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THE INFLUENCE OF TRANSVERSE SHEARING AND ROTATORY INERTIA ON THE VIBRATIONS OF FIBROUS COMPOSITE BEAM

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The aim of the paper was to determine the influence of transverse shear deformation and rotatory inertia on the natural frequencies and on the values of displacements of beams made of fibrous composites reinforced with layers of long fibres. It was assumed that the matrix of the composite beam possesses linear elastic and transversally isotropic properties. Moreover a reinforcement in the form of layers composed of long fibres symmetrically located in the cross section was considered. In order to describe the displacement and strain state of the matrix, Timoshenko theory was assumed. Using the complete analytical solutions obtained in the paper the accuracy analysis of results was performed in comparison with the theory of Bernoulli beam.

Keywords: dynamics of composite beam, transverse shear effect

1. INTRODUCTION

The fibrous composites are playing an increasing role as construction materials in a wide variety of applications. They are used in civil engineering and chemical, aerospace and shipbuilding industries. The composites composed of the matrix reinforced with long fibres (see Fig. 1), are characterized by high strength capability, lightness and significant transverse non-homogeneity.

Technical application of fibrous composite materials needs to take into considerations their shear deformation vulnerability in order to carry out the strength calculations [1, 2, 3, 6, 8, 10, 12]. Theoretical and experimental investigations show that the use of the classical assumption *about the non-deformability of the normal section* makes the values of the calculated displacements (deflections) lower. On the other hand it makes higher both the critical loads and the natural frequencies [10]. The errors connected with

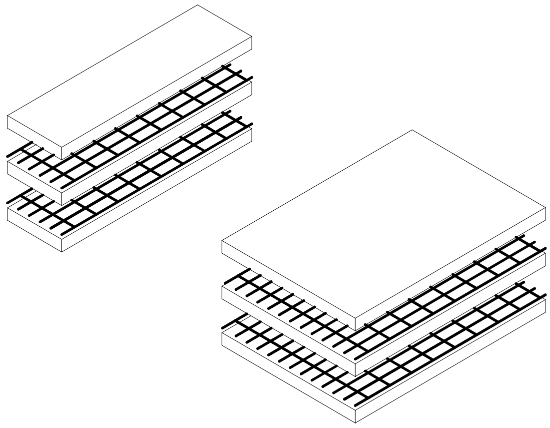

Fig. 1 Construction element reinforced with the layers of long fibres.

neglecting the influence of shear deformation on the vibrations of fibrous composite beam result not only from the relation h/l and the load type but also from the relation E^r/E (Young's modulus of the fibres to Young's modulus of the matrix) and from the fibre density and its location in cross-section [1, 2].

The aim of this study is to determine the influence of the transverse shear deformations and rotatory inertia on the natural frequencies and on the value of displacement field of beams made of fibrous composites reinforced by layers of long fibres.

The composite can be defined as a material consisting of at least two components. The first component constitutes the main phase (matrix). The second one, immersed in the matrix, constitutes the fibrous phase (2nd phase). The fibrous phase consists of any amount of *families*.

The *family* is a group of long fibres laying on the planes parallel to the neutral axis of the beam. The fibres belonging to the *family* are thin, straight and so densely packed that the continuous model can be assumed. We assume that the two phases meet the continuity criteria both in the sense of displacements and strains. As a consequence of the above assumptions we can take into consideration a theoretical model in the form of continuous double-phase medium. In such model the *continuum* of the 1st phase is immersed in the *continuum* of the 2nd phase. The idea of the model presented herein was taken from the works of Holnicki-Szulc [4] and Świtka [13].

The dynamic problem of beams and plates made of transversally isotropic material has been investigated by a number of authors e.g. Nowacki [9], Kączkowski [7], Szcześniak [11, 12], Jemielita [5]. For a wide literature review of the problem see [5, 7, 12].

2. FORMULATION OF THE PROBLEM

Let we analyse the transverse vibration problem of fibrous composite prismatic beam (cross section $b \times h$) in xz plane (see Fig. 2). Applying the Timoshenko theory, displacements of any point of the cross section can be described using the equations

$$\begin{aligned} &u_x(x,z,t) = u(x,t) + z\psi(x,t); \\ &u_y(x,z,t) = 0; \quad u_z(x,z,t) = w(x,t); \end{aligned} \tag{1}$$

where u and w denote respectively horizontal and vertical components of the displacement vector for points laying on the neutral axis. The ψ is the angle of rotation of the cross section.

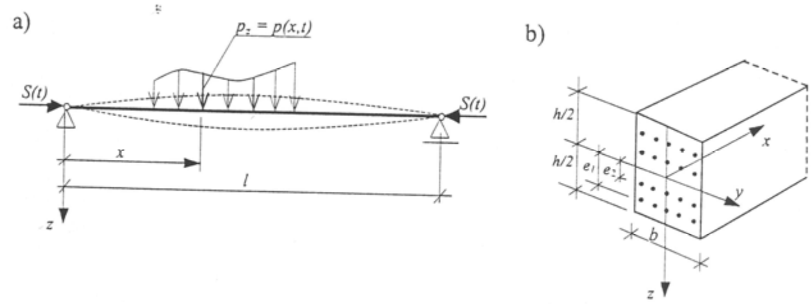

Fig. 2 Simply supported beam loaded by transverse load $p(x,t)$ and by axial load $S(t)$: a) model b) example of the symmetric reinforcement of the cross section with two pair of long fibre *families.*

The strains of the beam are given by

$$\varepsilon_x = \frac{\partial u_x}{\partial x} = \frac{\partial u}{\partial x} + z\frac{\partial \psi}{\partial x}; \qquad \gamma_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = \psi + \frac{\partial w}{\partial x}. \tag{2}$$

In this work we assume that the matrix is made of the transversally isotropic perfectly elastic material obeying Hooke's relations

$$\sigma_x = E\varepsilon_x; \qquad \tau_{xz} = G'\gamma_{xz}. \tag{3}$$

The fibre phase (reinforcement) consists of symmetrically located vertical layers of fibrous *families*. Each *family* consists of continuous, straight fibres to coincide with x-axis and laying in planes $z = z^r\,(r = 1,2,3...)$, $z^r \in (-h/2, h/2)$. The fibres of each *family* are thin, densely packed and support only axial loads. We assume that the fibres are made of linear elastic material which much higher strength coefficients in comparison with the coefficients of the matrix. The force in the r-th *family* is given by

$$S_x^r = j^r E^r A^r \left(\varepsilon_x^r - \varepsilon_x^{or}\right), \tag{4}$$

where $\varepsilon_x^r, \varepsilon_x^{or}, E^r, A^r$ and j^r mean respectively the unit elongation, the initial distortion, the Young's modulus, the cross-section area of the fibre and the amount of fibres in the *family*.

We assume in the paper the perfect adherence between the matrix surface and the fibres surfaces so the resultant internal forces in the composite beam can be calculated as a sum of forces in the beam's components.

$$N = \int_A \sigma_x dA + \sum_r S_x^r; \quad M = \int_A \sigma_x z dA + \sum_r S_x^r z^r; \quad T = \int_A \tau_{xz} dA. \tag{5}$$

Making use of equations (2), (3), (4) and assuming the amount of „i" equal pairs of fibre *families* to be symmetrically located in the cross-section at the distances $z^r = \pm e_1, \pm e_2, ... \pm e_i; \;\; e_i \in (0, h/2)$, and also neglecting the initial elongation of fibres, the equations (5) take the form

$$N = B\frac{\partial u}{\partial x}; \quad M = D\frac{\partial \psi}{\partial x}; \quad T = G' Ak\left(\psi + \frac{\partial w}{\partial x}\right), \tag{6}$$

where

$$B = EA + 2ij^r E^r A^r \qquad D = EJ + 2j^r E^r A^r \sum_i e_i^2 \tag{7}$$

represent respectively tension/compression stiffness of the beam and its bending stiffness [3]. Moreover $A = bh$; $J = bh^3/12$; G' - shear modulus of the matrix, $k = 5/6$.

We formulate the equations of motion of a straight prismatic beam based on the Hamilton principle. The assumption that the variations of displacements for the times t_0 and t_1 are equal to zero, gives the following variational equation

$$\int_{t_0}^{t_1}\left\{\int_0^l\left[-\left(\frac{\partial N}{\partial x}-\rho A\ddot{u}\right)\delta u-\left(\frac{\partial M}{\partial x}-T-\rho J\ddot{\psi}\right)\delta\psi-\left(\frac{\partial T}{\partial x}-S\frac{\partial^2 w}{\partial x^2}+p_z-\rho A\ddot{w}\right)\delta w\right]dx+ \right.$$
$$\left. +N\delta u\big|_0^l+M\delta\psi\big|_0^l+\left(T-S\frac{\partial w}{\partial x}\right)\delta w\big|_0^l\right\}dt=0, \tag{8}$$

to be satisfied for any value of functions δu, $\delta\psi$ and δw. In the above expression $p_z = p(x,t)$ denotes the external transverse distributed load, $S(t)$ denotes the external axial force, the symbol ρ denotes density and $\rho J\ddot{\psi}$ is the moment of rotatory inertia. Dots denote differentiation with respect to the time coordinate t.

The equation (8) implicates the system of three equations of motion

$$\begin{aligned} &\frac{\partial \mathrm{N}}{\partial \mathrm{x}} - \rho A\ddot{u} = 0, \\ &\frac{\partial M}{\partial x} - T - \rho J\ddot{\psi} = 0, \\ &\frac{\partial}{\partial x}\left(T - S\frac{\partial w}{\partial x}\right) - \rho A\ddot{w} = -p(x,t), \end{aligned} \tag{9}$$

and the appropriate natural boundary conditions. Analysing uncoupled problem of axial and transverse vibrations we obtain in the first case two combinations of possible conditions for each boundary. In the case of pure transverse vibrations the number of combinations of boundary conditions is equal to four. The initial conditions correspond with displacements u, ψ and w, and their velocities.

3. THE INFLUENCE OF THE ROTATORY INERTIA ON THE NATURAL FREQUENCIES

First of all let we determine the order of magnitude of the influence of the cross section rotatory inertia $\rho J\ddot{\psi}$ on the transverse natural frequencies of composite beam.

Using the equations of motion (9) we obtain, taking into consideration the constitutive equations (6) and eliminating the variable ψ, the following differential equation describing the eigenvalue problem

$$D\frac{\partial^4 w}{\partial x^4} + \rho A\ddot{w} - \rho J\frac{\partial^2 \ddot{w}}{\partial x^2} - \frac{\rho D}{G'k}\frac{\partial^2 \ddot{w}}{\partial x^2} + \frac{\rho^2 J}{G'k}\ddddot{w} = 0. \tag{10}$$

The 3rd and 5th component in equation (10) respects the influence of the roratory inertia and the 4th component respects the influence of the transverse shear deformation.

In the case of a simply supported beam, the equation (10) will be satisfied if

$$w(x,t) = A_n e^{-i\omega_n t} \sin \alpha_n x, \qquad n = 1,2,3,... \tag{11}$$

where A_n denotes the deflection amplitude, ω_n means the natural frequency and $\alpha_n = \frac{n\pi}{l}$.

Substituting (11) into (10) gives

$$\beta^2 \alpha_n^4 - \omega_n^2 - \frac{J}{A}\alpha_n^2 \omega_n^2 - \frac{D}{G^{'}kA}\alpha_n^2 \omega_n^2 + \frac{\rho J}{G^{'}kA}\omega_n^4 = 0, \tag{12}$$

where $\beta^2 = \frac{D}{\rho A}$.

If we take into consideration only the first two components in the equation (12) then we will obtain the formula to calculate the natural frequencies of a slender beam obeying the Bernoulli hypothesis

$$\omega_n^2 = \beta^2 \alpha_n^4, \qquad n = 1,2,3,... \tag{13}$$

In the expression (13) the influence of the shear deformations and the rotatory inertia effect wasn't taken into account.

Substituting (13) into the last component of (12), as a first approximation, we notice that this component can be treated as a small 2nd order term with respect to another components so it can be neglected [9].

Making use of the above remarks the equation (12) gives

$$\omega_n = \frac{\beta \alpha_n^2}{\sqrt{1 + \frac{J}{A}\alpha_n^2 \left(1 + \frac{D}{G^{'}kJ}\right)}} \approx \beta \alpha_n^2 \left[1 - \frac{1}{2}\frac{J}{A}\alpha_n^2 \left(1 + \frac{D}{G^{'}kJ}\right)\right], \tag{14}$$

If we take in (14) the value of inertia J to be equal zero we will obtain the formula to calculate the natural frequencies respecting only the influence of the shear deformation.

$$\omega_{np} = \frac{\beta \alpha_n^2}{\sqrt{1 + n^2 \pi^2 \zeta}} \approx \beta \alpha_n^2 \left(1 - \frac{1}{2} n^2 \pi^2 \zeta\right), \tag{15}$$

Taking $G^{'} = \infty$ we obtain the expression

$$\omega_{nb} \approx \beta \alpha_n^2 \left(1 - \frac{1}{2}\frac{J}{A}\alpha_n^2\right), \tag{16}$$

respecting only the influence of the rotatory inertia.

Let we apply the following coefficient in (15)

$$\zeta = \frac{D}{G^{'}kAl^2} \tag{17}$$

It characterises the shear deformability of the composite beam [3]. By using $(7)_2$ and taking $E/G^{'} = 2(1+\nu)$, the coefficient ζ becomes

$$\zeta = \frac{(1+\nu)h^2}{5l^2}\left(1 + 24n^r\mu^r\sum_i \frac{e_i^2}{h^2}\right). \tag{18}$$

The equation (18) shows that the coefficient ζ strongly depends on the parameters h/l, $n^r = E^r/E$ (Young's modulus of the fibres to Young's modulus of the matrix), $\mu^r = j^r A^r/A$ (density of fibre packages in the r-th *family*) and e_i/h (location of the *family* of fibres in the cross section). Figure 3 presents the diagram of the coefficient ζ as a function of the beam slenderness l/h and of the ratio E^r/E with $\nu = 0{,}30$; $\mu^r = 0{,}02$; $i = 2$, $e_1 = 0{,}45h$ and $e_2 = 0{,}35h$.

The relative errors ε_p and ε_b resulting from the neglecting the influence of shear deformations and rotatory inertia with relation to the natural frequency (13) of the slender composite beam are as follows, if we take into account (15) and (16)

$$\varepsilon_p = \frac{|\omega_n - \omega_{np}|}{\omega_n}\cdot 100\% = \frac{1}{2}n^2\pi^2\zeta\cdot 100\% \tag{19}$$

$$\varepsilon_b = \frac{|\omega_n - \omega_{nb}|}{\omega_n}\cdot 100\% = \frac{n^2\pi^2 J}{2l^2 A}\cdot 100\% \tag{20}$$

The relation

$$\frac{\varepsilon_p}{\varepsilon_b} = \frac{D}{G'kJ} = \frac{E}{G'k}\left(1 + 24\sum_i n^r\mu^r\frac{e_i^2}{h^2}\right) \tag{21}$$

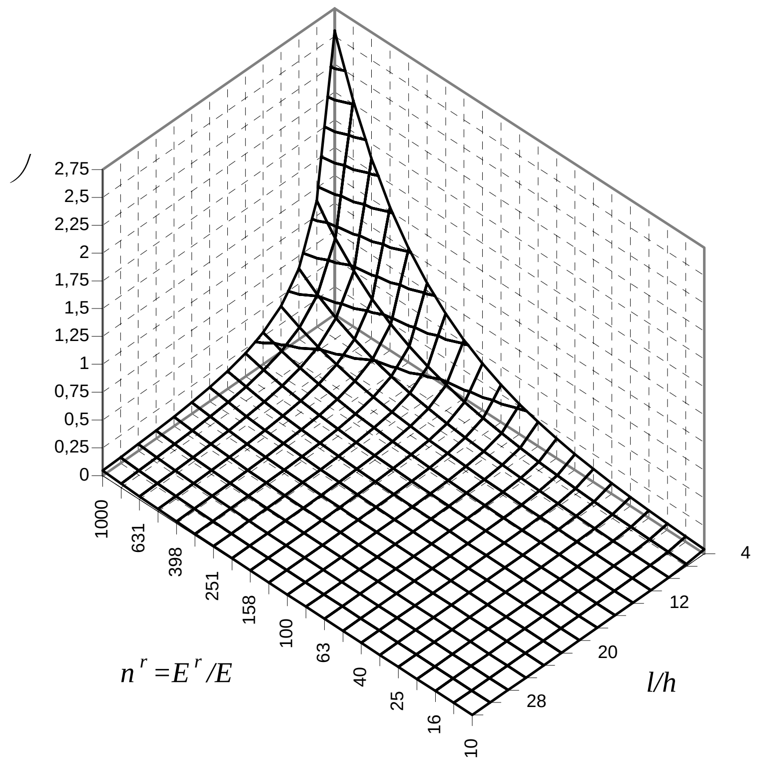

Fig. 3 The coefficient ζ as a function of the beam slenderness l/h and of the ratio of Young's moduli E^r/E.

states how much the influence of the shear deformation is greater than the influence of the rotatory inertia. Taking for example $E/G' = 2{,}6$; $i = 2$ (two pairs of identical fibre *families* in the cross section), $n^r = 20$; $\mu^r = 0{,}02$ (4% of reinforcement), $e_1 = 0{,}45h$; $e_2 = 0{,}35h$ we obtain $\varepsilon_p/\varepsilon_b = 12{,}85$. This leads to the conclusion that *for the composite beams with reinforcement by layers of long fibres the influence of shear deformation on the natural frequencies is at least one order of magnitude grater than the influence of rotatory inertia.*

Taking into account the above conclusion we will neglect the influence of the rotatory inertia of the cross section on the vibration of composite beams.

The relative error ε_p caused by neglecting the influence of shear deformation with the length of deformation wave $l/n = 10h$ and $5h$ (where h denotes the cross section height) is equal to 5,3% and 21,1% respectively (keeping remaining input values unchanged). So we can easily observe that the error is significant and increases in proportion to the coefficient ζ .

Thus, taking into account the influence of shear deformations only, gives the natural frequencies for simply supported composite beam in the form (15). The associated eigenmodes are expressed in the form

$$W_n(x) = A_n \sin \alpha_n x \quad ; \quad \Psi_n(x) = B_n \cos \alpha_n x \,. \tag{22}$$

4. HARMONICALLY FORCED VIBRATIONS

In the case of beam vibrations forced by transverse load $p(x,t) = p(x)e^{-i\omega t}$, neglecting the influence of axial loads and rotatory inertia, the system of equations (9) transforms into the system of uncoupled equations of motion

$$\begin{gathered} \frac{\partial^4 w}{\partial x^4} + \frac{\rho A}{D}\left(1 - \zeta\, l^2 \frac{\partial^2}{\partial x^2}\right)\ddot{w} = \frac{1}{D}\left(1 - \zeta\, l^2 \frac{\partial^2}{\partial x^2}\right)p(x,t), \\ \frac{\partial^4 \psi}{\partial x^4} + \frac{\rho A}{D}\left(1 - \zeta\, l^2 \frac{\partial^2}{\partial x^2}\right)\ddot{\psi} = -\frac{1}{D}\frac{\partial}{\partial x}\,p(x,t) \end{gathered} \tag{23}$$

As a result of the load to act harmonically the displacement $w(x,t)$ and the angle of rotation $\psi(x,t)$ varies harmonically

$$w(x,t) = W(x)\;\; e^{-i\omega t} \quad ; \quad \psi(x,t) = \Psi(x)\;\; e^{-i\omega t} \,. \tag{24}$$

Substituting (24) into (23) gives the following ordinary differential equations

$$\begin{gathered} \frac{d^4 W(x)}{dx^4} - \omega^2 \frac{\rho A}{D}\left(1 - \zeta\, l^2 \frac{d^2}{dx^2}\right)W(x) = \frac{1}{D}\left(1 - \zeta\, l^2 \frac{d^2}{dx^2}\right)p(x), \\ \frac{d^4 \Psi(x)}{dx^4} - \omega^2 \frac{\rho A}{D}\left(1 - \zeta\, l^2 \frac{d^2}{dx^2}\right)\Psi(x) = -\frac{1}{D}\frac{dp(x)}{dx} \end{gathered} \tag{25}$$

,

completed by the appropriate boundary conditions. For the simply supported beam we should use $W(0)=W(l)=0$ and $\frac{d\Psi(0)}{dx}=\frac{d\Psi(l)}{dx}=0$.

Taking

$$W(x)=\sum_{n=1}^{\infty} A_n \sin\alpha_n x \quad ; \quad \Psi(x)=\sum_{n=1}^{\infty} B_n \cos\alpha_n x \quad ;$$
$$p(x)=\sum_{n=1}^{\infty} p_n \sin\alpha_n x \tag{26}$$

and making use of Fourier transform [9] in order to solve the equations (25) eventually gives the following solution of the equations of motion (23)

$$w(x,t)=\frac{2}{l}\frac{e^{-i\omega t}}{\rho A}\sum_{n=1}^{\infty}\frac{\sin\alpha_n x}{\omega_n^2\left(1-\frac{\omega^2}{\omega_n^2}\right)}\int_0^l p(u)\sin\alpha_n u du \quad ,$$
$$\psi(x,t)=-\frac{2}{l}\frac{e^{-i\omega t}}{\rho A}\sum_{n=1}^{\infty}\frac{\alpha_n \cos\alpha_n x}{\left(1+n^2\pi^2\zeta\right)\omega_n^2\left(1-\frac{\omega^2}{\omega_n^2}\right)}\int_0^l p(u)\sin\alpha_n u du \,, \tag{27}$$

where ω denotes the frequency of excitation and ω_n denotes the natural frequency.

In the case of the load to be uniformly distributed along the beam $p(x,t)=pe^{-i\omega t}$ or for the concentrated load $p(x,t)=P\delta(x-\xi)e^{-i\omega t}$ acting in the section $x=\xi$ we obtain respectively

The solutions describing the harmonic motion problem for simply supported composite shearing-sensitive beam we obtained above can be used to evaluate the solutions of the slender reinforced beam problem. We just need to eliminate the shear deformation γ_{xz} substituting $G^{'}=\infty$ or $\zeta=0$ into equations (15), (17), (23), (25), (27), (28) and (29). If we assume additionally $A^r=0$ (elimination of the fibre phase) we will obtain appropriate solutions for the homogeneous beam [9].

$$w(x,t) = \frac{4pe^{-i\omega t}}{lD} \sum_{n=1,3,5,\ldots} \frac{\left(1 + n^2\pi^2\zeta\right)}{\alpha_n^5\left(1 - \frac{\omega^2}{\omega_n^2}\right)} \sin\alpha_n x\,,$$

$$\psi(x,t) = -\frac{4pe^{-i\omega t}}{lD} \sum_{n=1,3,5,\ldots} \frac{\cos\alpha_n x}{\alpha_n^4\left(1 - \frac{\omega^2}{\omega_n^2}\right)}\,, \tag{28}$$

and

$$w(x,t) = \frac{2Pe^{-i\omega t}}{lD} \sum_{n=1}^{\infty} \frac{\left(1 + n^2\pi^2\zeta\right)}{\alpha_n^4\left(1 - \frac{\omega^2}{\omega_n^2}\right)} \sin\alpha_n x \sin\alpha_n\xi\,,$$

$$\psi(x,t) = -\frac{2Pe^{-i\omega t}}{lD} \sum_{n=1}^{\infty} \frac{\cos\alpha_n x \sin\alpha_n\xi}{\alpha_n^3\left(1 - \frac{\omega^2}{\omega_n^2}\right)}\,, \tag{29}$$

The limit case when $\omega \to 0$ gives the static problem. Thus, considering the uniformly distributed load p or the concentrated load P acting in the mid-span of the beam we will obtain the following extreme values of displacement components using (28) and (29)

$$w(l/2) = \frac{5}{384}\frac{pl^4}{D}\left(1 + 9{,}6\zeta\right)\ ;\ \ \psi(0) = -\frac{pl^3}{24D} = -\psi(l), \tag{30}$$

$$w(l/2) = \frac{5}{384}\frac{pl^4}{D}\left(1 + 9{,}6\zeta\right)\ ;\ \ \psi(0) = -\frac{pl^3}{24D} = -\psi(l), \tag{31}$$

Taking additionally $\zeta = 0$ leads to the solution of the slender beam obeying Bernoulli hypothesis

5. PARAMETRIC STUDY

The aim of the analysis is to determine the influence of shear deformations on the value of deflections of the composite beam we deal with in this paper. As we mentioned before the girders made of fibrous composites are reinforced using fibres characterised by much better mechanical properties than the matrix properties. The fibres exhibit significant shear deformability. The use of Bernoulli hypothesis is suited for isotropic slender beams. Because of it a direct application of this hypothesis to solve the fibrous composite beam problem seems to be inappropriate and leads to significant errors.

The relative error connected with the omitting of shear deformations to be calculated for extreme deflections taking into account (30) and (31) becomes, in

$$\varepsilon = \frac{|w - w_B|}{|w_B|} \cdot 100\% \tag{32}$$

the case of uniformly distributed load

$$\varepsilon = 9{,}6\zeta \cdot 100\% \,. \tag{33}$$

For the concentrated load the relative error is 25% greater than the distributed

$$\varepsilon = 12\zeta \cdot 100\% \tag{34}$$

load error. In the equation (32) the symbol w_B denoting the deflection calculated in respect with the slender beams theory was used.

In order to demonstrate the influence of the beam slenderness changes l/h and of the ratio $n^r = E^r/E$ on the value of an error ε to be committed, let we analyse the following data $\nu = 0{,}30$, $\mu^r = 0{,}02$ (4% reinforcement), $i = 2$, $e_1 = 0{,}45h$, $e_2 = 0{,}35h$ assuming the distributed load problem.

The calculated values of the error ε are presented in Table 1 and visualised in Fig. 4.

Table 1.

ε %		l/h					
		25	20	15	10	8	4
$n^r = \frac{E^r}{E}$	10	1,02	1,60	2,84	6,39	9,98	39,9
	20	1,65	2,57	4,57	10,3	16,1	64,3
	50	3,52	5,49	9,77	22,0	34,3	137,3
	100	7,05	10,4	18,4	41,4	64,7	265,2
	200	12,88	20,1	35,7	80,4	125,6	502,3
	300	19,12	29,8	53,1	119,3	186,4	745,7

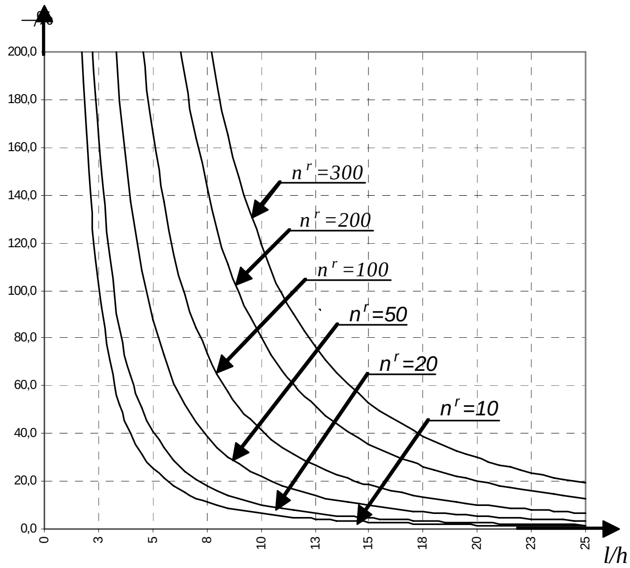

Fig. 4 The influence of the beam slenderness changes l/h and of the ratio $n^r = E^r/E$ on the value of the relative error ε caused by omitting transverse shear deformations effect.

6. CONCLUSIONS

The analytical complete results obtained in the paper as well as the analysis carried out show that considering the influence of the transverse shear deformations in the dynamic problem of fibrous composite beams reinforced by layers of long fibres strongly results in the natural frequencies and displacements to be calculated.

This influence mainly depends on the vulnerability parameter ζ which strongly depends on the parameters h^2/l^2, $n^r = E^r/E$, $\mu^r = j^r A^r/A$ (density of fibres' locations in the *r*-th *family*) and e_i/h (location of the *family* of fibres in the cross section) and on the way the load is distributed.

The influence of shear deformations on the behaviour of a homogenous beam (without reinforcement) with the ratio $l/h \geq 10$ is negligible. An important fact we presented in the paper is that for the composite beam possessing the same slenderness ratio this influence is significant and may reach the values grater than 100% (see Table 1).

However the influence of the rotatory inertia on the eigenvalues of composite beams is over ten times less than the influence of shear deformations. Thus it may be neglected.

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