s) = 0$ is called the* characteristic equation *of the system (1).*

Theorem 1. *The system matrix $\mathbf{A}(x,t)$ satisfies the equation*

$$\sum_{i=0}^{n} a_i \mathbf{A}^{i+k}(x,t) = \mathbf{0}_n, \quad k = 0,1,\dots \quad (a_n = 1). \tag{3}$$

Proof. It is easy to check that

$$\Big[\mathbf{I}s - \mathbf{A}(x,t)\Big]\Big[\mathbf{I}_n s^{-1} + \mathbf{A}(x,t)s^{-2} + \mathbf{A}^2(x,t)s^{-3} + \dots\Big] = \mathbf{I}_n. \tag{4}$$

Hence

$$\left[\mathbf{I}s - \mathbf{A}(x,t)\right]^{-1} = \mathbf{I}_n s^{-1} + \mathbf{A}(x,t)s^{-2} + \mathbf{A}^2(x,t)s^{-3} + \dots . \quad (5)$$

The substitution of (2) and (5) into the well-known equality (Gantmacher, 1974; Kaczorek, 1988):

$$\mathrm{Adj}\left[\mathbf{I}_n s - \mathbf{A}(x,t)\right] = \left[\mathbf{I}_n s - \mathbf{A}(x,t)\right]^{-1} \det\left[\mathbf{I}_n s - \mathbf{A}(x,t)\right]$$

yields

$$\mathrm{Adj}\left[\mathbf{I}_n s - \mathbf{A}(x,t)\right] = \left[\mathbf{I}_n s^{-1} + \mathbf{A}(x,t)s^{-2} + \mathbf{A}^2(x,t)s^{-3} + \dots\right] \times \left(s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0\right). \quad (6)$$

Note that the adjoint matrix $\mathrm{Adj}\,[\mathbf{I}_n s - \mathbf{A}(x,t)]$ is a polynomial matrix in s (a matrix with a nonnegative power of s).

Comparing the coefficient matrices at the same power $s^{-(k+1)}$ of (6), we obtain (3). ■

Remark 1. For $k = 0$, from (3) we have the extension of the classical Cayley-Hamilton theorem for the nonlinear system (1):

$$p(\mathbf{A}) = \mathbf{A}^n(x,t) + a_{n-1}\mathbf{A}^{n-1}(x,t) + \dots + a_1\mathbf{A}(x,t) + a_0\mathbf{I}_n = \mathbf{0}_n. \quad (7)$$

Example 1. Consider the nonlinear system (1) with

$$\mathbf{A} = \mathbf{A}(x,t) = \begin{bmatrix} x_1e^{-t} & -2x_2^2 \\ x_1e^{-t} & x_2^2e^t \end{bmatrix}, \quad (8)$$

where $x = [x_1 \ \ x_2]^T$.

The characteristic polynomial of (8) has the form

$$\det[\mathbf{I}_n s - \mathbf{A}] = \det\begin{bmatrix} s - x_1e^{-t} & 2x_2^2 \\ -x_1e^{-t} & s - x_2^2e^t \end{bmatrix} = s^2 - \left(x_1e^{-t} + x_2^2e^t\right)s + x_1x_2^2\left(1 + 2e^t\right). \quad (9)$$

In this case,

$$a_1(x,t) = -\left(x_1e^{-t} + x_2^2e^t\right), \quad a_0(x,t) = \left(1 + 2e^{-t}\right),$$

and using (3) we obtain, for $k = 0$,

$$\begin{bmatrix} x_1e^{-t} & -2x_2^2 \\ x_1e^{-t} & x_2^2e^t \end{bmatrix}^2 - \left(x_1e^{-t} + x_2^2e^t\right)\begin{bmatrix} x_1e^{-t} & -2x_2^2 \\ x_1e^{-t} & x_2^2e^t \end{bmatrix} + x_1x_2^2\left(1 + 2e^{-t}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and, for $k = 1$,

$$\begin{bmatrix} x_1e^{-t} & -2x_2^2 \\ x_1e^{-t} & x_2^2e^t \end{bmatrix}^3 - \left(x_1e^{-t} + x_2^2e^t\right)\begin{bmatrix} x_1e^{-t} & -2x_2^2 \\ x_1e^{-t} & x_2^2e^t \end{bmatrix}^2 + x_1x_2^2\left(1 + 2e^{-t}\right)\begin{bmatrix} x_1e^{-t} & -2x_2^2 \\ x_1e^{-t} & x_2^2e^t \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore, the matrix (8) satisfies Eqn. (3) for $k = 0, 1, \dots$. ♦

3. Rectangular System Matrices

Let us consider a rectangular matrix $\mathbf{A}(x,t)$ with the number of its columns m greater than its number of rows n, i.e. $m > n$,

$$\mathbf{A}(x,t) = \left[\mathbf{A}_1(x,t) \ \ \mathbf{A}_2(x,t)\right] \in \mathbb{R}^{n\times m}, \quad \mathbf{A}_1(x,t) \in \mathbb{R}^{n\times n}, \quad \mathbf{A}_2(x,t) \in \mathbb{R}^{n\times(m-n)}. \quad (10)$$

Let

$$p_1(s) = \det\left[\mathbf{I}_n s - \mathbf{A}_1(x,t)\right] = s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0, \quad (11)$$

where the coefficients $a_k = a_k(x,t)$, $k = 0, 1, \dots, n-1$ depend on x and t.

Theorem 2. *Let the characteristic polynomial of $\mathbf{A}_1(x,t)$ have the form (11). Then the matrix (10) satisfies the equation*

$$\sum_{i=0}^{n} a_i\left[\mathbf{A}_1^{m+i-n}(x,t),\ \mathbf{A}_1^{m+i-n-1}(x,t)\,\mathbf{A}_2(x,t)\right] = \mathbf{0}_{nm}, \qquad (a_n = 1), \quad (12)$$

where $\mathbf{0}_{nm}$ is the $n \times n$ zero matrix.

Proof. By induction it is easy to show that

$$\begin{bmatrix} \mathbf{A}_1(x,t) & \mathbf{A}_2(x,t) \\ 0 & 0 \end{bmatrix}^k = \begin{bmatrix} \mathbf{A}_1^k(x,t) & \mathbf{A}_1^{k-1}(x,t)\mathbf{A}_2(x,t) \\ 0 & 0 \end{bmatrix} \quad (13)$$

for $k = 0, 1, \ldots$. Using (11), we obtain

$$\det \begin{bmatrix} \mathbf{I}_n s - \mathbf{A}_1(x,t) & \mathbf{A}_2(x,t) \\ 0 & \mathbf{I}_n s \end{bmatrix}$$

$$= s^{m-n} \det \left[\mathbf{I}_n s - \mathbf{A}_1(x,t)\right] = \sum_{i=0}^{n} a_i s^{m+i-n}. \quad (14)$$

From the classical Cayley-Hamilton theorem for the matrix

$$\begin{bmatrix} \mathbf{A}_1(x,t) & \mathbf{A}_2(x,t) \\ 0 & 0 \end{bmatrix}$$

we have

$$\sum_{i=0}^{n} a_i \begin{bmatrix} \mathbf{A}_1(x,t) & \mathbf{A}_2(x,t) \\ 0 & 0 \end{bmatrix}^{m+i-n} = \mathbf{0}_m \quad (a_n = 1). \quad (15)$$

The substitution of (13) into (15) yields

$$\sum_{i=0}^{n} a_i \begin{bmatrix} \mathbf{A}_1^{m+i-n}(x,t) & \mathbf{A}_1^{m+i-n-1}(x,t)\mathbf{A}_2(x,t) \\ 0 & 0 \end{bmatrix} = \mathbf{0}_m, \quad (a_n = 1) \quad (16)$$

Taking into account only the first n rows of (16), we obtain (12). ∎

Remark 2. The matrix $\mathbf{A}_1(x,t)$ can be constructed from any n columns of the matrix $\mathbf{A}(x,t)$ (Kaczorek, 1988).

Theorem 3. *Let the characteristic polynomial of $\mathbf{A}_1(x,t)$ have the form (11). Then*

$$\sum_{i=0}^{n} a_i \left[\mathbf{A}_1^i(x,t) \quad \mathbf{A}_1^i(x,t)\mathbf{A}_2(x,t)\right] = \mathbf{0}_{nm}, \quad (a_n = 1). \quad (17)$$

Proof. From the classical Cayley-Hamilton theorem for the matrix $\mathbf{A}_1(x,t)$ we have

$$\sum_{i=0}^{n} a_i \mathbf{A}_1^i(x,t) = \mathbf{0}_n, \quad (a_n = 1). \quad (18)$$

The postmultiplication of (18) by the matrix $[\mathbf{I}_n \quad \mathbf{A}_2(x,t)]$ yields (17). ∎

Example 2. Consider the rectangular matrix

$$\mathbf{A}(x,t) = [\mathbf{A}_1(x,t) \quad \mathbf{A}_2(x,t)]$$

$$= \begin{bmatrix} e^{-t}\sin x_1 & e^{-2t}\cos x_2 & x_2 \sin x_1 \\ -e^{t}\cos x_2 & \sin x_1 & x_1 e^{-t} \end{bmatrix}, \quad (19)$$

where $x = [x_1 \quad x_2]^T$.

The characteristic polynomial of the matrix $\mathbf{A}_1(x,t)$ has the form

$$\begin{aligned} p_1(s) &= \det [\mathbf{I}_n s - \mathbf{A}_1(x,t)] \\ &= \det \begin{bmatrix} s - e^{-t}\sin x_1 & -e^{-2t}\cos x_2 \\ e^{t}\cos x_2 & s - \sin x_1 \end{bmatrix} \\ &= s^2 + a_1 s + a_0, \end{aligned} \quad (20)$$

where

$$a_1 = a_1(x,t) = -\left(1 + e^{-t}\right)\sin x_1,$$

$$a_0 = a_0(x,t) = e^{-t}\left(\sin^2 x_1 + \cos^2 x_2\right).$$

Using (12), we obtain the result given by (21). Equation (17) in this case has the form (22). Therefore, the matrix (19) satisfies (12) and (17). ♦

If $n > m$, then the matrix $\mathbf{A}(x,t)$ can be written in the form

$$\mathbf{A}(x,t) = \begin{bmatrix} \mathbf{A}_1(x,t) \\ \mathbf{A}_2(x,t) \end{bmatrix} \in \mathbb{R}^{n\times m}, \quad (23)$$

$$\mathbf{A}_1(x,t) \in \mathbb{R}^{m\times m}, \quad \mathbf{A}_2(x,t) \in \mathbb{R}^{(n-m)\times m}.$$

In much the same way as Theorem 1, the following dual theorem can be proved.

Theorem 4. *Let $\mathbf{A}(x,t)$ have the form (23) and*

$$\begin{aligned} \bar{p}_1(s) &= \det \left[\mathbf{I}_n s - \mathbf{A}_1(x,t)\right] \\ &= s^m + \bar{a}_{m-1}s^{m-1} + \cdots + \bar{a}_1 s + \bar{a}_0, \end{aligned} \quad (24)$$

where the coefficients $\bar{a}_i = \bar{a}_i(x,t)$, $i = 0, 1, \ldots, m-1$ are functions of x and t. Then

$$\sum_{i=0}^{m} \bar{a}_i \begin{bmatrix} \mathbf{A}_1^{n+i-m}(x,t) \\ \mathbf{A}_2(x,t)\mathbf{A}_1^{n+i-m+1}(x,t) \end{bmatrix} = \mathbf{0}_{nm} \quad (a_m = 1). \quad (25)$$

From the classical Cayley-Hamilton theorem for $\mathbf{A}_1(x,t)$ and (24), we have

$$\sum_{i=0}^{m} \bar{a}_i \mathbf{A}_1^i(x,t) = 0 \quad (\bar{a}_m = 1). \quad (26)$$

The premultiplication of (26) by the matrix

$$\begin{bmatrix} \mathbf{I}_m \\ \mathbf{A}_2(x,t) \end{bmatrix}$$

$$\Big[\mathbf{A}_1^3(x,t),\ \mathbf{A}_1^2(x,t)\mathbf{A}_2(x,t)\Big] + a_1(x,t)\Big[\mathbf{A}_1^2(x,t),\ \mathbf{A}_1(x,t)\mathbf{A}_2(x,t)\Big] + a_0(x,t)\Big[\mathbf{A}_1(x,t),\ \mathbf{A}_2(x,t)\Big]$$

$$= \left[\begin{matrix} e^{-3t}\sin^3 x_1 - 2e^{-2t}\sin x_1\cos^2 x_2 - e^{-t}\sin x_1\cos^2 x_2 \\ \cos^3 x_2 - e^{-t}\sin^2 x_1\cos x_2 - e^{-2t}\sin^2 x_1\cos x_2 - e^{t}\sin^2 x_1\cos x_2 \end{matrix}\right.$$

$$\begin{matrix} e^{-4t}\sin^2 x_1\cos x_2 - e^{-3t}\left(\cos^3 x_2 - \sin^2 x_1\cos x_2\right) + e^{-2t}\sin^2 x_1\cos x_2 \\ -\left(e^{-2t}+e^{-3t}\right)\sin x_1\cos^2 x_2 - e^{-t}\sin x_1\cos^2 x_2 + \sin^3 x_1 \end{matrix}$$

$$\left.\begin{matrix} e^{-4t}x_1\sin x_1\cos x_2 + e^{-3t}x_1\sin x_1\cos x_2 + e^{-2t}x_2\sin^3 x_1 - e^{-t}x_2\sin x_1\cos^2 x_2 \\ e^{-t}x_1\sin^2 x_1 - e^{-2t}x_1\cos^2 x_2 - e^{t}x_2\sin^2 x_1\cos x_2 - x_2\sin^2 x_1\cos x_2 \end{matrix}\right] \tag{21}$$

$$-\left(1+e^{-t}\right)\sin x_1 \left[\begin{matrix} e^{-2t}\sin^2 x_1 - e^{-t}\cos^2 x_2 & e^{-3t}\sin x_1\cos x_2 + e^{-2t}\sin x_1\cos x_2 \\ -\sin x_1\cos x_2 - e^{-t}\sin x_1\cos x_2 & \sin^2 x_1 - e^{-t}\cos^2 x_2 \end{matrix}\right.$$

$$\left.\begin{matrix} e^{-3t}x_1\cos x_2 + e^{-t}x_2\sin^2 x_1 \\ e^{-t}x_1\sin x_1 - e^{-t}x_2\sin x_1\cos x_2 \end{matrix}\right]$$

$$+e^{-t}\left(\sin^2 x_1 + \cos^2 x_2\right)\begin{bmatrix} e^{-t}\sin x_1 & e^{-2t}\cos x_2 & x_2\sin x_1 \\ -e^{t}\cos x_2 & \sin x_1 & x_1 e^{-t} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Big[\mathbf{A}_1^2(x,t),\ \mathbf{A}_1^2(x,t)\mathbf{A}_2(x,t)\Big] + a_1(x,t)\Big[\mathbf{A}_1(x,t),\ \mathbf{A}_1(x,t)\mathbf{A}_2(x,t)\Big] + a_0(x,t)\Big[\mathbf{I}_n,\ \mathbf{A}_2(x,t)\Big]$$

$$= \left[\begin{matrix} e^{-2t}\sin^2 x_1 - e^{-t}\cos^2 x_2 & e^{-3t}\sin x_1\cos x_2 + e^{-2t}\sin x_1\cos x_2 \\ -\sin x_1\cos x_2 - e^{t}\sin x_1\cos x_2 & e^{-t}\cos^2 x_2 + \sin^2 x_1 \end{matrix}\right.$$

$$\left.\begin{matrix} e^{-4t}x_1\sin x_1\cos x_2 + e^{-3t}\sin x_1\cos x_2 + e^{-2t}x_2\sin^3 x_1 - e^{-t}x_2\sin x_1\cos^2 x_2 \\ e^{-t}x_1\sin^2 x_1 - e^{-2t}x_1\cos^2 x_2 - e^{-t}x_2\sin^2 x_1\cos x_2 - x_2\sin^2 x_1\cos x_2 \end{matrix}\right] \tag{22}$$

$$-\left(1+e^{-t}\right)\sin x_1 \begin{bmatrix} e^{-t}\sin x_1 & e^{-2t}\cos x_2 & e^{-t}x_2\sin^2 x_1 + e^{-3t}x_1\cos x_2 \\ -e^{t}\cos x_2 & \sin x_1 & e^{-t}x_1\sin x_1 - e^{t}x_2\sin x_1\cos x_2 \end{bmatrix}$$

$$+e^{-t}\left(\sin^2 x_1 + \cos^2 x_2\right)\begin{bmatrix} 1 & 0 & x_2\sin x_1 \\ 0 & 1 & x_1 e^{-t} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

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yields

$$\sum_{i=0}^{m} \bar{a}_i \begin{bmatrix} \mathbf{A}_1^i(x,t) \\ \mathbf{A}_2(x,t)\mathbf{A}_1^i(x,t) \end{bmatrix} = \mathbf{0}_{mn}$$

$$(\bar{a}_m = 1). \tag{27}$$

Therefore we have proved the following theorem.

Theorem 5. *Let the characteristic polynomial of* $\mathbf{A}_1(x,t)$ *have the form (24). Then the matrix (23) satisfies Eqn. (27).*

Remark 3. Equation (12) can be obtained by the postmultiplication of (18) by the matrix

$$\Big[\mathbf{A}_1^{m-n}(x,t) \quad \mathbf{A}_1^{m-n-1}(x,t)\mathbf{A}_2(x,t)\Big]$$

and Eqn. (25) by the premultiplication of (26) by the matrix

$$\begin{bmatrix} \mathbf{A}_1^{n-m}(x,t) \\ \mathbf{A}_2(x,t)\mathbf{A}_1^{n-m-1}(x,t) \end{bmatrix}.$$

4. Concluding Remarks

The Cayley-Hamilton theorem has been extended for nonlinear time-varying systems with square (Theorem 1) and rectangular (Theorems 2–5) system matrices. It was shown that in both cases the system matrices satisfy many equations. For rectangular system matrices, starting from characteristic polynomials of square matrices, it is possible to obtain many different equations that are satisfied by these system matrices. Note that the equations are satisfied for all parameters of nonlinear systems.

The presented generalizations can be extended to block matrices and two-dimensional nonlinear time-varying systems.

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