Int. J. Appl. Math. Comput. Sci., 2010, Vol. 20, No. 4, 637–653
DOI: 10.2478/v10006-010-0047-x

ROBUST ADAPTIVE FUZZY FILTERS OUTPUT FEEDBACK CONTROL OF STRICT–FEEDBACK NONLINEAR SYSTEMS

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In this paper, an adaptive fuzzy robust output feedback control approach is proposed for a class of single input single output (SISO) strict-feedback nonlinear systems without measurements of states. The nonlinear systems addressed in this paper are assumed to possess unstructured uncertainties, unmodeled dynamics and dynamic disturbances, where the unstructured uncertainties are not linearly parameterized, and no prior knowledge of their bounds is available. In recursive design, fuzzy logic systems are used to approximate unstructured uncertainties, and K-filters are designed to estimate unmeasured states. By combining backstepping design and a small-gain theorem, a stable adaptive fuzzy output feedback control scheme is developed. It is proven that the proposed adaptive fuzzy control approach can guarantee the all the signals in the closed-loop system are uniformly ultimately bounded, and the output of the controlled system converges to a small neighborhood of the origin. The effectiveness of the proposed approach is illustrated by a simulation example and some comparisons.

Keywords: nonlinear systems, adaptive fuzzy control, backstepping, small-gain approach, K-filters.

1. Introduction

In the past decade, interest in adaptive control of nonlinear systems has been increasing, and many significant developments have been achieved. As a breakthrough in nonlinear control, adaptive backstepping control was introduced to achieve global stability and asymptotic tracking for a class of nonlinear systems in parametric strict-feedback form by Kanellakopopoulos *et al.* (1991). Later, the overparametrization problem was successfully eliminated by Kristic *et al.* (1992) through the tuning function method. In an effort to extend the backstepping control idea to larger classes of nonlinear systems, Kristic *et al.* (1995) and Qian *et al.* (2002) studied the adaptive control problem of parametric strict-feedback systems, obtained local stability results, and proposed several adaptive approaches to nonlinear systems with a triangular structure.

To accommodate uncertainties, a robust adaptive backstepping control has been developed for strict-feedback nonlinear systems with time-varying disturbances and static or dynamic uncertainties by Jiang *et al.* (1998; 1999) (to name a few). The advantages of backstepping methodology include the facts that: (i) global stability can be achieved with ease, (ii) the transient performance can be guaranteed and explicitly analyzed, and (iii) it has the flexibility to avoid unnecessary cancellation of useful nonlinearities compared with feedback linearization techniques. However, these schemes are only suitable for nonlinear systems with nonlinear dynamics models known exactly or with unknown parameters appearing linearly with respect to known nonlinear functions. If that kind of knowledge is not available *a priori*, these adaptive backstepping controllers cannot be applied.

Fuzzy logic systems have been widely used to model nonlinearities. A fuzzy logic system is a universal approximator which, with the increased size of fuzzy rules, can approximate any nonlinearities with arbitrary precision (Wang, 1994). Based on this capability, fuzzy logic systems are vastly adopted for nonlinear systems identification and control (Chen *et al.*, 1996; Denai *et al.*, 2002; Boukezzoula *et al.*, 2007; Qi *et al.*, 2009). Most of them use fuzzy logic systems as nonlinear models for the underlying nonlinearity. The stability issues for adaptive fuzzy controllers are addressed by Lyapunov functions. However, these adaptive fuzzy controllers are only applied to a relatively simple class of nonlinear systems. The key requirement is that the unknown nonlinearities must satisfy the matching conditions. If the unknown nonlinearities do not satisfy the matching conditions, the adaptive fuzzy controllers mentioned above cannot be implemented.

To handle the control problem of uncertain nonlinear systems without satisfying matching conditions, in recent years, many backstepping-based adaptive fuzzy controllers have been developed for nonlinear systems in strict-feedback form (Yang *et al.*, 2005; Wang *et al.*, 2007; Zou *et al.*, 2008; Chen *et al.*, 2005; 2007; Tong *et al.*, 2010a; 2010b). Among them, are those for single-input and single-output (SISO) nonlinear systems (Yang *et al.*, 2005; Wang *et al.*, 2007; Zou *et al.*, 2008; Tong *et al.*, 2010a; 2010b), those for multiple-input and multiple-output (MIMO) nonlinear systems (Chen *et al.*, 2005; 2007), and the ones for SISO nonlinear systems with dynamics and dynamical disturbances (Tong *et al.*, 2010a; 2010b).

In general, adaptive fuzzy backstepping control can provide a systematic methodology of solving tracking or regulation control problems, where fuzzy systems are used to approximate unknown nonlinear functions. Typically, adaptive fuzzy controllers are constructed recursively in the framework of the traditional backstepping design technique. The main features of these adaptive approaches include the facts that (i) they can deal with those nonlinear systems without satisfying the matching conditions, and (ii) they do not require unknown nonlinear functions being linearly parameterized (Kanellakopopoulos *et al.*, 1991; Kristic *et al.*, 1992; 1995; Qian *et al.*, 2002; Jiang *et al.*, 1998; 1999). Therefore, approximator-based adaptive fuzzy backstepping control has attracted great interest in the intelligent control community.

Despite these efforts regarding adaptive fuzzy backstepping control, the proposed adaptive fuzzy backstepping control methods are all based on the assumption that the states of the systems to be controlled can be measured directly. As noted by Wang (1994), in practice, state variables are often unmeasurable for many nonlinear systems. In such cases, some output feedback control schemes should be applied. It is worth mentioning that, in the case of linear systems, output-feedback control problems can be solved by combining state-feedback controllers with the state observer. However, the separation principle doses not hold for nonlinear systems (Kristic *et al.*, 1995; Qian *et al.*, 2002). Thus, the adaptive output feedback control design is more complex and difficult than the counterpart based on state feedback.

Motivated by the above observations, in this paper, a robust adaptive fuzzy output feedback control approach is proposed for a class of SISO strict-feedback nonlinear systems with modeled dynamics and dynamical disturbances, without measurements of states. Fuzzy logic systems are utilized to approximate unknown nonlinear functions, K-filters are used for estimating unmeasured states, and, combining the backstepping technique and the small-gain theorem, a new stable adaptive fuzzy output feedback robust control scheme is developed. The main advantages of the proposed control scheme are as follows: (i) by designing K-filters as a state observer, the proposed control method does not require that all the states of the system be measured directly, which is a common assumption in the existing adaptive fuzzy backstepping controller (Yang *et al.*, 2005; Wang *et al.*, 2007; Tong *et al.*, 2010a; 2010b); (ii) by combining backstepping design with input-to-state practically stability (ISpS) and the small-gain theorem, the proposed control method has a strong robustness to the modeled dynamics and dynamical disturbances, and the stability of entire closed-loop systems can be guaranteed by the small-gain theorem.

It is noted that, in recent years, several adaptive fuzzy backsteping control approaches have also been developed by Yang *et al.* (2005) and Tong *et al.* (2010a; 2010b) for some strict-feedback nonlinear systems based on small-gain theorem. However, the approach of Yang *et al.* (2005) can only control a class of nonlinear systems without unmodeled dynamics or dynamical disturbances and requires that the states of the controlled systems must be measured. Although the approaches of Tong *et al.* (2010a; 2010b) have addressed the same class of nonlinear systems as this paper, they also require that the states of the nonlinear systems must be measured. Therefore, they cannot be applied to nonlinear systems with unmeasured states.

2. Problem formulations and some preliminaries

2.1. Model description and basic assumptions. Consider a class of strict-feedback nonlinear systems with unmodeled dynamics and dynamical disturbances given by the following equations:

$$
\begin{aligned}
\dot{\zeta} &= q(\zeta, y),\\
\dot{x}_1 &= x_2 + f_{1,0}(y) + f_1(y) + \Delta_1(\zeta, y),\\
&\vdots\\
\dot{x}_{n-1} &= x_n + f_{n-1,0}(y) + f_{n-1}(y)\\
&\quad + \Delta_{n-1}(\zeta, y),\\
\dot{x}_n &= b_0\sigma(y)u + f_{n,0}(y) + f_n(y) + \Delta_n(\zeta, y),\\
y &= x_1,
\end{aligned}
\tag{1}
$$

where $x = [x_1, \dots, x_n]^T \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ is the control input, $y \in \mathbb{R}$ is the output; $\sigma(y)$ is a known smooth nonlinear function ($\sigma(y) \neq 0$), and $f_i(y)$, $1 \le i \le n$ is an unknown smooth nonlinear function; $f_{i,0}(y)$, $1 \le i \le n$ is a known smooth nonlinear function; ζ represents unmodeled dynamics and $\Delta_i(\zeta, y)$, $1 \le i \le n$ represents disturbances related to unmodeled dynamics; $b_0 \neq 0$ is an unknown constant and the sign of b_0 is known. In this paper, it is assumed that only $y = x_1$ is available for control design.

In the sequel, the following assumptions are imposed on the system (1):

Assumption 1. (Jiang *et al.*, 1998; 1999) For each $1 \leq i \leq n$, there exists an unknown positive constant p_i^* such that

$$|\Delta_i| \leq p_i^* \psi_{i1}(|y|) + p_i^* \psi_{i2}(|\zeta|), \tag{2}$$

where ψ_{i1} and ψ_{i2} are two known nonnegative smooth functions. Without loss of generality, it is assumed that $\psi_{i2}(0) = 0$.

It is worth mentioning that Assumption 1 implies that the allowed class of uncertainties $\Delta_i(x, \varsigma, t)$ satisfies the so-called triangular bounds condition in terms of x and ς. The same or similar assumptions can be found in recent works (Jiang *et al.*, 1998; 1999). Such a restriction is crucial in controller design.

Definition 1. (Kristic *et al.*, 1995) A continuous function γ: $[0, a) \to \mathbb{R}_+$ is said to belong to class κ if it is strictly increasing and $\gamma(0) = 0$. It is said to belong to class κ_∞ if $a = \infty$ and $\gamma(r) \to \infty$ as $r \to \infty$.

Assumption 2. (Jiang *et al.*, 1998; 1999) Unmodeled dynamics are input-to-state practically stable (ISpS), i.e., the system $\dot{\zeta} = q(\zeta, y)$ has an ISpS Lyapunov function $V_0(\zeta)$ such that

$$\begin{aligned} &\alpha_1(|\zeta|) \leq V_0(\zeta) \leq \alpha_2(|\zeta|), \\ &\frac{\partial V_0}{\partial \zeta} q(\zeta, y) \leq -\alpha_0(|\zeta|) + \gamma_0(|y|) + d_0, \end{aligned} \tag{3}$$

where α_0, α_1, α_2 and γ_0 are κ_∞-functions defined by Kristic *et al.* (1995), d_0 is a nonnegative constant.

Control objectives: The control task is to design an adaptive fuzzy controller using output y only of the form

$$\dot{\chi} = v(\chi, y), \quad u = \mu(\chi, y), \tag{4}$$

such that all the signals of the closed-loop systems (1) and (4) are uniformly ultimately bounded. Furthermore, the output can be forced to a small neighborhood of the origin.

Definition 2. (Coddington, 1989) Let f be a function defined for (x, y) in a set S. We say that f *satisfies locally the Lipschitz condition* on S if there exists a constant $M > 0$ such that

$$|f(x, y_1) - f(x, y_2)| \leq M\, |y_1 - y_2|$$

for all (x, y_1), (x, y_2) in S. The constant M is called a Lipschitz constant.

Lemma 1. (Jiang *et al.*, 1996) *Given the interconnected systems*

$$\dot{x}_1 = f_1(x_1, x_2, u_1), \tag{5}$$

$$\dot{x}_2 = f_2(x_1, x_2, u_2), \tag{6}$$

where, for $i = 1, 2$, $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$ *and* $f_i : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_i} \to \mathbb{R}^{n_i}$ *locally satisfies the Lipschitz condition.*

Assume that, for $i = 1, 2$, there exists an ISpS-Lyapunov function V_i for the x_i-subsystems such that the following holds:

1. there exist κ_∞-functions ϑ_{i1} and ϑ_{i2} such that

$$\vartheta_{i1}(|x_i|) \leq V_i(x_i) \leq \vartheta_{i2}(|x_i|), \quad \forall x_i \in \mathbb{R}^{n_i}, \tag{7}$$

2. there exist κ_∞-function α_i' and κ-functions χ_i, γ_i and some constant $c_i \geq 0$ such that, if

$$V_1(x_1) \geq \max\{\chi_1(V_2(x_2)), \gamma_1(|u_1|) + c_1\},$$

then

$$\nabla V_1(x_1) f_1(x_1, x_2, u_1) \leq -\alpha_1'(V_1), \tag{8}$$

and, if

$$V_2(x_2) \geq \max\{\chi_2(V_1(x_1)), \gamma_2(|u_2|) + c_2\},$$

then

$$\nabla V_2(x_2) f_2(x_1, x_2, u_2) \leq -\alpha_2'(V_2). \tag{9}$$

A nonlinear small-gain condition is given by Jiang *et al.* (1996), under which an ISpS-Lyapunov function for the interconnected systems (5)–(6) may be expressed in terms of ISpS-Lyapunov functions for the two subsystems.

Theorem 1. (Jiang *et al.*, 1996) *Assume that, for* $i = 1, 2$, *the* x_i*-subsystems have an ISpS-Lyapunov* V_i *satisfying (7)–(9). If there exists* $c_0 \geq 0$ *such that*

$$\chi_1 \circ \chi_2(r) < r, \quad \forall r > c_0, \tag{10}$$

then the interconnected system (5)–(6) is ISpS. Furthermore, if $c_0 = c_1 = c_2 = 0$*, then the system is ISS.*

2.2. Fuzzy logic systems and system modeling.

A fuzzy logic system (FLS) consists of four parts: a knowledge base, a fuzzifier, a fuzzy inference engine working on fuzzy rules, and a defuzzifier. The knowledge base for an FLS is composed of a collection of fuzzy If-then rules of the following form:

R^l : If x_1 is F_1^l and x_2 is F_2^l and … and x_n is F_n^l,
then y is G^l, $l = 1, 2, \ldots, N$,

where $x = [x_1, \ldots, x_n]^T$ and y are the FLS input and output, respectively; F_i^l and G^l are fuzzy sets, associated with the fuzzy functions $\mu_{F_i^l}(x_i)$ and $\mu_{G^l}(y)$; N is the rule number.

Through the singleton function, center average defuzzification and product inference, the FLS can be expressed as

$$y(x) = \frac{\sum_{l=1}^{N} \bar{y}_l \prod_{i=1}^{n} \mu_{F_i^l}(x_i)}{\sum_{l=1}^{N} \Big[\prod_{i=1}^{n} \mu_{F_i^l}(x_i) \Big]}, \tag{11}$$

where $\bar{y}_l = \max_{y \in R} \mu_{G^l}(y)$.

Define the fuzzy basis functions as

$$\varphi_l = \frac{\prod_{i=1}^{n} \mu_{F_i^l}(x_i)}{\sum_{l=1}^{N} (\prod_{i=1}^{n} \mu_{F_i^l}(x_i))}.$$

If we let $\theta = [\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_N]^T = [\bar{\theta}_1, \bar{\theta}_2, \ldots, \bar{\theta}_N]^T$ and $\varphi^T(x) = [\varphi_1(x), \ldots, \varphi_N(x)]$ then the FLS (11) can be rewritten as

$$y(x) = \theta^T \varphi(x). \tag{12}$$

It has been proved that the fuzzy logic system (12) can approximate any continuous function $f(x)$ over a compact set $\Omega \subset \mathbb{R}^q$ to any arbitrary accuracy as

$$f(x) = \theta^{*T} \varphi(x) + \varepsilon(x), \quad \forall x \in \Omega, \tag{13}$$

where θ^* is an ideal constant parameter, and $\varepsilon(x)$ is the fuzzy minimums approximation error, which is defined by Wang (1994) as

$$\theta^* = \arg \min_{\theta \in U} \{ \sup_{y \in \Omega} |f(x) - \theta^T \varphi(x)| \}.$$

By employing the FLS to approximate the unknown smooth function $f_i(y)$ in (1) and assuming that

$$f_i(y) = \theta_i^{*T} \varphi_i(y) + \varepsilon_i(y), \tag{14}$$

denote the fuzzy minimums approximation error vector as $\varepsilon(y) = [\ \varepsilon_1(y) \ \ \ldots \ \ \varepsilon_n(y)\ ]^T$.

Assumption 3. The fuzzy minimum approximation error vector $\varepsilon(y)$ satisfies $\|\varepsilon(y)\| \le \beta$, where β is an unknown positive constant, and $\|\cdot\|$ represents the 2-norm of a vector.

By substituting (14) into (1), the system (1) can be expressed as

$$\begin{aligned}
\dot{\zeta} &= q(\zeta, y), \\
\dot{x}_1 &= x_2 + f_{1,0}(y) + \theta_1^{*T} \varphi_1(y) \\
&\quad + \varepsilon_1(y) + \Delta_1(\zeta, y), \\
&\ \vdots \\
\dot{x}_{n-1} &= x_n + f_{n-1,0}(y) + \theta_{n-1}^{*T} \varphi_{n-1}(y) \\
&\quad + \varepsilon_{n-1}(y) + \Delta_{n-1}(\zeta, y), \\
\dot{x}_n &= b_0 \sigma(y) u + f_{n,0}(y) + \theta_n^{*T} \varphi_n(y) \\
&\quad + \varepsilon_n(y) + \Delta_n(\zeta, y), \\
y &= x_1.
\end{aligned} \tag{15}$$

Rewrite (15) as

$$\begin{aligned}
\dot{\zeta} &= q(\zeta, y), \\
\dot{x} &= Ax + f_0(y) + \Phi^T(y)\theta + \varepsilon(y) \\
&\quad + \Delta + [\ 0 \quad b_0,\ ]^T \sigma(y) u \\
y &= C_1^T x,
\end{aligned} \tag{16}$$

where

$$A = \begin{bmatrix} 0 & \\ \vdots & I_{n-1} \\ 0 & \cdots 0 \end{bmatrix},$$

$$f_0(y) = \begin{bmatrix} f_{1,0}(y) \\ \vdots \\ f_{n,0}(y) \end{bmatrix},$$

$$\Phi^T(y) = \begin{bmatrix} \varphi_1^T(y) & & \\ & \ddots & \\ & & \varphi_n^T(y) \end{bmatrix}_{n \times l},$$

$$\begin{aligned}
l &= l_1 + \cdots + l_n, \\
C_1 &= [1, 0, \ldots, 0]^T, \\
\theta^T &= [\ \theta_1^* \quad \cdots \quad \theta_n^*\ ]_{1 \times l}, \\
\Delta &= [\ \Delta_1 \quad \cdots \quad \Delta_n\ ]^T.
\end{aligned}$$

The system (16) is further rewritten as

$$\dot{\zeta} = q(\zeta, y), \tag{17}$$

$$\dot{x} = Ax + f_0(y) + G^T(y, u)\vartheta + \varepsilon(y) + \Delta, \tag{18}$$

$$y = C_1^T x,$$

where

$$\vartheta = \begin{bmatrix} b_0 \\ \theta \end{bmatrix}_{(l+1) \times 1},$$

$$G^T(y, u) = \left[\begin{bmatrix} 0_{(n-1) \times 1} \\ 1 \end{bmatrix} \sigma(y) u, \Phi^T(y) \right].$$

Choose a vector $k = [k_1, \cdots, k_n]^T$ so that the matrix $A_0 = A - kC_1^T$ is a strict Hurwitz matrix, i.e., given a positive definite matrix $Q = Q^T > 0$, there exists a positive definite matrix $P = P^T > 0$ such that

$$PA_0 + A_0^T P = -Q. \tag{19}$$

3. Adaptive fuzzy controller design and stability analysis

Note that in the system (1) or (16), the states $x_2, x_3, \ldots, x_n$ are an unmeasured, b_0 and θ are an unknown constant and unknown parameter vector, respectively. Thus, the states of the system (1), b_0 and θ should be estimated by using the filters given by Kristic *et al.* (1995) as well as Ye (2001). Define the virtual state estimate as

$$\hat{x} = \xi + \Omega^T \vartheta. \tag{20}$$

According to Kristic *et al.* (1995) and from (18), the K-filters may be defined as follows:

$$\begin{aligned}
\dot{\xi} &= A_0 \xi + ky + f_0(y), \\
\dot{\Omega}^T &= A_0 \Omega^T + G^T(y, u).
\end{aligned} \tag{21}$$

Note that the parameter vector ϑ is unknown, and, as such, it cannot be used in control design. Therefore, an estimate $\hat{\vartheta}$ of the parameter vector ϑ need to be obtained later. On the other hand, the virtual state estimate defined by (20) is not used in control design, and the actual state estimate should be $\hat{\hat{x}} = \xi + \Omega^T\hat{\vartheta}$. Denote by v_0 the first column of Ω^T. The vector v_0 is governed by

$$\dot{v}_0 = A_0 v_0 + C_n\sigma(y)u, \qquad (22)$$

where $C_n = [0, \cdots, 0, 1]^T$. In view of (21) and (22), Ω is expressed as

$$\Omega^T = [v_0, \Xi]. \qquad (23)$$

From (21), one obtains

$$\dot{\Xi} = A_0\Xi + \Phi^T(y). \qquad (24)$$

Define the observation error vector e as

$$e = [e_1, e_2, \cdots, e_n]^T = \frac{x - \hat{x}}{p^*}, \qquad (25)$$

where $p^* = \max\{p_i^*, p_i^{*2}, 1; 1 \le i \le n\}$ is an unknown constant. The time derivative of e can be expressed as

$$\dot{e} = A_0 e + \frac{\varepsilon(y) + \Delta}{p^*}. \qquad (26)$$

From the second equation in (16), one obtains

$$\dot{y} = x_2 + f_{1,0}(y) + \theta_1^{*T}\varphi_1(y) + \varepsilon_1(y) + \Delta_1. \qquad (27)$$

Since x_2 is unavailable, it is replaced by available filter signals. From (18), one has

$$\begin{aligned} x &= \xi + \Omega^T\vartheta + x - \hat{x} \\ &= \xi + \Omega^T\vartheta + p^* e. \end{aligned} \qquad (28)$$

Therefore, using (28), x_2 is expressed as

$$\begin{aligned} x_2 &= \xi_2 + \Omega_{(2)}^T\vartheta + p^* e_2 \\ &= b_0 v_{0,2} + \xi_2 + [0, \Xi_{(2)}]\vartheta + p^* e_2, \end{aligned} \qquad (29)$$

where $\Omega_{(2)}^T$ and $\Xi_{(2)}$ are the second rows of Ω^T and Ξ, respectively.

Substituting (29) into (27) yields

$$\dot{y} = b_0 v_{0,2} + \xi_2 + f_{1,0}(y) + \bar{\omega}^T\vartheta + p^* e_2 + \varepsilon_1(y) + \Delta_1, \qquad (30)$$

where the “regressor” ω and the “truncated regressor” $\bar{\omega}$ are defined by Kristic *et al.* (1995) as follows

$$\omega = [v_{0,2}, \Phi_{(1)}^T(y) + \Xi_{(2)}]^T, \qquad (31)$$

$$\bar{\omega} = [0, \Phi_{(1)}^T(y) + \Xi_{(2)}]^T. \qquad (32)$$

From (22), we obtain

$$\dot{v}_{0,i} = v_{0,i+1} - k_i v_{0,1}, \quad i = 2, \ldots, n, -1, \qquad (33)$$

$$\dot{v}_{0,n} = \sigma(y)u - k_n v_{0,1}. \qquad (34)$$

Define a change of coordinates as

$$z_1 = y\lambda'(y^2), \qquad (35)$$

$$z_i = v_{0,i} - \pi_{i-1}, \quad i = 2, \cdots, n, \qquad (36)$$

where $\lambda'(y^2)$ is the derivative of a smooth class κ_∞-function $\lambda(y^2)$, and $\lambda'(y^2) \neq 0$, which will be chosen later.

After the above preparations, adaptive fuzzy backstepping control design is given by the following procedures.

Step 0: Consider the following Lyapunov function:

$$V_0 = e^T P e. \qquad (37)$$

The time derivative of V_0 along (26) is

$$\dot{V}_0 = e^T(A_0^T P + P A_0)e + \frac{2}{p^*}e^T P(\varepsilon + \Delta). \qquad (38)$$

By Assumption 1 and Young’s inequality $2ab \le a^2 + b^2$ and $p^* \ge 1$, we have

$$\begin{aligned} \frac{2}{p^*}e^T P\Delta &\le \frac{2}{p^*}\|e\|\,\|P\|\,\|\Delta\| \\ &\le \frac{2}{p^*}\sum_{i=1}^{n}\|e\|\,\|P\|\,|\Delta_i| \\ &\le 2\,\|e\|\,\|P\|\,(\sum_{k=1}^{n}\psi_{k1}(|y|) + \sum_{k=1}^{n}\psi_{k2}(|\zeta|)), \end{aligned} \qquad (39)$$

$$2\,\|e\|\,\|P\|\sum_{k=1}^{n}\psi_{k1}(|y|) \le \|e\|^2 + \|P\|^2(\sum_{k=1}^{n}\psi_{k1}(|y|))^2. \qquad (40)$$

Since ψ_{i1} is a smooth function, using the same proof of Jiang (1999), we get

$$(\sum_{k=1}^{n}\psi_{k1}(|y|))^2 \le y^2\phi_1(y) + d_\psi^0, \qquad (41)$$

where ϕ_1 is a smooth nonnegative function, and $d_\psi^0 = (\sum_{i=1}^{n}\psi_{i1}(0))^2$ is a constant.

Substituting (41) into (40) yields

$$\begin{aligned} &2\,\|e\|\,\|P\|\sum_{k=1}^{n}\psi_{k1}(|y|) \\ &\le \|e\|^2 + \|P\|^2 y^2\phi_1(y) + \|P\|^2 d_\psi^0. \end{aligned} \qquad (42)$$

Using Young’s inequality, we have

$$\begin{aligned} &2\,\|e\|\,\|P\|\sum_{k=1}^{n}\psi_{k2}(|\zeta|) \\ &\le \|e\|^2 + \|P\|^2(\sum_{k=1}^{n}\psi_{k2}(|\zeta|))^2, \end{aligned} \qquad (43)$$

$$\frac{2}{p^*}e^T P\varepsilon \le 2\|e\|\,\|P\|\,\|\varepsilon\| \le \|e\|^2 + \|P\|^2\beta^2. \tag{44}$$

Substituting (42)–(44) into (38), we obtain

$$\begin{aligned}\dot V_1 \le &-[\lambda_{\min}(Q)-3]\|e\|^2\\ &+\|P\|^2(\sum_{k=1}^{n}\psi_{k2}(|\zeta|))^2 + \|P\|^2 y^2\phi_1(y)\\ &+\|P\|^2\beta^2 + \|P\|^2 d_\psi^0.\end{aligned} \tag{45}$$

Step 1: Consider the following Lyapunov function:

$$\begin{aligned}V_1 = &V_0 + \tfrac12\lambda(y^2) + \tfrac12\tilde\vartheta^T\Gamma^{-1}\tilde\vartheta\\ &+\tfrac12\gamma_1^{-1}\tilde\beta^2 + \tfrac12\gamma_2^{-1}\tilde p^2 + \tfrac12\gamma_3^{-1}|b_0|\tilde\kappa^2,\end{aligned} \tag{46}$$

where $\Gamma = \Gamma^T > 0$, $\gamma_1 > 0$, $\gamma_2 > 0$ and$\gamma_3 > 0$ are design constants; $\tilde\vartheta = \vartheta - \hat\vartheta$, $\tilde\beta = \beta - \hat\beta$, $\tilde p = p - \hat p$ and $\tilde\kappa = \kappa - \hat\kappa$; $\hat\vartheta$, $\hat\beta$, $\hat p$ and $\hat\kappa$ are the estimates of ϑ, β, p and κ, respectively. Here $\kappa = b_0^{-1}$ is an unknown constant. Define $\pi_1 = \hat\kappa\bar\pi_1$, where $\bar\pi_1$ is a stabilizing function to be designed later.

The time derivative of V_1 along (30) is

$$\begin{aligned}\dot V_1 &= \dot V_0 + y\lambda'(y^2)\dot y - \tilde\vartheta^T\Gamma^{-1}\dot{\hat\vartheta}\\ &\quad -\gamma_1^{-1}\tilde\beta\dot{\hat\beta} - \gamma_2^{-1}\tilde p\dot{\hat p} - \gamma_3^{-1}|b_0|\tilde\kappa\dot{\hat\kappa}\\ &= \dot V_0 + y\lambda'(y^2)(b_0 v_{0,2} + \xi_2\\ &\quad + f_{1,0}(y) + \bar\omega^T\vartheta + p^* e_2\\ &\quad + \varepsilon_1(y) + \Delta_1) - \tilde\vartheta^T\Gamma^{-1}\dot{\hat\vartheta} - \gamma_1^{-1}\tilde\beta\dot{\hat\beta}\\ &\quad - \gamma_2^{-1}\tilde p\dot{\hat p} - \gamma_3^{-1}|b_0|\tilde\kappa\dot{\hat\kappa}.\end{aligned} \tag{47}$$

Substituting (35) into (47) results in

$$\begin{aligned}\dot V_1 &= \dot V_0 + z_1(b_0 v_{0,2} + \xi_2 + f_{1,0}(y)\\ &\quad + \bar\omega^T\vartheta + p^* e_2 + \varepsilon_1(y) + \Delta_1)\\ &\quad - \tilde\vartheta^T\Gamma^{-1}\dot{\hat\vartheta} - \gamma_1^{-1}\tilde\beta\dot{\hat\beta} - \gamma_2^{-1}\tilde p\dot{\hat p}\\ &\quad - \gamma_3^{-1}|b_0|\tilde\kappa\dot{\hat\kappa}\\ &= \dot V_0 + b_0 z_1 z_2 - z_1 b_0\tilde\kappa\bar\pi_1\\ &\quad + z_1(\bar\pi_1 + \xi_2 + f_{1,0}(y) + \bar\omega^T\vartheta)\\ &\quad + y\lambda'(y^2)(p^* e_2 + \varepsilon_1(y) + \Delta_1)\\ &\quad - \tilde\vartheta^T\Gamma^{-1}\dot{\hat\vartheta} - \gamma_1^{-1}\tilde\beta\dot{\hat\beta}\\ &\quad - \gamma_2^{-1}\tilde p\dot{\hat p} - \gamma_3^{-1}|b_0|\tilde\kappa\dot{\hat\kappa}.\end{aligned} \tag{48}$$

Using Assumption 1 and Young's inequality, we have

$$\begin{aligned}&y\lambda'(y^2)(p^* e_2 + \Delta_1)\\ &\le p^*\left|y\lambda'(y^2)\right||e_2| + \left|y\lambda'(y^2)\right||\Delta_1|\\ &\le p^*\left|y\lambda'(y^2)\right||e_2| + p_1^*\left|y\lambda'(y^2)\right|(\psi_{11}(|y|)\\ &\quad - \psi_{11}(|0|)) + p_1^*\left|y\lambda'(y^2)\right|\psi_{12}(|\zeta|)\\ &\quad + p_1^*\left|y\lambda'(y^2)\right|\psi_{11}(|0|)\\ &\le |e_2|^2 + \frac{p_1^{*2}}{4}(y\lambda'(y^2))^2\\ &\quad + p_1^* y^2\lambda'(y^2)\bar\psi_{11}(|y|)\\ &\quad + \frac{p_1^{*2}}{2}(y\lambda'(y^2))^2 + \psi_{12}^2(|\zeta|) + \psi_{11}^2(0),\end{aligned} \tag{49}$$

where $\bar\psi_{11}(|y|) = \int_0^1 \psi'_{11}(s|y|)\,\mathrm{d}s$.

Using the proof of Jiang (1999), given any $d_{11} > 0$, there exists a smooth function $\hat\psi_{11}$ with $\hat\psi_{11}(0) = 0$, such that

$$|y|\,\bar\psi_{11}(|y|) \le y\hat\psi_{11}(y) + d_{11}, \quad \forall y \in \mathbb{R},$$

Therefore, (49) can be rewritten as

$$\begin{aligned}&y\lambda'(y^2)(p^* e_2 + \Delta_1)\\ &\quad \le \|e\|^2 + p\phi_{11}(y)(y\lambda'(y^2))^2\\ &\qquad + \psi_{12}^2(|\zeta|) + d_{11}^2 + \psi_{11}^2(0),\end{aligned} \tag{50}$$

where $p = (p^*)^2$ and

$$\phi_{11}(y) = 1 + 1/(2\lambda'(y^2)) + 1/(2\lambda'(y^2))\hat\psi_{11}^2(y)$$

is a smooth nonnegative function.

Note that, for $\forall\varsigma > 0$, the following inequality holds:

$$|r| - r\tanh(r/\varsigma) \le 0.2785\varsigma. \tag{51}$$

By (51), one has

$$\begin{aligned}&|\varepsilon_1(y)z_1| - z_1\eta_1\beta\tanh\left(\frac{z_1\eta_1}{\varsigma}\right)\\ &\quad \le \beta\Big(|z_1| - z_1\eta_1\tanh\left(\frac{z_1\eta_1}{\varsigma}\right)\Big)\\ &\quad \le 0.2785\varsigma\beta = \varsigma',\end{aligned} \tag{52}$$

where ς is an arbitrary small constant and $\eta_1 = -1$.

Substituting (45), (50) and (52) into (48) yields

$$\begin{aligned}\dot V_1 \le &-[\lambda_{\min}(Q)-4]\|e\|^2 + z_1\Big(\bar\pi_1 + \xi_2\\ &+ f_{1,0}(y) + \bar\omega\hat\vartheta + \frac{\|P\|^2}{\lambda'}y\phi_1(y)\\ &+ \hat p\phi_{11}(y)z_1 + \eta_1\hat\beta\tanh\left(\frac{z_1\eta_1}{\varsigma}\right)\Big)\\ &+ b_0 z_1 z_2 + \tilde\vartheta^T(\bar\omega z_1 - \Gamma^{-1}\dot{\hat\vartheta})\\ &+ \tilde\beta\Big(z_1\eta_1\beta\tanh\left(\frac{z_1\eta_1}{\varsigma}\right) - \gamma_1^{-1}\dot{\hat\beta}\Big)\\ &+ \tilde p(\phi_{11}(y)z_1^2 - \gamma_2^{-1}\dot{\hat p}) - \tilde\kappa(z_1 b_0\bar\pi_1\\ &+ \gamma_3^{-1}|b_0|\dot{\hat\kappa}) + \|P\|^2\Big(\sum_{k=1}^{n}\psi_{k2}(|\zeta|)\Big)^2\\ &+ \psi_{12}^2(|\zeta|) + \|P\|^2\beta^2 + \|P\|^2 d_\psi^0\\ &+ \psi_{11}^2(0) + \varsigma' + d_{11}^2.\end{aligned} \tag{53}$$

Choose the stabilizing control function $\bar{\pi}_1$, tuning functions and parameters adaptation laws as

$$\begin{aligned}\bar{\pi}_1 = &-y\rho(y^2) - \xi_2 - f_{1,0}(y) - \bar{\omega}\hat{\vartheta}\\ &- \frac{\|P\|^2}{\lambda'} y\varphi_1(y) - \hat{p}\phi_{11}(y)z_1\\ &- \eta_1\hat{\beta}\tanh\Big(\frac{z_1\eta_1}{\varsigma}\Big),\end{aligned} \tag{54}$$

$$\tau_1 = \bar{\omega}z_1, \tag{55}$$

$$\sigma_1 = z_1\eta_1\beta\tanh\Big(\frac{z_1\eta_1}{\varsigma}\Big), \tag{56}$$

$$\bar{\lambda}_1 = \phi_{11}(y)z_1^2, \tag{57}$$

$$\dot{\hat{\kappa}} = -\gamma_3(\mathrm{sgn}(b_0)\bar{\pi}_1 z_1 + \mu\hat{\kappa}), \tag{58}$$

where $\rho(y^2)$ is a smooth non-decreasing function with $\rho(0) > 0$, and $\mu > 0$ is a design parameter. Substituting (54)–(58) into (53) yields

$$\begin{aligned}\dot{V}_1 \le &-[\lambda_{\min}(Q) - 4]\,\|e\|^2 - z_1 y\rho(y^2)\\ &+ b_0 z_1 z_2 + \tilde{\vartheta}^T(\tau_1 - \Gamma^{-1}\dot{\hat{\vartheta}})\\ &+ \tilde{\beta}(\sigma_1 - \gamma_1^{-1}\dot{\hat{\beta}}) + \tilde{p}(\bar{\lambda}_1 - \gamma_2^{-1}\dot{\hat{p}})\\ &+ \mu\tilde{\kappa}\hat{\kappa} + \|P\|^2(\sum_{k=1}^{n}\psi_{k2}(|\zeta|))^2\\ &+ \psi_{12}^2(|\zeta|) + \|P\|^2\beta^2 + \|P\|^2 d_\psi^0\\ &+ \varsigma' + \psi_{11}^2(0) + d_{11}^2.\end{aligned} \tag{59}$$

Step 2: The time derivative of z_2 along (36) is

$$\begin{aligned}\dot{z}_2 = &\,v_{0,3} - k_2 v_{0,1} - \frac{\partial\pi_1}{\partial y}(\xi_2 + f_{1,0}(y)\\ &+ \omega^T\vartheta + p^*e_2 + \Delta_1 + \varepsilon_1(y))\\ &- \frac{\partial\pi_1}{\partial\xi}(A_0\xi + ky) - \frac{\partial\pi_1}{\partial\Xi}(A_0\Xi + \Phi^T(y))\\ &- \frac{\partial\pi_1}{\partial v_0}\dot{v}_0 - \frac{\partial\pi_1}{\partial\kappa}\dot{\kappa} - \frac{\partial\pi_1}{\partial\hat{\vartheta}}\Gamma(\tau_1 - \mu\hat{\vartheta})\\ &- \frac{\partial\pi_1}{\partial\hat{\beta}}\gamma_1(\sigma_1 - \mu\hat{\beta}) - \frac{\partial\pi_1}{\partial\hat{p}}\gamma_2(\bar{\lambda}_1 - \mu\hat{p})\\ &- \frac{\partial\pi_1}{\partial\hat{\vartheta}}(\dot{\hat{\vartheta}} - \Gamma\tau_1 + \Gamma\mu\hat{\vartheta})\\ &- \frac{\partial\pi_1}{\partial\hat{\beta}}(\dot{\hat{\beta}} - \gamma_1\sigma_1 + \gamma_1\mu\hat{\beta})\\ &- \frac{\partial\pi_1}{\partial\hat{p}}(\dot{\hat{p}} - \gamma_2\bar{\lambda}_1 + \gamma_2\mu\hat{p}).\end{aligned} \tag{60}$$

Consider the Lyapunov function

$$V_2 = V_1 + \frac{1}{2}z_2^2. \tag{61}$$

The time derivative of V_2 along the solutions of (60) is

$$\begin{aligned}\dot{V}_2 \le &\,\dot{V}_1 + z_2[z_3 + \pi_2 - k_2 v_{0,1} - \frac{\partial\pi_1}{\partial y}(\xi_2\\ &+ f_{1,0}(y) + \omega^T\vartheta) - \frac{\partial\pi_1}{\partial\xi}(A_0\xi + ky)\\ &- \frac{\partial\pi_1}{\partial v_0}\dot{v}_0 - \frac{\partial\pi_1}{\partial\Xi}(A_0\Xi + \Phi^T(y))\\ &- \frac{\partial\pi_1}{\partial\hat{\vartheta}}\Gamma(\tau_1 - \mu\hat{\vartheta}) - \frac{\partial\pi_1}{\partial\hat{\beta}}\gamma_1(\sigma_1 - \mu\hat{\beta})\\ &- \frac{\partial\pi_1}{\partial\hat{p}}\gamma_2(\bar{\lambda}_1 - \mu\hat{p})\\ &- \frac{\partial\pi_1}{\partial\hat{\vartheta}}(\dot{\hat{\vartheta}} - \Gamma\tau_1 + \Gamma\mu\hat{\vartheta})\\ &- \frac{\partial\pi_1}{\partial\hat{\beta}}(\dot{\hat{\beta}} - \gamma_1\sigma_1 + \gamma_1\mu\hat{\beta})\\ &- \frac{\partial\pi_1}{\partial\hat{p}}(\dot{\hat{p}} - \gamma_2\bar{\lambda}_1 + \gamma_2\mu\hat{p}) - \frac{\partial\pi_1}{\partial\kappa}\dot{\kappa}]\\ &+ \left|\frac{\partial\pi_1}{\partial y}p^*e_2z_2\right| + \left|\frac{\partial\pi_1}{\partial y}\Delta_1 z_2\right|\\ &+ \left|\frac{\partial\pi_1}{\partial y}\varepsilon_1(y)z_2\right|.\end{aligned} \tag{62}$$

By Assumption 1 and Young's inequality, using the similar derivations in Step 1, one obtains the following inequalities:

$$\left|\frac{\partial\pi_1}{\partial y}p^*e_2z_2\right| \le e^Te + p\Big(\frac{\partial\pi_1}{\partial y}\Big)^2 z_2^2, \tag{63}$$

$$\begin{aligned}&\left|\frac{\partial\pi_1}{\partial y}\Delta_1 z_2\right|\\ &\le \left|\frac{\partial\pi_1}{\partial y}\left(p_1^*\psi_{11}(|y|) + p_1^*\psi_{12}(|\zeta|)\right)z_2\right|\\ &\le p_1^*\left|\frac{\partial\pi_1}{\partial y}z_2\right|\psi_{11}(|y|)\\ &\quad + \frac{1}{4}p(\frac{\partial\pi_1}{\partial y})^2z_2^2 + \psi_{12}^2(|\zeta|)\\ &= p_1^*\left|\frac{\partial\pi_1}{\partial y}z_2\right|(\psi_{11}(|y|) - \psi_{11}(0))\\ &\quad + p_1^*\left|\frac{\partial\pi_1}{\partial y}z_2\right|\psi_{11}(0)\\ &\quad + \frac{1}{4}p(\frac{\partial\pi_1}{\partial y})^2z_2^2 + \psi_{12}^2(|\zeta|)\\ &\le p_1^*\left|\frac{\partial\pi_1}{\partial y}z_2\right||y|\,\bar{\psi}_{11}(|y|) + \frac{1}{2}p(\frac{\partial\pi_1}{\partial y})^2z_2^2\\ &\quad + \psi_{12}^2(|\zeta|) + \psi_{11}^2(0),\end{aligned} \tag{64}$$

$$\begin{aligned}&p_1^*\left|\frac{\partial\pi_1}{\partial y}z_2\right||y|\,\bar{\psi}_{11}(|y|)\\ &\le p_1^*\left|\frac{\partial\pi_1}{\partial y}z_2\right|(y\hat{\psi}_{11}(y) + d_{11})\\ &\le p\Big(\frac{\partial\pi_1}{\partial y}\Big)^2 z_2^2\hat{\psi}_{11}^2(y) + \frac{1}{4}y^2\\ &\quad + \frac{1}{2}p\Big(\frac{\partial\pi_1}{\partial y}\Big)^2 z_2^2 + \frac{1}{2}d_{11}^2,\end{aligned} \tag{65}$$

where $d_{11} > 0$ is a known constant, $\hat{\psi}_{11}$ is a known smooth function with $\hat{\psi}_{11}(0) = 0$.

Substituting (65) into (64) yields

$$\left|\frac{\partial \pi_1}{\partial y}\Delta_1 z_2\right| \le p(\frac{\partial \pi_1}{\partial y})^2 z_2^2(\hat{\psi}_{11}^2(y)+1) + \frac{1}{4}y^2 + \psi_{12}^2(|\zeta|) + \psi_{11}^2(0) + \frac{1}{2}d_{11}^2. \quad (66)$$

Note that

$$\left|\frac{\partial \pi_1}{\partial y}\varepsilon_1(y) z_2\right| - z_2\eta_2\beta\tanh\Big(\frac{z_2\eta_2}{\varsigma}\Big) \le 0.2785\varsigma\beta = \varsigma', \quad (67)$$

where $\eta_2 = -\partial\pi_1/\partial y$.

Substituting (63), (66) and (67) into (62), we obtain

$$\begin{aligned}
\dot{V}_2 \le & -[\lambda_{\min}(Q)-5]\,\|e\|^2 - z_1 y\rho(y^2) \\
& + \frac{1}{4}y^2 + z_2[z_3 + \pi_2 + \hat{b}_0 z_1 - k_2 v_{0,1} \\
& - \frac{\partial \pi_1}{\partial y}(\xi_2 + f_{1,0}(y) + \omega^T\hat{\vartheta}) + H_2] \\
& - z_2\frac{\partial \pi_1}{\partial \hat{\vartheta}}(\dot{\hat{\vartheta}} - \Gamma\tau_1 + \Gamma\mu\hat{\vartheta}) \\
& - z_2\frac{\partial \pi_1}{\partial \hat{\beta}}(\dot{\hat{\beta}} - \gamma_1\sigma_1 + \gamma_1\mu\hat{\beta}) \\
& - z_2\frac{\partial \pi_1}{\partial \hat{p}}(\dot{\hat{p}} - \gamma_2\bar{\lambda}_1 + \gamma_2\mu\hat{p}) \\
& + \tilde{\vartheta}^T(\tau_1 - \frac{\partial \pi_1}{\partial y}\omega z_2 + \ell z_2 - \Gamma^{-1}\dot{\hat{\vartheta}}) \\
& + \tilde{\beta}\Big(\sigma_1 + \eta_2 z_2\tanh\Big(\frac{z_2\eta_2}{\varsigma}\Big) - \gamma_1^{-1}\dot{\hat{\beta}}\Big) \\
& + \tilde{p}(\bar{\lambda}_1 + (\frac{\partial \pi_1}{\partial y})^2 z_2^2(\hat{\psi}_{11}^2(y)+2) - \gamma_2^{-1}\dot{\hat{p}}) \\
& + \mu\tilde{\kappa}\hat{\kappa} + \|P\|^2(\sum_{k=1}^{n}\psi_{k2}(|\zeta|))^2 \\
& + 2\psi_{12}^2(|\zeta|) + \|P\|^2\beta^2 + \|P\|^2 d_\psi^0 + 2\varsigma' \\
& + 2\psi_{11}^2(0) + \frac{3}{2}d_{11}^2,
\end{aligned} \quad (68)$$

where

$$\begin{aligned}
H_2 = & -\frac{\partial \pi_1}{\partial \xi}(A_0\xi + ky) - \frac{\partial \pi_1}{\partial \Xi}(A_0\Xi + \Phi^T(y)) \\
& - \frac{\partial \pi_1}{\partial v_0}\dot{v}_0 - \frac{\partial \pi_1}{\partial \kappa}\dot{\kappa} - \frac{\partial \pi_1}{\partial \hat{\vartheta}}\Gamma(\tau_1 - \mu\hat{\vartheta}) \\
& - \frac{\partial \pi_1}{\partial \hat{\beta}}\gamma_1(\sigma_1 - \mu\hat{\beta}) - \frac{\partial \pi_1}{\partial \hat{p}}\gamma_2(\bar{\lambda}_1 - \mu\hat{p}) \\
& - \hat{\beta}\eta_2\tanh\Big(\frac{z_2\eta_2}{\varsigma}\Big) + \hat{p}(\frac{\partial \pi_1}{\partial y})^2 z_2^2(\hat{\psi}_{11}^2(y)+2),
\end{aligned}$$

$\ell = [\ z_1 \quad 0 \quad \dots \quad 0\ ]^T.$

Choose the tuning functions and parameters adaptation laws as follows:

$$\tau_2 = \tau_1 - z_2\Big(\frac{\partial \pi_1}{\partial y}\omega - \ell\Big), \quad (69)$$

$$\tau_i = \tau_{i-1} - z_i\frac{\partial \pi_{i-1}}{\partial y}\omega, \quad i = 3,\dots,n, \quad (70)$$

$$\sigma_i = \sigma_{i-1} + z_i\eta_i\tanh\Big(\frac{z_i\eta_i}{\varsigma}\Big), \quad i = 2,\dots,n, \quad (71)$$

$$\bar{\lambda}_i = \bar{\lambda}_{i-1} + \Big(\frac{\partial \pi_{i-1}}{\partial y}\Big)^2 z_i^2(\hat{\psi}_{11}^2(y)+2), \quad i = 2,\dots,n, \quad (72)$$

$$\dot{\hat{\vartheta}} = \Gamma(\tau_n - \mu\hat{\vartheta}), \quad (73)$$

$$\dot{\hat{\beta}} = \gamma_1(\sigma_n - \mu\hat{\beta}), \quad (74)$$

$$\dot{\hat{p}} = \gamma_2(\bar{\lambda}_n - \mu\hat{p}), \quad (75)$$

where

$$\eta_i = -\frac{\partial \pi_{i-1}}{\partial y}, \quad i = 2,\dots,n.$$

Define

$$-\frac{\partial \pi_1}{\partial \hat{\vartheta}}(\dot{\hat{\vartheta}} - \Gamma\tau_1 + \Gamma\mu\hat{\vartheta}) = \sum_{j=2}^{n}\Delta_{1,j}z_j, \quad (76)$$

$$-\frac{\partial \pi_1}{\partial \hat{\beta}}(\dot{\hat{\beta}} - \gamma_1\sigma_1 + \gamma_1\mu\hat{\beta}) = \sum_{j=2}^{n}\Lambda_{1,j}z_j, \quad (77)$$

$$-\frac{\partial \pi_1}{\partial \hat{p}}(\dot{\hat{p}} - \gamma_2\bar{\lambda}_1 + \gamma_2\mu\hat{p}) = \sum_{j=2}^{n}\mathrm{A}_{1,j}z_j, \quad (78)$$

where

$$\begin{aligned}
\Delta_{1,j} &= \frac{\partial \pi_1}{\partial \hat{\vartheta}}\Gamma\frac{\partial \pi_{j-1}}{\partial y}\omega, \\
\Lambda_{1,j} &= \frac{\partial \pi_1}{\partial \hat{\beta}}\gamma_1\eta_j\tanh\Big(\frac{\eta_j z_j}{\varsigma}\Big), \\
\mathrm{A}_{1,j} &= \frac{\partial \pi_1}{\partial \hat{p}}\gamma_2\Big(\frac{\partial \pi_{j-1}}{\partial y}\Big)^2 z_j^2(\hat{\psi}_{11}^2(y)+2),
\end{aligned}$$

$i = 2,\dots,n.$

Choose the stabilizing control function π_2 as

$$\begin{aligned}
\pi_2 = & -\hat{b}_0 z_1 - c_2 z_2 + \frac{\partial \pi_1}{\partial y}(\xi_2 \\
& + f_{1,0}(y) + \omega^T\hat{\vartheta}) + k_2 v_{0,1} \\
& - (\Delta_{1,2} + \Lambda_{1,2} + \mathrm{A}_{1,2}) - H_2,
\end{aligned} \quad (79)$$

where $c_2 > 0$ is a design constant.

Substituting (69) and (71)–(79) into (68) yields

$$\begin{aligned}\dot{V}_2 \le &-[\lambda_{\min}(Q)-5]\,\|e\|^2 - z_1 y\rho(y^2)\\&+\frac{1}{4}y^2 + z_2z_3 + \tilde{\vartheta}^T(\tau_2-\Gamma^{-1}\dot{\hat{\vartheta}}) + \mu\tilde{\kappa}\hat{\kappa}\\&+\tilde{\beta}(\sigma_2-\gamma_1^{-1}\dot{\hat{\beta}}) + \tilde{p}(\bar{\lambda}_2-\gamma_2^{-1}\dot{\hat{p}})\\&+\|P\|^2(\sum_{k=1}^{n}\psi_{k2}(|\zeta|))^2 + 2\psi_{12}^2(|\zeta|)\\&+\|P\|^2\beta^2 + \sum_{j=3}^{n}(\Delta_{1,j}+\Lambda_{1,j}+\mathrm{A}_{1,j})z_2z_j\\&+\|P\|^2 d_\psi^0 + 2\varsigma' + 2\psi_{11}^2(0) + \frac{3}{2}d_{11}^2.\end{aligned} \tag{80}$$

Step i ($i=3,\dots,n-1$): A similar procedure in Step 2 is employed recursively for consemtive steps. The time derivative of z_i along (36) is

$$\begin{aligned}\dot{z}_i = &\,v_{0,i+1} - k_i v_{0,1} - \frac{\partial\pi_{i-1}}{\partial y}(\xi_2 + f_{1,0}(y)\\&+\omega^T\vartheta + p^*e_2 + \Delta_1 + \varepsilon_1(y))\\&-\frac{\partial\pi_{i-1}}{\partial\xi}(A_0\xi + ky)\\&-\frac{\partial\pi_{i-1}}{\partial\Xi}(A_0\Xi + \Phi^T(y)) - \frac{\partial\pi_{i-1}}{\partial v_0}\dot{v}_0\\&-\frac{\partial\pi_{i-1}}{\partial\kappa}\dot{\kappa} - \frac{\partial\pi_{i-1}}{\partial\hat{\vartheta}}\Gamma(\tau_{i-1}-\mu\hat{\vartheta})\\&-\frac{\partial\pi_{i-1}}{\partial\hat{\beta}}\gamma_1(\sigma_{i-1}-\mu\hat{\beta})\\&-\frac{\partial\pi_{i-1}}{\partial\hat{p}}\gamma_2(\bar{\lambda}_{i-1}-\mu\hat{p})\\&-\frac{\partial\pi_{i-1}}{\partial\hat{\vartheta}}(\dot{\hat{\vartheta}} - \Gamma\tau_{i-1} + \Gamma\mu\hat{\vartheta})\\&-\frac{\partial\pi_{i-1}}{\partial\hat{\beta}}(\dot{\hat{\beta}} - \gamma_1\sigma_{i-1} + \gamma_1\mu\hat{\beta})\\&-\frac{\partial\pi_{i-1}}{\partial\hat{p}}(\dot{\hat{p}} - \gamma_2\bar{\lambda}_{i-1} + \gamma_2\mu\hat{p}).\end{aligned} \tag{81}$$

Consider the following Lyapunov function:

$$V_i = V_{i-1} + \frac{1}{2}z_i^2. \tag{82}$$

The time derivative of V_i along the solutions of (81) is

$$\begin{aligned}\dot{V}_i \le &\,\dot{V}_{i-1} + z_i[z_{i+1} + \pi_i - k_i v_{0,1}\\&-\frac{\partial\pi_{i-1}}{\partial y}(\xi_2 + f_{1,0}(y) + \omega^T\vartheta)\\&-\frac{\partial\pi_{i-1}}{\partial\xi}(A_0\xi + ky)\\&-\frac{\partial\pi_{i-1}}{\partial\Xi}(A_0\Xi + \Phi^T(y)) - \frac{\partial\pi_{i-1}}{\partial v_0}\dot{v}_0\\&-\frac{\partial\pi_{i-1}}{\partial\kappa}\dot{\kappa} - \frac{\partial\pi_{i-1}}{\partial\hat{\vartheta}}\Gamma(\tau_{i-1}-\mu\hat{\vartheta})\\&-\frac{\partial\pi_{i-1}}{\partial\hat{\beta}}\gamma_1(\sigma_{i-1}-\mu\hat{\beta})\\&-\frac{\partial\pi_{i-1}}{\partial\hat{p}}\gamma_2(\bar{\lambda}_{i-1}-\mu\hat{p})\\&-\frac{\partial\pi_{i-1}}{\partial\hat{\vartheta}}(\dot{\hat{\vartheta}} - \Gamma\tau_{i-1} + \Gamma\mu\hat{\vartheta})\\&-\frac{\partial\pi_{i-1}}{\partial\hat{\beta}}(\dot{\hat{\beta}} - \gamma_1\sigma_{i-1} + \gamma_1\mu\hat{\beta})\\&-\frac{\partial\pi_{i-1}}{\partial\hat{p}}(\dot{\hat{p}} - \gamma_2\bar{\lambda}_{i-1} + \gamma_2\mu\hat{p})]\\&+\left|\frac{\partial\pi_{i-1}}{\partial y}p^*e_2z_i\right| + \left|\frac{\partial\pi_{i-1}}{\partial y}\Delta_1 z_i\right|\\&+\left|\frac{\partial\pi_{i-1}}{\partial y}\varepsilon_1(y)z_i\right|.\end{aligned} \tag{83}$$

By Young's inequality and Assumption 1, one obtains the following inequalities:

$$\left|\frac{\partial\pi_{i-1}}{\partial y}p^*e_2z_i\right| \le e^Te + p(\frac{\partial\pi_{i-1}}{\partial y})^2z_i^2, \tag{84}$$

$$\begin{aligned}&\left|\frac{\partial\pi_{i-1}}{\partial y}\Delta_1 z_i\right|\\&\le \left|\frac{\partial\pi_{i-1}}{\partial y}\left(p_1^*\psi_{11}(|y|) + p_1^*\psi_{12}(|\zeta|)\right)z_i\right|\\&\le p_1^*\left|\frac{\partial\pi_{i-1}}{\partial y}z_i\right|\psi_{11}(|y|)\\&\quad+\frac{1}{4}p(\frac{\partial\pi_{i-1}}{\partial y})^2z_i^2 + \psi_{12}^2(|\zeta|)\\&\le p_1^*\left|\frac{\partial\pi_{i-1}}{\partial y}z_i\right||y|\,\bar{\psi}_{11}(|y|)\\&\quad+\frac{1}{2}p(\frac{\partial\pi_{i-1}}{\partial y})^2z_i^2 + \psi_{12}^2(|\zeta|) + \psi_{11}^2(0)\end{aligned} \tag{85}$$

$$\begin{aligned}&p_1^*\left|\frac{\partial\pi_{i-1}}{\partial y}z_i\right||y|\,\bar{\psi}_{11}(|y|)\\&\le p_1^*\left|\frac{\partial\pi_{i-1}}{\partial y}z_i\right|(y\hat{\psi}_{11}(y) + d_{11})\\&\le p(\frac{\partial\pi_{i-1}}{\partial y})^2z_i^2\hat{\psi}_{11}^2(y)\\&\quad+\frac{1}{4}y^2 + \frac{1}{2}p(\frac{\partial\pi_{i-1}}{\partial y})^2z_i^2 + \frac{1}{2}d_{11}^2\end{aligned} \tag{86}$$

$$\begin{aligned}&\left|\frac{\partial\pi_{i-1}}{\partial y}\Delta_1 z_i\right|\\&\le p(\frac{\partial\pi_{i-1}}{\partial y})^2z_i^2(\hat{\psi}_{11}^2(y) + 1)\\&\quad+\frac{1}{4}y^2 + \psi_{12}^2(|\zeta|) + \psi_{11}^2(0) + \frac{1}{2}d_{11}^2\end{aligned} \tag{87}$$

$$\left|\frac{\partial \pi_{i-1}}{\partial y}\varepsilon_1(y)z_i\right| - z_i\eta_i\beta\tanh(\frac{z_i\eta_i}{\varsigma}) \le 0.2785\varsigma\beta = \varsigma', \tag{88}$$

where $\eta_i = -\partial\pi_{i-1}/\partial y$.

Substituting (84), (87) and (88) into (83) gives

$$\begin{aligned}\dot V_i \le & -[\lambda_{\min}(Q) - (i+3)]\,\|e\|^2 - z_1 y\rho(y^2) \\ & + \frac{i-1}{4}y^2 + z_i[z_{i+1} + \pi_i - k_i v_{0,1} \\ & - \frac{\partial\pi_{i-1}}{\partial y}(\xi_2 + f_{1,0}(y) + \omega^T\hat\vartheta) + H_i] \\ & - z_i\frac{\partial\pi_{i-1}}{\partial\hat\vartheta}(\dot{\hat\vartheta} - \Gamma\tau_{i-1} + \Gamma\mu\hat\vartheta) \\ & - z_i\frac{\partial\pi_{i-1}}{\partial\hat\beta}(\dot{\hat\beta} - \gamma_1\sigma_{i-1} + \gamma_1\mu\hat\beta) \\ & - z_i\frac{\partial\pi_{i-1}}{\partial\hat p}(\dot{\hat p} - \gamma_2\bar\lambda_{i-1} + \gamma_2\mu\hat p) \\ & + \tilde\vartheta^T(\tau_{i-1} - \frac{\partial\pi_{i-1}}{\partial y}\omega z_i - \Gamma^{-1}\dot{\hat\vartheta}) \\ & + \tilde\beta(\sigma_{i-1} + \eta_i z_i\tanh\left(\frac{z_i\eta_i}{\varsigma}\right) - \gamma_1^{-1}\dot{\hat\beta}) \\ & + \tilde p(\bar\lambda_{i-1} + \left(\frac{\partial\pi_{i-1}}{\partial y}\right)^2 z_i^2(\hat\psi_{11}^2(y)+2) \\ & - \gamma_2^{-1}\dot{\hat p}) + \mu\tilde\kappa\hat\kappa - \sum_{j=1}^{i-1}c_j z_j^2 \\ & + \|P\|^2(\sum_{k=1}^{n}\psi_{k2}(|\zeta|))^2 + i\psi_{12}^2(|\zeta|) \\ & + \|P\|^2\beta^2 + \|P\|^2 d_\psi^0 + i\varsigma' \\ & + i\psi_{11}^2(0) + \frac{i+1}{2}d_{11}^2,\end{aligned} \tag{89}$$

$$\begin{aligned}H_i = & -\frac{\partial\pi_{i-1}}{\partial\xi}(A_0\xi + ky) - \frac{\partial\pi_{i-1}}{\partial\Xi}(A_0\Xi + \Phi^T(y)) \\ & - \frac{\partial\pi_{i-1}}{\partial v_0}\dot v_0 - \frac{\partial\pi_{i-1}}{\partial\kappa}\dot\kappa - \frac{\partial\pi_{i-1}}{\partial\hat\vartheta}\Gamma(\tau_{i-1} - \mu\hat\vartheta) \\ & - \frac{\partial\pi_{i-1}}{\partial\hat\beta}\gamma_1(\sigma_{i-1} - \mu\hat\beta) - \frac{\partial\pi_{i-1}}{\partial\hat p}\gamma_2(\bar\lambda_{i-1} - \mu\hat p) \\ & - \hat\beta\eta_i\tanh\left(\frac{z_i\eta_i}{\varsigma}\right) + \hat p\left(\frac{\partial\pi_{i-1}}{\partial y}\right)^2 z_i^2(\hat\psi_{11}^2(y)+2).\end{aligned}$$

Define

$$-\frac{\partial\pi_{i-1}}{\partial\hat\vartheta}(\dot{\hat\vartheta} - \Gamma\tau_{i-1} + \Gamma\mu\hat\vartheta) = \sum_{j=i}^{n}\Delta_{i,j}z_j, \tag{90}$$

$$-\frac{\partial\pi_{i-1}}{\partial\hat\beta}(\dot{\hat\beta} - \gamma_1\sigma_{i-1} + \gamma_1\mu\hat\beta) = \sum_{j=i}^{n}\Lambda_{i,j}z_j, \tag{91}$$

$$-\frac{\partial\pi_{i-1}}{\partial\hat p}(\dot{\hat p} - \gamma_2\bar\lambda_{i-1} + \gamma\mu\hat p) = \sum_{j=i}^{n}\mathrm{A}_{i,j}z_j, \tag{92}$$

where

$$\Delta_{i,j} = \sum_{j=i}^{n}\frac{\partial\pi_{i-1}}{\partial\hat\vartheta}\Gamma\frac{\partial\pi_{j-1}}{\partial y}\omega,$$

$$\Lambda_{i,j} = \sum_{j=i}^{n}\frac{\partial\pi_{i-1}}{\partial\hat\beta}\gamma\eta_j\tanh\left(\frac{\eta_j z_j}{\varsigma}\right),$$

$$\mathrm{A}_{i,j} = \sum_{j=i}^{n}\frac{\partial\pi_{i-1}}{\partial\hat\vartheta}\gamma_2\left(\frac{\partial\pi_{j-1}}{\partial y}\right)^2 z_j^2(\hat\psi_{11}^2(y)+2).$$

Choose the stabilizing control function π_i as

$$\begin{aligned}\pi_i = & -z_{i-1} - c_i z_i + k_i v_{0,1} \\ & + \frac{\partial\pi_{i-1}}{\partial y}(\xi_2 + f_{1,0}(y) + \omega^T\hat\vartheta) \\ & - \sum_{k=2}^{i-1}(\Delta_{k,i} + \Lambda_{k,i} + \mathrm{A}_{k,i})z_k - H_i,\end{aligned} \tag{93}$$

where $c_i > 0$ is a design constant. Substituting (90)–(93) into (89) and repeating procedures in Step 2, we have

$$\begin{aligned}\dot V_i \le & -[\lambda_{\min}(Q) - (i+3)]\,\|e\|^2 - z_1 y\rho(y^2) \\ & + \frac{i-1}{4}y^2 - \sum_{j=1}^{i}c_j z_j^2 + \tilde\vartheta^T(\tau_i - \Gamma^{-1}\dot{\hat\vartheta}) \\ & + \tilde\beta(\sigma_i - \gamma_1^{-1}\dot{\hat\beta}) + \tilde p(\bar\lambda_i - \gamma_2^{-1}\dot{\hat p}) + \mu\tilde\kappa\hat\kappa \\ & + \sum_{j=i+1}^{n}\sum_{k=2}^{i}(\Delta_{k-1,j} + \Lambda_{k-1,j})z_k z_j \\ & + \|P\|^2(\sum_{k=1}^{n}\psi_{k2}(|\zeta|))^2 + z_i z_{i+1} \\ & + i\psi_{12}^2(|\zeta|) + \|P\|^2\beta^2 + \|P\|^2 d_\psi^0 + i\varsigma' \\ & + i\psi_{11}^2(0) + \frac{i+1}{2}d_{11}^2\end{aligned} \tag{94}$$

Step n: In the final design step, the actual control input u appears. Consider the overall Lyapunov function as

$$V_n = V_{n-1} + \frac{1}{2}z_n^2. \tag{95}$$

Using (33) and (34), the time derivative of V_n is

$$\begin{aligned}\dot V_n \le & -[\lambda_{\min}(Q) - (n+3)]\,\|e\|^2 - z_1 y\rho(y^2) \\ & + \frac{n-1}{4}y^2 + z_n[\sigma(y)u - k_n v_{0,1} \\ & - \frac{\partial\pi_{n-1}}{\partial y}(\xi_2 + f_{1,0}(y) + \omega^T\hat\vartheta) + H_n] \\ & - \frac{\partial\pi_{n-1}}{\partial\hat\vartheta}(\dot{\hat\vartheta} - \Gamma\tau_{n-1} + \Gamma\mu\hat\vartheta) \\ & - \frac{\partial\pi_{n-1}}{\partial\hat\beta}(\dot{\hat\beta} - \gamma_1\sigma_{n-1} + \gamma_1\mu\hat\beta)\end{aligned}$$

$$
\begin{aligned}
&- \frac{\partial \pi_{n-1}}{\partial \hat{p}}(\dot{\hat{p}} - \gamma_2 \bar{\lambda}_{n-1} + \gamma_2 \mu \hat{p}) \\
&+ \tilde{\vartheta}^T(\tau_{n-1} - \frac{\partial \pi_{n-1}}{\partial y}\omega z_n - \Gamma^{-1}\dot{\hat{\vartheta}}) \\
&+ \tilde{\beta}\Big(\sigma_{n-1} + \eta_n z_n \tanh\Big(\frac{z_n \eta_n}{\varsigma}\Big) - \gamma_1^{-1}\dot{\hat{\beta}}\Big) \\
&+ \tilde{p}(\bar{\lambda}_{n-1} + \Big(\frac{\partial \pi_{n-1}}{\partial y}\Big)^2 z_n^2(\hat{\psi}_{11}^2(y) + 2) \\
&- \gamma_2^{-1}\dot{\hat{p}}) + \mu\tilde{\kappa}\hat{\kappa} - \sum_{j=1}^{n-1} c_j z_j^2 \\
&+ \|P\|^2(\sum_{k=1}^{n}\psi_{k2}(|\zeta|))^2 + n\psi_{12}^2(|\zeta|) \\
&+ \|P\|^2\beta^2 + \|P\|^2 d_\psi^0 + n\varsigma' \\
&+ n\psi_{11}^2(0) + \frac{n+1}{2}d_{11}^2,
\end{aligned} \tag{96}
$$

where

$$
\begin{aligned}
H_n = &- \frac{\partial \pi_{n-1}}{\partial \xi}(A_0\xi + ky) - \frac{\partial \pi_{n-1}}{\partial \Xi}(A_0\Xi + \Phi^T(y)) \\
&- \frac{\partial \pi_{n-1}}{\partial v_0}\dot{v}_0 - \frac{\partial \pi_{n-1}}{\partial \kappa}\dot{\kappa} - \frac{\partial \pi_{n-1}}{\partial \hat{\vartheta}}\Gamma(\tau_{n-1} - \mu\hat{\vartheta}) \\
&- \frac{\partial \pi_{n-1}}{\partial \hat{\beta}}\gamma_1(\sigma_{n-1} - \mu\hat{\beta}) \\
&- \frac{\partial \pi_{n-1}}{\partial \hat{p}}\gamma_2(\bar{\lambda}_{n-1} - \mu\hat{p}) \\
&- \hat{\beta}\eta_n \tanh\Big(\frac{z_n\eta_n}{\varsigma}\Big) + \hat{p}\Big(\frac{\partial \pi_{n-1}}{\partial y}\Big)^2 z_n^2(\hat{\psi}_{11}^2(y) + 2).
\end{aligned}
$$

Choose the actual control u as

$$
\begin{aligned}
u = \frac{1}{\sigma(y)}\Big(&k_n v_{0,1} + \frac{\partial \pi_{n-1}}{\partial y}(\xi_2 + f_{1,0}(y) \\
&+ \omega^T\hat{\vartheta}) - c_n z_n - z_{n-1} - \sum_{k=2}^{n-1}(\Delta_{k-1,n} \\
&+ \Lambda_{k-1,n} + \mathrm{A}_{k-1,n})z_k - H_n\Big),
\end{aligned} \tag{97}
$$

where $c_n > 0$ is a design constant.

Substituting (73)–(75) and (97) into (96) yields

$$
\begin{aligned}
\dot{V}_n \le &- [\lambda_{\min}(Q) - (n+3)]\,\|e\|^2 - z_1 y\rho(y^2) \\
&+ \frac{n-1}{4}y^2 - \sum_{j=1}^{n} c_j z_j^2 + \mu\tilde{\vartheta}\hat{\vartheta} + \mu\tilde{\beta}\hat{\beta} \\
&+ \mu\tilde{p}\hat{p} + \mu\tilde{\kappa}\hat{\kappa} + \|P\|^2(\sum_{k=1}^{n}\psi_{k2}(|\zeta|))^2 \\
&+ n\psi_{12}^2(|\zeta|) + \|P\|^2\beta^2 + \|P\|^2 d_\psi^0 + n\varsigma' \\
&+ n\psi_{11}^2(0) + \frac{n+1}{2}d_{11}^2.
\end{aligned} \tag{98}
$$

By completing the squares,

$$
\begin{aligned}
\mu\tilde{\vartheta}^T\hat{\vartheta} = \mu\tilde{\vartheta}^T(\vartheta - \tilde{\vartheta}) &= \mu\tilde{\vartheta}^T\vartheta - \mu\Big\|\tilde{\vartheta}\Big\|^2 \\
&\le -\frac{1}{2}\mu\Big\|\tilde{\vartheta}\Big\|^2 + \frac{1}{2}\mu\|\vartheta\|^2,
\end{aligned} \tag{99}
$$

$$
\mu\tilde{\beta}\hat{\beta} \le -\frac{1}{2}\mu\Big\|\tilde{\beta}\Big\|^2 + \frac{1}{2}\mu\|\beta\|^2, \tag{100}
$$

$$
\mu\tilde{p}\hat{p} \le -\frac{1}{2}\mu\|\tilde{p}\|^2 + \frac{1}{2}\mu\|p\|^2, \tag{101}
$$

$$
\mu\tilde{\kappa}\hat{\kappa} \le -\frac{1}{2}\mu\|\tilde{\kappa}\|^2 + \frac{1}{2}\mu\|\kappa\|^2. \tag{102}
$$

Substituting (99)–(102) into (98) results in

$$
\begin{aligned}
\dot{V}_n \le &- [\lambda_{\min}(Q) - (n+3)]\,\|e\|^2 - \sum_{j=1}^{n} c_j z_j^2 \\
&- \frac{1}{2}\mu(\Big\|\tilde{\vartheta}\Big\|^2 + \Big\|\tilde{\beta}\Big\|^2 + \|\tilde{p}\|^2 + \|\tilde{\kappa}\|^2) \\
&- y^2(\lambda'(y^2)\rho(y^2) - \frac{n-1}{4}) \\
&+ \|P\|^2(\sum_{k=1}^{n}\psi_{k2}(|\zeta|))^2 + n\psi_{12}^2(|\zeta|) + D,
\end{aligned} \tag{103}
$$

where

$$
\begin{aligned}
D = &\frac{1}{2}\mu(\|\vartheta\|^2 + \|\beta\|^2 + \|p\|^2) + \|P\|^2\beta^2 \\
&+ \|P\|^2 d_\psi^0 + n\varsigma' + n\psi_{11}^2(0) + \frac{n+1}{2}d_{11}^2.
\end{aligned}
$$

Assume that

$$
\lambda_{\min}(Q) - (n+3) > 0.
$$

In the sequel, we are to robustify the adaptive fuzzy controller obtained in the preceding design procedures via the appropriate choice of design functions $\lambda(y^2)$ and $\rho(y^2)$ to check the conditions of small-gain Theorem 1.

Firstly, choose a smooth function $\rho(y^2)$ as introduced in Step 1 to satisfy

$$
y^2\Big[\lambda'(y^2)\rho(y^2) - \frac{n-1}{4}\Big] \ge c_1\lambda(y^2). \tag{104}
$$

Because $\lambda'(y^2) \ne 0$ for any y, as stated by Jiang (1999), such a smooth function always exists. Since each function ψ_{i2} is smooth and vanishes at the origin, there is a smooth class-κ_∞ function h such that

$$
\|P\|^2(\sum_{i=1}^{n}\psi_{i2}(|\zeta|))^2 + n\psi_{12}^2(|\zeta|) \le h(|\zeta|^2). \tag{105}
$$

Let

$$
\begin{aligned}
c = \min\Big\{&\frac{\lambda_{\min}(Q) - (n+3)}{\lambda_{\max}(P)}, 2c_j, \frac{\mu}{\lambda_{\max}(\Gamma^{-1})}, \\
&\mu\gamma_1,\ \mu\gamma_2,\ \mu\gamma_3; j = 1, \ldots, n\Big\}.
\end{aligned} \tag{106}
$$

Then (103) can be expressed as

$$\dot{V}_n \le -cV_n + h(|\zeta|^2) + D, \tag{107}$$

In order to use Lemma 1 and Theorem 1, one assumes that

$$\begin{aligned}\dot{V}_n &\le -d_1V_n + d_1V_n - cV_n + h(|\zeta|^2) + D\\ &\le -d_1V_n\end{aligned} \tag{108}$$

that is, for any $0 < d_1 < c$, (108) ensures that the following inequality holds:

$$d_1V_n - cV_n + h(|\zeta|^2) + D \le 0 \tag{109}$$

or, equivalently,

$$V_n \ge \frac{h(|\zeta|^2)}{c-d_1} + \frac{D}{c-d_1}. \tag{110}$$

From (3), one has

$$|\zeta| \le \alpha_1^{-1}(V_0(\zeta)), \tag{111}$$

$$|\zeta| \ge \alpha_2^{-1}(V_0(\zeta)). \tag{112}$$

From (111) and (110), we obtain

$$\frac{h(\alpha_1^{-1}(V_0(\zeta))^2)}{c-d_1} + \frac{D}{c-d_1} \ge \frac{h(|\zeta|^2)}{c-d_1} + \frac{D}{c-d_1}.$$

Therefore, (110) holds as long as the following inequality holds:

$$V_n \ge \frac{h(\alpha_1^{-1}(V_0(\zeta))^2)}{c-d_1} + \frac{D}{c-d_1}. \tag{113}$$

On the other hand,

$$\begin{aligned}\max&\left\{\frac{2h(\alpha_1^{-1}(V_0(\zeta))^2)}{c-d_1}, \frac{2D}{c-d_1}\right\}\\ &\ge \frac{h(\alpha_1^{-1}(V_0(\zeta))^2)}{c-d_1} + \frac{D}{c-d_1}.\end{aligned} \tag{114}$$

Therefore, if

$$V_n \ge \max\{\frac{2h(\alpha_1^{-1}(V_0(\zeta))^2)}{c-d_1}, \frac{2D}{c-d_1}\}, \tag{115}$$

it follows that (108) holds.

Secondly, in order to invoke Theorem 1 (the small-gain theorem), the function $\lambda(y^2)$ needs to be chosen appropriately such that for arbitrary $d_2 > 0$, the following inequality holds:

$$\gamma^{-1} \circ \gamma_0(|y|) \le \frac{1}{4}\lambda(y^2) + d_2 \le \frac{1}{2}V_n + d_2, \tag{116}$$

where the notation $\circ$ stands for the composition operator between two functions. Since γ is a κ_∞-function and γ is an increasing function, we have

$$\gamma(\frac{1}{2}V_n + d_2) \le \gamma(V_n) + \gamma(2d_2). \tag{117}$$

Substituting (116) and (117) into (3) results in

$$\begin{aligned}\frac{\partial V_0}{\partial \zeta}q(\zeta, y) &\le -\alpha_0(|\zeta|) + \gamma(\gamma^{-1} \circ \gamma_0(|y|)) + d_0\\ &\le -\alpha_0(|\zeta|) + \gamma(V_n) + \gamma(2d_2) + d_0.\end{aligned} \tag{118}$$

For any given $0 < d_3 < 1$, by (109) and (115), we obtain

$$\begin{aligned}\dot{V}_0 &\le -\alpha_0(|\zeta|) + \gamma(V_n) + \gamma(2d_2) + d_0\\ &\le -d_3\alpha_0 \circ \alpha_2^{-1}(V_0(\zeta))\\ &\quad + d_3\alpha_0 \circ \alpha_2^{-1}(V_0(\zeta)) - \alpha_0 \circ \alpha_2^{-1}(V_0(\zeta))\\ &\quad + \gamma(V_n) + \gamma(2d_2) + d_0.\end{aligned} \tag{119}$$

The following inequality holds:

$$\dot{V}_0 \le -d_3\alpha_0 \circ \alpha_2^{-1}(V_0(\zeta)), \tag{120}$$

as long as

$$\begin{aligned}&d_3\alpha_0 \circ \alpha_2^{-1}(V_0(\zeta)) - \alpha_0 \circ \alpha_2^{-1}(V_0(\zeta))\\ &\qquad + \gamma(V_n) + \gamma(2d_2) + d_0 \le 0\end{aligned} \tag{121}$$

or, equivalently,

$$\alpha_0 \circ \alpha_2^{-1}(V_0(\zeta)) \ge \frac{\gamma(V_n)}{1-d_3} + \frac{\gamma(2d_2)+d_0}{1-d_3}. \tag{122}$$

From (122), we get

$$V_0 \ge \alpha_2 \circ \alpha_0^{-1}\left\{\frac{\gamma(V_n)}{1-d_3} + \frac{\gamma(2d_2)+d_0}{1-d_3}\right\}. \tag{123}$$

Since

$$\begin{aligned}\max&\left\{\alpha_2 \circ \alpha_0^{-1} \circ \frac{2\gamma(V_n)}{1-d_3}, \alpha_2 \circ \alpha_0^{-1} \circ \frac{2\gamma(2d_2)+2d_0}{1-d_3}\right\}\\ &\ge \alpha_2 \circ \alpha_0^{-1}\left\{\frac{\gamma(V_n)}{1-d_3} + \frac{\gamma(2d_2)+d_0}{1-d_3}\right\},\end{aligned}$$

for any given $d_4 > 0$, as long as

$$\begin{aligned}d_4V_0 \ge \max&\left\{d_4\alpha_2 \circ \alpha_0^{-1} \circ \frac{2\gamma(V_n)}{1-d_3},\right.\\ &\left. d_4\alpha_2 \circ \alpha_0^{-1} \circ \frac{2\gamma(2d_2)+2d_0}{1-d_3}\right\},\end{aligned} \tag{124}$$

it is sufficient to guarantee that the inequality (120) holds, that is, $\dot{V}_0 \le -d_3\alpha_0 \circ \alpha_2^{-1}(V_0)$. From (108) and (115), the condition (8) is satisfied. The class κ_∞-function for the $(x_1, v_0, z_1, \cdots, z_n, \hat{\beta}, \hat{p}, \hat{\vartheta}, \kappa)$ system with input d_4V_0 and output V_n is given as

$$\chi_1(s) = \frac{2h(\alpha_1^{-1}(\frac{1}{d_4}s)^2)}{c-d_1}. \tag{125}$$

Similarly, from (120) and (124), the condition (9) is also satisfied for the z-system with input V_n and output d_4V_0. The gain function is

$$\chi_2(s) = d_4\alpha_2 \circ \alpha_0^{-1} \circ \frac{2\gamma(s)}{1-d_3}. \tag{126}$$

Finally, to check the conditions of Theorem 1 (Jiang *et al.*, 1996), for any given $s > 0$, we select any function γ of class κ_∞ such that

$$\gamma(s) < \frac{1-d_3}{2}\alpha_0 \circ \alpha_2^{-1} \circ \alpha_1\Big(\sqrt{h^{-1}(\frac{c-d_1}{2}s)}\Big). \quad (127)$$

From (125), (126) and (127), we obtain

$$\chi_1 \circ \chi_2(s) < s. \quad (128)$$

Therefore, by (128) and Theorem 1, we conclude that the closed-loop system is ISpS, and the variables $x_i(t)$, $e(t)$, ϑ, β, p, κ and $u(t)$ are uniformly ultimately bounded.

The above analysis and small-gain design are summarized in the following theorem.

Theorem 2. *For the nonlinear system (1), under Assumptions 1–3, after the application of the above design procedures, the proposed adaptive fuzzy output feedback control scheme can guarantee that all the signals of the closed-loop system are uniformly ultimately bounded.*

From the previous synthesis, we can obtain the following controller design procedure:

Step 1: Define the fuzzy IF-THEN rules and the membership functions, determine the fuzzy basis functions, and establish the fuzzy logic systems (11).

Step 2: Specify the observer gain vector K, such that A_0 is a strict Hurwitz matrix.

Step 3: Specify a positive definite matrix $Q = Q^T > 0$, such that $\lambda_{\min}(Q) - (n + 3) > 0$, solving the matrix equation (19) to obtain positive definite matrix P.

Step 4: Select the appropriate design parameter $c_1 > 0$, by using (104), (116) and (127), construct functions $\rho(y^2)$ and $\lambda(y^2)$, then obtain the adaptation functions $\hat{\kappa}$ in (58) and the stabilizing function $\bar{\pi}_1$ in (54), i.e., the intermediate control function $\pi_1 = \hat{\kappa}\bar{\pi}_1$.

Step 5: Select the appropriate design parameters $c_i > 0$ $(i = 2, \dots, n)$, $\Gamma = \Gamma^T > 0$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\gamma_3 > 0$, $\mu > 0$. Compute the partial derivations of the stabilizing function π_i, for $i = 1, \dots, n-1$, then obtain the intermediate control function π_i in (93). Finally, obtain u in (97) and adaptation functions $\hat{\beta}$, $\hat{p}$ and $\hat{\vartheta}$, which are expressed by (73)–(75), respectively.

4. Simulation

In this section, a simulation example and simulation comparisons are presented to show the effectiveness of the proposed adaptive backstepping control scheme.

Example 1. Consider the second-order nonlinear system with unmodeled dynamics and dynamical disturbances

$$\begin{aligned}\dot{\zeta} &= -\zeta + 0.125y^2,\\ \dot{x}_1 &= x_2 + f_{1,0}(y) + f_1(y) + \Delta_1,\\ \dot{x}_2 &= b_0u + f_{2,0}(y) + f_2(y) + \Delta_2,\\ y &= x_1,\end{aligned} \quad (129)$$

where $f_{1,0}(y) = 0$ and $f_{2,0}(y) = 0$ are known functions. $f_1(y) = 5y^2$ and $f_2(y) = -0.5y$ are unknown functions. $\dot{\zeta} = -\zeta + 0.125y^2$ stands for unmodeled dynamics, $\Delta_1 = \zeta^2$ and $\Delta_2 = 0.2\zeta^2$ are nonlinear dynamical disturbances and $b_0 = -4$.

Based on the following inequalities:

$$|\Delta_1| \le |\zeta|^2 = 0.|x_1|^2 + 1.|\zeta|^2 \le p_1^*|x_1|^2 + p_1^*|\zeta|^2,$$
$$|\Delta_2| \le |\zeta|^2 = 0.|x_1|^2 + 0.2.|\zeta|^2 \le p_2^*|x_1|^2 + p_2^*|\zeta|^2,$$

set $\psi_{11}(s) = 0$, $\psi_{21}(s) = 0$, $\psi_{12}(s) = s^2$, $\psi_{22}(s) = s^2$. Then Assumption 1 is satisfied.

Taking $V_0 = \zeta^2$, the time derivative of V_0 along (129) is

$$\begin{aligned}\dot{V}_0 &= 2\zeta\dot{\zeta} = -2\zeta^2 + 0.25\zeta y^2\\ &\le -1.875\zeta^2 + 0.125y^4\\ &\le -\zeta^2 + 0.125y^4.\end{aligned}$$

Defining $\alpha_0(s) = s^2$, $\gamma_0(s) = 0.125s^4$, $\alpha_1(s) = 0.5s^2$, $\alpha_2(s) = 1.5s^2$ and $d_0 = 0$, Assumption 2 holds.

According to the controller design procedures in the above section, the controller design is as follows:

Step 1: Since the functions $f_1(y)$ and $f_1(y)$ in (129) are zero at the point $y = 0$, it usually takes a symmetrical interval about the origin $[-a, a]$. Then choose the fuzzy membership function to cover the interval $[-a, a]$ uniformly. In this way, the fuzzy logic system can achieve better approximating results. In this paper, we take $[-a, a]$ as $[-6, 6]$ for the variable y. Therefore, define fuzzy membership functions as

$$\mu_{F_1^l}(y) = \exp[-(y - 6 + 2l)^2], \quad l = 1, \dots, 5,$$
$$\mu_{F_2^l}(y) = \exp[-(y - 6 + 2l)^2] \cdot \exp[-(y - 12 + l4)^2], \quad l = 1, \dots, 5.$$

The fuzzy basis functions are expressed as

$$\varphi_{1j}(y) = \frac{\exp[-(y-6+2j)^2]}{\sum_{n=1}^{5}\exp[-(y-6+2n)^2]}, \quad l = 1, \dots, 5,$$

$$\varphi_{2j}(y) = \frac{\exp[-(y-6+2j)^2] \times \exp[-(y-12+4j)^2]}{\sum_{n=1}^{5}\exp[-(y-6+2n)^2] \times \exp[-(y-12+4n)^2]}, \quad l = 1, \dots, 5.$$

Therefore, the unknown functions $f_i(y)$ $(i = 1,\ 2)$ can be approximated as $f_1(y) = \theta_1^{*T}\varphi_1(y) + \varepsilon_1(y)$ and $f_2(y) = \theta_2^{*T}\varphi_2(y) + \varepsilon_2(y)$.

Step 2: Specify the observer gain vector

$$K = [k_1, k_2]^T = [15, 15]^T,$$

such that A_0 is a strict Hurwitz matrix.

Step 3: Given a positive definite matrix $Q = 8I$, such that $\lambda_{\min}(Q) - (n+3) > 0$, by solving the Lyapunov equation (19) , a positive definite matrix

$$P = \begin{bmatrix} 4.2667 & -4 \\ -4 & 4.2844 \end{bmatrix}$$

is obtained.

Step 4: Selecting the design parameter $c_1 = 2$ and design function $\lambda(y^2) = 10y^8 + 0.1y^2$, we obtain the function

$$\phi_{11}(y) = 1 + \frac{1}{2} \cdot \frac{1}{40y^6 + 0.1}.$$

Using (104), we choose function $\rho(y^2) = y^2 + 4.85$. Using (105) and (127), respectively, the design functions are $h(s) = 280s^2$, $\gamma(s) = 0.007s^{\frac{1}{2}}$ and $d_1 = 0.0001$. If we choose $d_3 = 0.0001$, $d_2 = d_4 = 0.1$ and $c = 1$, then the conditions of the small-gain-theorem are satisfied.

Define the parameter vector as

$$\vartheta = \begin{bmatrix} b_0 & \theta_1^{*T} & \theta_2^{*T} \end{bmatrix}^T.$$

The filters are given as

$$\dot{\xi} = A_0\xi + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} y,$$

$$\dot{\Xi} = A_0\Xi + \Phi, \quad \dot{v}_0 = A_0 v_0 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$

where

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}, \qquad A_0 = \begin{bmatrix} -k_1 & 1 \\ -k_2 & 0 \end{bmatrix},$$
$$\Xi = \begin{bmatrix} \Xi_{(1)} \\ \Xi_{(2)} \end{bmatrix}, \qquad \Phi = \begin{bmatrix} \varphi_1^T & 0 \\ 0 & \varphi_2^T \end{bmatrix},$$
$$v_0 = \begin{bmatrix} v_{01} \\ v_{02} \end{bmatrix}.$$

The stabilization functions and the control laws are given by

$$\begin{aligned} \pi_1 &= \hat{\kappa}\bar{\pi}_1, \\ \bar{\pi}_1 &= -y\rho(y^2) - \xi_2 - f_{1,0}(y) - \bar{\omega}\hat{\vartheta} \\ &\quad - \frac{\|P\|^2}{\lambda'} y\phi_1(y) - \hat{p}\phi_{11}(y)z_1 - \eta_1\hat{\beta}\tanh\Big(\frac{z_1\eta_1}{\varsigma}\Big), \end{aligned}$$

where

$$\begin{aligned} \omega &= [v_{0,2}, \Phi_{(1)}^T(y) + \Xi_{(2)}]^T, \\ \bar{\omega} &= [0, \Phi_{(1)}^T(y) + \Xi_{(2)}]^T. \end{aligned}$$

Step 5: Controller u and parameter adaptation laws are chosen as

$$\begin{aligned} u = \frac{1}{\sigma(y)}\Big(&- \hat{b}_0 z_1 - c_2 z_2 + \frac{\partial \pi_1}{\partial y}(\xi_2 + f_{1,0}(y) + \omega^T\hat{\vartheta}) \\ &+ k_2 v_{0,1} - (\Delta_{1,2} + \Lambda_{1,2} + \mathrm{A}_{1,2}) - H_2\Big) \end{aligned}$$

$$\begin{aligned} \dot{\hat{\vartheta}} &= \Gamma(\tau_2 - \mu\hat{\vartheta}), & \dot{\hat{\beta}} &= \gamma_1(\sigma_2 - \mu\hat{\beta}), \\ \dot{\hat{p}} &= \gamma_2(\bar{\lambda}_2 - \mu\hat{p}), & \dot{\hat{\kappa}} &= -\gamma_3(\mathrm{sgn}(b_0)\bar{\pi}_1 z_1 + \mu\hat{\kappa}), \end{aligned}$$

where

$$\begin{aligned} \tau_1 &= \bar{\omega}z_1, \\ \sigma_1 &= z_1\eta_1\beta\tanh\Big(\frac{z_1\eta_1}{\varsigma}\Big), \\ \bar{\lambda}_1 &= \phi_{11}(y)z_1^2, \\ \tau_2 &= \bar{\omega}z_1 + \ell z_2 - z_2\frac{\partial \pi_1}{\partial y}\omega, \\ \sigma_2 &= z_1\eta_1\tanh\Big(\frac{z_1\eta_1}{\varsigma}\Big) + z_2\eta_2\tanh\Big(\frac{z_2\eta_2}{\varsigma}\Big), \\ \lambda_2 &= \phi_{11}(y)z_1^2 + \Big(\frac{\partial \pi_1}{\partial y}\Big)^2 z_2^2(\hat{\psi}_{11}^2(y) + 2), \end{aligned}$$

$$\begin{aligned} H_2 = &-\frac{\partial \pi_1}{\partial \xi}(A_0\xi + ky) - \frac{\partial \pi_1}{\partial \Xi}(A_0\Xi + \Phi^T(y)) \\ &- \frac{\partial \pi_1}{\partial v_0}\dot{v}_0 - \frac{\partial \pi_1}{\partial \kappa}\dot{\kappa} - \frac{\partial \pi_1}{\partial \hat{\vartheta}}\Gamma(\tau_1 - \mu\hat{\vartheta}) \\ &- \frac{\partial \pi_1}{\partial \hat{\beta}}\gamma_1(\sigma_1 - \mu\hat{\beta}) - \frac{\partial \pi_1}{\partial \hat{p}}\gamma_2(\bar{\lambda}_1 - \mu\hat{p}) \\ &- \hat{\beta}\eta_2\tanh\Big(\frac{z_2\eta_2}{\varsigma}\Big) + \hat{p}\Big(\frac{\partial \pi_1}{\partial y}\Big)^2 z_2^2(\hat{\psi}_{11}^2(y) + 2). \end{aligned}$$

The design parameters in the controller and adaptation laws are chosen as $c_2 = 0.5$, $\mu = 0.9$, $\Gamma = 1.2I$, $\varsigma = 0.1$, $\gamma_1 = 1.2$, $\gamma_2 = 1.2$, $\gamma_3 = 1.2$.

If the initial conditions are given as

$$\begin{aligned} &x_1(0) = 0.12, & \zeta(0) &= 0.1, \\ &\hat{\kappa}(0) = 0.1, & \hat{p}(0) &= 1.5, \\ &\hat{\vartheta}(0) = [0.8, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]^T, \\ &\xi(0) = \Xi(0) = v_0(0) = \hat{\beta}(0) = 0. \end{aligned}$$

The simulation results are shown in Figs. 1–3, where Figs. 1 and 2 show the trajectories of x_1, x_2, $\hat{x}_1$ and $\hat{x}_2$, respectively, while Fig. 3. shows the trajectory of control input u.

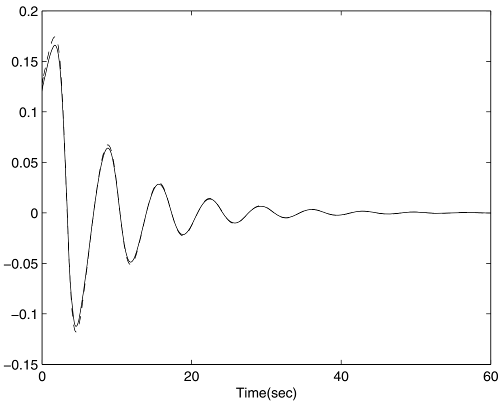

Fig. 1. Trajectories of x_1 (dashed) and $\hat{x}_1$ (solid).

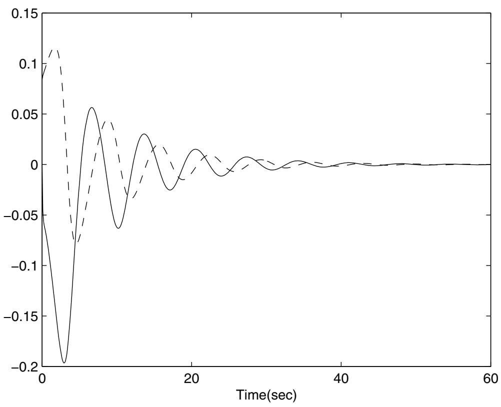

Fig. 2. Trajectories of x_2 (dashed) and $\hat{x}_2$ (solid).

Case 1: Consider the nonlinear system (129) without unmodeled dynamics and dynamical disturbances, and assume that x_1 and x_2 are both measured.

For this case, we use the same fuzzy membership functions and fuzzy logic systems as in Example 1 to approximate the unknown functions $f_1(y)$ and $f_2(y)$, respectively. According to Yang *et al.* (2005), the stabilization functions, the controller and adaptation laws are given by

$$\alpha_1 = -k_1 z_1 - \frac{1}{4\gamma_1^2} z_1 \lambda_1 \xi_1^T(x_1)\xi_1(x_1) - \hat{\theta}_1 \tanh\Big(\frac{\hat{\theta}_1 z_1}{\delta_1}\Big),$$

$$\dot{\lambda}_1 = \Gamma_{11}\Big[\frac{1}{4\gamma_1^2}\lambda_1 \xi_1^T(x_1)\xi_1(x_1) z_1^2 - \sigma_{11}(\lambda_1 - \lambda_1^0)\Big],$$

$$\dot{\hat{\theta}}_1 = \Gamma_{12}[\|z_1\| - \sigma_{12}(\hat{\theta}_1 - \theta_1^0)],$$

$$u = -k_2 z_2 - z_1 - \frac{1}{4\gamma_2^2} z_2 \lambda_2 \xi_2^T \xi_2 - \hat{\theta}_2 z_2 \tanh\Big(\frac{\hat{\theta}_2 z_2}{\delta_2}\Big),$$

$$\dot{\lambda}_2 = \Gamma_{21}\Big[\frac{1}{4\gamma_2^2}\lambda_2 \xi_2^T \xi_2 z_2^2 - \sigma_{21}(\lambda_2 - \lambda_2^0)\Big],$$

$$\dot{\hat{\theta}}_2 = \Gamma_{22}\Big[\|z_2\| - \sigma_{22}(\hat{\theta}_2 - \theta_2^0)\Big].$$

Design parameters in the controller and adaptation laws are chosen as $k_1 = 0.2$, $k_2 = 0.8$, $\gamma_1 = 1$, $\gamma_2 = 2$, $\mu = 0.9$, $\Gamma_{11} = 1$, $\Gamma_{12} = 1$, $\Gamma_{21} = 1$, $\Gamma_{22} = 1$, $\delta_1 = 0.01$, $\delta_2 = 0.02$, $\lambda_1^0 = 0$, $\lambda_2^0 = 0$, $\theta_1^0 = 0$, $\theta_2^0 = 0$, $\sigma_{11} = 0.01$, $\sigma_{12} = 0.01$, $\sigma_{21} = 0.01$, $\sigma_{22} = 0.01$ if the initial conditions are given as $x_1(0) = 0.12$, $x_2(0) = 0$, $\lambda_1(0) = 0$, $\lambda_2(0) = 0$, $\hat{\theta}_1(0) = 0$, $\hat{\theta}_2(0) = 0$.

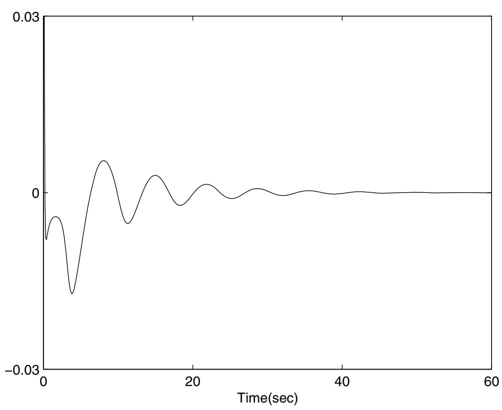

Fig. 3. System control input u.

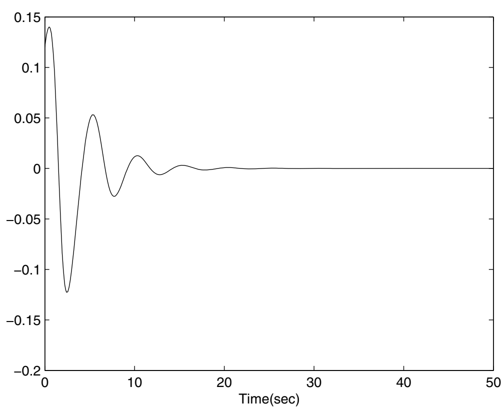

Fig. 4. Trajectory of x_1.

Example 2. In order to further illustrate the effectiveness of the proposed control method, use the control scheme by Yang *et al.* (2005) to control the nonlinear system in Example 1.

The simulation results are shown in Figs. 4–6, where Figs. 4 and 5 show the trajectories of x_1, x_2, respectively, while Fig. 6 shows the trajectory of control input u. From the above simulation results, it is concluded that the adaptive fuzzy control method (Yang *et al.*, 2005) can

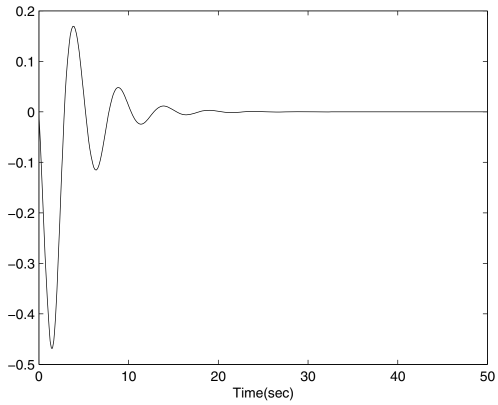

Fig. 5. Trajectory of x_2.

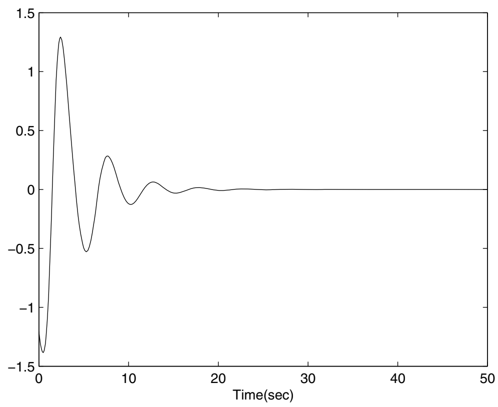

Fig. 6. System control input u.

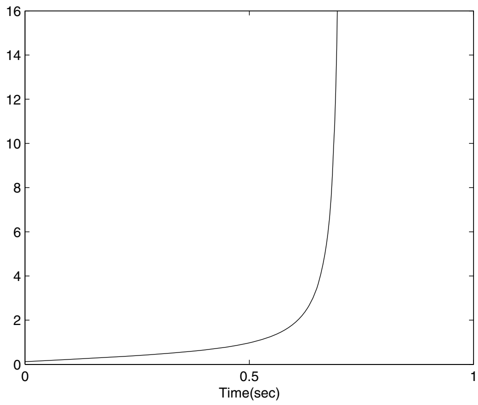

Fig. 7. Trajectory of x_1.

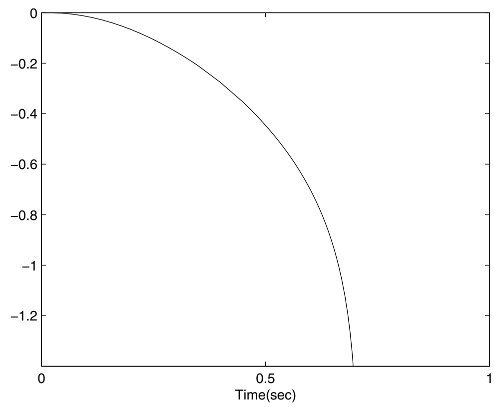

Fig. 8. Trajectory of x_2.

guarantee that all variables of the closed-loop systems are bounded and can achieve better control performance under the assumptions that the controlled nonlinear system does not contain unmodeled dynamics or dynamical disturbances, and the states are measured directly.

Case 2: Consider the nonlinear system (129) with unmodeled dynamics and dynamical disturbances, and assume that x_1 and x_2 are both measured.

For this case, use the same control scheme and the initial conditions as in Case 1, and we obtain the simulation results, which are shown by Figs. 7–8. From these one can conclude that the control scheme of Yang *et al.* (2005) cannot guarantee that the variables x_1, x_2 and u are bounded if the nonlinear system considered contains unmodeled dynamics and dynamical disturbances. The above simulation results in Example 1 and the simulation comparison in Example 2 demonstrate that the proposed adaptive fuzzy control approach can guarantee that all the signals in the closed-loop system are uniformly ultimately bounded and achieve better control performance even if the controlled system contains unmeasured states, unmodeled dynamics and dynamical disturbances.

5. Conclusion

In this paper, an adaptive fuzzy output feedback robust control approach was developed for a class of SISO strict-feedback nonlinear systems by combining backstepping design, K-filters and a small-gain theorem. The proposed control approach not only guarantees that all variables of the closed-loop system are uniformly ultimately bounded, but it also has a strong robustness to unmodeled dynamics and dynamical disturbances. Meanwhile, it cancels the restrictive condition given in recent works (Yang *et al.*, 2005; Tong *et al.*, 2010a; 2010b) that the states of controlled systems must be measured directly. Therefore, this paper has extended the existing results for the adaptive fuzzy backstepping control to nonlinear systems.

Acknowledgment

This work was supported by the National Natural Science Foundation of China (No. 61074014), the Outstanding Youth Funds of the Liaoning Province (No. 2005219001) and the Educational Department of the Liaoning Province (No. 2006R29 and No. 2007T80).

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Received: 2 March 2010
Revised: 1 September 2010
Re-revised: 26 September 2010