Int. J. Appl. Math. Comput. Sci., 2012, Vol. 22, No. 2, 339–351
DOI: 10.2478/v10006-012-0025-6

LMI OPTIMIZATION PROBLEM OF DELAY–DEPENDENT ROBUST STABILITY CRITERIA FOR STOCHASTIC SYSTEMS WITH POLYTOPIC AND LINEAR FRACTIONAL UNCERTAINTIES

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This paper studies an LMI optimization problem of delay-dependent robust stability criteria for stochastic systems with polytopic and linear fractional uncertainties. The delay is assumed to be time-varying and belong to a given interval, which means that lower and upper bounds of this interval time-varying delay are available. The uncertainty under consideration includes polytopic-type uncertainty and linear fractional norm-bounded uncertainty. Based on the new Lyapunov–Krasovskii functional, some inequality techniques and stochastic stability theory, delay-dependent stability criteria are obtained in terms of Linear Matrix Inequalities (LMIs). Moreover, the derivative of time delays is allowed to take any value. Finally, four numerical examples are given to illustrate the effectiveness of the proposed method and to show an improvement over some results found in the literature.

Keywords: delay-dependent stability, linear matrix inequality, Lyapunov–Krasovskii functional, stochastic systems.

1. Introduction

Time-delay occurs in many dynamical systems such as biological systems, chemical, metallurgical processing and nuclear reactors, long-transmission lines in pneumatic and hydraulic systems as well as electrical networks (Kolmanoskii and Myshkis, 1992). Frequently, it has been a source of the oscillations, instability and poor performance. Considerable effort has been applied to different aspects of linear time-delay systems in recent years (Hale and Verduyn lunel, 1993; Huang and Zhou, 2000; Ivanesu *et al.*, 2000; Mahmoud and Al-Muthairi, 1994; Liu, 2005; Xue and Qiu, 2000; Xia and Jia, 2003). Moreover, the stability analysis of interval time-varying systems has been focused on as a topic of theoretical and practical importance (He *et al.*, 2006; Jiang and Han, 2008; Yue, 2006; Jiang and Han, 2006; Kwon and Park, 2008). Systems with interval time-varying delays mean that the lower bound of the time delay which guarantees the stability of the system is not restricted to zero. A typical example of dynamical systems with interval time-varying delays is a networked control system (Yue, 2006).

Uncertainties are frequently encountered in various engineering and communication systems. The characteristics of dynamic systems are significantly affected by the presence of uncertainties, even to the extent of instability in an extreme situation (Zhou *et al.*, 2006).

In engineering applications, it is very common that one does not know exactly the system under investigation; that is, the system contains some elements (blocks) that are uncertain. Usually it is known that these uncertain elements belong to some specific admissible domains, which in turn depend on the nature of the elements and also on the information available about the system. In other words, it is known only that the system belongs to the family of systems that arises when the uncertain elements (blocks) range over the admissible domains and therefore one may treat the family as a new object for analysis. This family is referred to as an uncertain system. When it is possible to show that all systems of the family are stable, the stability of the original system that is a particular member of the family is guaranteed. The robust stability problem considers the stability problem of systems that contain some uncertainties.

Over the past decade, much effort has been spent on the analysis and synthesis of uncertain systems with time-delay (see, e.g., the works of Chen (2002) Kim (2001), Liu and Zhang (2005) and the references therein).

In this paper, the stability analysis for stochastic system is investigated under polytopic type uncertainty and linear fractional uncertainty. First, polytopic uncertainties can arise when the uncertain matrix in norm-bounded uncertainties provides some prior known structures of uncertainties. Therefore the polytopic type uncertainty can be regarded as an important class of parameter uncertainty. Recently, the problem of robust stability and stabilization for delayed systems with polytopic uncertainties have been studied and LMI-based approaches have been developed (Li *et al.*, 2008; Xia and Jia, 2002; He *et al.*, 2004; Geromel and Colaneri, 2006; Chesi *et al.*, 2007; Ramos and Peres, 2001; Xu *et al.*, 2004). As is well known, usually fractional uncertainties are more general than norm-bounded uncertainties. Recently, some results on the stability of systems with linear fractional uncertainty have been reported (Li *et al.*, 2007; Balasubramaniam *et al.*, 2009; Gu *et al.*, 2003; Balasubramaniam and Lakshmanan, 2011).

In recent years, increasing efforts have been made to study stochastic systems with time-delays. The stability and control problem for uncertain stochastic delayed systems has been extensively investigated by a considerable number of researchers (Miyamura and Aihara, 2004; Yan *et al.*, 2009; Zhang *et al.*, 2009; 2008; Chen *et al.*, 2005; Yue and Han, 2005; He *et al.*, 2010). Tian *et al.* (2010), dealt with the problem of robust H_∞ control design for nonlinear networked control systems, which are presented in the form of a T–S fuzzy model with a time-varying input delay and the whole variation interval of the delay is divided into two subintervals of equal length, which is different from the existing method. Recently, only few authors have discussed stability criteria for a system with linear fractional and polytopic uncertainties.

Based on the above motivations, this paper aims to develop an LMI optimization problem of delay-dependent robust stability criteria for stochastic systems by constructing a new Lyapunov–Krasovskii functional with integral terms involving lower and upper bounds of interval time-varying delays such as

$$\int_{t-\frac{\tau_m}{2}}^{t} \begin{bmatrix} x(s) \\ x(s-\frac{\tau_m}{2}) \end{bmatrix}^T \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix} \begin{bmatrix} x(s) \\ x(s-\frac{\tau_m}{2}) \end{bmatrix} \mathrm{d}s,$$

$$\int_{t-\frac{\tau_M}{2}}^{t} \begin{bmatrix} x(s) \\ x(s-\frac{\tau_M}{2}) \end{bmatrix}^T \begin{bmatrix} Q_4 & Q_5 \\ Q_5^T & Q_6 \end{bmatrix} \begin{bmatrix} x(s) \\ x(s-\frac{\tau_M}{2}) \end{bmatrix} \mathrm{d}s,$$

$$\frac{\tau_m}{2}\int_{-\frac{\tau_m}{2}}^{0} \mathrm{d}s \int_{t+s}^{t} y^T(\theta)R_4 y(\theta)\,\mathrm{d}\theta,$$

$$\frac{\tau_M}{2}\int_{-\frac{\tau_M}{2}}^{0} \mathrm{d}s \int_{t+s}^{t} y^T(\theta)R_5 y(\theta)\,\mathrm{d}\theta,$$

$$\delta\int_{-\tau_M}^{-\tau_m} \mathrm{d}s \int_{t+s}^{t} y^T(\theta)R_6 y(\theta)\,\mathrm{d}\theta.$$

by employing some analytical techniques, sufficient conditions are derived for the stochastic systems considered in terms of LMIs, which can be easily calculated by the MATLAB LMI Control Toolbox. Moreover, a polytopic-type and linear fractional uncertainty which includes as a norm-bounded uncertainty as a special case is discussed. Some numerical examples are given to illustrate the effectiveness and conservativeness of the proposed method.

Notation. Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ denote, respectively, the n-dimensional Euclidean space and the set of all $n \times n$ real matrices. The superscript T denotes the transposition and the notation $X \geq Y$ (respectively $X > Y$), where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively positive definite). I_n is the $n \times n$ identity matrix. $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. Moreover, let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (that is, the filtration contains all $\mathcal{P}$-null sets and is right continuous). The asterisk $*$ always denotes the symmetric block in one symmetric matrix. Sometimes, the arguments of a function or a matrix will be omitted in the analysis when no confusion can arise.

2. Problem description and preliminaries

Consider the following stochastic system with state delay:

$$\begin{aligned} \mathrm{d}x(t) &= [A(t)x(t) + B(t)x(t-\tau(t))]\,\mathrm{d}t \\ &\quad + [C(t)x(t) + D(t)x(t-\tau(t))]\,\mathrm{d}w(t), \\ x(t) &= \phi(t), \quad t \in [-\tau_M, 0], \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $A(t) = A + \Delta A(t), B(t) = B + \Delta B(t), C(t) = C + \Delta C(t)$ and $D(t) = D + \Delta D(t)$, A, B, C and D are known real matrices of appropriate dimensions, $\Delta A(t)$, $\Delta B(t)$, $\Delta C(t)$ and $\Delta D(t)$ are unknown real matrices of appropriate dimensions representing system time-varying parameter uncertainties; $\omega(t)$ denotes one dimensional Brownian motion satisfying $E\{\mathrm{d}\omega(t)\} = 0$ and $E\{\mathrm{d}\omega(t)^2\} = \mathrm{d}t$. It is defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with a natural filtration $\{\mathcal{F}_t\}$. Here $\phi(t)$ is any given initial data in $L^2_{\mathcal{F}_0}([-\tau_M, 0]; \mathbb{R}^n)$. Furthermore, $\tau(t)$ denotes the time varying bounded delay and is assumed to satisfy

$$0 \leq \tau_m \leq \tau(t) \leq \tau_M, \quad \dot{\tau}(t) \leq \mu, \tag{2}$$

where τ_m, τ_M and μ are constants. We consider robust stability of the system described by (1) and (2) subject to polytopic uncertainty. For polytopic uncertainty, matrices A, B, C and D in (1) can be expressed as

$$\begin{bmatrix} A & B & C & D \end{bmatrix} = \sum_{i=1}^{r} \lambda_i \begin{bmatrix} A^{(i)} & B^{(i)} & C^{(i)} & D^{(i)} \end{bmatrix}, \quad (3)$$

where

$$\sum_{i=1}^{r} \lambda_i = 1, \quad 0 \le \lambda_i \le 1.$$

Next, we address the linear fractional norm-bounded uncertainty. Suppose that matrices A, B, C and D have parameter perturbations $\Delta A(t)$, $\Delta B(t)$, $\Delta C(t)$ and $\Delta D(t)$, which are in the form of

$$\begin{bmatrix} \Delta A(t) & \Delta B(t) & \Delta C(t) & \Delta D(t) \end{bmatrix} = H\Lambda(t) \begin{bmatrix} E & E_1 & E_2 & E_3 \end{bmatrix}, \quad (4)$$

where E, E_1, E_2 and E_3 are given matrices. The class of parametric uncertainties $\Lambda(t)$ that satisfy

$$\Lambda(t) = [I - F(t)J]^{-1} F(t) \quad (5)$$

is said to be admissible, where J is also a known matrix satisfying

$$I - JJ^T > 0, \quad (6)$$

and $F(t)$ is an uncertain matrix satisfying

$$F^T(t)F(t) \le I. \quad (7)$$

Definition 1. The stochastic time-delay system (1) is said to be *robustly stochastically stable* if there exists a positive scalar $c > 0$ such that for all admissible uncertainties

$$\lim_{T \longrightarrow \infty} \mathbb{E} \int_0^T x^T(t)x(t)\,\mathrm{d}t \le c \sup_{s \in [-\tau_M, 0]} \mathbb{E}\|\phi(s)\|^2.$$

In obtaining the main results of this paper, the following lemmas will be essential for the proof.

Lemma 1. (Boyd *et al.*, 1994) (Schur complement) *Given constant matrices Ω_1, Ω_2 and Ω_3 of appropriate dimensions, where $\Omega_1^T = \Omega_1$ and $\Omega_2^T = \Omega_2 > 0$, we have*

$$\Omega_1 + \Omega_3^T \Omega_2^{-1} \Omega_3 < 0$$

if and only if

$$\begin{bmatrix} \Omega_1 & \Omega_3^T \\ * & -\Omega_2 \end{bmatrix} < 0, \quad \text{or} \quad \begin{bmatrix} -\Omega_2 & \Omega_3 \\ * & \Omega_1 \end{bmatrix} < 0.$$

Lemma 2. (Gu, 2000) *For any constant matrix $M > 0$, any scalars a and b with $a < b$ and a vector function $x(t) : [a, b] \longrightarrow \mathbb{R}^n$ such that the integrals concerned are well defined, we have*

$$\Big[\int_a^b x(s)\,\mathrm{d}s\Big]^T M \Big[\int_a^b x(s)\,\mathrm{d}s\Big] \le (b-a)\Big[\int_a^b x^T(s) M x(s)\,\mathrm{d}s\Big].$$

Lemma 3. (Li *et al.*, 2007) *Suppose $\Lambda(t)$ is given by (5)–(7). Given matrices $M = M^T$, S and N of appropriate dimensions, the inequality*

$$M + S\Lambda(t)N + N^T \Lambda^T(t) S^T < 0$$

holds for $F(t)$ such that $F^T(t)F(t) \le I$ if, and only if, for some $\epsilon > 0$

$$\begin{bmatrix} M & S & \epsilon N^T \\ S^T & -\epsilon I & \epsilon J^T \\ \epsilon N & \epsilon J & -\epsilon I \end{bmatrix} < 0.$$

3. Main results

Consider the uncertain stochastic system (1) with time varying delays. We take

$$y(t) = A(t)x(t) + B(t)x(t - \tau(t)), \quad (8)$$

$$g(t) = C(t)x(t) + D(t)x(t - \tau(t)). \quad (9)$$

Then the system (1) becomes

$$\mathrm{d}x(t) = y(t)\mathrm{d}t + g(t)\,\mathrm{d}\omega(t). \quad (10)$$

Moreover, the following equality holds:

$$x(t) - x(t - \tau(t)) = \int_{t-\tau(t)}^{t} y(s)\,\mathrm{d}s + \int_{t-\tau(t)}^{t} g(s)\,\mathrm{d}\omega(s). \quad (11)$$

The above equality is used in the proof of the main result. Now the following theorem will discussed without uncertainties.

Theorem 1. *For given scalars τ_M, τ_m and μ, the equilibrium point of the stochastic system (1) is asymptotically stable in the mean square sense if there exist matrices $P > 0$,*

$$\begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix} > 0, \quad \begin{bmatrix} Q_4 & Q_5 \\ Q_5^T & Q_6 \end{bmatrix} > 0,$$

$R_k > 0$, $k = 1 \ldots, 6$, such that for any matrices N_i and M_i ($i = 1, \ldots, 8$), the following LMI is feasible:

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \ldots & \Omega_{18} \\ * & \Omega_{22} & \ldots & \Omega_{28} \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \ldots & \Omega_{88} \end{bmatrix} < 0, \quad (12)$$

where

$$\Omega_{11} = Q_4 + Q_1 + R_1 + R_2 + R_3 - R_4 - R_5 + N_1A + A^TN_1^T + M_1C + C^TM_1^T,$$
$$\Omega_{12} = N_1B + A^TN_2^T + M_1D + C^TM_2^T,$$
$$\Omega_{13} = Q_2 + R_4 + A^TN_3^T + C^TM_3^T,$$
$$\Omega_{14} = A^TN_4^T + C^TM_4^T,$$
$$\Omega_{15} = Q_5 + R_5 + A^TN_5^T + C^TM_5^T,$$
$$\Omega_{16} = A^TN_6^T + C^TM_6^T,$$
$$\Omega_{17} = P + A^TN_7^T + C^TM_7^T - N_1,$$
$$\Omega_{18} = A^TN_8^T + C^TM_8^T - M_1,$$
$$\Omega_{22} = -R_6 - (1-\mu)R_1 + N_2B + B^TN_2^T + M_2D + D^TM_2^T,$$
$$\Omega_{23} = B^TN_3^T + D^TM_3^T,$$
$$\Omega_{24} = R_6 + B^TN_4^T + D^TM_4^T,$$
$$\Omega_{25} = B^TN_5^T + D^TM_5^T,$$
$$\Omega_{26} = B^TN_6^T + D^TM_6^T,$$
$$\Omega_{27} = B^TN_7^T + D^TM_7^T - N_2,$$
$$\Omega_{28} = B^TN_8^T + D^TM_8^T - M_2,$$
$$\Omega_{33} = Q_3 - Q_1 - R_4, \quad \Omega_{34} = -Q_2,$$
$$\Omega_{35} = \Omega_{36} = 0, \quad \Omega_{37} = -N_3,$$
$$\Omega_{38} = -M_3, \quad \Omega_{44} = -Q_3 - R_2 - R_6,$$
$$\Omega_{45} = \Omega_{4,6} = 0, \quad \Omega_{47} = -N_4, \quad \Omega_{48} = -M_4,$$
$$\Omega_{55} = Q_6 - Q_4 - R_5, \quad \Omega_{56} = -Q_5,$$
$$\Omega_{57} = -N_5, \quad \Omega_{58} = -M_5,$$
$$\Omega_{66} = -Q_6 - R_3, \quad \Omega_{67} = -N_6,$$
$$\Omega_{68} = -M_6,$$
$$\Omega_{77} = \left(\frac{\tau_m}{2}\right)^2 R_4 + \left(\frac{\tau_M}{2}\right)^2 R_5 + \delta^2 R_6 - N_7 - N_7^T,$$
$$\Omega_{78} = -N_8^T - M_7,$$
$$\Omega_{88} = P - M_8 - M_8^T,$$
$$N = \left[N_1^T\ N_2^T\ N_3^T\ N_4^T\ N_5^T\ N_6^T\ N_7^T\ N_8^T\right]^T,$$
$$M = \left[M_1^T\ M_2^T\ M_3^T\ M_4^T\ M_5^T\ M_6^T\ M_7^T\ M_8^T\right]^T,$$
$$\delta = \tau_M - \tau_m.$$

Proof. Consider the Lyapunov–Krasovskii functional

$$V(x_t,t) = V_1(x_t,t) + V_2(x_t,t) + V_3(x_t,t) + V_4(x_t,t), \quad (13)$$

where

$$V_1(x_t,t) = x^T(t)Px(t),$$
$$V_2(x_t,t) = \int_{t-\tau(t)}^{t} x^T(s)R_1x(s)\,\mathrm{d}s + \int_{t-\tau_m}^{t} x^T(s)R_2x(s)\,\mathrm{d}s + \int_{t-\tau_M}^{t} x^T(s)R_3x(s)\,\mathrm{d}s,$$
$$V_3(x_t,t) = \int_{t-\frac{\tau_m}{2}}^{t} \begin{bmatrix} x(s) \\ x(s-\frac{\tau_m}{2})\end{bmatrix}^T \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3\end{bmatrix} \begin{bmatrix} x(s) \\ x(s-\frac{\tau_m}{2})\end{bmatrix} \mathrm{d}s + \int_{t-\frac{\tau_M}{2}}^{t} \begin{bmatrix} x(s) \\ x(s-\frac{\tau_M}{2})\end{bmatrix}^T \begin{bmatrix} Q_4 & Q_5 \\ Q_5^T & Q_6\end{bmatrix} \begin{bmatrix} x(s) \\ x(s-\frac{\tau_M}{2})\end{bmatrix} \mathrm{d}s,$$
$$V_4(x_t,t) = \frac{\tau_m}{2}\int_{-\frac{\tau_m}{2}}^{0} \mathrm{d}s \int_{t+s}^{t} y^T(\theta)R_4y(\theta)\,\mathrm{d}\theta + \frac{\tau_M}{2}\int_{-\frac{\tau_M}{2}}^{0} \mathrm{d}s \int_{t+s}^{t} y^T(\theta)R_5y(\theta)\,\mathrm{d}\theta + \delta\int_{-\tau_M}^{-\tau_m} \mathrm{d}s \int_{t+s}^{t} y^T(\theta)R_6y(\theta)\,\mathrm{d}\theta.$$

■

Then it can be obtained by Itō's formula that

$$\mathrm{d}V(x_t,t) = LV(x_t,t)\,\mathrm{d}t + 2x^T(t)Pg(t)\,\mathrm{d}\omega(t), \quad (14)$$

where

$$LV_1(x_t,t) = 2x^T(t)Py(t) + g^T(t)Pg(t),$$
$$LV_2(x_t,t) \le x^T(t)R_1x(t) + x^T(t)R_2x(t) - (1-\mu)x^T(t-\tau(t))R_1x(t-\tau(t)) - x^T(t-\tau_m)R_2x(t-\tau_m) + x^T(t)R_3x(t) - x^T(t-\tau_M)R_3x(t-\tau_M),$$
$$LV_3(x_t,t) = \begin{bmatrix} x(t) \\ x(t-\frac{\tau_m}{2})\end{bmatrix}^T \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3\end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\frac{\tau_m}{2})\end{bmatrix} - \begin{bmatrix} x(t-\frac{\tau_m}{2}) \\ x(t-\tau_m)\end{bmatrix}^T \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3\end{bmatrix} \begin{bmatrix} x(t-\frac{\tau_m}{2}) \\ x(t-\tau_m)\end{bmatrix} + \begin{bmatrix} x(t) \\ x(t-\frac{\tau_M}{2})\end{bmatrix}^T \begin{bmatrix} Q_4 & Q_5 \\ Q_5^T & Q_6\end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\frac{\tau_M}{2})\end{bmatrix} - \begin{bmatrix} x(t-\frac{\tau_M}{2}) \\ x(t-\tau_M)\end{bmatrix}^T \begin{bmatrix} Q_4 & Q_5 \\ Q_5^T & Q_6\end{bmatrix} \begin{bmatrix} x(t-\frac{\tau_M}{2}) \\ x(t-\tau_M)\end{bmatrix},$$

$$
\begin{aligned}
&LV_4(x_t,t)\\
&= y^T(t)\Big[\Big(\frac{\tau_m}{2}\Big)^2 R_4 + \Big(\frac{\tau_M}{2}\Big)^2 R_5 + \delta^2 R_6\Big]y(t)\\
&\quad - \frac{\tau_m}{2}\int_{t-\frac{\tau_m}{2}}^{t} y^T(s)R_4 y(s)\,\mathrm{d}s\\
&\quad - \frac{\tau_M}{2}\int_{t-\frac{\tau_M}{2}}^{t} y^T(s)R_5 y(s)\,\mathrm{d}s\\
&\quad - \delta\int_{t-\tau_M}^{t-\tau_m} y^T(s)R_6 y(s)\,\mathrm{d}s,\\
&LV_4(x_t,t)\\
&\le y^T(t)\Big[\Big(\frac{\tau_m}{2}\Big)^2 R_4 + \Big(\frac{\tau_M}{2}\Big)^2 R_5 + \delta^2 R_6\Big]y(t)\\
&\quad - \frac{\tau_m}{2}\int_{t-\frac{\tau_m}{2}}^{t} y^T(s)R_4 y(s)\,\mathrm{d}s\\
&\quad - \frac{\tau_M}{2}\int_{t-\frac{\tau_M}{2}}^{t} y^T(s)R_5 y(s)\,\mathrm{d}s\\
&\quad - \delta\int_{t-\tau(t)}^{t-\tau_m} y^T(s)R_6 y(s)\,\mathrm{d}s.
\end{aligned}
$$

Using the note discussed by Kwon *et al.* (2010), we have

$$
-1 = -(\tau_m)^{-1}\Big(\frac{\tau_m}{2}\Big) - \Big(1-(\tau_m)^{-1}\Big(\frac{\tau_m}{2}\Big)\Big). \tag{15}
$$

Using (11), (15) and Lemma 2, an upper bound of the integral term

$$
-\int_{t-\frac{\tau_m}{2}}^{t} y^T(s)R_4 y(s)\,\mathrm{d}s
$$

can be obtained as

$$
\begin{aligned}
&-\int_{t-\frac{\tau_m}{2}}^{t} y^T(s)R_4 y(s)\,\mathrm{d}s\\
&= -(\tau_m)^{-1}\Big(\frac{\tau_m}{2}\Big)\int_{t-\frac{\tau_m}{2}}^{t} y^T(s)R_4 y(s)\,\mathrm{d}s\\
&\quad - \Big[1-(\tau_m)^{-1}\Big(\frac{\tau_m}{2}\Big)\Big]\int_{t-\frac{\tau_m}{2}}^{t} y^T(s)R_4 y(s)\,\mathrm{d}s\\
&= -(\tau_m)^{-1}\Big(\frac{\tau_m}{2}\Big)\int_{t-\frac{\tau_m}{2}}^{t} y^T(s)R_4 y(s)\,\mathrm{d}s\\
&\quad - (\frac{\tau_m}{2}\Big)^{-1}\Big(1-\Big(\tau_m\Big)^{-1}\Big(\frac{\tau_m}{2})\Big)\Big(\frac{\tau_m}{2}\Big)\\
&\quad \times \int_{t-\frac{\tau_m}{2}}^{t} y^T(s)R_4 y(s)\,\mathrm{d}s\\
&\le -(\tau_m)^{-1}\Big[\int_{t-\frac{\tau_m}{2}}^{t} y^T(s)\,\mathrm{d}s\Big]R_4\Big[\int_{t-\frac{\tau_m}{2}}^{t} y(s)\,\mathrm{d}s\Big]\\
&\quad - \Big(\frac{\tau_m}{2}\Big)^{-1}\Big(1-(\tau_m)^{-1}\Big(\frac{\tau_m}{2}\Big)\Big)\\
&\quad \times \Big[\int_{t-\frac{\tau_m}{2}}^{t} y^T(s)\,\mathrm{d}s\Big]R_4\Big[\int_{t-\frac{\tau_m}{2}}^{t} y(s)\,\mathrm{d}s\Big]\\
&\le -\Big(\frac{\tau_m}{2}\Big)^{-1}\Big[\int_{t-\frac{\tau_m}{2}}^{t} y^T(s)\,\mathrm{d}s\Big]R_4\Big[\int_{t-\frac{\tau_m}{2}}^{t} y(s)\,\mathrm{d}s\Big]\\
&= -\Big(\frac{\tau_m}{2}\Big)^{-1}\Big[x(t)-x(t-\frac{\tau_m}{2})-\int_{t-\frac{\tau_m}{2}}^{t} g(s)\,\mathrm{d}\omega(s)\Big]^T\\
&\quad \times R_4\Big[x(t)-x\Big(t-\frac{\tau_m}{2}\Big)-\int_{t-\frac{\tau_m}{2}}^{t} g(s)\,\mathrm{d}\omega(s)\Big]\\
&= -\Big(\frac{\tau_m}{2}\Big)^{-1}\Big(\Big[x(t)-x\Big(t-\frac{\tau_m}{2}\Big)\Big]^T\\
&\quad \times R_4\Big[x(t)-x\Big(t-\frac{\tau_m}{2}\Big)\Big]\\
&\quad - 2\Big[x(t)-x\Big(t-\frac{\tau_m}{2}\Big)\Big]^T R_4\Big[\int_{t-\frac{\tau_m}{2}}^{t} g(s)\,\mathrm{d}\omega(s)\Big]\\
&\quad + \Big[\int_{t-\frac{\tau_m}{2}}^{t} g(s)\,\mathrm{d}\omega(s)\Big]^T R_4\Big[\int_{t-\frac{\tau_m}{2}}^{t} g(s)\,\mathrm{d}\omega(s)\Big]\Big).
\end{aligned} \tag{16}
$$

By using

$$
\begin{aligned}
-1 &= -(\tau_M)^{-1}\Big(\frac{\tau_M}{2}\Big) - \Big(1-(\tau_M)^{-1}\Big(\frac{\tau_M}{2}\Big)\Big),\\
-1 &= -(\tau_M-\tau_m)^{-1}(\tau_M-\tau_m)\\
&\quad - \Big(1-(\tau_M-\tau_m)^{-1}(\tau_M-\tau_m)\Big)
\end{aligned}
$$

and Lemma 2, upper bounds of the terms

$$
-\int_{t-\frac{\tau_M}{2}}^{t} y^T(s)R_5 y(s)\,\mathrm{d}s
$$

and

$$
-\int_{t-\tau(t)}^{t-\tau_m} y^T(s)R_6 y(s)\,\mathrm{d}s
$$

are respectively obtained as

$$
\begin{aligned}
&-\int_{t-\frac{\tau_M}{2}}^{t} y^T(s)R_5 y(s)\,\mathrm{d}s\\
&\le -(\frac{\tau_M}{2})^{-1}\Big(\Big[x(t)-x(t-\frac{\tau_M}{2})\Big]^T\\
&\quad \times R_5\Big[x(t)-x\Big(t-\frac{\tau_M}{2}\Big)\Big] - 2\Big[x(t)-x\Big(t-\frac{\tau_M}{2})\Big]^T
\end{aligned}
$$

$$\times R_5\Big[\int_{t-\frac{\tau_M}{2}}^{t} g(s)\,\mathrm{d}\omega(s)\Big] + \Big[\int_{t-\frac{\tau_M}{2}}^{t} g(s)\,\mathrm{d}\omega(s)\Big]^T$$

$$R_5\Big[\int_{t-\frac{\tau_M}{2}}^{t} g(s)\,\mathrm{d}\omega(s)\Big]\Big), \tag{17}$$

$$-\int_{t-\tau(t)}^{t-\tau_m} y^T(s)R_6 y(s)\,\mathrm{d}s$$
$$\leq -(\tau_M-\tau_m)^{-1}\Big(\Big[x(t-\tau_m)-x(t-\tau(t))\Big]^T$$
$$\times R_6\Big[x(t-\tau_m)-x(t-\tau(t))\Big]$$
$$-2\Big[x(t-\tau_m)-x(t-\tau(t))\Big]^T$$
$$\times R_6\Big[\int_{t-\tau(t)}^{t-\tau_m} g(s)\,\mathrm{d}\omega(s)\Big]$$
$$+\Big[\int_{t-\tau(t)}^{t-\tau_m} g(s)\,\mathrm{d}\omega(s)\Big]^T R_6\Big[\int_{t-\tau(t)}^{t-\tau_m} g(s)\,\mathrm{d}\omega(s)\Big]\Big). \tag{18}$$

From (8) and (9), for any matrices N and M we have

$$0 = 2\xi^T(t)N\Big[A(t)x(t)+B(t)x(t-\tau(t))-y(t)\Big], \tag{19}$$

$$0 = 2\xi^T(t)M\Big[C(t)x(t)+D(t)x(t-\tau(t))-g(t)\Big]. \tag{20}$$

Substituting (16)–(18) into (14) and adding (19)–(20), we have

$$LV(x_t,t) \leq \xi^T(t)\Omega\xi(t) + 2(\zeta(t)\,\mathrm{d}\omega(t)), \tag{21}$$

where

$$\xi^T(t) = \Big[x^T(t)\; x^T\Big(t-\tau(t)\Big)\; x^T(t-\frac{\tau_m}{2})\; x^T\Big(t-\tau_m\Big)$$
$$\times x^T\Big(t-\frac{\tau_M}{2}\Big)\; x^T(t-\tau_M)\; y^T(t)\; g^T(t)\Big].$$

$$(\zeta(t)\mathrm{d}\omega(t))$$
$$= \frac{1}{\tau_m}\Big[x(t)-x(t-\frac{\tau_m}{2}\Big]^T R_4\Big[\int_{t-\frac{\tau_m}{2}}^{t} g(s)\,\mathrm{d}\omega(s)\Big]$$
$$+\frac{1}{\tau_M}\Big[x(t)-x(t-\frac{\tau_M}{2}\Big]^T R_5\Big[\int_{t-\frac{\tau_M}{2}}^{t} g(s)\,\mathrm{d}\omega(s)\Big]$$
$$+\frac{1}{\tau_M-\tau_m}\Big[x(t-\tau_m)-x(t-\tau(t))\Big]^T$$
$$\times R_6\Big[\int_{t-\tau(t)}^{t-\tau_m} g(s)\,\mathrm{d}\omega(s)\Big] + x^T Pg(t)\,\mathrm{d}\omega(t).$$

Taking expectation on both sides of (21), we have

$$\mathbb{E}\{LV(x_t,t)\} \leq \mathbb{E}\{\xi^T(t)\Omega\xi(t)\}. \tag{22}$$

Now we proceed to prove that the system (1) is stochastically stable by using a similar method as Chen *et al.* (2004). Set $\lambda_0 = \lambda_{\min}(-\Omega)$, then $\lambda_0 > 0$ follows from (12). From (22) and by using Itō's formula (Shi and Boukas, 1997),

$$\mathbb{E}V(t) - \mathbb{E}V(\tau_M) = \mathbb{E}\int_{\tau_M}^{t} LV(s)\,\mathrm{d}s$$
$$\leq -\lambda_0\mathbb{E}\int_{\tau_M}^{t} \|x(s)\|^2\,\mathrm{d}s.$$

It follows that

$$\mathbb{E}\int_{\tau_M}^{t} \|x(s)\|^2\,\mathrm{d}s \leq \frac{1}{\lambda_0}\mathbb{E}V(\tau_M).$$

For the system (1), following Chen *et al.* (2004) it is easy to prove that there exists a positive scalar $c_1 \geq 1$ such that

$$\mathbb{E}\|x(s)\|^2 \leq c_1 \sup_{s\in[-\tau_M,\,0]} \mathbb{E}\|\phi(s)\|^2, \qquad t\in[0,\tau_M].$$

Therefore, by the definitions of $V(t)$ and $x(t)$, there always exists a scalar $c > 0$ such that

$$\lim_{T\longrightarrow\infty} \mathbb{E}\int_{0}^{T} \|x(s)\|^2\,\mathrm{d}s \leq c \sup_{s\in[-\tau_M,0]} \mathbb{E}\|\phi(s)\|^2,$$

which means that the system (1) is stochastically stable by Definition 1. This completes the proof.

Remark 1. Theorem 1 provides delay-dependent stability criteria for the stochastic system (1). Such criteria are derived based on the assumption that the time-varying delay is differentiable and the value of μ is known. The conditions in Theorem 1 are formulated in terms of the solvability of LMIs (Boyd *et al.*, 1994) and can be easily solved using MATLAB LMI Control Toolbox. It is worth noting that, by applying convex optimization algorithms, we can conclude that the maximum allowable upper bound of the interval time-varying delay, that is, τ_M, guarantees the feasibility of the presented LMIs.

We can obtain the maximum allowable upper bound τ_M by solving the following optimization problem:

$$\begin{cases} \text{Max } \tau_M \\ \text{subject to the LMIs} \quad P > 0, \\ \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix} > 0, \begin{bmatrix} Q_4 & Q_5 \\ Q_5^T & Q_6 \end{bmatrix} > 0, \\ R_k > 0\,(k = 1\ldots,6), N_i, M_i\,(i = 1,\ldots,8) \\ \text{and } \Omega < 0. \end{cases}$$

Remark 2. In this paper, in order to derive the stability criterion, we employ a new Lyapunov–Krasovskii functional (13), which is mainly based on the information about $\tau_m/2$, $\tau_M/2$ and $(\tau_M - \tau_m)$; some suitable free-weight matrices are also introduced.

Theorem 2. *For given scalars τ_M, τ_m and μ, the equilibrium point of the stochastic system (1) subject to the polytopic uncertainty (3) is asymptotically stable in the mean square if there exist matrices $P > 0$,*

$$\begin{bmatrix} Q_1^{(i)} & Q_2^{(i)} \\ Q_2^{(i)T} & Q_3^{(i)} \end{bmatrix} > 0,$$

$$\begin{bmatrix} Q_4^{(i)} & Q_5^{(i)} \\ Q_5^{(i)T} & Q_6^{(i)} \end{bmatrix} > 0,$$

(i=1,2,...,r), $R_k > 0$, $k = 1\dots,6$ such that for any matrices N_i and $M_i, (i = 1,\dots,8)$ the following LMI is feasible:

$$\bar{\Omega}_i < 0, \tag{23}$$

for $i = 1, 2, \dots, r$, where

$$\bar{\Omega}_i = \begin{bmatrix} \bar{\Omega}_{11,i} & \bar{\Omega}_{12,i} & \dots & \bar{\Omega}_{18,i} \\ * & \bar{\Omega}_{22,i} & \dots & \bar{\Omega}_{28,i} \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & \bar{\Omega}_{88,i} \end{bmatrix} < 0,$$

with

$$\begin{aligned} \bar{\Omega}_{11,i} &= Q_4^{(i)} + Q_1^{(i)} + R_1 + R_2 + R_3 - R_4 - R_5 \\ &\quad + N_1 A^{(i)} + A^{(i)T} N_1^T + M_1 C^{(i)} + C^{(i)T} M_1^T, \end{aligned}$$

$$\bar{\Omega}_{12,i} = N_1 B^{(i)} + A^{(i)T} N_2^T + M_1 D^{(i)} + C^{(i)T} M_2^T,$$

$$\bar{\Omega}_{13,i} = Q_2^{(i)} + R_4 + A^{(i)T} N_3^T + C^{(i)T} M_3^T,$$

$$\bar{\Omega}_{14,i} = A^{(i)T} N_4^T + C^{(i)T} M_4^T,$$

$$\bar{\Omega}_{15,i} = Q_5^{(i)} + R_5 + A^{(i)T} N_5^T + C^{(i)T} M_5^T,$$

$$\bar{\Omega}_{16,i} = A^{(i)T} N_6^T + C^{(i)T} M_6^T,$$

$$\bar{\Omega}_{17,i} = P + A^{(i)T} N_7^T + C^{(i)T} M_7^T - N_1,$$

$$\bar{\Omega}_{18,i} = A^{(i)T} N_8^T + C^{(i)T} M_8^T - M_1,$$

$$\begin{aligned} \bar{\Omega}_{22,i} &= -R_6 - (1-\mu) R_1 + N_2 B^{(i)} + B^{(i)T} N_2^T \\ &\quad + M_2 D^{(i)} + D^{(i)T} M_2^T, \end{aligned}$$

$$\bar{\Omega}_{23,i} = B^{(i)T} N_3^T + D^{(i)T} M_3^T,$$

$$\bar{\Omega}_{24,i} = R_6 + B^{(i)T} N_4^T + D^{(i)T} M_4^T,$$

$$\bar{\Omega}_{25,i} = B^{(i)T} N_5^T + D^{(i)T} M_5^T,$$
$$\bar{\Omega}_{26,i} = B^{(i)T} N_6^T + D^{(i)T} M_6^T,$$

$$\bar{\Omega}_{27,i} = B^{(i)T} N_7^T + D^{(i)T} M_7^T - N_2,$$

$$\bar{\Omega}_{28,i} = B^{(i)T} N_8^T + D^{(i)T} M_8^T - M_2,$$

$$\bar{\Omega}_{33,i} = Q_3^{(i)} - Q_1^{(i)} - R_4, \quad \bar{\Omega}_{34,i} = -Q_2^{(i)},$$

$$\bar{\Omega}_{35,i} = \Omega_{36,i} = 0, \quad \bar{\Omega}_{37,i} = -N_3, \quad \bar{\Omega}_{38,i} = -M_3,$$

$$\bar{\Omega}_{44,i} = -Q_3^{(i)} - R_2 - R_6, \quad \bar{\Omega}_{45,i} = \bar{\Omega}_{4,6,i} = 0,$$

$$\bar{\Omega}_{47,i} = -N_4, \quad \bar{\Omega}_{48,i} = -M_4,$$

$$\bar{\Omega}_{55,i} = Q_6^{(i)} - Q_4^{(i)} - R_5,$$

$$\bar{\Omega}_{56,i} = -Q_5^{(i)}, \quad \bar{\Omega}_{57,i} = -N_5, \quad \bar{\Omega}_{58,i} = -M_5,$$

$$\bar{\Omega}_{66,i} = -Q_6^{(i)} - R_3, \ \bar{\Omega}_{67,i} = -N_6, \quad \bar{\Omega}_{68,i} = -M_6,$$

$$\bar{\Omega}_{77,i} = \left(\frac{\tau_m}{2}\right)^2 R_4 + \left(\frac{\tau_M}{2}\right)^2 R_5 + \delta^2 R_6 - N_7 - N_7^T,$$

$$\bar{\Omega}_{78,i} = -N_8^T - M_7, \quad \bar{\Omega}_{88,i} = P - M_8 - M_8^T,$$

Proof. By Schur's complement, the matrix inequality (23) implies

$$\sum_{i=1}^{r} \lambda_i \bar{\Omega}_i < 0$$

or $\Omega < 0$, where

$$A = \sum_{i=1}^{r} \lambda_i A^{(i)}, \qquad B = \sum_{i=1}^{r} \lambda_i B^{(i)},$$
$$C = \sum_{i=1}^{r} \lambda_i C^{(i)}, \qquad D = \sum_{i=1}^{r} \lambda_i D^{(i)},$$
$$Q = \sum_{i=1}^{r} \lambda_i Q_j^{(i)},$$

$j = 1, 2, \dots, 6$. This completes the proof.

Similarly to the proof of Theorem 1, we can establish the following result. ■

Theorem 3. *For given scalars τ_M, τ_m and μ, the equilibrium point of the stochastic system (1) subject to the linear fractional norm-bounded uncertainty (4) is robustly asymptotically stable in the mean square if there exist scalars $\epsilon_1 > 0$, $\epsilon_2 > 0$, matrices $P > 0$,*

$$\begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix} > 0, \quad \begin{bmatrix} Q_4 & Q_5 \\ Q_5^T & Q_6 \end{bmatrix} > 0,$$

$R_k > 0$, $k = 1\dots,6$, such that for any matrices N_i and M_i $(i = 1,\dots,8)$, the following LMI is feasible:

$$\begin{bmatrix} \Omega & NH & \epsilon_1 S_1^T & MH & \epsilon_2 S_2^T \\ * & -\epsilon_1 I & \epsilon_1 J^T & 0 & 0 \\ * & * & -\epsilon_1 I & 0 & 0 \\ * & * & * & -\epsilon_2 I & \epsilon_2 J^T \\ * & * & * & * & -\epsilon_2 I \end{bmatrix} < 0, \tag{24}$$

Ω being defined in (12).

Proof. Assume that the inequality (24) holds. It can be seen that (24) can be rewritten as

$$\Psi = \begin{bmatrix} \Omega & NH & \epsilon_1 S_1^T & MH & \epsilon_2 S_2^T \\ * & -\epsilon_1 I & \epsilon_1 J^T & 0 & 0 \\ * & * & -\epsilon_1 I & 0 & 0 \\ * & * & * & -\epsilon_2 I & \epsilon_2 J^T \\ * & * & * & * & -\epsilon_2 I \end{bmatrix} < 0,$$

$$S_1 = \begin{bmatrix} E & E_1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$

$$S_2 = \begin{bmatrix} E_2 & E_3 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

Thus,

$$\Psi = \Omega + N\Lambda(t)S_1 + S_1^T\Lambda(t)N^T + M\Lambda(t)S_2 + S_2^T\Lambda(t)M^T < 0$$

holds according to Lemma 3. It can be verified that Ψ is exactly the same as Ω of (12) when A, B, C and D are replaced respectively by $A + H\Lambda(t)E$, $B + H\Lambda(t)E_1$, $C + H\Lambda(t)E_2$ and $D + H\Lambda(t)E_3$ in (12).

Now, without considering the terms $\tau_M/2$ and $\tau_m/2$ in the Lyapunov–Krasovskii functional, the corresponding result is discussed in the following corollary. ■

Corollary 1. *For given scalars τ_M, τ_m and μ, the equilibrium point of the stochastic system (1) subject to the linear fractional norm-bounded uncertainty (4) is robustly asymptotically stable in the mean square if there exist scalars $\epsilon_1 > 0$, $\epsilon_2 > 0$, matrices $P > 0$, $R_1 > 0$, $k = 1, 2, 3$, $R_6 > 0$ for any matrices $\bar{N}_i$ and $\bar{M}_i$ $(i = 1, \dots, 6)$ the following LMI is feasible:*

$$\begin{bmatrix} \Xi & \bar{N}H & \epsilon_1 \bar{S}_1^T & \bar{M}H & \epsilon_2 \bar{S}_2^T \\ * & -\epsilon_1 I & \epsilon_1 J^T & 0 & 0 \\ * & * & -\epsilon_1 I & 0 & 0 \\ * & * & * & -\epsilon_2 I & \epsilon_2 J^T \\ * & * & * & * & -\epsilon_2 I \end{bmatrix} < 0, \tag{25}$$

where $\Xi = (\Xi_{n\times m})_{6\times 6}$ with

$$\Xi_{11} = R_1 + R_2 + R_3 + N_1 A + A^T N_1^T + M_1 C + C^T M_1^T,$$

$$\Xi_{12} = N_1 B + A^T N_2^T + M_1 D + C^T M_2^T,$$

$$\Xi_{13} = A^T N_3^T + C^T M_3^T,$$

$$\Xi_{14} = A^T N_4^T + C^T M_4^T,$$

$$\Xi_{15} = P + A^T N_5^T + C^T M_5^T - N_1,$$

$$\Xi_{16} = A^T N_6^T + C^T M_6^T - M_1,$$

$$\Xi_{22} = -(1-\mu)R_1 - R_6 + N_2 B + B^T N_2^T + M_2 D + D^T M_2^T,$$

$$\Xi_{23} = B^T N_3^T + D^T M_3^T + R_6,$$

$$\Xi_{24} = B^T N_4^T + D^T M_4^T,$$

$$\Xi_{25} = B^T N_5^T + D^T M_5^T - N_2,$$

$$\Xi_{26} = B^T N_6^T + D^T M_6^T - M_2,$$

$$\Xi_{33} = -R_2 - R_6, \quad \Omega_{34} = 0, \quad \Xi_{35} = -N_3,$$

$$\Xi_{36} = -M_3,$$

$$\Omega_{44} = -R_3,\ \Xi_{45} - N_4, \quad \Xi_{46} = -M_4,,$$

$$\Xi_{55} = \delta^2 R_6 - N_5 - N_5^T, \quad \Xi_{56} = -N_6^T - M_5,$$

$$\Omega_{66} = P - M_6 - M_6^T,$$

$$\bar{N} = \begin{bmatrix} N_1^T & N_2^T & N_3^T & N_4^T & N_5^T & N_6^T \end{bmatrix}^T,$$

$$\bar{M} = \begin{bmatrix} M_1^T & M_2^T & M_3^T & M_4^T & M_5^T & M_6^T \end{bmatrix}^T,$$

$$\bar{S}_1 = \begin{bmatrix} E & E_1 & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$

$$\bar{S}_2 = \begin{bmatrix} E_2 & E_3 & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

Proof. Consider the Lyapunov–Krasovskii functional

$$V(x_t, t) = V_1(x_t, t) + V_2(x_t, t) + V_3(x_t, t),$$

where

$$V_1(x_t, t) = x^T(t) P x(t),$$

$$V_2(x_t, t) = \int_{t-\tau(t)}^{t} x^T(s) R_1 x(s)\,\mathrm{d}s + \int_{t-\tau_m}^{t} x^T(s) R_2 x(s)\,\mathrm{d}s + \int_{t-\tau_M}^{t} x^T(s) R_3 x(s)\,\mathrm{d}s,$$

$$V_3(x_t, t) = \delta \int_{-\tau_M}^{-\tau_m} \mathrm{d}s \int_{t+s}^{t} y^T(\theta) R_6 y(\theta)\,\mathrm{d}\theta.$$

Then, it can be obtained by Itō's formula that

$$\mathrm{d}V(x_t, t) = LV(x_t, t)\,\mathrm{d}t + 2x^T(t) P g(t)\,\mathrm{d}\omega(t),$$

where

$$LV_1(x_t,t) = 2x^T(t)Py(t) + g^T(t)Pg(t),$$

$$\begin{aligned} LV_2(x_t,t) &\le x^T(t)R_1x(t) \\ &\quad - (1-\mu)x^T(t-\tau(t))R_1x(t-\tau(t))x^T(t)R_2x(t) \\ &\quad - x^T(t-\tau_m)R_2x(t-\tau_m) + x^T(t)R_3x(t) \\ &\quad - x^T(t-\tau_M)R_3x(t-\tau_M), \end{aligned}$$

$$\begin{aligned} LV_3(x_t,t) &= y^T(t)\delta^2R_6y(t) \\ &\quad - \delta\int_{t-\tau_M}^{t-\tau_m} y^T(s)R_6y(s)\,\mathrm{d}s \end{aligned}$$

$$\begin{aligned} LV_3(x_t,t) &\le y^T(t)\delta^2R_6y(t) \\ &\quad - \delta\int_{t-\tau(t)}^{t-\tau_m} y^T(s)R_6y(s)\,\mathrm{d}s. \end{aligned}$$

■

From (8) and (9) for any matrices N and M we have

$$0 = 2\xi_1^T(t)N\Big[A(t)x(t) + B(t)x(t-\tau(t)) - y(t)\Big],$$
$$0 = 2\xi_1^T(t)M\Big[C(t)x(t) + D(t)x(t-\tau(t)) - g(t)\Big].$$

where

$$\xi_1^T(t) = \Big[x^T(t)\ x^T(t-\tau(t))\ x^T(t-\tau_m)\ x^T(t-\tau_M)\ y^T(t)\ g^T(t)\Big].$$

Following similar arguments as in the proof of Theorem 3, we can obtain the desired result immediately, and hence the detailed proof is omitted.

Remark 3. It is easy to see that, while setting $J = 0$, the linear fractional norm-bounded uncertainty reduces to a routine norm-bounded uncertainty. Therefore, one can easily derive a corresponding result for the routine norm-bounded uncertainty from Theorem 3.

4. Numerical examples

In this section, we will give four examples to show the effectiveness of the established theoretical results.

4.1. Example. Consider the system (1) and (3) with the following matrices:

$$A^1 = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \qquad A^2 = \begin{bmatrix} -2 & -1 \\ 0 & -2 \end{bmatrix},$$
$$B^1 = \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix}, \qquad B^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$
$$C^1 = \begin{bmatrix} -0.2 & 0 \\ 0 & -0.2 \end{bmatrix}, \qquad C^2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix},$$
$$D^1 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.7 \end{bmatrix}, \qquad D^2 = \begin{bmatrix} 0.2 & 0 \\ 0 & -0.1 \end{bmatrix}.$$

It was reported by Li *et al.* (2008) that the above system is robustly asymptotically stable in the mean square when the constant delay $\tau_M = 5$. However, by Theorem 2 and using Matlab LMI Toolbox for $\mu = 0$, $\tau_m = 0$, it is found that the equilibrium point of an uncertain stochastic system is asymptotically stable in the mean square for any constant allowable upper bounds. This implies that our stability criterion gives a less conservative result than the methods discussed by Li *et al.* (2008).

4.2. Example. Consider the system (1) and (3) with the following matrices:

$$A^1 = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \qquad A^2 = \begin{bmatrix} -1.5 & -1 \\ 0 & -2 \end{bmatrix},$$
$$B^1 = \begin{bmatrix} 1 & 0 \\ -0.1 & 0.85 \end{bmatrix}, \qquad B^2 = \begin{bmatrix} -1 & 1 \\ 1 & 0.85 \end{bmatrix},$$
$$C^1 = \begin{bmatrix} -0.2 & 0 \\ 0 & -0.2 \end{bmatrix}, \qquad C^2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix},$$
$$D^1 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.7 \end{bmatrix}, \qquad D^2 = \begin{bmatrix} 0.2 & 0 \\ 0 & -0.1 \end{bmatrix}.$$

It was reported by Li *et al.* (2008) that the above system is robustly asymptotically stable in the mean square for a constant delay of $\tau_M = 2.4019$. However, by our Theorem 2 and using Matlab LMI Toolbox for $\mu = 0$, $\tau_m = 0$, it is found that the equilibrium point of an uncertain stochastic system is asymptotically stable in the mean square for any constant allowable upper bounds. This implies that our stability criterion gives a less conservative result than the methods discussed by Li *et al.* (2008).

4.3. Example. Consider the uncertain stochastic system with time-varying delay described by

$$\begin{aligned} &\mathrm{d}x(t) \\ &= \Big[(A+\Delta A(t))x(t) + (B+\Delta B(t))x(t-\tau(t))\Big]\,\mathrm{d}t \\ &\quad + \Big[\Delta C(t)x(t) + \Delta D(t)x(t-\tau(t))\Big]\,\mathrm{d}w(t), \end{aligned}$$

where

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix},$$

$$\|\Delta A(t)\| \leq 0.2, \qquad \|\Delta B(t)\| \leq 0.2,$$

$$\|\Delta C(t)\| \leq 0.2, \qquad \|\Delta D(t)\| \leq 0.2.$$

Then take the uncertainties as described by (4) as follows

$$H = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix},$$

$$E = E_1 = E_2 = E_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

According to Theorem 3, the upper bounds on the time delay to guarantee that the system is robustly stochastically stable are listed in Tables 1 and 2. Table 1 also lists the upper bounds obtained from the criterion by Miyamura and Aihara (2004), Yan *et al.* (2009), Zhang *et al.* (2009) and Zhang *et al.* (2008). Hence the method proposed in this paper gives less conservative results than the existing results found in the literature (Miyamura and Aihara, 2004; Yan *et al.*, 2009; Zhang *et al.*, 2009; Zhang *et al.*, 2008).

4.4. Example. Consider the uncertain stochastic system with time-varying delay described by

$$\begin{aligned} &\mathrm{d}x(t) \\ &= \Big[(A + \Delta A(t))x(t) + (B + \Delta B(t))x(t - \tau(t))\Big]\,\mathrm{d}t \\ &\quad + \Big[Cx(t) + Dx(t - \tau(t))\Big]\mathrm{d}w(t), \end{aligned}$$

where

$$A = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ -0.5 & -1 \end{bmatrix},$$

$$\|\Delta A(t)\| \leq 0.1, \quad \|\Delta B(t)\| \leq 0.1,$$

$$C = D = \mathrm{diag}(\sqrt{0.1}, \sqrt{0.1}),$$

$$E = E_1 = \mathrm{diag}(0.1, 0.1),$$

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

According to Theorem 3, the upper bounds on the time delay to guarantee that the system is robustly stochastically stable are listed in Table 3, which also gives the upper bounds obtained from the criterion by Yue and Han (2005) as well as He *et al.* (2010). Hence the method proposed in this paper gives less conservative results than the results found in the literature (Yue and Han, 2005; He *et al.*, 2010).

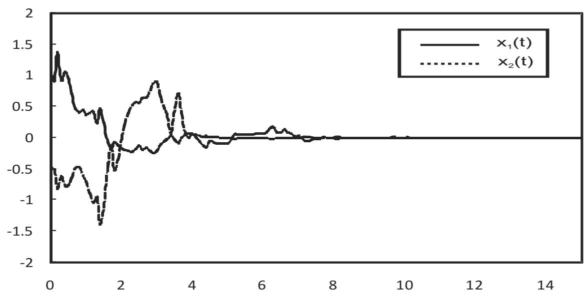

Fig. 1. State trajectories of x_1 and x_2 for Example 4.3 with different initial conditions $(1, -0.5)$.

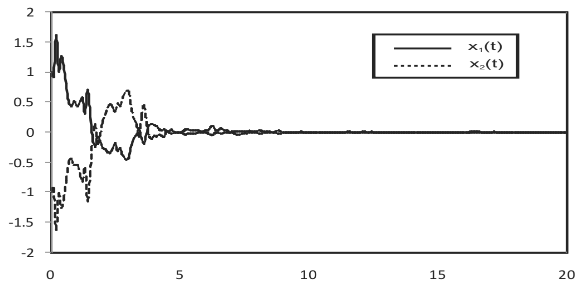

Fig. 2. State trajectories of x_1 and x_2 for Example 4.4 with different initial conditions $(1, -1)$.

5. Conclusion

Several sufficient conditions ensuring an LMI optimization problem of delay-dependent robust stability criteria for stochastic systems with polytopic and linear fractional uncertainties have been proposed. By choosing a suitable Lyapunov–Krasovskii functional and the free-weighting matrix method, some less conservative stability criteria have been obtained. The restriction that the derivative of the time-varying delay is less than one has been removed. In numerical comparisons, significant improvements over the recent existing results have been observed.

Acknowledgment

The authors are very much thankful to the referees for their valuable comments and suggestions for improving this manuscript.

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Table 1. Maximum allowable upper bound of h_2 with different μ, $h_1 = 0$ and $J = 0$.

μ	0	0.3	0.5	0.9	1
Miyamura and Aihara, 2004	0.3555	–	–	–	–
Yan *et al.*, 2009	1.0660	0.7288	0.5252	0.1489	–
Zhang *et al.*, 2009	–	1.2950	1.1006	0.9434	0.9424
Zhang *et al.*, 2008	1.8684	–	1.1304	0.9402	–
Theorem 3	2.4670	1.6880	1.3428	0.9656	0.9428

Table 2. Maximum allowable upper bound of h_2 with different μ, h_1 and J.

μ	0	0.3	0.5	1	2
$J = 0, h_1 = 0.5$	2.1952	1.5607	1.2709	0.9359	0.9359
$J = 0, h_1 = 1$	2.1929	1.6065	1.3593	1.3149	1.3149
$J = 0.5, h_1 = 0.5$	1.6603	1.2994	1.0947	0.8828	0.8828

Table 3. Maximum allowable upper bound of h_2. $(J = 0, h_1 = 0), (J = 0, h_1 = 0.5), (J = 0, h_1 = 1), (J = 0.5, h_1 = 0.5)$ are denoted by $a*, b*, c*, d*$, respectively.

μ	0	0.3	0.5	0.9	1	2
Yue and Han, 2005	–	–	0.8502	0.4606	0.5001	–
He *et al.*, 2010	–	–	0.9010	0.7312	0.7312	–
$a*$	8.9548	2.4868	1.7940	0.9849	0.8264	0.8264
$b*$	8.9585	2.4955	1.8055	1.1016	1.1016	1.1016
$c*$	8.9725	2.5354	1.8602	1.4679	1.4679	1.4679
$d*$	6.6322	2.4472	1.8017	1.1188	1.1188	1.1188

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Received: 2 August 2010
Revised: 24 January 2011
Re-revised: 10 October 2011