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FAULT TOLERANT CONTROL OF SWITCHED NONLINEAR SYSTEMS WITH TIME DELAY UNDER ASYNCHRONOUS SWITCHING

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This paper investigates the problem of fault tolerant control of a class of uncertain switched nonlinear systems with time delay under asynchronous switching. The systems under consideration suffer from delayed switchings of the controller. First, a fault tolerant controller is proposed to guarantee exponentially stability of the switched systems with time delay. The dwell time approach is utilized for stability analysis and controller design. Then the proposed approach is extended to take into account switched time delay systems with Lipschitz nonlinearities and structured uncertainties. Finally, a numerical example is given to illustrate the effectiveness of the proposed method.

Keywords: time delay, fault tolerant control, switched nonlinear systems, asynchronous switching.

1. Introduction

Switched systems belong to a special class of hybrid control systems that comprises a collection of subsystems together with a switching rule which specifies the switching among the subsystems. Many practical systems are inherently multimodal in the sense that several dynamical systems are required to describe their behavior, which may depend on various environmental factors. Besides, switched systems are widely applied in many fields, including mechanical systems, automotive industry, aircraft and air traffic control, and many other domains (Varaiya, 1993; Wang and Brockett, 1997; Tomlin *et al.*, 1998).

During the last decades there have been many studies on stability analysis and the design of stabilizing feedback controllers for switched systems. The interest in this direction is reflected by numerous works (Sun, 2004; 2006; Cheng *et al.*, 2005; Liberzon, 2003; Lin and Antsaklis, 2009). As an important analytic tool, the multiple Lyapunov function approach has been employed to analyze the stability of switched systems, which has been shown to be very efficient (Zhai *et al.*, 2007; Hespanha, 2004; Hespanha *et al.*, 2005). Based on the dwell time method, stability analysis and stabilization for switched systems have also been investigated (De Persis *et al.*, 2002; Wang and Zhao, 2007; Sun *et al.*, 2006a; De Persis *et al.*, 2003).

The time delay phenomenon is very common in many kinds of engineering systems, for instance, long-distance transportation systems, hydraulic pressure systems, networked control systems and so on, so time delay systems have also received increased attention in the control community (Guo and Gao, 2007; Guan and Gao, 2007). Many valuable results have been obtained for systems of this type (Zhang *et al.*, 2007a; Gao *et al.*, 2008; Xiang and Wang, 2009a; Sun *et al.*, 2006b; Zhang *et al.*, 2007b). On the other hand, actuators may be subjected to failures in a real environment. Therefore, it is of practical interest to investigate a control system which can tolerate faults of actuators. Several approaches to the design of reliable controllers have been proposed (Lien *et al.*, 2008; Yao and Wang, 2006; Abootalebi *et al.*, 2005; Liu *et al.*, 1998; Yu, 2005). A reliable controller is designed for switched nonlinear systems using the multiple Lyapunov function approach by Wang *et al.* (2007).

However, there inevitably exists asynchronous switching between the controller and the system in actual operation, which deteriorates the performance of systems. Therefore, it is important to investigate the problem of the stabilization of switched systems under asynchronous switching (Xie and Wang, 2005; Xie *et al.*, 2001; Ji *et al.*, 2007; Hetel *et al.*, 2007; Mhaskar *et al.*, 2008; Xiang and Wang, 2009b).

In this paper, we are interested in the problem of fault tolerant control for a class of uncertain nonlinear switched systems with time delay and actuator failures under asynchronous switching. The remainder of the paper is organized as follows. In Section 2, problem formulation and some necessary lemmas are given. In Section 3, based on the dwell time approach and the linear matrix inequality (LMI) technique, we first consider the design of a fault tolerant controller and a switching signal for a switched system with time delay under asynchronous switching. Sufficient conditions for the existence of the controller are obtained in terms of a set of LMIs. Then the design approach to the controller for a switched nonlinear system with time delay under asynchronous switching is presented. A numerical example is given to illustrate the effectiveness of the proposed design approach in Section 4. Concluding remarks are given in Section 5.

Notation. Throughout this paper, the superscript 'T' denotes the transpose, $\|\cdot\|$ denotes the Euclidean norm. $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote the maximum and minimum eigenvalues of matrix P, respectively, I is an identity matrix of appropriate dimensions. The asterisk '$*$' in a matrix is used to denote a term that is induced by symmetry. The set of positive integers is represented by $\mathbb{Z}^+$.

2. System description and preliminaries

Let us consider the following switched system with time delay and an actuator failure:

$$\begin{aligned}\dot{x}(t) &= \hat{A}_{\sigma(t)}x(t) + \hat{A}_{d\sigma(t)}x(t-d) \\ &\quad + B_{\sigma(t)}u^f(t) + D_{\sigma(t)}f_{\sigma(t)}(x(t),t), \quad (1)\\ x(t) &= \phi(t), \quad t \in [t_0 - d, t_0], \quad (2)\end{aligned}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u^f(t) \in \mathbb{R}^l$ is the input of an actuator fault, d denotes the state delay, $\phi(t)$ is a continuous vector-valued function. The function $\sigma(t) : [t_0, \infty) \to \underline{N} = \{1, 2, \ldots, N\}$ is the system switching signal, and N denotes the number of the subsystems. The switching signal $\sigma(t)$ discussed in this paper is time-dependent, i.e., $\sigma(t) : \{(t_0, \sigma(t_0)), (t_1, \sigma(t_1)), \cdots\}$, where t_0 is the initial time, and t_k denotes the k-th switching instant. $\hat{A}_i, \hat{A}_{di}$ for $i \in \underline{N}$ are uncertain real-valued matrices with appropriate dimensions which satisfy

$$\hat{A}_i = A_i + H_iF_i(t)E_{1i}, \quad \hat{A}_{di} = A_{di} + H_iF_i(t)E_{di}, \quad (3)$$

where $A_i, A_{di}, H_i, E_{1i}, E_{di}$ are known real constant matrices with proper dimensions imposing the structure of the uncertainties. Here $F_i(t)$ for $i \in \underline{N}$ are unknown time-varying matrices which satisfy

$$F_i^T(t)F_i(t) \le I, \quad (4)$$

D_i and B_i for $i \in \underline{N}$ are known real constant matrices, and $f_i(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ for $i \in \underline{N}$ are unknown nonlinear functions satisfying the following Lipschitz conditions:

$$\|f_i(x(t),t)\| \le \|U_i x(t)\|, \quad (5)$$

where U_i are known real constant matrices.

However, there inevitably exists asynchronous switching between the controller and the system in actual operation. Suppose that the i-th subsystem is activated at the switching instant t_{k-1}, the j-th subsystem is activated at the switching instant t_k, and the corresponding switching controller is activated at the switching instants $t_{k-1} + \Delta_{k-1}$ and $t_k + \Delta_k$, respectively. The case that the switching instants of the controller experience delays with respect to those of the system can be shown as in Fig. 1. There we can see that controller K_i corresponding to the i-th subsystem operates the i-th subsystem in $[t_{k-1} + \Delta_{k-1}, t_k)$, and operates the j-th subsystem in $[t_k, t_k + \Delta_k)$.

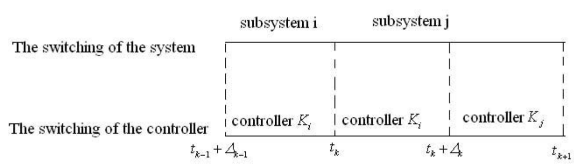

Fig. 1. Diagram of asynchronous switching.

Denoting by $\sigma'(t)$ the switching signal of the controller, the corresponding switching instants can be written as

$$t_1 + \Delta_1, t_2 + \Delta_2, \ldots, t_k + \Delta_k, \ldots, k \in \mathbb{Z}^+,$$

where $\Delta_k (|\Delta_k| < d)$ represents the period that the switching instant of the controller lags behind the one of the system, and the period is said to be mismatched.

Remark 1. The mismatched period

$$\Delta_k < \inf_{k \ge 0}(t_{k+1} - t_k)$$

guarantees that there always exists a period $[t_{k-1} + \Delta_{k-1}, t_k)$. This period is said to be matched in what follows.

The input of an actuator fault is described as

$$u^f(t) = M_{\sigma'(t)}u(t), \tag{6}$$

where M_i for $i \in \underline{N}$ are actuator fault matrices,

$$M_i = \mathrm{diag}\{m_{i1}, m_{i2}, \dots, m_{il}\}$$

$$0 \le \underline{m}_{ik} \le m_{ik} \le \overline{m}_{ik}, \quad \overline{m}_{ik} \ge 1, k = 1, 2, \dots, l. \tag{7}$$

For simplicity, we introduce the following notation:

$$M_{i0} = \mathrm{diag}\{\tilde{m}_{i1}, \tilde{m}_{i2}, \dots, \tilde{m}_{il}\}, \tag{8}$$

$$J_i = \mathrm{diag}\{j_{i1}, j_{i2}, \dots, j_{il}\}, \tag{9}$$

$$L_i = \mathrm{diag}\{l_{i1}, l_{i2}, \dots, l_{il}\}, \tag{10}$$

where

$$\tilde{m}_{ik} = \frac{1}{2}(\overline{m}_{ik} + \underline{m}_{ik}),$$
$$j_{ik} = \frac{\overline{m}_{ik} - \underline{m}_{ik}}{\overline{m}_{ik} + \underline{m}_{ik}},$$
$$l_{ik} = \frac{m_{ik} - \tilde{m}_{ik}}{\tilde{m}_{ik}}.$$

By (8)–(10), we have

$$M_i = M_{i0}(I + L_i), \quad |L_i| \le J_i \le I, \tag{11}$$

where

$$|L_i| = \mathrm{diag}\{|l_{i1}|, |l_{i2}|, \dots, |l_{il}|\}.$$

Remark 2. Note that $m_{ik} = 1$ means normal operation of the k-th actuator signal of the i-th subsystem. When $m_{ik} = 0$, it covers the case of the complete failure of the k-th actuator signal of the i-th subsystem. When $\underline{m}_{ik} > 0$ and $m_{ik} \neq 1$, it corresponds to the case of a partial failure of the k-th actuator signal of the i-th subsystem. The system (1)–(2) without uncertainties can be described as

$$\dot{x}(t) = A_{\sigma(t)}x(t) + A_{d\sigma(t)}x(t-d) + B_{\sigma(t)}u^f(t) + D_{\sigma(t)}f_{\sigma(t)}(x(t), t), \tag{12}$$

$$x(t) = \phi(t), \quad t \in [t_0 - d, t_0]. \tag{13}$$

The system (12)–(13) without nonlinear terms can be written as

$$\dot{x}(t) = A_{\sigma(t)}x(t) + A_{d\sigma(t)}x(t-d) + B_{\sigma(t)}u^f(t), \tag{14}$$

$$x(t) = \phi(t), \qquad t \in [t_0 - d, t_0]. \tag{15}$$

Definition 1. If there exists a switching signal $\sigma(t)$, such that the trajectory of the system (1)–(2) satisfies $\|x(t)\| \le \alpha \|x(t_0)\| e^{-\beta(t-t_0)}$, where $\alpha \ge 1$, $\beta > 0$, $t \ge t_0$, then the system (1)–(2) is said to be *exponentially stable*.

The following lemmas play an important role in our further developments.

Lemma 1. (Halanay, 1966) *Let $r \ge 0$, $a > b > 0$. If there exists a real-value continuous function $u(t) \ge 0$, $t \ge t_0$ such that the differential inequality*

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} \le -au(t) + b \sup_{t-r \le \theta \le t} u(\theta), \quad t \ge t_0$$

holds, then

$$u(t) \le \sup_{-r \le \theta \le 0} u(t_0 + \theta)e^{-\mu(t-t_0)}, \quad t \ge t_0,$$

where $\mu > 0$, and

$$\mu - a + be^{\mu r} = 0$$

is satisfied.

Lemma 2. (Xiang and Wang, 2009a) *For matrices X, Y with appropriate dimensions and a matrix $Q > 0$, we have*

$$X^TY + Y^TX \le X^TQX + Y^TQ^{-1}Y.$$

Lemma 3. (Petersen, 1987) *For matrices R_1, R_2 with appropriate dimensions, there exists a positive scalar $\beta > 0$ such that*

$$R_1\Sigma(t)R_2 + R_2^T\Sigma^{\mathrm{T}}(t)R_1^T \le \beta R_1UR_1^T + \beta^{-1}R_2^TUR_2,$$

where $\Sigma(t)$ is a time-varying diagonal matrix, U is a known real-value matrix satisfying $|\Sigma(t)| \le U$.

Lemma 4. (Xiang and Wang, 2009a) *Let U, V, W and X be real matrices of appropriate dimensions with X satisfying $X = X^T$. Then for all $V^TV \le I$ we have*

$$X + UVW + W^TV^TU^T < 0$$

if and only if there exists a scalar $\varepsilon > 0$ such that

$$X + \varepsilon UU^T + \varepsilon^{-1}W^TW < 0.$$

Lemma 5. (Boyd, 1994, Schur Complement) *For a given matrix*

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix}$$

with $S_{11} = S_{11}^T$, $S_{22} = S_{22}^T$, the following condition is equivalent:

(1) $S < 0$

(2) $S_{22} < 0, S_{11} - S_{12}S_{22}^{-1}S_{12}^T < 0.$

The objective of this paper is to design a fault tolerant controller such that the system (1)–(2) under asynchronous switching is robust exponentially stable.

3. Main results

To obtain our main results, consider the system (12)–(13) with the asynchronous switching controller $u(t) = K_{\sigma'(t)}x(t)$. The corresponding closed-loop system is given by

$$\begin{aligned}\dot{x}(t) &= (A_{\sigma(t)} + B_{\sigma(t)}M_{\sigma'(t)}K_{\sigma'(t)})x(t)\\ &\quad + A_{d\sigma(t)}x(t-d) + D_{\sigma(t)}f_{\sigma(t)}(x(t),t), \quad (16)\\ x(t) &= \phi(t), \quad t\in[t_0-d, t_0]. \quad (17)\end{aligned}$$

Lemma 6. *Consider the system (12)–(13), for given positive scalars $\alpha, \eta > 0$, if there exist symmetric positive definite matrices $X_i > 0, P_{ij} > 0$ and matrices Y_i for fault matrix M_i, such that for $i, j \in \underline{N}$*

$$\begin{bmatrix} \Xi_i & A_{di}X_i & D_i & X_iU_i^T\\ * & -X_i & 0 & 0\\ * & * & -I & 0\\ * & * & * & -I\end{bmatrix} < 0, \quad (18)$$

$$\begin{bmatrix} \Xi_{ij} & P_{ij}A_{dj} & P_{ij}D_j\\ * & -P_{ij} & 0\\ * & * & -I\end{bmatrix} < 0 \quad (19)$$

and the dwell time satisfies $\inf_{k\geq 0}(t_{k+1}-t_k) \geq T$. Then there exists a controller

$$u(t) = K_{\sigma'(t)}x(t), \quad K_i = Y_iX_i^{-1}, \quad (20)$$

which can guarantee that the closed-loop system is exponentially stable, where

$$\begin{aligned}\Xi_i &= (A_iX_i + B_iM_iY_i)^T + A_iX_i + B_iM_iY_i\\ &\quad + (1+\alpha)X_i,\\ \Xi_{ij} &= (A_j + B_jM_iY_iX_i^{-1})^TP_{ij} + P_{ij}(A_j + B_jM_iY_iX_i^{-1})\\ &\quad + (1+\eta)P_{ij} + U_j^TU_j,\end{aligned}$$

$$T > 2d + \frac{\ln\rho_1\rho_2}{\mu},$$

$$\rho_1 = \max_{\substack{i,j\in\underline{N}\\ i\neq j}}\left\{\frac{\lambda_{\max}(X_j^{-1})}{\lambda_{\min}(P_{ij})}\right\},$$

$$\rho_2 = \max_{\substack{i,j\in\underline{N}\\ i\neq j}}\left\{\frac{\lambda_{\max}(P_{ij})}{\lambda_{\min}(X_i^{-1})}\right\},$$

μ satisfies $\mu + e^{\mu d} = 1 + \min\{\alpha, \eta\}$.

Proof. See Appendix. ■

The following theorem presents sufficient conditions for the existence of a fault tolerant controller for the system (1)–(2) under asynchronous switching.

Theorem 1. *Consider the system (1)–(2). For given positive scalars $\alpha, \eta > 0$, if there exist symmetric positive definite matrices $X_i > 0, P_{ij} > 0$, positive scalars $\varepsilon_i, \beta_i, \zeta_i, \theta_i$, and matrices Y_i, such that for $i, j \in \underline{N}$*

$$\begin{bmatrix} \Theta_i & A_{di}X_i & D_i & X_iU_i^T & Y_iM_{i0}J_i^{1/2} & X_iE_{1i}^T\\ * & -X_i & 0 & 0 & 0 & X_iE_{di}^T\\ * & * & -I & 0 & 0 & 0\\ * & * & * & -I & 0 & 0\\ * & * & * & * & -\varepsilon_iI & 0\\ * & * & * & * & * & -\beta_iI\end{bmatrix} < 0, \quad (21)$$

$$\begin{bmatrix} \Theta_{ij} & P_{ij}A_{dj} & P_{ij}D_j & \zeta_jX_i^{-1}Y_i^TM_{i0}J_i^{1/2} & P_{ij}B_jJ_i^{1/2} & \theta_jE_{1j}^T & P_{ij}H_j\\ * & -P_{ij} & 0 & 0 & 0 & \theta_jE_{dj}^T & 0\\ * & * & -I & 0 & 0 & 0 & 0\\ * & * & * & -\zeta_jI & 0 & 0 & 0\\ * & * & * & * & -\zeta_jI & 0 & 0\\ * & * & * & * & * & -\theta_jI & 0\\ * & * & * & * & * & * & -\theta_jI\end{bmatrix} < 0 \quad (22)$$

and the dwell time satisfies $\inf_{k\geq 0}(t_{k+1}-t_k) \geq T$, then there exists a controller

$$u(t) = K_{\sigma'(t)}x(t), \quad K_i = Y_iX_i^{-1}, \quad (23)$$

which can guarantee that the closed-loop system is exponentially stable, where

$$\begin{aligned}\Theta_i &= (A_iX_i + B_iM_{i0}Y_i)^T + A_iX_i + B_iM_{i0}Y_i\\ &\quad + (1+\alpha)X_i + \beta_iH_{1i}H_{1i}^T + \varepsilon_iB_iJ_iB_i^T,\\ \Theta_{ij} &= (A_j + B_jM_{i0}Y_iX_i^{-1})^TP_{ij}\\ &\quad + P_{ij}(A_j + B_jM_{i0}Y_iX_i^{-1})\\ &\quad + (1+\eta)P_{ij} + U_j^TU_j,\end{aligned}$$

$$T > 2d + \frac{\ln\rho_1\rho_2}{\mu},$$

$$\rho_1 = \max_{\substack{i,j\in\underline{N}\\ i\neq j}}\left\{\frac{\lambda_{\max}(X_j^{-1})}{\lambda_{\min}(P_{ij})}\right\},$$

$$\rho_2 = \max_{\substack{i,j\in\underline{N}\\ i\neq j}}\left\{\frac{\lambda_{\max}(P_{ij})}{\lambda_{\min}(X_i^{-1})}\right\},$$

μ satisfies $\mu + e^{\mu d} = 1 + \min\{\alpha, \eta\}$.

Proof. Consider the system (1)–(2) with the controller $u(t) = K_{\sigma'(t)}x(t)$. The corresponding closed-loop system is given by

$$\begin{aligned}\dot{x}(t) &= (\hat{A}_{\sigma(t)} + B_{\sigma(t)}M_{\sigma'(t)}K_{\sigma'(t)})x(t)\\ &\quad + \hat{A}_{d\sigma(t)}x(t-d) + D_{\sigma(t)}f_{\sigma(t)}(x(t),t), \quad (24)\\ x(t) &= \phi(t), \quad t \in [t_0 - d, t_0]. \quad (25)\end{aligned}$$

Write

$$T_i = \begin{bmatrix} \Lambda_i & \hat{A}_{di}X_i & D_i & X_iU_i^T & Y_i^T M_{i0} J_i^{1/2} \\ * & -X_i & 0 & 0 & 0 \\ * & * & -I & 0 & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -\varepsilon_i I \end{bmatrix}, \quad (26)$$

where

$$\Lambda_i = (\hat{A}_iX_i + B_iM_{i0}Y_i)^T + \hat{A}_iX_i + B_iM_{i0}Y_i + (1+\alpha)X_i + \varepsilon_i B_i J_i B_i^T.$$

Substituting (11) to (26) and using Lemma 4, it is easy to see that (21) is equivalent to $T_i < 0$.

Write

$$Z_{ij} = \begin{bmatrix} \Lambda_{ij} & P_{ij}\hat{A}_{dj} & P_{ij}D_j & \zeta_j X_i^{-1}Y_i^T M_{i0} J_i^{1/2} & P_{ij}B_jJ_i^{1/2} \\ * & -P_{ij} & 0 & 0 & 0 \\ * & * & -I & 0 & 0 \\ * & * & * & -\zeta_j I & 0 \\ * & * & * & * & -\zeta_j I \end{bmatrix},$$

where

$$\Lambda_{ij} = (\hat{A}_j + B_jM_{i0}Y_iX_i^{-1})^T P_{ij} + P_{ij}(\hat{A}_j + B_jM_{i0}Y_iX_i^{-1}) + (1+\eta)P_{ij} + U_j^TU_j.$$

Following a similar proof line, we have $Z_{ij} < 0$ from (22). From Lemma 6 we conclude that Theorem 1 holds. The proof is completed. ■

Remark 3. Note that the matrix inequalities (21) and (22) are mutually constrained. Therefore, we can first solve the linear matrix inequality (21) to obtain matrices X_i and Y_i. Then we solve (22) by substituting X_i and Y_i into (22). By adjusting the parameter α, η appropriately, feasible solutions X_i, Y_i, and P_{ij} can be found such that the matrix inequalities (21) and (22) hold.

From Theorem 1, we can easily obtain the following results.

Corollary 1. *Consider the system (14)–(15). For given positive scalars α, η, if there exist symmetric positive definite matrices $X_i > 0, P_{ij} > 0$, matrices Y_i and positive scalars $\varepsilon_i > 0$, $\zeta_i > 0$, such that for $i, j \in \underline{N}$*

$$\begin{bmatrix} \Gamma_i & A_{di}X_i & Y_i^T M_{i0} J_i^{1/2} \\ * & -X_i & 0 \\ * & * & -\varepsilon_i I \end{bmatrix} < 0, \quad (27)$$

$$\begin{bmatrix} \Gamma_{ij} & P_{ij}A_{dj} & \zeta_j X_i^{-1}Y_i^T M_{i0} J_i^{1/2} & P_{ij}B_jJ_i^{1/2} \\ * & -P_{ij} & 0 & 0 \\ * & * & -\zeta_j I & 0 \\ * & * & * & -\zeta_j I \end{bmatrix} < 0 \quad (28)$$

and the dwell time satisfies $\inf_{k\geq 0}(t_{k+1} - t_k) \geq T$*, then there exists a controller*

$$u(t) = K_{\sigma'(t)}x(t), \quad K_i = Y_iX_i^{-1}, \quad (29)$$

which can guarantee that the closed-loop system is exponentially stable, where

$$\begin{aligned}\Gamma_i &= (A_iX_i + B_iM_{i0}Y_i)^T + A_iX_i + B_iM_{i0}Y_i\\ &\quad + (1+\alpha)X_i + \varepsilon_i B_i J_i B_i^T,\\ \Gamma_{ij} &= (A_j + B_jM_{i0}Y_iX_i^{-1})^T P_{ij}\\ &\quad + P_{ij}(A_j + B_jM_{i0}Y_iX_i^{-1}) + (1+\eta)P_{ij},\end{aligned}$$

$$T > 2d + \frac{\ln \rho_1\rho_2}{\mu},$$

$$\rho_1 = \max_{\substack{i,j\in\underline{N}\\ i\neq j}} \left\{ \frac{\lambda_{\max}(X_j^{-1})}{\lambda_{\min}(P_{ij})} \right\},$$

$$\rho_2 = \max_{\substack{i,j\in\underline{N}\\ i\neq j}} \left\{ \frac{\lambda_{\max}(P_{ij})}{\lambda_{\min}(X_i^{-1})} \right\},$$

μ satisfies $\mu + e^{\mu d} = 1 + \min\{\alpha, \eta\}$.

Corollary 2. *Consider the system (12)–(13). For given positive scalars α, η, if there exist symmetric positive definite matrices $X_i > 0, P_{ij} > 0$, matrices Y_i and positive scalar $\varepsilon_i > 0$, $\zeta_i > 0$, such that for $i, j \in \underline{N}$*

$$\begin{bmatrix} \Sigma_i & A_{di}X_i & D_i & X_iU_i^T & Y_i^T M_{i0} J_i^{1/2} \\ * & -X_i & 0 & 0 & 0 \\ * & * & -I & 0 & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -\varepsilon_i I \end{bmatrix} < 0 \quad (30)$$

$$\begin{bmatrix} \Sigma_{ij} & P_{ij}A_{dj} & P_{ij}D_j & \zeta_j X_i^{-1}Y_i^T M_{i0} J_i^{1/2} \\ & -P_{ij} & 0 & 0 \\ * & * & -I & 0 \\ * & * & * & -\zeta_j I \\ * & * & * & * \end{bmatrix}$$

$$\begin{bmatrix} \cdots & P_{ij}B_jJ_i^{1/2} \\ & 0 \\ & 0 \\ & 0 \\ & -\zeta_j I \end{bmatrix} < 0 \quad (31)$$

and the dwell time satisfies $\inf_{k\geq 0}(t_{k+1}-t_k)\geq T$, *then there exists a controller*

$$u(t) = K_{\sigma'(t)}x(t), \quad K_i = Y_iX_i^{-1}, \quad (32)$$

which can guarantee that the closed-loop system is exponentially stable, where

$$\begin{aligned}\Sigma_i &= (A_iX_i + B_iM_{i0}Y_i)^T + A_iX_i + B_iM_{i0}Y_i \\ &\quad + (1+\alpha)X_i + \varepsilon_iB_iJ_iB_i^T, \\ \Sigma_{ij} &= (A_j + B_jM_{i0}Y_iX_i^{-1})^TP_{ij} \\ &\quad + P_{ij}(A_j + B_jM_{i0}Y_iX_i^{-1}) \\ &\quad + (1+\eta)P_{ij} + U_j^TU_j,\end{aligned}$$

$$T > 2d + \frac{\ln \rho_1\rho_2}{\mu},$$

$$\rho_1 = \max_{\substack{i,j\in \underline{N} \\ i\neq j}} \left\{ \frac{\lambda_{\max}(X_j^{-1})}{\lambda_{\min}(P_{ij})} \right\},$$

$$\rho_2 = \max_{\substack{i,j\in \underline{N} \\ i\neq j}} \left\{ \frac{\lambda_{\max}(P_{ij})}{\lambda_{\min}(X_i^{-1})} \right\},$$

μ *satisfies* $\mu + e^{\mu d} = 1 + \min\{\alpha, \eta\}$.

4. Numerical example

In this section, an example is given to illustrate the effectiveness of the proposed method. Consider the system (1)–(2) with the following parameters:

$$2A_1 = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.2 & 0 \\ 0 & -0.3 \end{bmatrix},$$

$$A_{d1} = \begin{bmatrix} -0.2 & 0 \\ 0 & -0.1 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} -0.2 & 0.3 \\ 0 & -0.1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} -8 & 0 \\ 0 & 7 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -3 & 0 \\ 0 & 6 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 0.3 & -0.2 \\ 0 & -0.1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} -0.1 & 0.1 \\ -0.1 & 0.2 \end{bmatrix},$$

$$U_1 = \begin{bmatrix} -0.1 & 0 \\ 0 & 0 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 0 & -0.1 \\ 0 & 0 \end{bmatrix},$$

$$H_1 = \begin{bmatrix} 0.1 & 0.1 \\ 0 & 0.3 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0.4 & 0 \\ 0.2 & 0 \end{bmatrix},$$

$$E_{11} = \begin{bmatrix} 0 & 0.6 \\ 0 & 0 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0.7 & 0.3 \\ 0.1 & 0.2 \end{bmatrix},$$

$$E_{d1} = \begin{bmatrix} 0.1 & 0.3 \\ 0.9 & 0.6 \end{bmatrix}, \quad E_{d2} = \begin{bmatrix} 0.2 & 0.4 \\ 0.3 & 0.7 \end{bmatrix},$$

$$d = 1.2,$$

$$f_1(x(t),t) = \begin{bmatrix} 0.1\sin x_1 \\ 0 \end{bmatrix},$$

$$f_2(x(t),t) = \begin{bmatrix} 0 \\ 0.1\sin x_2 \end{bmatrix}.$$

The fault matrices are as follows:

$$\begin{aligned} 0.1 &\leq m_{11} \leq 0.5, \\ 0.2 &\leq m_{12} \leq 0.8, \\ 0.2 &\leq m_{21} \leq 0.4, \\ 0.3 &\leq m_{22} \leq 0.9., \end{aligned}$$

that is,

$$2M_{10} = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad M_{20} = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.6 \end{bmatrix},$$

$$J_1 = \begin{bmatrix} 0.67 & 0 \\ 0 & 0.6 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0.33 & 0 \\ 0 & 0.5 \end{bmatrix}.$$

Choosing $\alpha = 3$, $\eta = 2$, by solving the LMIs in Theorem 1, we have

$$K_1 = \begin{bmatrix} 6.6186 & 0.8014 \\ -0.4395 & -3.5108 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} 5.8192 & 0.9082 \\ -0.6788 & -4.8117 \end{bmatrix},$$

and

$$X_1 = \begin{bmatrix} 1.4270 & -0.1871 \\ -0.1871 & 1.3903 \end{bmatrix},$$

$$X_2 = \begin{bmatrix} 1.8435 & -0.2754 \\ -0.2754 & 1.5919 \end{bmatrix},$$

$$P_{12} = \begin{bmatrix} 1.8322 & 0.3021 \\ 0.3021 & 1.4479 \end{bmatrix},$$

$$P_{21} = \begin{bmatrix} 3.9085 & 0.3765 \\ 0.3765 & 2.8780 \end{bmatrix},$$

$\rho_1 = 0.6390$, $\rho_2 = 8.1459$, $T > 4.7$. Choose the switching signal as follows

$$\sigma(t) = \begin{cases} 1, & 2k\tau^* \leq t < (2k+1)\tau^*, \\ 2, & (2k+1)\tau^* \leq t < (2k+2)\tau^*, \end{cases}$$

where $k = 0, 1, 2, \ldots, \tau^* = 5$.

The state response of the closed-loop system is shown in Fig. 2, where $\Delta_k = 1 (k = 1, 2)$ and the initial condition is

$$x(t) = \begin{bmatrix} 2 & -1 \end{bmatrix}^T, \quad t \in [-1.2, 0].$$

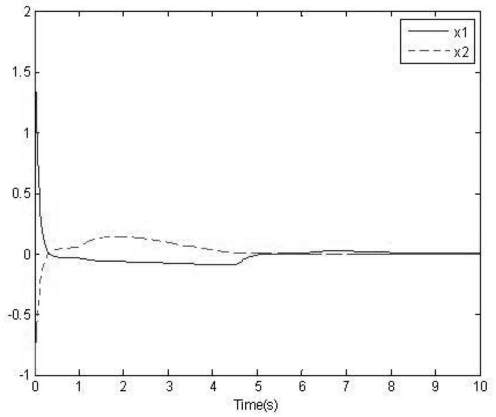

Fig. 2. State response of the closed-loop system.

5. Conclusion

This paper investigates the problem of fault tolerant control for a class of uncertain switched nonlinear systems with time delay and actuator failures under asynchronous switching. Sufficient conditions for the existence of a fault tolerant control law were derived. The proposed controller can be obtained by solving a set of LMIs. A numerical example was provided to show the effectiveness of the proposed approach.

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Appendix

Proof of Lemma 6. Without loss of generality, we assume the initial time $t_0 = 0$.

When $t \in [t_{k-1} + \Delta_{k-1}, t_k)$, the closed-loop system (16)–(17) can be written as

$$\begin{aligned}\dot{x}(t) &= (A_i + B_i M_i K_i)x(t) + A_{di}x(t-d) \\ &\quad + D_i f_i(x(t), t).\end{aligned} \tag{33}$$

Consider the following Lyapunov functional candidate:

$$V_i(t) = x^T(t)P_i x(t).$$

Along the trajectory of the system (33), the time derivative of $V_i(t)$ is given by

$$\begin{aligned}\dot{V}_i(t) &= 2\dot{x}^T(t)P_i x(t) \\ &= x^T(t)\Big[(A_i + B_i M_i K_i)^T P_i \\ &\quad + P_i(A_i + B_i M_i K_i)\Big]x(t) \\ &\quad + x^T(t)P_i A_{di}x(t-d) + x^T(t-d)A_{di}^T P_i x(t) \\ &\quad + 2x^T(t)P_i D_i f(x(t), t).\end{aligned}$$

From Lemma 2 and (5), we have

$$\begin{aligned}\dot{V}_i(t) \le x^T(t)\Big[&(A_i+B_iM_iK_i)^TP_i+P_i(A_i+B_iM_iK_i)\\&+P_iA_{di}P_i^{-1}A_{di}^TP_i+P_iD_iD_i^TP_i\Big]x(t)\\&+x^T(t-d)P_ix(t-d)\\&+f_i^T(x(t),t)f_i(x(t),t)\\\le x^T(t)\Big[&(A_i+B_iM_iK_i)^TP_i\\&+P_i(A_i+B_iM_iK_i)+U_i^TU_i\\&+P_iD_iD_i^TP_i+P_iA_{di}P_i^{-1}A_{di}^TP_i\Big]x(t)\\&+x^T(t-d)P_ix(t-d).\end{aligned}$$

By Lemma 5, (18) is equivalent to

$$\begin{aligned}(A_iX_i+B_iM_iY_i)^T+A_iX_i+B_iM_iY_i+(1+\alpha)X_i\\+A_{di}X_iA_{di}^T+D_iD_i^T+X_iU_i^TU_iX_i<0.\end{aligned} \tag{34}$$

Substituting $X_i = P_i^{-1}, K_i = Y_iX_i^{-1}$ to (34) and using P_i, pre- and postmultiply the left term of (34) to obtain

$$\begin{aligned}(A_i+B_iM_iK_i)^TP_i+P_i(A_i+B_iM_iK_i)+U_i^TU_i\\+P_iD_iD_i^TP+P_iA_{di}P_i^{-1}A_{di}^TP_i+(1+\alpha)P_i<0.\end{aligned} \tag{35}$$

Then, by (35), we have

$$\begin{aligned}\dot{V}_i(t) &\le -x^T(t)(1+\alpha)P_ix(t)+x^T(t-d)P_ix(t-d)\\&\le -(1+\alpha)V_i(t)+\sup_{-d\le\theta_1\le0}V_i(t+\theta_1).\end{aligned} \tag{36}$$

By Lemma 1, we have

$$\begin{aligned}&V_i(t)\\&\le \sup_{-d\le\theta_1\le0}V_i(t_{k-1}+\Delta_{k-1}+\theta_1)e^{-\mu_1(t-t_{k-1}-\Delta_{k-1})},\end{aligned} \tag{37}$$

where $\mu_1>0$, and satisfies $\mu_1+e^{\mu_1 d}=1+\alpha$.

Let

$$\kappa_1=\frac{\lambda_{\max}(P_i)}{\lambda_{\min}(P_i)}.$$

We have

$$\begin{aligned}\|x(t)\| \le \kappa_1^{\frac12}&\sup_{-d\le\theta_1\le0}\|x(t_{k-1}+\Delta_{k-1}+\theta_1)\|\\&\cdot e^{-\frac12\mu_1(t-t_{k-1}-\Delta_{k-1})}.\end{aligned} \tag{38}$$

When $t\in[t_k, t_k+\Delta_k)$, the closed-loop system (16)–(17) can be written as

$$\dot{x}(t)=(A_j+B_jM_iK_i)x(t)+A_{dj}x(t-d)+D_jf_j(x(t),t). \tag{39}$$

Consider the following Lyapunov functional candidate:

$$V_{ij}(t)=x^T(t)P_{ij}x(t).$$

Repeating the above proof line, from (19) we have

$$V_{ij}(t)\le\sup_{-d\le\theta_2\le0}V_{ij}(t_k+\theta_2)e^{-\mu_2(t-t_k)}, \tag{40}$$

where $\mu_2>0$, and satisfies $\mu_2+e^{\mu_2 d}=1+\eta$.

Let

$$\kappa_2=\frac{\lambda_{\max}(P_{ij})}{\lambda_{\min}(P_{ij})}.$$

We have

$$\|x(t)\|\le\kappa_2^{\frac12}\sup_{-d\le\theta_2\le0}\|x(t_k+\theta_2)\|\,e^{-\frac12\mu_2(t-t_k)}. \tag{41}$$

Choosing $\mu=\min\{\mu_1,\mu_2\}$, we have

$$\begin{aligned}V_{\sigma(t_{k-1})}(t)\le&\sup_{-d\le\theta_1\le0}V_{\sigma(t_{k-1})}(t_{k-1}+\Delta_{k-1}+\theta_1)\\&\cdot e^{-\mu(t-t_{k-1}-\Delta_{k-1})},\quad t\ge t_{k-1}+\Delta_{k-1},\end{aligned} \tag{42}$$

$$\begin{aligned}V_{\sigma(t_{k-1})\sigma(t_k)}(t)\le&\sup_{-d\le\theta_2\le0}V_{\sigma(t_{k-1})\sigma(t_k)}(t_k+\theta_2)\\&\cdot e^{-\mu(t-t_k)},\quad t\ge t_k.\end{aligned} \tag{43}$$

Let

$$\rho_1=\max_{\substack{i,j\in\underline{N}\\i\ne j}}\left\{\frac{\lambda_{\max}(P_j)}{\lambda_{\min}(P_{ij})}\right\}.$$

Then we have

$$V_{\sigma(t_k)}(t)\le\rho_1V_{\sigma(t_{k-1})\sigma(t_k)}(t). \tag{44}$$

Let

$$\rho_2=\max_{\substack{i,j\in\underline{N}\\i\ne j}}\left\{\frac{\lambda_{\max}(P_{ij})}{\lambda_{\min}(P_i)}\right\},$$

for $\theta_2\in[-d,0]$. We have

$$\begin{aligned}&V_{\sigma(t_{k-1})\sigma(t_k)}(t_k+\theta_2)\\&\le\rho_2V_{\sigma(t_{k-1})}(t_k+\theta_2)\\&\le\rho_2e^{\mu d}\sup_{-d\le\theta_1\le0}V_{\sigma(t_{k-1})}(t_{k-1}+\Delta_{k-1}+\theta_1)\\&\quad\cdot e^{-\mu(t_k-t_{k-1})}e^{\mu\Delta_{k-1}}.\end{aligned} \tag{45}$$

Notice that $-d\le\Delta_k+\theta_1\le d$, so we can obtain

$$\begin{aligned}&V_{\sigma(t_{k-1})}(t_{k-1}+\Delta_{k-1}+\theta_1)\\&\le\rho_1\rho_2e^{2\mu d}\sup_{-d\le\theta_1\le0}V_{\sigma(t_{k-2})}(t_{k-2}+\Delta_{k-2}+\theta_1)\\&\quad\cdot e^{-\mu[(t_{k-1}+\Delta_{k-1})-(t_{k-2}+\Delta_{k-2})]}\\&\le(\rho_1\rho_2e^{2\mu d})^{k-1}e^{-\mu(t_{k-1}-t_0)}e^{-\mu(\Delta_{k-1}-\Delta_0)}\\&\quad\sup_{-d\le\theta_1\le0}V_{\sigma(t_0)}(t_0+\Delta_0+\theta_1),\end{aligned} \tag{46}$$

which leads to

$$\begin{aligned} &V_{\sigma(t_{k-1})}(t) \\ &\le (\rho_1\rho_2 e^{2\mu d})^{k-1} e^{-\mu(t_{k-1}-t_0)} e^{-\mu(t-t_{k-1}-\Delta_{k-1})} \\ &\quad \cdot e^{-\mu(\Delta_{k-1}-\Delta_0)} \sup_{-d\le\theta_1\le 0} V_{\sigma(t_0)}(t_0+\Delta_0+\theta_1). \end{aligned} \tag{47}$$

From $t_{k+1} - t_k \ge T$, we have

$$t - t_0 - \Delta_0 \ge (k-1)T - d. \tag{48}$$

Let

$$T > 2d + \frac{\ln \rho_1\rho_2}{\mu},$$

$$\nu = -\frac{1}{2}\left(\frac{\ln \rho_1\rho_2 + 2d\mu}{T} - \mu\right) > 0.$$

Then

$$\begin{aligned} V_{\sigma(t_{k-1})}(t) \le &\sup_{-d\le\theta_1\le 0} V_{\sigma(t_0)}(t_0+\Delta_0+\theta_1) \\ &\cdot e^{(\frac{\ln \rho_1\rho_2+2\mu d}{T}-\mu)(t-t_0-\Delta_0)}. \end{aligned} \tag{49}$$

Similarly, we have

$$\begin{aligned} &V_{\sigma(t_{k-1})\sigma(t_k)}(t) \\ &\le \rho_1^{-1}(\rho_1\rho_2 e^{2\mu d})^{\frac{d}{T}} \sup_{-d\le\theta_1\le 0} V_{\sigma(t_0)}(t_0+\Delta_0+\theta_1) \\ &\quad \cdot e^{(\frac{\ln \rho_1\rho_2+2\mu d}{T}-\mu)(t-t_0-\Delta_0)} \end{aligned} \tag{50}$$

The proof is completed.

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