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ANALYTIC SOLUTION OF TRANSCENDENTAL EQUATIONS

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A decomposition technique of the solution of an n-th order linear differential equation into a set of solutions of 2-nd order linear differential equations is presented.

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1. Introduction

Let us consider the differential equation determining the transient error in a linear control system of the n-th order with lumped and constant parameters a_i, $i = 1, \ldots, n$:

$$\frac{\mathrm{d}^n x(t)}{\mathrm{d}t^n} + a_1 \frac{\mathrm{d}^{n-1} x(t)}{\mathrm{d}t^{n-1}} + \cdots + a_{n-1} \frac{\mathrm{d}x(t)}{\mathrm{d}t} + a_n x(t) = 0, \tag{1}$$

with the initial conditions which, in general, are different from zero:

$$x^{(i-1)}(0) = c_i \neq 0 \quad \text{for} \quad i = 1, 2, \ldots, n.$$

The solution of Eqn. (1) takes the following form:

$$x(t) = \sum_{k=1}^{n} A_k e^{s_k t}, \tag{2}$$

where s_k are the simple roots of the characteristic equation

$$s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0 \ . \tag{3}$$

In order to obtain an explicit form for A_k, we need higher derivatives of $x(t)$:

$$\frac{\mathrm{d}^p x(t)}{\mathrm{d}t^p} = \sum_{k=1}^{n} s_k^p A_k e^{s_k t}, \quad p = 1, 2, \ldots, n-1. \tag{4}$$

The formulae (2) and (4) represent a system of n linear equations with respect to unknown terms $A_k e^{s_k t}$. Its matrix of coefficients is the Vandermonde matrix

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \\ s_1 & s_2 & \ldots & s_n \\ \vdots & \vdots & & \vdots \\ s_1^{n-1} & s_2^{n-1} & \ldots & s_n^{n-1} \end{pmatrix}. \tag{5}$$

According to the assumption that $s_i \neq s_j$ for $i \neq j$, the matrix (5) has an inverse and the system (2) and (4) can be solved. We denote by V the Vandermonde determinant of the matrix (5) and by V_k the Vandermonde determinant of order $(n-1)$ of the variables $s_1, \ldots, s_{k-1}, s_{k+1}, \ldots, s_n$.

We also denote by $\Phi_r^{(k)}$ the fundamental symmetric function of the r-th order of $(n-1)$ variables $s_1, \ldots, s_{k-1}, s_{k+1}, \ldots, s_n$ for $r = 0, 1, \ldots, n-1$:

$$\left.\begin{aligned} \Phi_0^{(k)} &= 1, \\ \Phi_1^{(k)} &= s_1 + s_2 + \cdots + s_{k-1} + s_{k+1} \cdots + s_n \\ &= -a_1 - s_k, \\ \Phi_2^{(k)} &= s_1 s_2 + s_1 s_3 + \cdots + s_1 s_{k-1} + s_1 s_{k+1} \\ &\quad \cdots + s_2 s_3 + \cdots + s_2 s_{k-1} + s_2 s_{k+1} \\ &\quad \cdots + s_2 s_n + \ldots \\ &= a_2 - s_1 s_k - s_2 s_k - \cdots - s_n s_k, \\ \Phi_n^{(k)} &= \prod_{i=1, i\neq k}^{n} s_i = (-1)^n \frac{a_n}{s_k}. \end{aligned}\right\} \tag{6}$$

It can be shown that the elements of the matrix inverse to the matrix (5) have the form

$$\alpha_{ik} = \frac{(-1)^{i+k}}{V} \cdot \Phi_{n-k}^{(i)} V_k. \tag{7}$$

The solution of the system (2) and (4) is as follows:

$$\begin{aligned} A_k e^{s_k t} &= \sum_{j=1}^{n} \alpha_{kj} x^{(j-1)}(t) \\ &= \sum_{j=1}^{n} \frac{(-1)^{k+j}}{V} \cdot \Phi_{n-j}^{(k)} V_k x^{(j-1)}(t) \end{aligned} \tag{8}$$

or

$$A_k e^{s_k t} = \frac{(-1)^k V_k}{V} \sum_{j=1}^{n} (-1)^j \Phi_{n-j}^{(k)} x^{(j-1)}(t), \quad k = 1, 2, \ldots, n\ . \tag{9}$$

For $t = 0$, we know $x^{(j-1)}(0) = c_j$, and the substitution $t = 0$ into (9) gives

$$A_k = \frac{(-1)^k V_k}{V} \sum_{j=1}^{n} (-1)^j \Phi_{n-j}^{(k)} c_j. \tag{10}$$

Using the relation (6), we can formulate the following theorem.

Theorem 1. *The explicit form of the coefficient A_1 is as follows:*

$$\begin{aligned} A_1 = {} & \frac{c_n - \Big(\sum_{j=1}^{n} s_j\Big) c_{n-1} + \Big(\sum_{i,j \neq i \neq 1}^{n} s_j s_i\Big) c_{n-2}}{(s_n - s_1)(s_{n-1} - s_1)\ldots(s_2 - s_1)} \\ & - \frac{\Big(\sum_{i,j,k \neq 1}^{n} s_i s_j s_k\Big) c_{n-3} + \ldots}{(s_n - s_1)(s_{n-1} - s_1)\ldots(s_2 - s_1)} \\ & + \frac{\cdots + (-1)^{n-1} \prod_{i \neq 1}^{n} s_i c_1}{(s_n - s_1)(s_{n-1} - s_1)\ldots(s_2 - s_1)}. \end{aligned} \tag{11}$$

Then the coefficients $A_2, A_3, \ldots, A_n$ can be obtained by the sequential change of the indices of s_i according to the scheme

$$s_1 \longrightarrow s_2 \longrightarrow s_3 \longrightarrow \ldots \longrightarrow s_{n-1} \longrightarrow s_n \longrightarrow s_1.$$

Example 1. The solution of the 3-rd order equation is as follows:

$$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + A_3 e^{s_3 t}.$$

The coefficient

$$A_1 = \frac{(-1) V_1}{V} \sum_{j=1}^{3} (-1)^j \Phi_{3-j}^{(1)} c_j,$$

where

$$V_1 = \begin{vmatrix} 1 & 1 \\ s_2 & s_3 \end{vmatrix} = s_3 - s_2,$$

$$V = \begin{vmatrix} 1 & 1 & 1 \\ s_1 & s_2 & s_3 \\ s_1^2 & s_2^2 & s_3^2 \end{vmatrix} = (s_2 - s_1)(s_3 - s_2)(s_3 - s_1),$$

$$\Phi_0^{(1)} = 1, \qquad \Phi_1^{(1)} = s_2 + s_3, \qquad \Phi_2^{(1)} = s_2 s_3,$$

$$\begin{aligned} A_1 = {} & \frac{(-1)(s_3 - s_2)}{(s_2 - s_1)(s_3 - s_2)(s_3 - s_1)} \\ & \cdot \Big[(-1)\Phi_2^{(1)} c_1 + \Phi_1^{(1)} c_2 - \Phi_0^{(1)} c_3\Big] \\ = {} & \frac{(-1)\big[-s_2 s_3 c_1 + (s_2 + s_3) c_2 - c_3\big]}{(s_2 - s_1)(s_3 - s_1)} \\ = {} & \frac{c_3 - (s_2 + s_3) c_2 + s_2 s_3 c_1}{(s_2 - s_1)(s_3 - s_1)}, \\ A_2 = {} & \frac{c_3 - (s_3 + s_1) c_2 + s_3 s_1 c_1}{(s_3 - s_2)(s_1 - s_2)}, \\ A_3 = {} & \frac{c_3 - (s_1 + s_2) c_2 + s_1 s_2 c_1}{(s_1 - s_3)(s_2 - s_3)}. \end{aligned}$$

After the substitution of (10) into (9), we obtain

$$\begin{aligned} & e^{s_k t} \frac{(-1)^k V_k}{V} \sum_{j=1}^{n} (-1)^j \Phi_{n-j}^{(k)} x^{(j-1)}(0) \\ & \qquad = \frac{(-1)^k V_k}{V} \sum_{j=1}^{n} (-1)^j \Phi_{n-j}^{(k)} x^{(j-1)}(t) \end{aligned}$$

and, finally, for $k = 1, 2, \ldots, n$, we have

$$e^{s_k t} \sum_{j=1}^{n} (-1)^j \Phi_{n-j}^{(k)} c_j = \sum_{j=1}^{n} (-1)^j \Phi_{n-j}^{(k)} x^{(j-1)}(t). \tag{12}$$

Premultiplying both sides of (12) and using Viete's relations between the roots s_i and the coefficient a_1 of the characteristic equation,

$$\sum_{k=1}^{n} s_k = -a_1, \tag{13}$$

we obtain the main result formulated as Theorem 2.

Theorem 2. (Górecki and Turowicz, 1968) *The relation between cofficients a_i, $i = 1, 2, \ldots, n$, the initial values c_j, $j = 1, 2, \ldots, n$ and solutions $x^{(j)}(t)$ is as follows:*

$$\begin{aligned} & e^{-a_1 t} \prod_{k=1}^{n} \sum_{j=1}^{n} (-1)^j \Phi_{n-j}^{(k)} c_j \\ & \qquad = \prod_{k=1}^{n} \sum_{j=1}^{n} (-1)^j \Phi_{n-j}^{(k)} x^{(j-1)}(t). \end{aligned} \tag{14}$$

Both the sides of Eqn. (2) are composed of symmetric polynomials of variables $s_1, \dots, s_n$. Accordingly, it is possible to express these terms as polynomials of the coefficiets $a_1, \dots, a_n$. Using Vieta's relations, it is possible to replace the roots s_k by the coefficients a_i and to avoid calculating the roots by solving algebraic equations.

Example 2. For $n = 3$, we have

$$\begin{aligned}
&e^{-a_1 t}\Big\{a_3^2 c_1^3 + 2a_2 a_3 c_2 c_1^2 + (a_1 a_3 + a_2^2)\, c_2^2 c_1 \\
&+(a_1 a_2 - a_3)\, c_2^3 + (a_1 a_2 + 3a_3)\, c_1 c_2 c_3 + a_1 a_3 c_1^2 c_3 \\
&+a_2 c_1 c_3^2 + (a_1^2 + a_2)\, c_2^2 c_3 + 2a_1 c_2 c_3^2 + c_3^3\Big\} \\
&\quad = a_3^2 \big[x(t)\big]^3 + 2a_2 a_3 x^{(1)}(t) \big[x(t)\big]^2 + (a_1 a_3 + a_2^2) \\
&\quad \cdot \big[x^{(1)}(t)\big]^2 x(t) + (a_1 a_2 - a_3) \big[x^{(1)}(t)\big]^3 \\
&\quad + (a_1 a_2 + 3a_3)\, x(t) x^{(1)}(t) x^{(2)}(t) \\
&\quad + a_1 a_3 \big[x(t)\big]^2 x^{(2)}(t) + a_2 x(t) \big[x^{(2)}(t)\big]^2 \\
&\quad + (a_1^2 + a_2) \big[x^{(1)}(t)\big]^2 x^{(2)}(t) + 2a_1 x^{(1)}(t) \\
&\quad \cdot \big[x^{(2)}(t)\big]^2 + \big[x^{(2)}(t)\big]^3,
\end{aligned} \tag{15}$$

where

$$\begin{aligned}
a_1 &= -(s_1 + s_2 + s_3), \\
a_2 &= s_1 s_2 + s_1 s_3 + s_2 s_3, \\
a_3 &= -s_1 s_2 s_3.
\end{aligned}$$

Example 3. For $n = 4$, similarly

$$\begin{aligned}
&e^{-a_1 t}\Big[a_4^3 c_1^4 + 3a_3 a_4^2 c_1^3 c_2 + 2a_2 a_4^2 c_1^3 c_3 + a_1 a_4^2 c_1^3 c_4 \\
&\quad +(3a_3^2 a_2 a_4) a_4 c_1^2 c_2^2 + (4a_2 a_3 + 3a_1 a_4) a_4 c_1^2 c_2 c_3 \\
&\quad +2(a_1 a_3 + 2a_4) a_4 c_1^2 c_2 c_4 \\
&\quad +\big(a_2^2 + a_1 a_3 + 2a_4\big) a_4 c_1^2 c_3^2 \\
&\quad +\big(a_1 a_2 + 3a_3\big) a_4 c_1^2 c_3 c_4 + a_2 a_4 c_1^2 c_4^2 \\
&\quad +(a_3^3 + 2a_2 a_3 a_4 - a_1 a_4^2) c_1 c_2^3 + 2(a_2 a_3^2 + a_2^2 a_4 \\
&\quad +2a_1 a_3 a_4 - 2a_4^2) c_1 c_2^2 c_3 \\
&\quad +(a_1 a_3^2 + a_1 a_2 a_4 + 5a_3 a_4) c_1 c_2^2 c_4 \\
&\quad +(a_2^2 a_3 + a_1 a_3^2 + 5a_1 a_2 a_4 - a_3 a_4) c_1 c_2 c_3^2 \\
&\quad +(a_1 a_2 a_3 + 3a_1^2 a_4 + 3a_3^2 + 4a_2 a_4) c_1 c_2 c_3 c_4 \\
&\quad +(a_2 a_3 + 3a_1 a_4) c_1 c_2 c_4^2 \\
&\quad +\big(a_1 a_2 a_3 + a_1^2 a_4 - a_3^2 + 2a_2 a_4\big) c_1 c_3^3 \\
&\quad +\big(a_1^2 a_3 + a_2 a_3 + 5a_1 a_4\big) c_1 c_3^2 c_4 \\
&\quad +2\big(a_1 a_3 + 2a_4\big) c_1 c_3 c_4^2 + a_3 c_1 c_4^3 \\
&\quad +(a_2 a_3^2 - a_1 a_3 a_4 + a_4^2) c_2^4 \\
&\quad +(2a_2^2 a_3 + a_1 a_3^2 - a_1 a_2 a_4 - a_3 a_4) c_2^3 c_3 \\
&\quad +(a_1 a_2 a_3 - a_1^2 a_4 + a_3^2 - 2a_2 a_4) c_2^3 c_4 \\
&\quad +a_2(a_2^2 + 3a_1 a_3 - 3a_4) c_2^2 c_3^2 \\
&\quad +(a_1 a_2^2 + a_1^2 a_3 + 5a_2 a_3 - a_1 a_4) c_2^2 c_3 c_4 \\
&\quad +(a_2^2 + a_1 a_3 + 2a_4) c_2^2 c_4^2 \\
&\quad +(2a_1 a_2^2 + a_1^2 a_3 - a_2 a_3 - a_1 a_4) c_2 c_3^3 \\
&\quad +2(a_1^2 a_2 + a_2^2 + 2a_1 a_3 - 2a_4) c_2 c_3^2 c_4 \\
&\quad +(4a_1 a_2 + 3a_3) c_2 c_3 c_4^2 + 2a_2 c_2 c_4^3 \\
&\quad \big(a_1^2 a_2 - a_1 a_3 + a_4\big) c_3^4 + \big(a_1^3 + 2a_1 a_2 - a_3\big) c_3^3 c_4 \\
&\quad +\big(3a_1^2 + a_2\big) c_3^2 c_4^2 + 3a_1 c_3 c_4^3 + c_4^4\Big] \\
&= a_4^3 x(t)^4 + 3a_3 a_4^2 x(t)^3 x^{(1)}(t) \\
&\quad +2a_2 a_4^2 x(t)^3 x^{(2)}(t) + a_1 a_4^2 x(t)^3 x^{(3)}(t) \\
&\quad +\big(3a_3^2 + a_2 a_4\big) a_4 x(t)^2 \big[x^{(1))}(t)\big]^2 \\
&\quad +\big(4a_2 a_3 + 3a_1 a_4\big) a_4 x(t)^2 x^{(1)}(t) x^{(2)}(t) \\
&\quad +2\big(a_1 a_3 + 2a_4\big) a_4 x(t)^2 x^{(1)}(t) x^{(3)}(t) \\
&\quad +\big(a_2^2 + a_1 a_3 + 2a_4\big) a_4 x(t)^2 \big[x^{(2)}(t)\big]^2 \\
&\quad +\big(a_1 a_2 + 3a_3\big) a_4 x(t)^2 x^{(2)}(t) x^{(3)}(t) \\
&\quad +a_2 a_4 x(t)^2 \big[x^{(3)}(t)\big]^2 \\
&\quad +\big(a_3^3 + 2a_2 a_3 a_4 - a_1 a_4^2\big) x(t) \big[x^{(1)}(t)\big]^3 \\
&\quad +2\big(a_2 a_3^2 + a_2^2 a_4 + 2a_1 a_3 a_4 - 2a_4^2\big) x(t) \\
&\quad \cdot\big[x^{(1)}(t)\big]^2 x^{(2)}(t) + \big(a_1 a_3^2 + a_1 a_2 a_4 + 5a_3 a_4\big) \\
&\quad \cdot x(t) \big[x^{(1)}(t)\big]^2 x^{(3)}(t) \\
&\quad +\big(a_2^2 a_3 + a_1 a_3^2 + 5a_1 a_2 a_4 - a_3 a_4\big) x(t) x^{(1)}(t) \\
&\quad \cdot\big[x^{(2)}(t)\big]^2 \\
&\quad +\big(a_1 a_2 a_3 + 3a_1^2 a_4 + 3a_3^2 + 4a_2 a_4\big) \\
&\quad \cdot x(t) x^{(1)}(t) x^{(2)}(t) x^{(3)}(t) \\
&\quad +\big(a_2 a_3 + 3a_1 a_4\big) x(t) x^{(1)}(t) \big[x^{(3)}(t)\big]^2 \\
&\quad +\big(a_1 a_2 a_3 + a_1^2 a_4 - a_3^2 + 2a_2 a_4\big) x(t) \big[x^{(2)}(t)\big]^3 \\
&\quad +\big(a_1^2 a_3 + a_2 a_3 + 5a_1 a_4\big) x(t) \big[x^{(2)}(t)\big]^2 x^{(3)}(t) \\
&\quad +2\big(a_1 a_3 + 2a_4\big) x(t) x^{(2)}(t) \big[x^{(3)}(t)\big]^2 \\
&\quad +a_3 x(t) \big[x^{(3)}(t)\big]^3 \\
&\quad +\big(a_2 a_3^2 - a_1 a_3 a_4 + a_4^2\big) \big[x^{(1)}(t)\big]^4 \\
&\quad \big(2a_2^2 a_3 + a_1 a_3^2 - a_1 a_2 a_4 - a_3 a_4\big) \big[x^{(1)}(t)\big]^3 x^{(2)}(t) \\
&\quad +\big(a_1 a_2 a_3 - a_1^2 a_4 + a_3^2 + 2a_2 a_4\big) \big[x^{(1)}(t)\big]^3 x^{(3)}(t) \\
&\quad +a_2 \big(a_2^2 + 3a_1 a_3 - 3a_4\big) \big[x^{(1)}(t)\big]^2 \big[x^{(2)}(t)\big]^2 \\
&\quad +\cdots + \big(a_1 a_2^2 + a_1^2 a_3 + 5a_2 a_3 - a_1 a_4\big) \big[x^{(1)}(t)\big]^2 \\
&\quad \cdot x^{(2)}(t) x^{(3)}(t) \\
&\quad +\big(a_2^2 + a_1 a_3 + 2a_4\big) \big[x^{(1)}(t)\big]^2 \big[x^{(3)}(t)\big]^2 \\
&\quad +\big(2a_1 a_2^2 + a_1^2 a_3 - a_2 a_3 - a_1 a_4\big) x^{(1)}(t) \big[x^{(2)}(t)\big]^3 \\
&\quad 2\big(a_1^2 a_2 + a_2^2 + 2a_1 a_3 - 2a_4\big) x^{(1)}(t) \big[x^{(2)}(t)\big]^2 x^{(3)}(t)
\end{aligned}$$

$$
\begin{aligned}
&+\big(4a_1a_2+3a_3\big)x^{(1)}(t)x^{(2)}(t)\big[x^{(3)}(t)\big]^2\\
&+2a_2c_2\big[x^{(3)}(t)\big]^3\\
&+\big(a_1^2a_2-a_1a_3+a_4\big)\big[x^{(2)}(t)\big]^4\\
&+\big(a_1^3+2a_1a_2-a_3\big)\big[x^{(2)}(t)\big]^3x^{(3)}(t)\\
&+\big(3a_1^2+a_2\big)\big[x^{(2)}(t)\big]^2\big[x^{(3)}(t)\big]^2\\
&+3a_1x^{(2)}(t)\big[x^{(3)}(t)\big]^3+\big[x^{(3)}(t)\big]^4.
\end{aligned}\tag{16}
$$

2. Analytical method of determining zeroes and extremal values of the solution $x(t)$ described by the relation (2)

2.1. Basic results. The general relation analogous to the formulae (15) or (16) for the equation of the n-th order is very complicated. For that reason, we illustrate the method on examples of equations of the 3-rd and 4-th orders. We assume that at the extremal point t_e, or at the zero t_0 of the solution (2), the second derivative $\mathrm{d}^2x/\mathrm{d}t^2 \neq 0$. We can write the relation (15) in the following form:

$$
\begin{aligned}
&\big[x^{(2)}\big]^3\Big\{\Big[\frac{x}{x^{(2)}}\Big]^3a_3^2+\Big(2a_2a_3\frac{x^{(1)}}{x^{(2)}}+a_1a_3\Big)\Big[\frac{x}{x^{(2)}}\Big]^2\\
&+\Big[(a_1a_3+a_2^2)\Big(\frac{x^{(1)}}{x^{(2)}}\Big)^2+(a_1a_2+3a_3)\frac{x^{(1)}}{x^{(2)}}+a_2\Big]\frac{x}{x^{(2)}}\\
&+\Big[(a_1a_2-a_3)\Big(\frac{x^{(1)}}{x^{(2)}}\Big)^3+(a_1^2+a_2)\Big(\frac{x^{(1)}}{x^{(2)}}\Big)^2\\
&+2a_1\frac{x^{(1)}}{x^{(2)}}+1\Big]\Big\}\\
&=e^{-a_1t}c_3^3\Big\{a_3^2\Big(\frac{c_1}{c_2}\Big)^3+\Big(2a_2a_3\frac{c_2}{c_3}+a_1a_3\Big)\Big(\frac{c_1}{c_3}\Big)^2\\
&\quad+\Big[(a_1a_3+a_2^2)\Big(\frac{c_2}{c_3}\Big)^2+(a_1a_2+3a_3)\frac{c_2}{c_3}+a_2\Big]\frac{c_1}{c_3}\\
&\quad+\Big[(a_1a_2-a_3)\Big(\frac{c_2}{c_3}\Big)^3+(a_1^2+a_2)\Big(\frac{c_2}{c_3}\Big)^2\\
&\quad+2a_1\frac{c_2}{c_3}+1\Big]\Big\}.
\end{aligned}\tag{17}
$$

Setting

$$\frac{x}{x^{(2)}}=\frac{c_1}{c_3}=u,\tag{18}$$

$$\frac{x^{(1)}}{x^{(2)}}=\frac{c_2}{c_3}=v,\tag{19}$$

we can write the relations (17) in the following form:

$$
\begin{aligned}
&\Big\{\big[x^{(2)}\big]^3-e^{-a_1t}c_3^3\Big\}\\
&\cdot\Big\{\big[a_3^2u^3+(2a_2a_3v+a_1a_3)u^2\\
&\quad+\big[(a_1a_3+a_2^2)v+a_2\big]u\\
&\quad+\big[(a_1a_2-a_3)v^3+(a_1^2+a_2)v^2+2a_1v+1\big]\Big\}=0.
\end{aligned}\tag{20}
$$

If we assume that $c_2 = 0$, then from (19) we have $x^{(1)}(t_e) = 0$ and $v = 0$. It is a necessary condition for extremum. In this case the equation (20) has a simple form:

$$\Big\{\big[x^{(2)}\big]^3-e^{-a_1t_e}c_3^3\Big\}\big[a_3^2u^3+a_1a_3u^2+a_2u+1\big]=0.\tag{21}$$

If we assume $c_1 = 0$, then from (18) we obtain that $x(t_0) = 0$ and $u = 0$. It is a necessary condition for $x(t)$ to be zero. In this case, Eqn. (20) has the following form:

$$
\begin{aligned}
&\Big\{\big[x^{(2)}\big]^3-e^{-a_1t_0}c_3^3\Big\}\\
&\cdot\big[a_1a_2-a_3)v^3+(a_1^2+a_2)v^2+2a_1v+1\big]=0.
\end{aligned}\tag{22}
$$

It is possible to find the relations between the roots of the equation

$$a_3^2u^3+a_1a_3u^2+a_2u+1=0,\tag{23}$$

and the roots s_1, s_2 and s_3 of the characteristic equation

$$s^3+a_1s^2+a_2s+a_3=0.\tag{24}$$

Setting

$$u=\frac{y}{\sqrt[3]{a_3^2}}\tag{25}$$

in Eqn. (23), we obtain the following:

$$y^3+\frac{a_1}{\sqrt[3]{a_3}}y^2+\frac{a_2}{\sqrt[3]{a_3^2}}y+1=0.\tag{26}$$

Similarly, setting

$$s=\sqrt[3]{a_3}\,z\tag{27}$$

in Eqn. (24), we obtain that

$$z^3+\frac{a_1}{\sqrt[3]{a_3}}z^2+\frac{a_2}{\sqrt[3]{a_3^2}}z+1=0.\tag{28}$$

Equations (26) and (28) are identical. As a result, we have that

$$y=z\quad\text{or}\quad\sqrt[3]{a_3^2}\,u=\frac{s}{\sqrt[3]{a_3}}.\tag{29}$$

Finally, from (29), we conclude that

$$u=\frac{s}{a_3}.\tag{30}$$

Returning to (19), we find that, if $x^{(1)} = c_2 = 0$, at the extremum point t_e the following relations hold:

$$\frac{x(t_e)}{x^{(2)}(t_e)}=\frac{c_1}{c_3}=\frac{s_i}{a_3},\quad i=1,2,3.\tag{31}$$

Taking into account in (31) that $a_3 = -s_1s_2s_3$, we finally obtain that

$$
\left.
\begin{aligned}
&\frac{x(t_e)}{x^{(2)}(t_e)}=\frac{c_1}{c_3}=-\frac{1}{s_2s_3}\quad\text{or}\\
&\frac{x(t_e)}{x^{(2)}(t_e)}=\frac{c_1}{c_3}=-\frac{1}{s_3s_1}\quad\text{or}\\
&\frac{x(t_e)}{x^{(2)}(t_e)}=\frac{c_1}{c_3}=-\frac{1}{s_1s_2}.
\end{aligned}
\right\}\tag{32}
$$

Theorem 3. *From the relations (32) it is possible to determine extrema (if they exist) using the relations*

$$\left.\begin{aligned} s_2 s_3 x(t_e) + x^{(2)}(t_e) = 0,\\ s_3 s_1 x(t_e) + x^{(2)}(t_e) = 0,\\ s_1 s_2 x(t_e) + x^{(2)}(t_e) = 0, \end{aligned}\right\} \tag{33}$$

under the constraints that c_1 *and* c_3 *fullfil the same relations.*

Following a similar procedure with the equation

$$(a_1 a_2 - a_3)v^3 + (a_1^2 + a_2)v^2 + 2a_1 v + 1 = 0, \tag{34}$$

we can find that in the article by Górecki and Szymkat (1983) it is proved that the roots of the equation

$$8r^3 + 8a_1 r^2 + 2(a_2 + a_1^2)r + a_1 a_2 - a_3 = 0 \tag{35}$$

are as follows:

$$r_1 = \frac{s_1 + s_2}{2}, \qquad r_2 = \frac{s_2 + s_3}{2}, \qquad r_3 = \frac{s_3 + s_1}{2}.$$

Setting $2r = p$ in (35), we obtain the equation

$$p^3 + 2a_1 p^2 + (a_1^2 + a_2)p + a_1 a_2 - a_3 = 0. \tag{36}$$

whose roots are $p_1 = s_1 + s_2$, $p_2 = s_2 + s_3$, $p_3 = s_3 + s_1$. Let

$$p = \frac{1}{q}. \tag{37}$$

Then Eqn. (36) has the following form:

$$(a_1 a_2 - a_3)q^3 + (a_1^2 + a_2)q^2 + 2a_1 q + 1 = 0, \tag{38}$$

and its roots are

$$q_1 = \frac{1}{s_1 + s_2}, \quad q_2 = \frac{1}{s_2 + s_3}, \quad q_3 = \frac{1}{s_3 + s_1}. \tag{39}$$

Finally, from (19) and (39), we obtain that

$$\begin{aligned} \frac{x^{(1)}(t_0)}{x^{(2)}(t_0)} &= \frac{c_2}{c_3} = \frac{1}{s_1 + s_2},\\ \frac{x^{(1)}(t_0)}{x^{(2)}(t_0)} &= \frac{c_2}{c_3} = \frac{1}{s_2 + s_3},\\ \frac{x^{(1)}(t_0)}{x^{(2)}(t_0)} &= \frac{c_2}{c_3} = \frac{1}{s_3 + s_1}. \end{aligned} \tag{40}$$

Theorem 4. *From the relation (40), it is possible to determine the zeros of* $x(t_0)$ *(if they exist) using the relations*

$$\left.\begin{aligned} x^{(1)}(t_0)(s_1 + s_2) - x^{(2)}(t_0) = 0,\\ x^{(1)}(t_0)(s_2 + s_3) - x^{(2)}(t_0) = 0,\\ x^{(1)}(t_0)(s_3 + s_1) - x^{(2)}(t_0) = 0, \end{aligned}\right\} \tag{41}$$

under the constraints that c_2 *and* c_3 *fullfil the same relations.*

A generalization of these result relations (33) and (41) to higher order equations may be obtained directly, due to the following remark.

Remark 1. The relations (33) and (41) may be obtained directly from the following propositions.

Let the coefficients A_i of the solution $x(t)$ fullfil the relations

$$\left.\begin{aligned} A_1 = 0 \quad A_2 \neq 0, \quad A_3 \neq 0,\\ A_2 = 0 \quad A_1 \neq 0, \quad A_3 \neq 0,\\ A_3 = 0 \quad A_1 \neq 0, \quad A_2 \neq 0. \end{aligned}\right\} \tag{42}$$

In this way, we obtain equations which contain only two exponential terms, and such equations can be solved in analytical form.

The relations (42) are more general than (33) and (41) because they are also valid when $c_2 \neq 0$ or $c_1 \neq 0$. Moreover, they also hold for higher order equations. For such equations, to obtain only two exponential terms, it is necessary to assume more than one coefficient A_i equal to zero.

3. Basic result

Theorem 5. *The equation*

$$x(t) = \sum_{i=1}^{n} A_i e^{s_i t} \tag{43}$$

or

$$x^{(1)}(t) = \sum_{i=1}^{n} s_i A_i e^{s_i t} \tag{44}$$

can be decomposed into a system of equations containing a set of equations composed of only two terms. The set contains

$$\binom{n}{n-2} = \frac{1}{2}\, n(n-1)$$

equations with two exponential terms.

Example 4. For $n = 3$, we have the following equations:

$$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + A_3 e^{s_3 t}, \tag{45}$$

$$x^{(1)}(t) = A_1 s_1 e^{s_1 t} + A_2 s_2 e^{s_2 t} + A_3 s_3 e^{s_3 t}, \tag{46}$$

where

$$A_1 = \frac{c_3 - (s_2 + s_3)c_2 + s_2 s_3 c_1}{(s_1 - s_2)(s_1 - s_3)}, \tag{47}$$

$$A_2 = \frac{c_3 - (s_3 + s_1)c_2 + s_3 s_1 c_1}{(s_2 - s_3)(s_2 - s_1)}, \tag{48}$$

$$A_3 = \frac{c_3 - (s_1 + s_2)c_2 + s_1 s_2 c_1}{(s_3 - s_1)(s_3 - s_2)}. \tag{49}$$

Looking for an extremum, we use Eqn. (46), where the necessary condition is $x^{(1)}(t) = 0$. Assuming that

$$x^{(1)}(t) = 0, \qquad A_1 = 0, \tag{50}$$

after eliminating the initial condition c_1 from (50), we obtain that

$$e^{(s_2-s_3)t_e} = \frac{c_3 - s_2c_2}{c_3 - s_3c_2}, \tag{51}$$

where

$$e^{(s_3-s_1)t_e} = \frac{c_3 - s_3c_2}{c_3 - s_1c_2}, \tag{52}$$

$$e^{(s_1-s_2)t_e} = \frac{c_3 - s_1c_2}{c_3 - s_2c_2}, \tag{53}$$

$$c_1 = \frac{1}{s_2s_3}\Big[(s_2+s_3)c_2 - c_3\Big].$$

Similarly, asumming $A_2 = 0$, we obtain, after eliminating c_2, that

$$e^{(s_2-s_3)t_e} = \frac{c_3 - s_2^2c_1}{c_3 - s_3^2c_1}\,\frac{s_3}{s_2}, \tag{54}$$

$$e^{(s_3-s_1)t_e} = \frac{c_3 - s_3^2c_1}{c_3 - s_1^2c_1}\,\frac{s_1}{s_3}, \tag{55}$$

$$e^{(s_1-s_2)t_e} = \frac{c_3 - s_1^2c_1}{c_3 - s_2^2c_1}\,\frac{s_2}{s_1}. \tag{56}$$

Finally, asumming $A_3 = 0$, after eliminating c_3, we obtain

$$e^{(s_2-s_3)t_e} = \frac{c_2 - s_2c_1}{c_2 - s_3c_1}\,\frac{s_3}{s_2}, \tag{57}$$

$$e^{(s_3-s_1)t_e} = \frac{c_2 - s_3c_1}{c_2 - s_1c_1}\,\frac{s_1}{s_3}, \tag{58}$$

$$e^{(s_1-s_2)t_e} = \frac{c_2 - s_1c_1}{c_2 - s_2c_1}\,\frac{s_2}{s_1}. \tag{59}$$

Similarly, the equation

$$x(t) = A_1e^{s_1t} + A_2e^{s_2t} + A_3e^{s_3t} + A_4e^{s_4t} = 0$$

can be decomposed into the following set of equations:

$$\begin{aligned}
A_1e^{s_1t} + A_2e^{s_2t} &= 0, \text{ where } A_3 = 0 \quad\text{and}\quad A_4 = 0,\\
A_1e^{s_1t} + A_3e^{s_3t} &= 0, \text{ where } A_2 = 0 \quad\text{and}\quad A_4 = 0,\\
A_1e^{s_1t} + A_4e^{s_4t} &= 0, \text{ where } A_2 = 0 \quad\text{and}\quad A_3 = 0,\\
A_2e^{s_2t} + A_3e^{s_3t} &= 0, \text{ where } A_1 = 0 \quad\text{and}\quad A_4 = 0,\\
A_2e^{s_2t} + A_4e^{s_4t} &= 0, \text{ where } A_1 = 0 \quad\text{and}\quad A_3 = 0,\\
A_3e^{s_3t} + A_4e^{s_4t} &= 0, \text{ where } A_1 = 0 \quad\text{and}\quad A_2 = 0.
\end{aligned}$$

It is a set of

$$\binom{4}{2} = \frac{3\cdot 4}{2} = 6$$

equations with only two exponential terms.

Remark 2. It is evident that looking for $x(t_0) = 0$ instead of $x^{(1)}(t_e) = 0$, we must multiply the relations (51)–(59) a propriately by s_i/s_j. For example,

$$e^{(s_j-s_i)t_0} = \frac{c_3 - s_jc_2}{c_3 - s_ic_2}\,\frac{s_i}{s_j},$$

and so on.

Remark 3. If Eqn. (3) has repeated roots, then the relations (2) and (11) must be transformed by properly passing to the limit.

In the particular case, when $s_1 = s_2 = \cdots = s_n = s$, we obtain

$$x(t) = e^{st}\sum_{k=1}^{n} A_k t^{k-1},$$

$$A_k = \sum_{i=0}^{k} \frac{x^{(k)}(0)(-1)^i s^i}{i!(k-i)!}, \quad k = 1, 2, \dots, n.$$

The necessary condition for the existence of the local extremum of the solution (2) is $x^{(1)}(t) = 0$, and the problem is reduced to an algebraic one,

$$\sum_{k=1}^{n} A_k\big[st_e^{(k-1)} + (k-1)t_e^{k-2}\big] = 0.$$

4. Conclusion

It was shown that every differential equation of the n-th order can be decomposed into a set of $\frac{1}{2}n(n-1)$ equations of the 2-nd order, which can be solved in analytical form.

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