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NOVEL OPTIMAL RECURSIVE FILTER FOR STATE AND FAULT ESTIMATION OF LINEAR STOCHASTIC SYSTEMS WITH UNKNOWN DISTURBANCES

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This paper studies recursive optimal filtering as well as robust fault and state estimation for linear stochastic systems with unknown disturbances. It proposes a new recursive optimal filter structure with transformation of the original system. This transformation is based on the singular value decomposition of the direct feedthrough matrix distribution of the fault which is assumed to be of arbitrary rank. The resulting filter is optimal in the sense of the unbiased minimum-variance criteria. Two numerical examples are given in order to illustrate the proposed method, in particular to solve the estimation of the simultaneous actuator and sensor fault problem and to make a comparison with the existing literature results.

Keywords: Kalman filtering, minimum-variance estimation, state estimation, fault estimation, unknown disturbances, linear discrete-time systems.

1. Introduction

Joint fault and state estimation of linear time-varying stochastic systems with unknown disturbances is considered. This problem is solved by using an unknown input filtering approach to produce unknown disturbance decoupled joint fault-state estimation. This may be achieved by making use of unbiased minimum-variance estimation. The proposed filter can play a significant role in several applications, e.g., the model-based Fault Detection and Isolation (FDI) problem (Ben Hmida *et al.*, 2010; Chen and Patton, 1996; 1999) and the Fault Tolerant Control (FTC) problem (Blanke *et al.*, 2006).

The subject of Unknown Input Filtering (UIF) has been studied extensively in the past three decades. A first approach was developed by Kitanidis (1987) using an Unbiased Minimum-Variance (UMV) estimation technique. This approach assumes that no prior knowledge about the dynamical evolution of the unknown inputs is available. Since then, many authors have extended Kitanidis's filter, (e.g., Cheng *et al.*, 2009; Gillijns and Moor, 2007b).

A parameterizing technique is used to obtain an optimal filter under less restrictive conditions as in the work of Darouach and Zasadzinski (1997). Their results are only limited to treat the state estimation problem. However, Hsieh (2000) as well as Gillijns and Moor (2007a) investigated the joint input and state estimation problem using an innovation filtering technique. All the previous works are, nevertheless, limited to linear systems without direct feedthrough of the unknown input to the output.

In the case where the unknown inputs or disturbances affect both the system state and the output, the state estimation problem was addressed by Hou and Patton (1998), Darouach *et al.* (2003) as well as Cheng *et al.* (2009). Hou and Patton (1998) developed a straightforward state estimation method to design an optimal filter. A parameterizing technique was used by Darouach *et al.* (2003) to derive an Optimal Estimator Filter (OEF). Recently, Cheng *et al.* (2009) developed a recursive optimal filter with global optimality in the sense of unbiased minimum-variance. In addition, they established a necessary and sufficient condition for the convergence and stability of this filter. However, all these results are solely limited to the estimation of the state, not the input.

The joint input and state estimation problem for linear systems with direct feedthrough was treated by Gillijns and Moor (2007b) as well as Hsieh (2009). In the former work the Recursive Three Step Filter (RTSF) is derived where the input estimation is based on the Weighted Least-Squares (WLS) approach and the state estimation problem is solved by Kitanidis's method. The RTSF is usable if only the direct feedthrough matrix has full rank. Recently, the case of arbitrary rank was proposed by Hsieh (2009). The designed optimal filter, known as the ERTSF (Extended RTSF), made it possible to estimate an UMV state and input. But, in certain cases, the filter suffers from a degradation of the input estimates.

The FDI problem for linear systems without and with unknown disturbances has been thoroughly investigated (Nikoukhah, 1994; Keller, 1998; 1999; Keller and Sauter 2011; Jamouli *et al.*, 2003, Chen and Patton, 1996; 1999; Ben Hmida *et al.*, 2010). Keller (1999) designed a full-order Kalman filter under a particular eigenstructure assignment. A new state filtering strategy was developed to detect and isolate multiple faults appearing simultaneously or sequentially. Keller and Sauter (2011) presented a restricted diagonal detection filter for multiple fault detection and isolation. The proposed filter makes it possible to generate a minimum variance white innovation sequence having directional properties in response to each fault. For detection and isolation of multiple impulsive faults, a bank of Generalised Likelihood Ratio (GLR) tests is applied to the innovation sequence generated by the detection filter.

Unfortunately, these works do not attack the case where unknown disturbances may affect the systems and produce a false alarm. To deal with this case for systems with unknown disturbances, we can, according to Nikoukhah (1994), develop a robust fault detection and isolation in continuous time by using the error innovation technique to generate unbiased white residual signals. The fault is diagnosed by a statistical testing of the whiteness, mean and covariance of residuals.

A new method was developed by Keller (1998) to detect and isolate multiple faults appearing simultaneously or sequentially in Linear Time-Invariant (LTI) stochastic discrete-time systems with unknown inputs. That method consists of generating directional residuals using an isolation filter. Jamouli *et al*. (2003) presented a Fault Isolation Filter (FIF) for fault detection in linear stochastic systems affected by disturbances. The filter will generate a directional residual decoupled from the unknown input allowing the treatment of multiple faults appearing simultaneously or sequentially.

Chen and Patton (1996; 1999) studied the optimal filtering and robust fault diagnosis problem for stochastic systems with unknown disturbances. An optimal observer was proposed, which can produce disturbance decoupled state estimation with minimum-variance for linear time-varying systems with both noise and unknown disturbances. The output estimation error with disturbance decoupling is used as a residual signal. After that, a statistical testing procedure is applied to examine the residual and to diagnose faults. Nevertheless, the simultaneous actuator and sensor faults problem has not been studied.

Recently, Ben Hmida *et al*. (2010) presented a new recursive filter to joint fault and state estimation of linear time-varying discrete-time systems in the presence of unknown disturbances. The method is based on the assumption that no prior knowledge about the dynamical evolution of the fault and the unknown disturbances is available. The filter has two advantages: it considers an arbitrary direct feedthrough matrix of the fault and allows for estimation of multiple faults. However, in certain cases the obtained filter may suffer from poor fault estimation quality.

In this paper, we extend the version of Ben Hmida *et al*. (2010) to further propose a new recursive optimal filter structure. The fault affects both the state and output equations where the direct feedthrough matrix has arbitrary rank, whereas the unknown disturbances only affect the state system equation, without any prior information about their dynamical evolution. This filter has been obtained through a linear transformation of the original system. Next, we use a linear Unbiased Minimum-Variance (UMV) estimation. The estimation of the fault is obtained by using a weighted least-squares approach. Thus, the resulting filter will be applied to solve a simultaneous actuator and sensor fault estimation.

The remainder of this paper is organized as follows. Section 2 states the problem of interest. In Section 3 we design the proposed filter. Thus Section 4 presents an illustrative example.

2. Problem statement

Consider a linear stochastic discrete-time system in the following form:

$$x_{k+1} = A_k x_k + B_k u_k + F_k^x f_k + E_k^x d_k + w_k, \quad (1)$$

$$y_k = H_k x_k + F_k^y f_k + v_k, \quad (2)$$

where $x_k \in \mathbb{R}^n$ is the state vector, $y_k \in \mathbb{R}^m$ is the observation vector, $u_k \in \mathbb{R}^r$ is the known input vector, $f_k \in \mathbb{R}^p$ is the additive fault vector and $d_k \in \mathbb{R}^q$ is the unknown disturbances vector. Here w_k and v_k are uncorrelated white noise sequences with zero means, and the covariances matrices $Q_k = \mathcal{E}\left[w_k w_k^T\right] \geq 0$ and $R_k = \mathcal{E}\left[v_k v_k^T\right] > 0$, respectively. The matrices A_k, H_k, F_k^x, E_k^x and F_k^y are known and have appropriate dimensions. We assume that (A_k, H_k) is observable, $m \geq p + q$ and the initial state is uncorrelated with the white noise processes w_k and v_k. The initial state x_0 is a Gaussian random variable with $\mathcal{E}\left[x_0\right] = \hat{x}_0$ and $\mathcal{E}\left[(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^T\right] = P_0^x$, where $\mathcal{E}\left[\cdot\right]$ denotes the expectation operator.

The aim of this paper is to extend the results of Ben Hmida *et al.* (2010) in order to derive a new recursive optimal filter structure to obtain a better fault and state estimation when $0 < \operatorname{rank}(F_k^y) \le p$ in spite of the presence of the unknown disturbances d_k.

Initially, we seek to change the coordinates of the system (1) and (2) by using the technique developed by Cheng *et al.* (2009). Let $r_k = \operatorname{rank}(F_k^y) < p$. Then the singular value decomposition of the matrix F_k^y is given by

$$F_k^y = \begin{bmatrix} U_{1,k} \; U_{2,k} \end{bmatrix} \begin{bmatrix} \Sigma_k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{1,k}^T \\ V_{2,k}^T \end{bmatrix}, \tag{3}$$

where we have $\Sigma_k \in \mathbb{R}^{r_k \times r_k}$, $U_{1,k} \in \mathbb{R}^{m \times r_k}$, $U_{2,k} \in \mathbb{R}^{m \times (m-r_k)}$, $V_{1,k} \in \mathbb{R}^{p \times r_k}$ and $V_{2,k} \in \mathbb{R}^{p \times (p-r_k)}$. $\begin{bmatrix} U_{1,k} \; U_{2,k} \end{bmatrix}$ and $\begin{bmatrix} V_{1,k} \; V_{2,k} \end{bmatrix}$ are unitary matrices.

There exists a transformation matrix of the from $T_k = \begin{bmatrix} T_{1,k}^T & T_{2,k}^T \end{bmatrix}^T$ such that the system (1) and (2) is written as (Cheng *et al.*, 2009)

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k + F_{1,k}^x f_{1,k} + F_{2,k}^x f_{2,k} \\ &\quad + E_k^x d_k + w_k, && (4) \\ z_{1,k} &= H_{1,k} x_k + \Sigma_k f_{1,k} + v_{1,k}, && (5) \\ z_{2,k} &= H_{2,k} x_k + v_{2,k}, && (6) \end{aligned}$$

where

$$\begin{aligned} v_{1,k} &= T_{1,k} v_k, & v_{2,k} &= T_{2,k} v_k, \\ z_{1,k} &= T_{1,k} y_k \in \mathbb{R}^{r_k}, & z_{2,k} &= T_{2,k} y_k \in \mathbb{R}^{(m-r_k)}, \\ F_{1,k}^x &= F_k^x V_{1,k}, & F_{2,k}^x &= F_k^x V_{2,k}, \\ H_{1,k} &= T_{1,k} H_k, & H_{2,k} &= T_{2,k} H_k, \end{aligned}$$

with the assumption that

$$\operatorname{rank}(H_{2,k} F_{2,k-1}^x) = \operatorname{rank}(F_{2,k-1}^x), \tag{7a}$$
$$\operatorname{rank}(H_{2,k} E_{k-1}^x) = \operatorname{rank}(E_{k-1}^x). \tag{7b}$$

Since $[V_{1,k} \; V_{2,k}]$ is a unitary matrix, it follows that the fault must be reconstructed through its two components $f_{1,k}$ and $f_{2,k}$ according to

$$f_k = V_{1,k} f_{1,k} + V_{2,k} f_{2,k}. \tag{8}$$

The transformation matrix T_k is given by

$$T_k = \begin{bmatrix} I_{r_k} & -U_{1,k}^T R_k U_{2,k} \left(U_{2,k}^T R_k U_{2,k} \right)^{-1} \\ 0_{(m-r_k) \times r_k} & I_{(m-r_k)} \end{bmatrix} \times \begin{bmatrix} U_{1,k}^T \\ U_{2,k}^T \end{bmatrix}. \tag{9}$$

Cheng *et al.* (2009) define $R_{1,k}$ and $R_{2,k}$ as the variance of $v_{1,k}$ and $v_{2,k}$, respectively, and $R_{12}(k,i)$ as their covariance. Then it follows that

$$\begin{aligned} R_{1,k} &= \mathcal{E}[v_{1,k} v_{1,k}^T], \\ &= U_{1,k}^T R_k U_{1,k} - U_{1,k}^T R_k U_{2,k} \\ &\quad \times \left(U_{2,k}^T R_k U_{2,k} \right)^{-1} U_{2,k}^T R_k U_{1,k}, \\ R_{2,k} &= \mathcal{E}[v_{2,k} v_{2,k}^T] = U_{2,k}^T R_k U_{2,k}, \\ R_{12}(k,k) &= \mathcal{E}[v_{1,k} v_{2,k}^T] = 0, \\ R_{12}(k,i) &= \mathcal{E}[v_{1,k} v_{2,i}^T] = 0 \quad \text{for} \quad k \neq i. \end{aligned}$$

Moreover, Cheng *et al.* (2009) show the following relations:

- $\operatorname{cov}[v_{1,k}, w_i] = 0$ and $\operatorname{cov}[v_{2,k}, w_i] = 0$ for $k \neq i$.
- $\operatorname{cov}[v_{1,k}, x_0] = 0$ and $\operatorname{cov}[v_{2,k}, x_0] = 0$.

Under the system equations (4)–(6), we note that we can estimate the second component of the fault not at the step k but at the step $k-1$, because $f_{2,k-1}$ will be estimated from the state. The proposed state and fault filter has the following structure:

$$\begin{aligned} \hat{x}_{k/k-1} &= A_{k-1} \hat{x}_{k-1/k-1} + B_{k-1} u_{k-1} \\ &\quad + F_{1,k-1}^x \hat{f}_{1,k-1}, && (10) \\ \hat{f}_{2,k-1} &= K_k^{f_2} \left(z_{2,k} - H_{2,k} \hat{x}_{k/k-1} \right), && (11) \\ \hat{x}_{k/k-1}^* &= \hat{x}_{k/k-1} + F_{2,k-1}^x \hat{f}_{2,k-1} \\ &\quad + K_k^{x*} (z_{2,k} - H_{2,k} \hat{x}_{k/k-1}), && (12) \\ \hat{f}_{1,k} &= K_k^{f_1} \left(z_{1,k} - H_{1,k} \hat{x}_{k/k-1}^* \right), && (13) \\ \hat{f}_k &= V_{1,k} \hat{f}_{1,k} + V_{2,k} \hat{f}_{2,k-1}, && (14) \\ \hat{x}_{k/k} &= \hat{x}_{k/k-1}^* + K_k^x \left(z_{2,k} - H_{2,k} \hat{x}_{k/k-1}^* \right), && (15) \end{aligned}$$

where the gain matrices $K_k^{f_1} \in \mathbb{R}^{r_k \times r_k}$, $K_k^{f_2} \in \mathbb{R}^{(p-r_k) \times (m-r_k)}$, $K_k^{x*} \in \mathbb{R}^{n \times (m-r_k)}$ and $K_k^x \in \mathbb{R}^{n \times (m-r_k)}$ still have to be determined to satisfy the following criteria:

- Unbiasedness: The estimator must satisfy

$$\mathcal{E}[\tilde{f}_k] = \mathcal{E}[f_k - \hat{f}_k] = 0, \tag{16}$$
$$\mathcal{E}[\tilde{x}_{k/k}] = \mathcal{E}[x_k - \hat{x}_{k/k}] = 0. \tag{17}$$

- Minimum-variance: The estimator is determined such that
  - the mean square errors $\mathcal{E}[(\tilde{f}_k)^T \tilde{f}_k]$ is minimized subject to the constraint (16),
  - $\operatorname{trace}\left\{ P_{k/k}^x = \mathcal{E}[\tilde{x}_{k/k} (\tilde{x}_{k/k})^T] \right\}$ is minimized subject to the constraints (16) and (17).

3. Filter design

In this section, the gain matrices $K_k^{f_1}$, $K_k^{f_2}$, K_k^x and K_k^{x*} will be determined so that the filter (10)–(15) yields robust estimates of f_k and x_k in spite of the presence of the unknown disturbances d_k. Next, the UMV fault and state estimation will be demonstrated.

3.1. Fault estimation. The estimation errors of $f_{1,k}$ and $f_{2,k-1}$ are given by

$$\begin{aligned}\tilde{f}_{1,k} &= f_{1,k} - \hat{f}_{1,k}\\ &= (I_{r_k} - K_k^{f_1}\Sigma_k)f_{1,k} - K_k^{f_1}e_{1,k}, \quad (18)\\ \tilde{f}_{2,k-1} &= f_{2,k-1} - \hat{f}_{2,k-1}\\ &= (I_{p-r_k} - K_k^{f_2}H_{2,k}F_{2,k-1}^x)f_{2,k-1}\\ &\quad - K_k^{f_2}H_{2,k}E_{k-1}^x d_{k-1} - K_k^{f_2}e_{2,k}, \quad (19)\end{aligned}$$

where

$$\begin{aligned}e_{1,k} &= H_{1,k}\tilde{x}_{k/k-1}^* + v_{1,k}, \quad (20)\\ \tilde{x}_{k/k-1}^* &= x_k - \hat{x}_{k/k-1}^*,\\ e_{2,k} &= H_{2,k}\tilde{\bar{x}}_{k/k-1} + v_{2,k}, \quad (21)\\ \tilde{\bar{x}}_{k/k-1} &= A_{k-1}\tilde{x}_{k-1/k-1}\\ &\quad + F_{1,k-1}^x\tilde{f}_{1,k-1} + w_{k-1}, \quad (22)\\ \tilde{x}_{k/k} &= x_k - \hat{x}_{k/k}.\end{aligned}$$

Referring to (18) and (19), the estimator $\hat{f}_k$ given by (14) is unbiased if and only if $K_k^{f_1}$ and $K_k^{f_2}$ satisfy the following constraints:

$$K_k^{f_1}\Sigma_k = I_{r_k}, \quad (23)$$
$$K_k^{f_2}G_k = \mathcal{F}_k, \quad (24)$$

where

$$G_k = [H_{2,k}F_{2,k-1}^x \quad H_{2,k}E_{k-1}^x]$$

and

$$\mathcal{F}_k = [I_{p-r_k} \quad 0_{(p-r_k)\times q}].$$

Let $\tilde{x}_{k/k-1}^*$, $\tilde{x}_{k-1/k-1}$ and $\tilde{f}_{1,k-1}$ be unbiased. The covariance matrices of $e_{1,k}$ and $e_{2,k}$ are defined respectively by

$$\begin{aligned}\tilde{R}_{1,k} &= \mathcal{E}\left[e_{1,k}e_{1,k}^T\right]\\ &= H_{1,k}P_{k/k-1}^{x^*}H_{1,k}^T + R_{1,k}, \quad (25)\\ \tilde{R}_{2,k} &= \mathcal{E}\left[e_{2,k}e_{2,k}^T\right]\\ &= H_{2,k}\overline{P}_{k/k-1}^x H_{2,k}^T + R_{2,k}, \quad (26)\end{aligned}$$

where

$$P_{k/k-1}^{x^*} = \mathcal{E}\left[\tilde{x}_{k/k-1}^*\tilde{x}_{k/k-1}^{*T}\right]$$

and

$$\overline{P}_{k/k-1}^x = \mathcal{E}\left[\tilde{\bar{x}}_{k/k-1}\tilde{\bar{x}}_{k/k-1}^T\right].$$

Since neither $e_{1,k}$, nor $e_{2,k}$ has unit variances, the Least-Squares (LS) solutions do not have a minimum-variance. For that, $f_{1,k}$ and $f_{2,k}$ can be obtained by weighted least-squares estimation (Kailath *et al.*, 2000; Gillijns and Moor, 2007b) with two weighting matrices $\tilde{R}_{1,k}^{-1}$ and $\tilde{R}_{2,k}^{-1}$. Then, to have unbiased fault estimates, the matrices gain $K_k^{f_1}$ and $K_k^{f_2}$ are obtained as follows:

$$K_k^{f_1} = \left(\Sigma_k^T\tilde{R}_{1,k}^{-1}\Sigma_k\right)^{-1}\Sigma_k^T\tilde{R}_{1,k}^{-1}, \quad (27)$$
$$K_k^{f_2} = \mathcal{F}_k G_k^*, \quad (28)$$

where

$$G_k^* = \left(G_k^T\tilde{R}_{2,k}^{-1}G_k\right)^+ G_k^T\tilde{R}_{2,k}^{-1} \quad (29)$$

is the generalized inverse of the matrix G_k.

The variances of the WLS solutions (18) and (19) are respectively given by

$$P_k^{f_1} = \mathcal{E}\left[\tilde{f}_{1,k}(\tilde{f}_{1,k})^T\right] = \left(\Sigma_k^T\tilde{R}_{1,k}^{-1}\Sigma_k\right)^{-1}, \quad (30)$$
$$P_k^{f_2} = \mathcal{E}\left[\tilde{f}_{2,k-1}(\tilde{f}_{2,k-1})^T\right] = K_k^{f_2}\tilde{R}_{2,k}(K_k^{f_2})^T, \quad (31)$$

Referring to Eqns. (8), (14), (18) and (19), the fault error estimate $\tilde{f}_k$ has the following form:

$$\tilde{f}_k = \begin{bmatrix}V_{1,k} & V_{2,k}\end{bmatrix}\begin{bmatrix}\tilde{f}_{1,k}\\ \tilde{f}_{2,k}\end{bmatrix}. \quad (32)$$

Using (32), the covariance matrix P_k^f is given by

$$P_k^f = \begin{bmatrix}V_{1,k} & V_{2,k}\end{bmatrix}\begin{bmatrix}P_k^{f_1} & P_k^{f_{12}}\\ P_k^{f_{21}} & P_k^{f_2}\end{bmatrix}\begin{bmatrix}V_{1,k}^T\\ V_{2,k}^T\end{bmatrix}, \quad (33)$$

where

$$\begin{aligned}P_k^{f_{12}} &= (P_k^{f_{21}})^T = \mathcal{E}[\tilde{f}_{1,k}\,\tilde{f}_{2,k-1}^T] \quad (34)\\ &= K_k^{f_1}H_{1,k}\left[\bar{P}_{k/k-1}^x H_{2,k}^T + \tilde{S}_k\right](K_k^{f_2})^T\end{aligned}$$

with

$$\tilde{S}_k = \mathcal{E}[\tilde{x}_{k/k-1}^* v_{2,k}^T] = -(F_{k-1}^x K_k^{f_2} + K_k^{x*})\tilde{R}_{2,k}.$$

3.2. State estimation. Referring to Eqns. (1) and (12), the state estimation error $\tilde{x}_{k/k-1}^*$ is defined as

$$\begin{aligned}\tilde{x}_{k/k-1}^* &= \tilde{\bar{x}}_{k/k-1} - (K_k^{x*} + F_{2,k-1}^x K_k^{f_2})e_{2,k}\\ &\quad + (E_{k-1}^x - K_k^{x*}H_{2,k}E_{k-1}^x)d_{k-1} \quad (35)\\ &\quad - K_k^{x*}H_{2,k}F_{2,k-1}^x f_{2,k-1}.\end{aligned}$$

The estimator $\hat{x}_{k/k-1}^*$ (12) is unbiased if K_k^{x*} satisfies the following constraint to eliminate the terms $f_{2,k-1}$ and d_{k-1} from the error estimate (35):

$$K_k^{x*}G_k = \mathcal{F}_k^*, \quad (36)$$

where

$$G_k = [H_{2,k}F^x_{2,k-1} \quad H_{2,k}E^x_{k-1}]$$

and

$$\mathcal{F}^*_k = \left[0_{n\times(p-r_k)} \quad E^x_{k-1}\right].$$

Lemma 1. *A necessary and sufficient condition for the estimators (4) and (5) to be unbiased is that matrix G_k has full column rank, i.e.,*

$$\mathrm{rank}(G_k) = \mathrm{rank}(F^x_{2,k-1}) + \mathrm{rank}(E^x_{k-1}). \qquad (37)$$

Proof. Equations (24) and (36) can be written as

$$\begin{bmatrix} K^{f_2}_k \\ K^{x*}_k \end{bmatrix} G_k = \begin{bmatrix} \mathcal{F}_k \\ \mathcal{F}^*_k \end{bmatrix}. \qquad (38)$$

A necessary and sufficient condition for the existence of the solution to (38) is

$$\mathrm{rank}\begin{bmatrix} \mathcal{F}_k \\ \mathcal{F}^*_k \\ G_k \end{bmatrix} = \mathrm{rank}(G_k) \qquad (39)$$

We expand (39) and obtain

$$\begin{aligned}\mathrm{rank}\begin{bmatrix} \mathcal{F}_k \\ \mathcal{F}^*_k \\ G_k \end{bmatrix} &= \mathrm{rank}\begin{bmatrix} I_{p-r_k} & 0_{(p-r_k)\times q} \\ 0_{n\times(p-r_k)} & E^x_{k-1} \\ H_{2,k}F^x_{2,k-1} & H_{2,k}E^x_{k-1} \end{bmatrix} \\ &= \mathrm{rank}[H_{2,k}F^x_{2,k-1} \quad H_{2,k}E^x_{k-1}] \\ &= \mathrm{rank}(H_{2,k}F^x_{2,k-1}) + \mathrm{rank},(H_{2,k}E^x_{k-1})\end{aligned}$$

Finally, referring to the assumption (7), we will have

$$\mathrm{rank}(G_k) = \mathrm{rank}(F^x_{2,k-1}) + \mathrm{rank}(E^x_{k-1}).$$

However, this can be easily justified by considering that the faults and the unknown disturbances have independent influences. ∎

Referring to (35) and (36), the covariance matrix $P^{x*}_{k/k-1}$ has the following form:

$$\begin{aligned}P^{x*}_{k/k-1} &= \mathcal{E}\left[\tilde{x}^*_{k/k-1}(\tilde{x}^*_{k/k-1})^T\right] \\ &= (I_n - F^x_{2,k-1}K^{f_2}_k H_{2,k} - K^{x*}_k H_{2,k})\overline{P}^x_{k/k-1} \\ &\quad \times (I_n - F^x_{2,k-1}K^{f_2}_k H_{2,k} - K^{x*}_k H_{2,k})^T \\ &\quad + (F^x_{2,k-1}K^{f_2}_k + K^{x*}_k)R_{2,k}(F^x_{2,k-1}K^{f_2}_k + K^{x*}_k)^T.\end{aligned}$$

The gain matrix K^{x*}_k is determined by minimizing the trace of the covariance matrix $P^{x*}_{k/k-1}$ (37) such that (36) is satisfied. Using the Kitanidis method (Kitanidis, 1987), we obtain

$$\begin{bmatrix} \tilde{R}_{2,k} & -G_k \\ G^T_k & 0 \end{bmatrix}\begin{bmatrix} (K^{x*}_k)^T \\ (\Lambda^*_k)^T \end{bmatrix} = \begin{bmatrix} -\tilde{R}_{2,k}(K^{f_2}_k)^T(F^x_{2,k-1})^T + H_{2,k}\bar{P}^x_{k/k-1} \\ \mathcal{F}^*_k \end{bmatrix}, \qquad (40)$$

where Λ^*_k is the matrix of Lagrange multipliers.

Equation (38) will have a unique solution. Accordingly, the gain matrix K^{x*}_k is given by

$$\begin{aligned}K^{x*}_k &= \left[\bar{P}^x_{k/k-1}H^T_{2,k} - F^x_{2,k-1}K^{f_2}_k\tilde{R}_{2,k}\right] \\ &\quad \times \tilde{R}^{-1}_{2,k}(I - G_kG^*_k) + \mathcal{F}^*_kG^*_k, \end{aligned} \qquad (41)$$

where G^*_k was defined in (29).

Using (1) and (15), the state estimation error $\tilde{x}_{k/k}$ has the following form:

$$\begin{aligned}\tilde{x}_{k/k} &= x_k - \hat{x}_{k/k} \\ &= (I - K^x_kH_{2,k})\tilde{x}^*_{k/k-1} - K^x_kv_{2,k}. \end{aligned} \qquad (42)$$

Considering (40), the covariance matrix $P^x_{k/k}$ is determined as follows:

$$\begin{aligned}P^x_{k/k} &= \mathcal{E}\left[\tilde{x}_{k/k}\tilde{x}^T_{k/k}\right] \qquad (43) \\ &= P^{x*}_{k/k-1} + K^x_kR^*_kK^{xT}_k - V^*_kK^{xT}_k - K^x_kV^{*T}_k,\end{aligned}$$

where

$$\begin{aligned}R^*_k &= H_{2,k}P^{x*}_{k/k-1}H^T_{2,k} + R_{2,k} + H_{2,k}S^*_k \\ &\quad + (H_{2,k}S^*_k)^T, \qquad (44) \\ V^*_k &= P^{x*}_{k/k-1}(H_{2,k})^T + S^*_k, \qquad (45) \\ S^*_k &= \mathcal{E}\left[\tilde{x}^*_{k/k-1}v^T_{2,k}\right] \\ &= -(F^x_{2,k-1}K^{f2}_k + K^{x*}_k)R_{2,k}. \qquad (46)\end{aligned}$$

In order to obtain a minimum-variance estimate, we have to minimize the trace of (41). Thus, the gain matrix K^x_k is given by

$$K^x_k = (P^{x*}_{k/k-1}H^T_{2,k} + S^*_k)\alpha^T_k(\alpha_kR^*_k\alpha^T_k)^{-1}\alpha_k. \qquad (47)$$

where α_k is an arbitrary matrix which has to be chosen such that $\alpha_kR^*_k\alpha^T_k$ has full rank (Gillijns and Moor, 2007a; Darouach *et al.*, 2003).

3.3. Filter time update.

From Eqn. (22), the covariance matrix $\overline{P}^x_{k/k-1}$ has the following form:

$$\begin{aligned}\overline{P}^x_{k+1/k} &= \mathcal{E}\left[\tilde{\bar{x}}_{k+1/k}(\tilde{\bar{x}}_{k+1/k})^T\right] \qquad (48) \\ &= \left[A_k \quad F^x_{1,k}\right]\begin{bmatrix} P^x_{k/k} & P^{xf_1}_k \\ (P^{xf_1}_k)^T & P^{f_1}_k \end{bmatrix}\begin{bmatrix} A^T_k \\ F^{xT}_{1,k} \end{bmatrix} + Q_k,\end{aligned}$$

where $P^{xf_1}_k$ is calculated by using (18) and (40). Then, we obtain

$$\begin{aligned}P^{xf_1}_k &= \mathcal{E}\left[\tilde{x}_{k/k}(\tilde{f}_{1,k})^T\right] \\ &= (K^x_k(V^*_k)^T - P^{x*}_{k/k-1})(K^{f1}_kH_{1,k}))^T. \end{aligned} \qquad (49)$$

4. Applications

In this section, we propose the use of the resulting filter to solve the problem of estimating simultaneous actuator and sensor faults and to make a comparison with the existing literature results, in particular the ones of Ben Hmida *et al.* (2010).

4.1. Robust estimation of simultaneous actuator and sensor faults. We consider the numerical example used by Chen and Patton (1996; 1999). The linearized model of a simplified longitudinal flight control system is the following:

$$\begin{aligned} x_{k+1} &= (A_k + \Delta A_k)x_k + (B_k + \Delta B_k)u_k \\ &\quad + F_k^a f_k^a + w_k, \\ y_k &= H_k x_k + F_k^s f_k^s + v_k, \end{aligned}$$

where the state variables are the pitch angle δ_z, the pitch rate ω_z and the normal velocity η_y. The control input u_k is the elevator control signal. F_k^a and F_k^s denote the matrices defining the distribution of the actuator fault f_k^a and the sensor fault f_k^s, respectively. The presented system equations can be rewritten as follows:

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k + F_k^x f_k + E_k^x d_k + w_k, \\ y_k &= H_k x_k + F_k^y f_k + v_k, \end{aligned}$$

where F_k^x and F_k^y represent the matrices defining the injection of the faults vector in the state and measurement equations:

$$F_k^x = [F_k^a \quad 0], \quad F_k^y = [0 \quad F_k^s].$$

The term $E_k^x d_k$ represents the parameter perturbation in matrices the A_k and B_k:

$$E_k^x d_k = \Delta A_k x_k + \Delta B_k u_k.$$

The system parameter matrices are

$$A_k = \begin{bmatrix} 0.9944 & -0.1203 & -0.4302 \\ 0.0017 & 0.9902 & -0.0747 \\ 0 & 0.8187 & 0 \end{bmatrix},$$

$$B_k = \begin{bmatrix} 0.4252 \\ -0.0082 \\ 0.1813 \end{bmatrix},$$

$$H_k = I_{3\times 3}, x_k = [\eta_y \quad \omega_z \quad \delta_z]^T,$$

$$Q_k = \operatorname{diag}\left\{0.1^2, 0.1^2, 0.01^2\right\}, \quad R_k = 0.1^2 I_{3\times 3}$$

We inject simultaneously two faults in the system,

$$\begin{bmatrix} f_k^a \\ f_k^s \end{bmatrix} = \begin{bmatrix} 4u_s(k-20) - 4u_s(k-60) \\ -2u_s(k-30) + 2u_s(k-70) \end{bmatrix},$$

where $u_s(k)$ is the unit-step function. The first fault, f_k^a, occurs in the actuator between time instants 20 and 60, and the second fault, f_k^s, occurs in the sensor for δ_z between time instants 30 and 70. Thus, we consider the cases of single and multiples faults. The unknown disturbance is given by

$$E_k^x d_k = E_k^x \Big\{ [\Delta a_{21} \quad \Delta a_{22} \quad \Delta a_{23}]\, x_k + \Delta b_2 u_k \Big\},$$

where Δa_{ij} and Δb_i $(i = 1 \text{and} j = 1,2,3)$ are perturbations in aerodynamic and control coefficients. The matrices defining injection of the fault and the unknown disturbances are

$$E_k^x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad F_k^a = \begin{bmatrix} 0.4252 \\ -0.0082 \\ 0.1813 \end{bmatrix}, \quad F_k^s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

In the simulation, the aerodynamic coefficients are perturbed by $\pm 50\%$, i.e., $\Delta a_{ij} = -0.5 a_{ij}$ and $\Delta b_i = 0.5 b_i$. In addition, we set $u_k = 10, x_0 = 0, P_0 = 0.1^2 I_{3\times 3}$.

In Fig. 1, the actual and estimated values of the state vector $x_k = [\eta_y\ \omega_z\ \delta_z]^T$ are displayed. Figure 2 presents the actual and estimated values of the first element and the second element of the fault vector $f_k = [f_k^a\ f_k^s]^T$, respectively. In Table 1, the Root Mean Square Errors

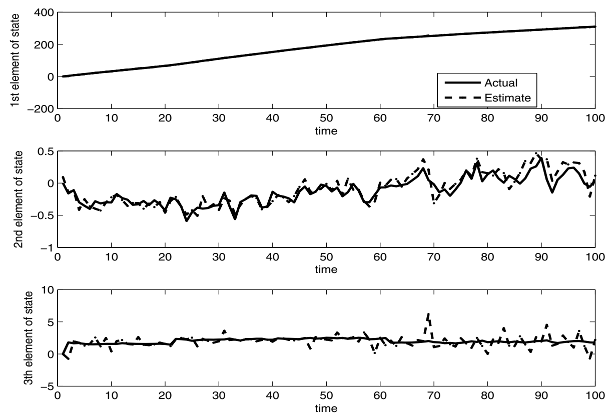

Fig. 1. Actual state x_k and the estimated state $\hat{x}_k$.

(RMSEs) of the states $x_k = [\eta_y\ \omega_z\ \delta_z]^T$ and the faults $f_k = \left[f_k^a \quad f_k^s\right]^T$ are given along with the traces of their steady-input and state estimation error covariances. According to the simulation results (Figs. 1 and 2 and Table 1), it can be seen that the proposed filter produces better estimates of the state and faults. We primarily focus on simultaneous estimation of actuator and sensor faults in spite of the presence of unknown disturbances.

Table 1. Performance of the proposed filter.

x^1	x^2	x^3	f^a	f^s
0.7467	0.1059	0.8966	1.1198	0.2159

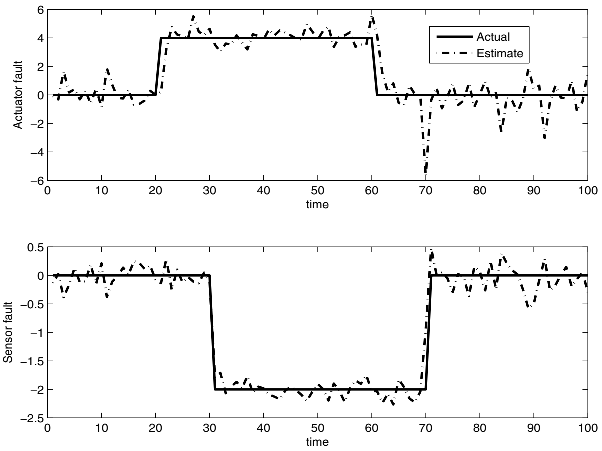

Fig. 2. Actual fault f_k and the estimated fault $\hat{f}_k$.

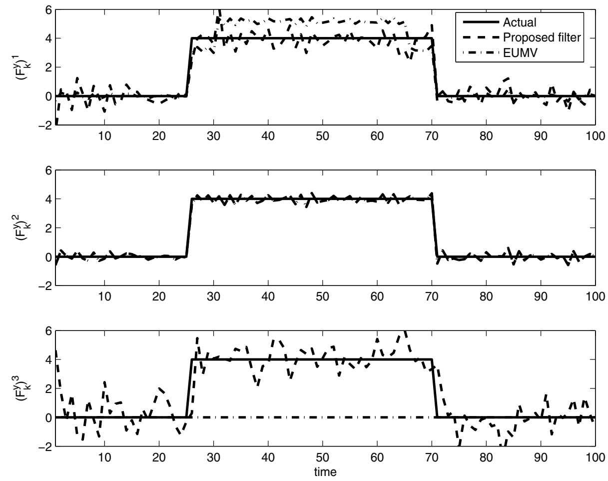

Fig. 3. Actual fault f_k^1 and the estimated fault $\hat{f}_k^1$.

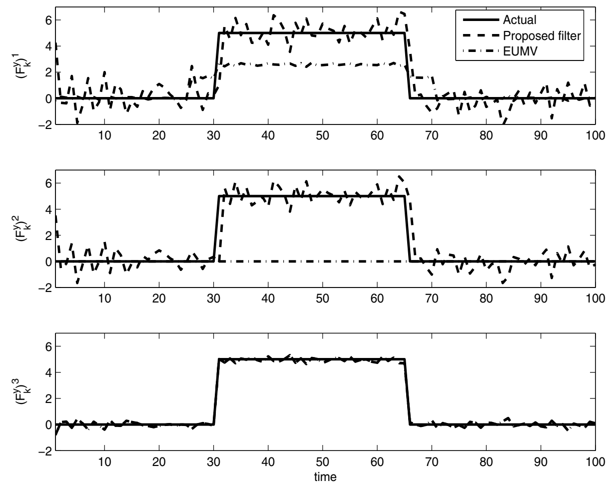

Fig. 4. Actual fault f_k^2 and the estimated fault $\hat{f}_k^2$.

4.2. Comparative study. We consider the numerical example used by Ben Hmida *et al*. (2010). In this simulation, three cases of F_k^y will be studied:

$$(F_k^y)^1 = \begin{bmatrix} 2 & 1 \\ 0.6 & 0.3 \\ 0.2 & 0.1 \end{bmatrix}, \quad (F_k^y)^2 = \begin{bmatrix} 2 & 0 \\ 0.6 & 0 \\ 0.2 & 0 \end{bmatrix},$$

$$(F_k^y)^3 = \begin{bmatrix} 0 & 1.4 \\ 0 & 0.3 \\ 0 & 1.6 \end{bmatrix},$$

dealing with particular ranks of the matrix F_k^y.

We assume that the fault and the disturbance are given by

$$\begin{bmatrix} f_k^1 \\ f_k^2 \end{bmatrix} = \begin{bmatrix} 4u_s(k-25) - 4u_s(k-70) \\ 5u_s(k-30) - 5u_s(k-65) \end{bmatrix},$$

$$d_k = 4u_s(k-15) - 4u_s(k-55),$$

where $u_s(k)$ is the unit-step function.

Figures 3 and 4 present the actual and estimated values of the first and second elements of the fault vector $f_k = [f_k^1 \ f_k^2]^T$, respectively. In Tables 2 and 3, the RMSE value of the states $x_k = [x_k^1 \ x_k^2 \ x_k^3]$ and the fault $f_k = [f_k^1 \ f_k^2]^T$ are given along with the traces of their steady-fault and state estimation error covariance matrices. According to the simulation results in Figs. 3 and 4 and Tables 2 and 3, we may conclude with the following results:

1. In all cases, the EUMV filter (Ben Hmida *et al*., 2010) and the proposed filter give the same values of the RMSE of the state estimation error. On the other hand, the proposed filter gives smaller RMSE values of the fault estimation errors.
2. When referring to the traces of the steady-fault estimation error covariance, it appears that

$$\begin{aligned} &\text{trace}(P^f(\text{EUMV filter})) \\ &\quad = \text{trace}(P^{f_1}(\text{Proposed filter})) \\ &\quad \le \text{trace}(P^f(\text{Proposed filter})). \end{aligned}$$

Clearly, the fault estimation obtained by the EUMV filter is in fact only the first component estimate $\hat{f}_{1,k}$ in our proposed filter. Dealing with the other component of the fault $f_{2,k}$, our proposed filter improves the overall performance, despite a superior trace of the steady-fault estimation error covariance as, logically,

Table 2. Evaluation of the RMSE values.

F_k^y	Filter	x_k^1	x_k^2	x_k^3	f_k^1	f_k^2
$(F_k^y)^1$	EUMV filter	0.5248	0.1346	0.1863	0.7577	1.5179
	Proposed filter	0.5248	0.1346	0.1863	0.6588	1.0942
$(F_k^y)^2$	EUMV filter	0.6301	0.1542	0.2129	0.2565	2.9580
	Proposed filter	0.6301	0.1542	0.2129	0.2565	1.0434
$(F_k^y)^3$	EUMV filter	0.3378	0.0938	0.3647	2.6833	0.2103
	Proposed filter	0.3378	0.0938	0.3647	1.1992	0.2103

Table 3. Trace of steady fault and state estimation error covariances.

F_k^y	Filter	trace$(P^x_{k/k})$	trace$(P^{f1}_{k/k})$	trace$(P^{f2}_{k/k})$	trace$(P^{f}_{k/k})$
$(F_k^y)^1$	EUMV filter	0.2645	–	–	0.0301
	Proposed filter	0.2645	0.0301	0.8068	0.8370
$(F_k^y)^2$	EUMV filter	0.3732	–	–	0.0525
	Proposed filter	0.3732	0.0525	0.4361	0.4885
$(F_k^y)^3$	EUMV filter	0.1977	–	–	0.0338
	Proposed filter	0.1977	0.0338	0.8350	0.8688

$$
\begin{aligned}
&\mathrm{trace}(P^f(\text{Proposed filter})) \\
&\quad = \mathrm{trace}(P^f(\text{EUMV filter})) \\
&\qquad + \mathrm{trace}(P^{f2}(\text{Proposed filter})).
\end{aligned}
$$

5. Conclusion

In this paper, the problem of joint state and fault estimation for linear time-varying stochastic systems with unknown disturbances was solved. To achieve this objective, a new recursive optimal unbiased minimum-variance filter was proposed when the direct feedthrough matrix of the fault has arbitrary rank. The advantages of this filter are especially important in the case when we do not have any prior information about the fault and the unknown disturbances. This filter is applied efficiently to solve two problems. Firstly, it estimates the actuator and sensor faults simultaneously. Secondly, it establishes a comparative study with the existing literature results.

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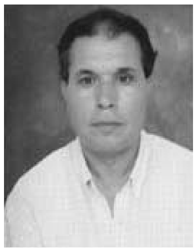

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